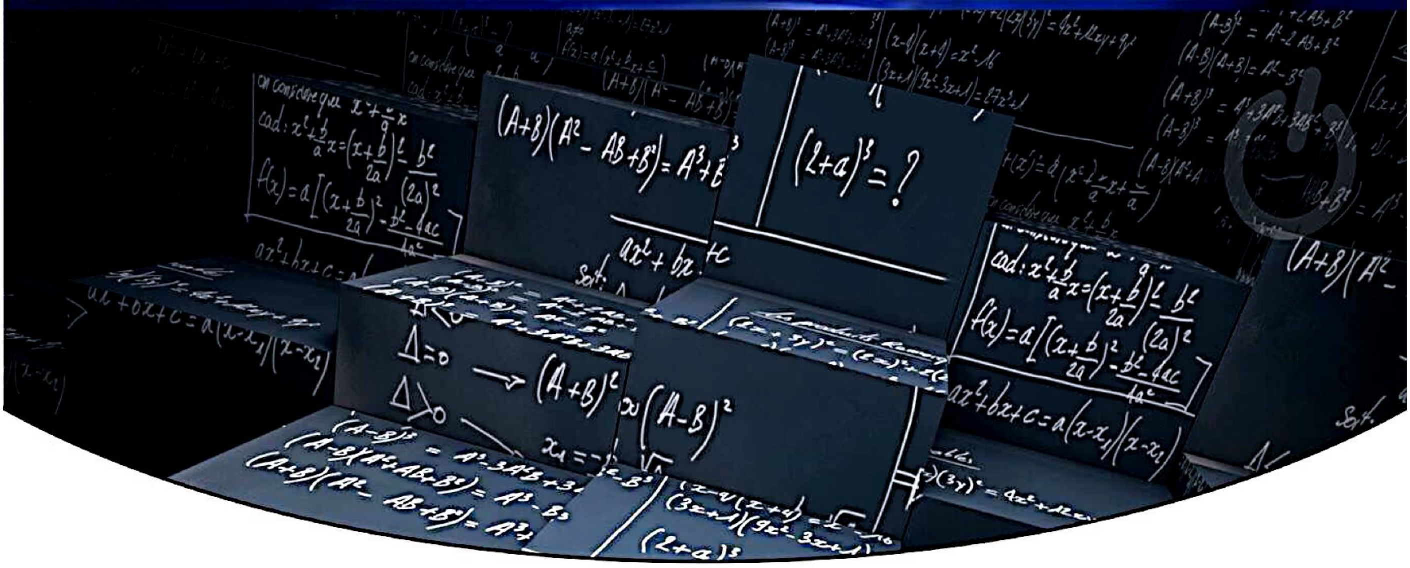

Probability

Summer017

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Probability & Random Variables

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Notebook

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* * *

* CHAPTER(1): Probability:

* Set Definitions:

Def: a set is a collection of objects (these objects are called elements).

for example: set $A = \{a, b, c, d, e\}$

given upper case $A, B, C, \ldots$

given lower case.

$a \in A$: a is an element in set A.

$a \notin A$: a is NOT element in set A.

* Two methods to specify sets:

→ Tabular Method.

→ Rule Method.

1) Tabular Method:

for e.g: $A = \{2, 4, 6, 8\}$ $\qquad B = \{-1 < a < 3\}$

2) Rule Method:

for e.g: A = { set of even Numbers between 2 & 8 }.

B = { All real Numbers between -1 & 3; -1 & 3 Not included. }

* Sets could be classified as:

→ Countable.

→ Uncountable.

1) Countable Set:

for e.g: $A = \{1, 3, 5, 7\}$ $\qquad B = \{2, 4, 6, 8, 10, 12, \ldots\}$

1 2 3 4 $\qquad$ 1 2 3 4 5 6, ...

2) Uncountable Set:

for e.g: $C = \{0.5 < b < 10\}$

*** Another Classification for set:**
→ finite.
→ infinite.

1) Finite Set:
- has finite Number of elements. ⇒ $A = \{1, 5, 7\}$.

