Probability & Random Variables

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Notebook

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* Probability *

SUMMARY

※ CHAPTER (1):

* number of subsets in set $= 2^N$ → number of elements.

- $\bar{A} = S - A$
- $A = S - \bar{A}$
- $A \cap \bar{A} = \phi$
- $A \cup \bar{A} = S$
- $\bar{S} = \phi$
- $\bar{\phi} = S$

* De Morgan's Law: $\boxed{\overline{A \cup B} = \bar{A} \cap \bar{B}}$ & $\boxed{\overline{A \cap B} = \bar{A} \cup \bar{B}}$

* Axioms: ① $\boxed{\underbrace{0}_{P(\phi)} \leq P(A) \leq \underbrace{1}_{P(S)}}$ ② for N disjoint events: $\boxed{P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} P(A_i)}$

* Joint Probability:
$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$ ⟹ if A & B disjoint: $\boxed{P(A \cup B) = P(A) + P(B)}$

* Conditional Probability:
$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$ ⟹ if A & B disjoint: $P(A/B) = \text{Zero}$.

* Independent Events:
- $P(A/B) = P(A)$
- $P(B/A) = P(B)$

$\boxed{P(A \cap B) = P(A) \cdot P(B)}$

* Total Probability: $\boxed{P(A) = \sum_{i=1}^{N} P(A/B_i)\, P(B_i)}$
↳ for N disjoint events. $B_1, B_2, \dots, B_N$

* Bayes Rule:
$\boxed{P(B_i/A) = \frac{P(A/B_i)\, P(B_i)}{P(A)}}$

* Combined Expirements:
for two-sub-exp. S_1 & S_2: $\boxed{S = S_1 \overset{\text{combined.}}{\times} S_2}$
↳ * for probability: events $\underbrace{A}_{\subseteq S_1}$ & $\underbrace{B}_{\subseteq S_2}$

$\boxed{P(A \times B) = P(A) \cdot P(B)}$

* Permutations: $\boxed{P_r^n = \frac{n!}{(n-r)!}}$ * Combinations: $\boxed{C_r^n = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}}$

* Bernouli Trial: $\boxed{P(K) = \binom{N}{K} p^K (1-p)^{N-K}}$ for N Bernouli Trials.

※ CHAPTER (2):

* CDF:

$$\boxed{F_X(x) = P(X \leq x)}$$

$$\boxed{F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)}$$

* Proparties:

- $F_X(-\infty) = 0$
- $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$
- $F_X(\infty) = 1$
- $0 \leq F_X(x) \leq 1$
- $F_X(x^+) = F_X(x)$
- for $x_1 \leq x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$

* for Continuous R.V $P(X = x_1) = 0$

* PDF:

$$\boxed{f_X(x) = \frac{d F_X(x)}{dx}}$$

$$\boxed{f_X(x) = \sum_{i=1}^{N} P(x_i)\, \delta(x - x_i)}$$

* Proparties:

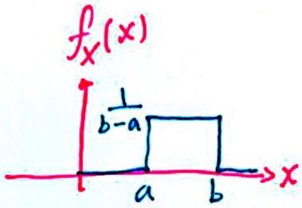
- $f_X(x) \geq 0$
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
- $F_X(x) = \int_{-\infty}^{x} f_X(z)\, dz$
- $P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(z)\, dz$

* Important R.V:

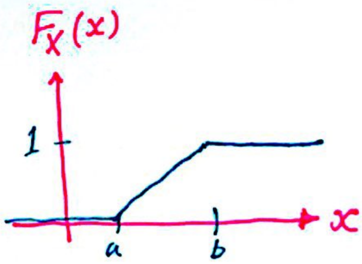
(1) Uniform R.V:

$X \sim U(a,b)$

- PDF: $f_X(x) = \begin{cases} \frac{1}{b-a} & , a \leq x \leq b \\ 0 & , o.w \end{cases}$



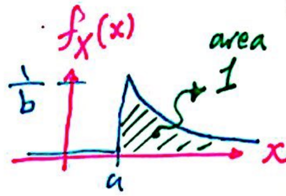
- CDF: $F_X(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x < b \\ 1 & , b \leq x \end{cases}$