2) Infinite Set:
⇒ $B = \{1, 2, 3, \dots\}$ $C = \{1 < a < 10\}$.

*** Empty Set:** $\emptyset$, $\{\}$.
⇒ the set with NO elements.

*** Universal Set:** S
⇒ the set that contains all other sets (in certain situation).
→ $A \subseteq B$: A is contained in B.
or A is Subset in B.

__Example:__ $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$A = \{1, 3, 5\}$ ⇒ $A \subseteq B$
$C = \{2, 4\}$ ⇒ $C \subseteq B$
$D = \{1, 7, 8, 9\}$ ⇒ $D \not\subseteq B$

*** __Rule__:** If set A has N elements, then there are 2^N subsets of A.

e.g: $A = \{a, b, c\}$
__N=3__ → $2^3 = 8$ subsets.
$\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \emptyset$

__Note:__ $\{a,b\}$ same as $\{b,a\}$

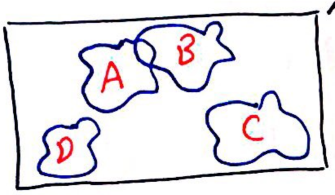
⇒ in this e.g we can consider A as S.

* Sets A & B are said to be disjoint or mutually exclusive
$\Rightarrow$ if they have NO common elements.

from the last e.g: $\{a, b\}$ & $\{c\}$ are disjoint.

※ Set Operations:

Venn Diagram.



for. eg: D & C are disjoint.

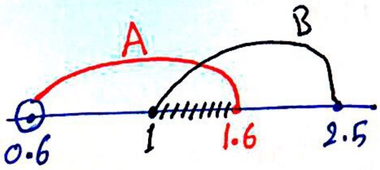
1] Set Equality:

$A = B$; when $A \subseteq B$ & $B \subseteq A$.

2] The difference between two sets:

$A - B$: All elements in A but NOT in B.

Example: $A = \{0.6 < a \leq 1.6\}$, $B = \{1 \leq b \leq 2.5\}$

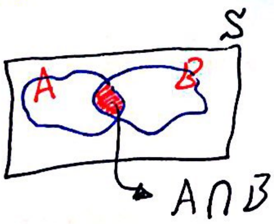


remove the shaded Area. from A:

$A - B = \{0.6 < c < 1\}$

3] Intersection:

$A \cap B$: the set of the common elements between A & B.



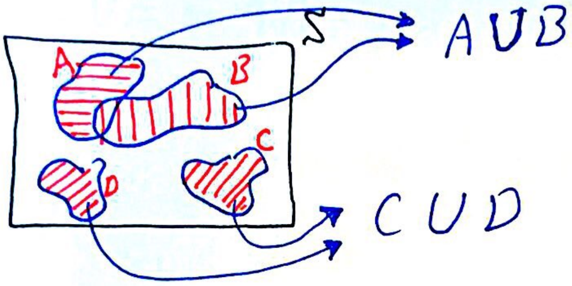
$A \cap B \rightarrow$ And , together.

from the previous example:

$A \cap B = \{1 \leq d \leq 1.6\}$

4] Union:

$A \cup B$: all elements in A & B.



e.g: $A = \{1, 2, 4, 6, 8\}$
$B = \{2, 5, 6, 9, 10\}$

$A \cap B = \{2, 6\}$

$A \cup B = \{1, 2, 4, 6, 8, 5, 9, 10\}$

Don't take the frequant elements.

* For N sets:

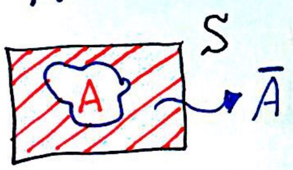
$A_1, A_2, \dots, A_N$

$$\bigcap_{i=1}^{N} A_i = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_N.$$

$$\bigcup_{i=1}^{N} A_i = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_N.$$

5] Complement:

$\bar{A}$: the set all elements NOT in A.



- $\bar{A} = S - A$
- $A = S - \bar{A}$
- $A \cap \bar{A} = \emptyset \Rightarrow A$ & $\bar{A}$ are disjoint.
- $A \cup \bar{A} = S$
- $\bar{S} = \emptyset$
- $\bar{\emptyset} = S$

* Algebra of sets:

(1) Commutative law:
- $A \cap B = B \cap A$
- $A \cup B = B \cup A$

(2) Distribution Law:
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(3) Associative Law:
- $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$
- $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$

※ De Morgan's Law:

1) $\boxed{\overline{A \cup B} = \bar{A} \cap \bar{B}}$

2) $\boxed{\overline{A \cap B} = \bar{A} \cup \bar{B}}$

※ Mathematical Model of Expirements:

{1} Sample Space:

S: The set of all possible outcomes of the expirements.

e.g: exp: Roll a die & record the appeared number:

Solution: all possible outcomes 1, 2, 3, 4, 5, 6

$\Rightarrow S = \{1, 2, 3, 4, 5, 6\}$

e.g: exp: flip a coin: $\Rightarrow S = \{H, T\}$

{2} Events:

Subsets of S.

e.g: exp: Roll a die.

Define event A "the appeared number is even".

Solution: exp $\Rightarrow S = \{1, 2, 3, 4, 5, 6\}$

event $\Rightarrow A = \{2, 4, 6\}$

{3} Assign Probabilities:

P(A): Probability of event A.

↳ Non-negative number.

* $0 \leq P(A) \leq 1$

※ Probability Axioms:

1) $0 \leq P(A) \leq 1$

$P(\phi)$ — $P(S)$

ϕ: impossible event.

S: Certain event.

2) For N disjoint events: $(A_i \cap A_j = \phi)$ $\forall\ i \neq j$

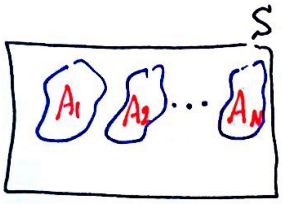
$A_1, A_2, A_3, \dots, A_N$

Then:

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} P(A_i)$$

$\Rightarrow$

$$\Rightarrow P\left(\bigcup_{i=1}^{N} A_i\right) = P(A_1 \cup A_2 \cup A_3 \cdots \cup A_N)$$
$$= P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_N)$$



$$\Rightarrow P\left(\bigcup_{i=1}^{N} A_i\right) = \frac{\text{Sum of Areas } (A_1 + A_2 + \cdots + A_N)}{\text{Total Area}}$$

Example: Roll two die and observe the appeared numbers.

a) Find the sample space?

b) Determine the events: $A = \{sum = 7\}$, $B = \{8 < sum \leq 11\}$, $C = \{sum > 10\}$

c) Find $P(A)$, $P(B)$, $P(C)$?

d) Find $P(A \cup B)$, $P(B \cup C)$, $P(A \cap B)$, $P(B \cap C)$?

Solution:

a) $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \cdots\cdots (6,6)\}$

b) $A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$B = \{(3,6), (4,5), (5,4), (6,3), (5,5), (5,6), (6,5), (4,6), (6,4)\}$

$C = \{(6,6), (5,6), (6,5)\}$

c) $P(A) = \frac{6}{36} = \boxed{\frac{1}{6}}$ $\quad P(B) = \frac{9}{36} = \boxed{\frac{1}{4}}$ $\quad P(C) = \frac{3}{36} = \boxed{\frac{1}{12}}$

d) $P(A \cup B) = P(A) + P(B) = \frac{6}{36} + \frac{9}{36} = \boxed{\frac{15}{36}}$

Notice that A & B are disjoint, so $P(A \cup B) = P(A) + P(B)$.

$P(B \cup C) = \boxed{\frac{10}{36}}$ Notice that $P(B \cup C) \neq P(B) + P(C)$.