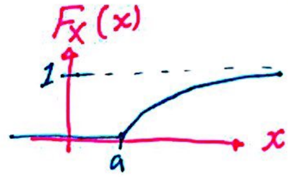
② Exponential R.V :

$X \sim \exp(a,b)$

• PDF: $f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & , x \geq a \\ 0 & , x < a \end{cases}$



• CDF: $F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & , x \geq a \\ 0 & , x < a \end{cases}$



③ Gaussian R.V :

general:

$X \sim N(a_x, \sigma_x^2) \xrightarrow{} $ special case: $X \sim N(0,1)$

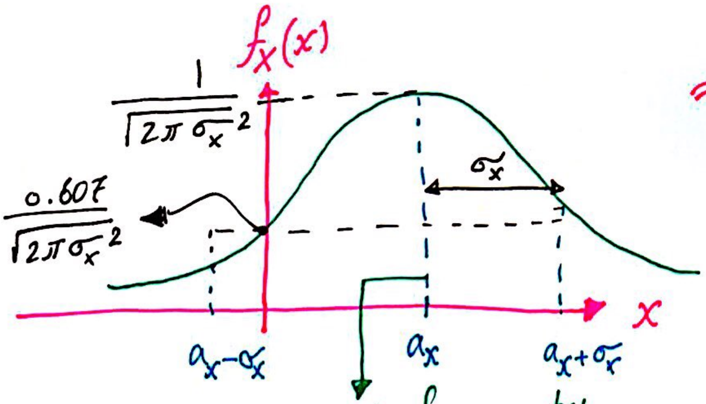
$F_X(x)$ $\qquad$ $F(x)$

* for the special case: • PDF: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

• CDF: $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx'$ ⇒ found by Table.

* for the general form: • PDF: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$

• CDF: $F_X(x) = F\left(\frac{x-a_x}{\sigma_x}\right)$ ⇒ by Table



① $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx = 1$

② $\int_{a_x}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx = \int_{-\infty}^{a_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx$

$= \frac{1}{2}$

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☺ First Material ☺

④ Bernouli R.V:

$X \in \{0, 1\}$

• PDF: $f_X(x) = (1-p)\,\delta(x) + p\,\delta(x-1)$

• CDF: $F_X(x) = (1-p)\,u(x) + p\,u(x-1)$

⑤ Binomial R.V:

$X \in \{1, 2, \dots, N\}$

• PDF: $f_X(x) = \sum_{i=1}^{N+1} \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i} \delta(x - x_i)$

• CDF: $F_X(x) = \sum_{i=1}^{N+1} \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i} u(x - x_i)$

* Conditional Density & Distribution Functions:

• For $F_X(x) = P\{X \le x\}$: $F_X(x/B) = P\{X \le x / B\} = \frac{P\{X \le x \cap B\}}{P\{B\}}$

$f_X(x|B) = \frac{d F_X(x/B)}{dx}$

• For $F_X(x) = P\{X \le x\}$ with $B = \{X \le b\}$:

$$F_X(x|B) = \begin{cases} \frac{F_X(x)}{F_X(b)} & , b > x \\ 1 & , b < x \end{cases}$$

⇒ depend on the condition $B = \{X \le b\}$ if it is changed $F_X(x|B)$ will change.

※ CHAPTER(3):

* Expectation in general:

$$\boxed{\begin{array}{l} \bullet \text{ Discrete: } \sum_{i=1}^{N} x_i \, P\{X = x_i\} = \overline{X} \\ \bullet \text{ Continuous: } \overline{X} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \end{array}}$$

	Uniform.	Exponential.	Gaussian.	Bernouli.	Binomial.
$\overline{X}$	$\frac{a+b}{2}$	$a+b$	a_x	p	Np

* Epectation of a function of R.V:

- Discrete: $E[g(X)] = \sum_{i=1}^{N} g(x_i) P\{X = x_i\}$
- Continuous: $E[g(X)] = \int_{-\infty}^{\infty} g(x_i) f_X(x)\, dx$