$P(A \cap B) = P(\phi) = \underline{\underline{Zero}}$

$P(B \cap C) = \boxed{\frac{2}{36}}$

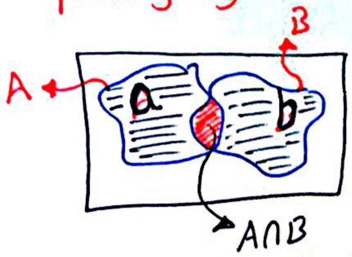
※ Joint Probability:

The joint probability between two events A & B is donated by "$P(A\cap B)$" → probability of the occurrence of A & B together.

$$\boxed{P(A\cap B) = P(A) + P(B) - P(A\cup B)}$$

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$

proving by venn diagram:



$A = a \cup (A\cap B)$

$B = b \cup (A\cap B)$

$\Rightarrow P(A) + P(B) - P(A\cup B) = \cancel{P(a)} + P(A\cap B) + \cancel{P(b)} + \cancel{P(A\cap B)} - (\cancel{P(a)} + \cancel{P(A\cap B)} + \cancel{P(b)})$

$\Longrightarrow P(A) + P(B) - P(A\cup B) = P(A\cap B)$ #

***As a special case:**

if A & B are disjoint, Then: $\boxed{P(A\cup B) = P(A) + P(B)}$

$\Rightarrow P(A\cup B) \leq P(A) + P(B)$ → equal when $A\cap B = \emptyset$ (disjoint).

※ Conditional Probability:

$P(A/B)$: The probability of A given that B has occurred.

given that (condition on).

$$\boxed{P(A/B) = \frac{P(A\cap B)}{P(B)}}$$

$$\boxed{P(A\cap B) = P(A/B)\,P(B)}$$

* if $A\cap B = \emptyset \Longrightarrow P(A/B) = $ Zero.

Example: In a box there is 100 resistors with resistances & tolerances as:

Tolerance	22Ω	47Ω	100Ω	Total
5%	10	28	24	62
10%	14	16	8	38
Total Number	24	44	32	100

⟹

exp.: Draw out one resistor.

Define the events:

A : the resistor is 47Ω.
B : the resistor is with 5% Tolerance.
C : the resistor is 100Ω.

Find: a) $P(A), P(B), P(C)$
b) $P(A/B), P(A/C), P(B/C)$

Solution:

a) $P(A) = \frac{44}{100}$, $P(B) = \frac{62}{100}$, $P(C) = \frac{32}{100}$

b) $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{28/100}{62/100} = \boxed{\frac{28}{62}}$.

$P(A/C) = \frac{P(A\cap C)}{P(C)} = \frac{P(\emptyset)}{P(C)} = \boxed{\text{Zero}}$.

$P(B/C) = \frac{P(B\cap C)}{P(C)} = \frac{24/100}{32/100} = \boxed{\frac{24}{32}}$.

Example:

10Ω	22Ω	27Ω	47Ω
18	12	33	17

"80 resistors"

Exp#1: Draw out one resistor from the box.

P(the resistor is 10Ω) = 18/80
P(the resistor is 22Ω) = 12/80
P(the resistor is 27Ω) = 33/80
P(the resistor is 47Ω) = 17/80

Exp#2: Draw out two resistors without replacement.

$P(\underbrace{2^{nd} \text{ is } 10\Omega}_{A} \cap \underbrace{1^{st} \text{ is } 22\Omega}_{B}) \Rightarrow P(A\cap B) = P(A/B)\, P(B)$

$= \frac{18}{79} * \frac{12}{80} = \boxed{\frac{27}{790}}$

* with replacement: $P(A/B) = P(A) = \frac{18}{80}$

※ Independent Events:

The two events A & B are said to be independent if:

$P(A/B) = P(A)$ A is Not affected by B.

$P(B/A) = P(B)$ B is Not affected by A.

* Consequance of independance:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = P(A) \Rightarrow P(A \cap B) = P(A) \cdot P(B)$$

* If A_1 & A_2 are independent, Then:

1] A_1 & $\bar{A}_2$ are independent.

2] $\bar{A}_1$ & A_2 are independent.

3] $\bar{A}_1$ & $\bar{A}_2$ are independent.