* If $E[X] = \bar{X}$, Then for $E[aX+b]$: $a\bar{X}+b$

* Moments about the Origin: $\boxed{m_n = E[X^n]}$

- $m_0 = 1$
- $m_1 = \bar{X}$
- $m_2 = E[X^2]$
  - Total Averagepower

$\boxed{Var(X) = \sigma_X^2 = m_2 - m_1^2}$

* For $X \sim U(a,b)$:

- $\bar{X}$: $\bar{X} = \frac{a+b}{2}$
- Var: $Var(X) = \frac{(b-a)^2}{12}$

* For $X \sim exp(a,b$:

- $\bar{X}$: $\bar{X} = a+b$
- Var: $Var(X) = b^2$

⇒ For any distribution you just need to Know: $\bar{X}$, Var
From them you find m_2 by using: $m_2 = Var + \bar{X}^2$

* Moments about the mean: $\boxed{\mu_n = E[(X-\bar{X})^n]}$

- $\mu_0 = 1$
- $\mu_1 = 0$
- $\mu_2 = E[(X-\bar{X})^2] = \sigma_X^2 = Var(X)$

$\boxed{Var(X) = \mu_2 = m_2 - m_1^2}$

* For $X \sim N(a_X, \sigma_X^2)$:

- $\bar{X} = a_X$
- $Var(X) = \sigma_X^2$

$\Rightarrow E[X^2] = \sigma_X^2 + a_X^2$

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* Characteristic Function $\phi_X(\omega)$:

$$\boxed{\phi_X(\omega) = E[e^{j\omega X}]}$$

$$\boxed{\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx}$$

$$\boxed{f_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x} \phi_X(\omega)\,d\omega}$$

$$\boxed{m_n = (-j)^n \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right]_{@\omega=0}}$$

* Moments Generating Function:

$$\boxed{M_X(v) = E[e^{vX}] = \int_{-\infty}^{\infty} e^{vx} f_X(x)\,dx} \quad , \quad \boxed{m_n = \left.\frac{d^n M_X(v)}{dv^n}\right]_{@v=0}}$$

* Transformation:

- Monotonic
  - Increasing

$$\boxed{\begin{aligned} F_Y(y) &= F_X(T^{-1}(y)) \\ f_Y(y) &= f_X(T^{-1}(y)) \cdot \frac{dT^{-1}(y)}{dy} \end{aligned}}$$

  - Decreasing

$$\boxed{\begin{aligned} F_Y(y) &= 1 - F_X(T^{-1}(y)) \\ f_Y(y) &= - f_X(T^{-1}(y)) \cdot \frac{dT^{-1}(y)}{dy} \end{aligned}}$$

  - * In general: $\boxed{f_Y(y) = f_X(T^{-1}(y)) \left|\frac{dT^{-1}(y)}{dy}\right|}$

- Non-monotonic

$$\boxed{f_Y(y) = \sum_n \frac{f_X(x_n)}{\left| \left.\frac{dT(x)}{dx}\right]_{x=x_n} \right|}}$$

* For $X \sim N(a_x, \sigma_x^2)$, $Y = T(X) = ax+b$

To find $f_Y(y)$: find the new mean a_y & the new variance σ_y^2

Then use the main relation: $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(x-a_y)^2}{2\sigma_y^2}}$

※ CHAPTER (4):

* $$\boxed{F_{X,Y}(x,y) = P\{X \leq x, Y \leq y\}}$$

* **Proparties:**

- $F_{X,Y}(-\infty,-\infty) = 0$
- $F_{X,Y}(\infty,\infty) = 1$
- $0 \leq F_{X,Y}(x,y) \leq 1$
- $F_{X,Y}(-\infty,y) = 0$
- $F_{X,Y}(\infty,y) = F_Y(y)$
- $F_{X,Y}(x,y)$ Non-decreasing
- $F_{X,Y}(x,-\infty) = 0$
- $F_{X,Y}(x,\infty) = F_X(x)$
- $P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = F_{X,Y}(x_2,y_2) + F_{X,Y}(x_1,y_1) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)$

* $$\boxed{f_{X,Y}(x,y) = \frac{d^2 F_{X,Y}(x,y)}{dx\,dy}}$$

* **proparties:**

- $f_{X,Y} \geq 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$
- $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x',y')\,dx'\,dy'$
- $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,y)\,dy\,d\xi_1$
- $F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(x,\xi_2)\,dx\,d\xi_2$
- $P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy$
- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$
- $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$

* **Note:** $F_{X,Y}(x,y)$ called : Joint Distribution Function.
$f_{X,Y}(x,y)$ called: Joint Density Function.
$F_X(x)$, $F_Y(y)$ called : Marginal Distribution of X, Y.
$f_X(x)$, $f_Y(y)$ called: Marginal Density of X, Y.

* If X & Y are Independent, Then:

$$\boxed{\begin{aligned} F_{X,Y}(x,y) &= F_X(x)\cdot F_Y(y) \\ f_{X,Y}(x,y) &= f_X(x)\cdot f_Y(y) \end{aligned}}$$

* In General for $W = X+Y$: $F_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_{X,Y}(x,y)\,dx\,dy$

* In Case X & Y indep. : $f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy = f_Y(y) \star f_X(x)$

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* **Central Limit Theorem:** $W \sim N(a_\omega, \sigma_\omega^2)$

for $W = X_1 + \cdots + X_N$

$$a_\omega = \bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_N$$
$$\sigma_\omega^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$$

$$\Rightarrow \boxed{f_W(\omega) = \frac{1}{\sqrt{2\pi\sigma_\omega^2}} e^{-\frac{(x-a_\omega)^2}{2\sigma_\omega^2}}}$$

※ **CHAPTER (5):**

** $\boxed{E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy}$

* if $g(x,y) = g(x)$:

$$E[g(x,y)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

* For $X = \alpha_1 X_1 + \cdots + \alpha_N X_N \Rightarrow \boxed{\bar{X} = \alpha_1 \bar{X}_1 + \cdots + \alpha_N \bar{X}_N}$

* For $X = \alpha_1 g_1(X_1) + \cdots + \alpha_N g_N(X_N) \Rightarrow \boxed{\bar{X} = \alpha_1 \overline{g_1(X_1)} + \cdots + \alpha_N \overline{g_N(X_N)}}$

* **Joint Moments:**

$$\boxed{m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\,dx\,dy}$$

- 0^{th} order: $\boxed{m_{00} = 1}$
- 1^{st} order: $\boxed{\begin{array}{l} m_{10} = E[X] \\ m_{01} = E[Y] \end{array}}$ → Center of gravity. $(\bar{x}, \bar{y})$
- 2^{nd} order: $\boxed{\begin{array}{l} m_{20} = E[X^2] \\ m_{02} = E[Y^2] \\ m_{11} = E[XY] = R_{XY} \end{array}}$ → Correlation

* **Joint Central Moments:**

$$\boxed{\mu_{nk} = E[(x-\bar{x})^n (y-\bar{y})^k]}$$

- 0^{th} order: $\boxed{\mu_{00} = 1}$
- 1^{st} order: $\boxed{\begin{array}{l} \mu_{10} = 0 \\ \mu_{01} = 0 \end{array}}$
- 2^{nd} order: $\boxed{\begin{array}{l} \mu_{20} = E[(x-\bar{x})^2] = \sigma_X^2 \\ \mu_{02} = E[(y-\bar{y})^2] = \sigma_Y^2 \\ \mu_{11} = E[(x-\bar{x})(y-\bar{y})] = C_{XY} \end{array}}$ → Covariance.

$$C_{XY} = R_{XY} - \bar{X}\bar{Y}$$

- If X & Y are Orthogonal $\Rightarrow$ $\underline{R_{XY} = 0}$ & $\underline{C_{XY} = -\bar{X}\bar{Y}}$
- If X&Y are Indep. $\Rightarrow$ Then they are UnCorrelated $\Rightarrow$ $\underline{R_{XY} = \bar{X}\bar{Y}}$ & $\underline{C_{XY} = 0}$
- $\underline{C_{XX} = \sigma_X^2}$ & $\underline{C_{YY} = \sigma_Y^2}$