Example: Given that A_1 & A_2 are two independent events and $P(A_1) = 0.6$, $P(A_2) = 0.3$

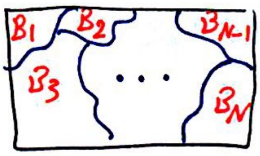
Find: a) $P(A_1 \cap \bar{A}_2) = P(A_1) \cdot P(\bar{A}_2) = 0.6 * (1-0.3) = \boxed{0.42}$.

b) $P(\bar{A}_2/\bar{A}_1) = P(\bar{A}_2) = 1 - 0.3 = \boxed{0.7}$.

* Total Probability:

$B_i \cap B_j = \phi \;\forall\; i \neq j$

Suppose we have N disjoint events $B_1, B_2, \ldots, B_N$ which satisfy $\bigcup_{i=1}^{N} B_i = S \Rightarrow P\left(\bigcup_{i=1}^{N} B_i\right) = 1$

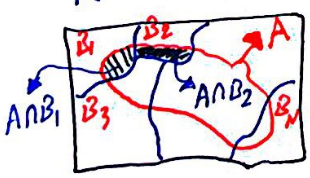


$$\Rightarrow P(B_1) + P(B_2) + \cdots + P(B_N) = 1$$

$$P(A) = \sum_{i=1}^{N} P(A/B_i)\, P(B_i)$$

Proof:

$$P(A) = P(A \cap S) = P\left(A \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) = P\big((A \cap B_1) \cup (A \cap B_2) \cup \cdots (A \cap B_N)\big)$$



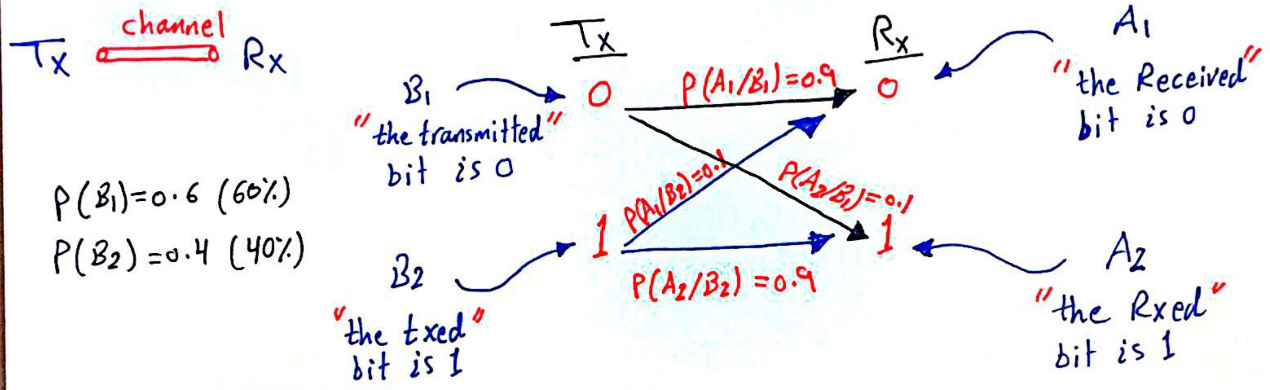
$$= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_N)$$

$$= \sum_{i=1}^{N} P(A \cap B_i) = \sum_{i=1}^{N} P(A/B_i)\, P(B_i) \quad \#$$

* Bayes Rule:

$$P(B_i/A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A/B_i)\,P(B_i)}{P(A)}$$

Example: * Binary Communication Channel * "(BCC)".



$P(B_1) = 0.6$ (60%)

$P(B_2) = 0.4$ (40%)

find the following:

a) $P(A_1)$, $P(A_2)$.

b) $\underbrace{P(B_1/A_1),\ P(B_2/A_1)}_{sum=1}$, $\underbrace{P(B_1/A_2),\ P(B_2/A_2)}_{sum=1}$.

solution:

a) • $P(A_1) = P\big((A_1 \cap B_1) \cup (A_1 \cap B_2)\big) = P(A_1 \cap B_1) + P(A_1 \cap B_2)$

$= P(A_1/B_1)\,P(B_1) + P(A_1/B_2)\,P(B_2)$ → from Total Probability Law.

$= (0.9) * (0.6) + (0.1) * (0.4) \Rightarrow \boxed{P(A_1) = 0.58}$.