* Correlation Coefficient: $\boxed{\rho_{xy} = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{xy}}{\sigma_x \sigma_y}}$

- If x & y are indep. $\Rightarrow$ uncorrelated $\Rightarrow$ $C_{xy}=0 \Rightarrow \rho_{xy}=0$
- for $C_{xx}=\sigma_x^2$ or $C_{yy}=\sigma_y^2$ when $X=Y$: $\rho_{xy}=1$

$-1 \leq \rho_{xy} \leq 1$

$0 \leq |\rho_{xy}| \leq 1 \longrightarrow$ Highly Correlated.

$0 \longrightarrow$ Uncorrelated.

* Variance for Multiple R.V's:

for: $X = \sum_{i=1}^{N} \alpha_i X_i \Rightarrow \boxed{\sigma_x^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j C_{x_i x_j}}$

$i \neq j$

*** if the R.V's Uncorrelated:

$C_{x_i x_j} = 0 \Rightarrow \boxed{\sigma_x^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2}$

* Joint Gaussian R.V's:

* for 2-RV's X_1 & X_2: $f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi \sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_{x_1}^2} - \frac{2\rho(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{x_2}^2}\right]}$

* for N R.V's: $f_{X_1,\dots,X_N}(x_1,\dots,x_N) = \frac{\left|[C_x]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} \cdot e^{-\frac{[x-\bar{x}]^T [C_x]^{-1} [x-\bar{x}]}{2}}$

$\Rightarrow$ They will be given in the Exam.

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Second Material

* Transformation of Multiple R.V's:

- We find at first $X_1 = V_1(y_1, \dots, y_N)$
  $\vdots$
  $X_N = V_N(y_1, \dots, y_N)$

- Then we find J : $J = \begin{vmatrix} \frac{dV_1}{dy_1} & \frac{dV_1}{dy_2} & \cdots & \frac{dV_1}{dy_N} \\ \vdots & & & \\ \frac{dV_N}{dy_1} & \cdots & & \frac{dV_N}{dy_N} \end{vmatrix}$ → Determinent.

- Then find $f_{Y_1,\dots,Y_N}(y_1,\dots,y_N)$ using:

$$f_{Y_1,\dots,Y_N}(y_1,\dots,y_N) = f_{X_1,\dots,X_N}(V_1,\dots,V_N) \cdot |J|$$

→ Absolute.

※ CHAPTER (6):

* R.P Mean : $\boxed{m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x;t)\,dx}$

* DC value for X(t) : $\boxed{DC\text{ value} = A\left[E[X(t)]\right]}$

A → Time Average.

- Note:

$$A[\text{any signal}] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\text{any signal}]\,dt.$$

* Auto-Correlation : $\boxed{R_{XX}(t,t+\tau) = E[X(t)\,X(t+\tau)]}$; when $\tau = 0$ $R_{XX}(t,t) = E[X^2(t)]$

* R.P Variance : $\boxed{\sigma_X^2(t) = E[X^2(t)] - m_X^2(t)}$

* R.P Auto Covariance : $\boxed{C_{XX}(t,t+\tau) = R_{XX}(t,t+\tau) - m_X(t)\,m_X(t+\tau)}$

; when $\tau = 0$ $C_{XX}(t,t) = \sigma_X^2(t)$

* Stationary:

- First order :

$f_X(x,t_i) = f_X(x,t_j)$

$\Rightarrow m_X(t_i) = m_X(t_j) = \overline{X}$ → Constant Mean.

$\overline{X}$: represent here the DC value $= E[X(t)]$.

$\Rightarrow$ To find DC value:
- ① $m_X(t) = E[X(t)]$.
- ② $A[m_X(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} m_X(t)\,dt$ = DC value. $\to = \bar{X}$ if stationary.

• Second Order: $\tau = t_2 - t_1$

$$\boxed{\begin{aligned} f_X(x_1,x_2;t_1,t_2) &= f_X(x_1,x_2;t_1+\Delta,t_2+\Delta) \\ R_{XX}(t,t+\tau) &= R_{XX}(\tau) \end{aligned}}$$

* Wide Sense Stationary (WSS) R.P:

R.P $X(t)$ is WSS if:

① $E[X(t)] = m_X(t) = \bar{X}$

② $R_{XX}(t,t+\tau) = R_{XX}(\tau)$

$\to \sigma_X^2(t) = \sigma_X^2$

$\Rightarrow$ Constant Mean & Constan Variance.

*Note: $\bar{X}$: DC value. $\bar{X}^2$: DC power. σ_X^2: AC power.

* Proparties of $R_{XX}(\tau)$:

① $|R_{XX}(\tau)| \leq R_{XX}(0)$

② $R_{XX}(-\tau) = R_{XX}(\tau)$

③ $R_{XX}(0) = E[X^2(t)]$ $\to$ Total average Power.

④ $\lim_{\tau\to\infty} R_{XX}(\tau) = \bar{X}^2 \to$ DC average power.

*Auto-Covariance for WSS: $\boxed{C_{XX}(t,t+\tau) = C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2}$

* Cross-Correlation function:

$$\boxed{R_{XY}(t,t+\tau) = E[X(t)\,Y(t+\tau)]}$$

• for $R_{XY}(t,t+\tau) = 0 \Rightarrow X(t) \perp Y(t)$

• for $R_{XY}(t,t+\tau) = E[X(t)]\cdot E[Y(t+\tau)]$

$\to$ X & Y are statistically indep.

* Cross-Covariance function:

$$\boxed{C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)\,m_Y(t+\tau)}$$

* $X(t)$ & $Y(t)$ are joint WSS If:

① $X(t)$ is WSS.

② $Y(t)$ is WSS.

③ $R_{XY}(t,t+\tau) = R_{XY}(\tau)$

$\to C_{XY}(t,t+\tau) = C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$

* Gaussian R.P:

$$f_X(x_1,\dots,x_N) = \frac{|[C_X]^{-1}|^{1/2}}{(2\pi)^{N/2}}\, e^{-\frac{1}{2}[x-\bar{x}]^T [C_X]^{-1} [x-\bar{x}]}$$

$C_{ij} = C_{XX}(t_i,t_j) = R_{XX}(t_i,t_j) - m_X(t_i)\,m_X(t_j)$ $\to$ if WSS: $C_{ij} = C_{XX}(t_j - t_i) = R_{XX}(t_j - t_i) - \bar{X}^2$

*Estimation of $m_X(t)$ & $R_{XX}(t, t+\tau)$:

$$\boxed{\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)} \qquad \boxed{\hat{R}_{XX}(t, t+\tau) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)\, x_i(t+\tau)}$$

* Ergodicity:

WSS R.P is said to be ergodic if:

① $\bar{X} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$ ② $R_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\,dt$

* Measuring $R_{XX}(\tau)$:

- $x(t)$ given.
- we find $R_o(2T)$ by: $R_o(2T) = \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\,dt$
- Then we have: $R_{XX}(\tau) \simeq R_o(2T)$ → Contain the exact value ± Error value.

※ CHAPTER (7):

$g(t) \xleftrightarrow{FT}$ $\boxed{G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\,dt.}$

	Time-Domain	Frequancy-Domain
* Average Power in $X_T^i(t)$:	$P_T^i = \frac{1}{2T}\int_{-T}^{T} x_T^{i\,2}(t)\,dt = \frac{1}{2T}\int_{-T}^{T} x^{i\,2}(t)\,dt$	$P_T^i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\lvert X_T^i(\omega)\rvert^2}{2T}\,d\omega$
* Average Power in $x^i(t)$:	$P^i = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^{i\,2}(t)\,dt$	$P^i = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\lvert X_T^i(\omega)\rvert^2}{2T}\,d\omega$
* Average Power of $X(t)$:	$P_{XX} = E\{P^i\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\,dt$	$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[\lvert X_T(\omega)\rvert^2]}{2T}\,d\omega$

*Power-Spectrum-Density:

$$\boxed{\rho_{XX}(\omega) = \lim_{T\to\infty} \frac{E[\lvert X_T(\omega)\rvert^2]}{2T}} \Rightarrow P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{XX}(\omega)\,d\omega \quad \text{OR} \quad P_{XX} = A[E[X^2(t)]]$$

If WSS: $\underline{P_{XX} = E[X^2(t)]}$

Note: $$\delta(\alpha) = \lim_{T\to\infty} \frac{T}{\pi}\left[\frac{\sin(\alpha T)}{(\alpha T)}\right]^2$$

* $\rho_{XX}(\omega)$ proparties:

① $\rho_{XX}(\omega) \geq 0$ ② $\rho_{XX}(-\omega) = \rho_{XX}(\omega)$ ③ $\rho_{XX}(\omega)$ is Real. ④ $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{XX}(\omega)\,d\omega$

⑤ $\rho_{\dot{X}\dot{X}}(\omega) = \omega^2 \rho_{XX}(\omega)$ ⑥ $\rho_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]\, e^{-j\omega\tau}\, d\tau$

⑦ $A[R_{XX}(t,t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{XX}(\omega)\, e^{+j\omega\tau}\, d\tau$

*** special case (if ⑥ & ⑦ $X(t)$ is WSS):

- $\rho_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$
- $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{XX}(\omega)\, e^{+j\omega\tau}\, d\omega$

* Bandwidth:

$$\boxed{\rho_{XX}^{norm}(\omega) = \frac{\rho_{XX}(\omega)}{\int_{-\infty}^{\infty}\rho_{XX}(\omega)\,d\omega}}$$

- for Base band:

$$\boxed{W_{rms}^{bb} = \sqrt{\frac{\int_{-\infty}^{\infty}\omega^2 \rho_{XX}(\omega)\,d\omega}{\int_{-\infty}^{\infty}\rho_{XX}(\omega)\,d(\omega)}}}$$

- for Bandpass:

$$W_{rms}^{bp} = 2\sqrt{\frac{\int_{-\infty}^{\infty}(\omega-\bar{\omega}_0)^2 \rho_{XX}(\omega)\,d\omega}{\int_{-\infty}^{\infty}\rho_{XX}(\omega)\,d\omega}}$$

* for R.P: $W(t) = X(t) + Y(t)$:

- $R_{WW}(t,t+\tau) = R_{XX}(t,t+\tau) + R_{YY}(t,t+\tau) + R_{XY}(t,t+\tau) + R_{YX}(t,t+\tau)$
- $\rho_{WW}(\omega) = \rho_{XX}(\omega) + \rho_{YY}(\omega) + \rho_{XY}(\omega) + \rho_{YX}(\omega)$
- $P_{WW} = P_{XX} + P_{YY} + P_{XY} + P_{YX}$

* for $W(t) = X(t) + Y(t)$ if $X(t)$ & $Y(t)$ are orthogonal:

$$\boxed{\begin{aligned} &\bullet\ R_{WW}(t, t+\tau) = R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau) \\ &\bullet\ \rho_{WW}(\omega) = \rho_{XX}(\omega) + \rho_{YY}(\omega) \\ &\bullet\ P_{WW} = P_{XX} + P_{YY} \end{aligned}}$$

** $\rho^{*}_{XY} = \rho_{YX}$

* White Noise: $\rho_{NN}(\omega) = \frac{N_o}{2}$, $R_{NN}(\tau) = FT^{-1}\{\frac{N_o}{2}\} = \frac{N_o}{2}\delta(\tau)$

$P_{NN} = \infty$

$N(t)$ Always WSS.

* for a LPF:

$r(t) = X(t) + N(t)$

$N^{i}(t) \longrightarrow \boxed{LPF} \longrightarrow N^{o}(t)$

$N^{o}(t) = N^{i}(t) * h(t)$

$$\boxed{\rho^{th}_{N^oN^o}(\omega) = \rho^{th}_{N^iN^i}(\omega)\ |H(f)|^2}$$

* * *

☺ Full Material. ☺

* Good Luck *