• $P(A_2) = 1 - P(A_1) \Rightarrow \boxed{P(A_2) = 0.42}$.

OR $P(A_2) = P(A_2/B_1)\,P(B_1) + P(A_2/B_2)\,P(B_2) = 0.1 * 0.6 + 0.9 * 0.4$

$\Rightarrow \boxed{P(A_2) = 0.42}$.

b) • $P(B_1/A_1) = \frac{P(A_1/B_1)\,P(B_1)}{P(A_1)} = \frac{(0.9)*(0.6)}{(0.58)} \Rightarrow \boxed{P(B_1/A_1) = 0.931}$.

• $P(B_2/A_1) = \frac{P(A_1/B_2)\,P(B_2)}{P(A_1)} \Rightarrow \boxed{P(B_2/A_1) = 0.069}$. OR $P(B_2/A_1) = 1 - P(B_1/A_1)$

• $P(B_1/A_2) = \frac{P(A_2/B_1)\,P(B_1)}{P(A_2)} \Rightarrow \boxed{P(B_1/A_2) = 0.143}$.

• $P(B_2/A_2) = 1 - 0.143$

$\boxed{P(B_2/A_2) = 0.857}$.

※ Combined Expirements:

⇒ Expirement formed by combining sub-exp.

Example:

exp: Flip a coin & roll a die.

ⓐ • Sample space $S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)$
$(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

OR: exp. → sub-exp1: flip a coin. ⇒ $S_1 = \{H, T\}$
→ sub-exp2: Roll a die. ⇒ $S_2 = \{1, 2, 3, 4, 5, 6\}$

$$\Rightarrow S = \underbrace{S_1 \times S_2}_{\text{means combination.}} = \begin{Bmatrix} (H,1), (H,2), (H,3), (H,4), (H,5), (H,6) \\ (T,1), (T,2), (T,3), (T,4), (T,5), (T,6) \end{Bmatrix}$$

ⓑ • Define event C: "the appeared face is head & the appeared Number is even"

event A ⊆ S_1 ; event B ⊆ S_2

$\Rightarrow C = \{(H,2), (H,4), (H,6)\}$

OR: $A = \{H\}$ (from sub-exp1) ⇒ $C = A \overset{\text{combined.}}{\times} B$
$B = \{2, 4, 6\}$ (from sub-exp2) $= \{(H,2), (H,4), (H,6)\}$

ⓒ • $P(C) = \frac{3}{12}$ OR: $P(C) = P(A \times B) = P(A) \cdot P(B) = \frac{1}{2} * \frac{3}{6} = \frac{3}{12}$

Example:

Flip 3 coins at once, find: P(all faces are head)?

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$\Rightarrow \boxed{P(\{HHH\}) = 1/8}$.

OR:

$P(\{HHH\}) = \underset{\text{coin 1}}{P(H)} \cdot \underset{\text{coin 2}}{P(H)} \cdot \underset{\text{coin 3}}{P(H)} = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \boxed{\frac{1}{8}}$.

※ Permutations:

⇒ all possible sequences of ordering r elements (The order is important.) taken from n elements without replacement.

Example:

4-cards.

$\boxed{a}\ \boxed{b}\ \boxed{c}\ \boxed{d}$

$n=4$

⇒ order 2 cards $(r=2)$

place 1	place 2
a	b
a	c
a	d
b	a
b	c
b	d
c	a
c	b
c	d
d	a
d	b
d	c

⇒ permutations.

# of permutations = 12

$\Rightarrow \underbrace{4}_{\text{draw the 1st card.}} * \underbrace{3}_{\text{draw the 2nd card.}} = 12$

Example: 5-cards $\boxed{a}\ \boxed{b}\ \boxed{c}\ \boxed{d}\ \boxed{e}$ ⇒ $n=5$ & $r=3$

P1 P2 P3

$5 * 4 * 3 = 60$

$$\boxed{\#\text{of permutations} \triangleq P_r^n = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}}$$

※ Combinations:

⇒ Same as Permutations but the order is NOT important.

Example: 4-cards $\boxed{a}\ \boxed{b}\ \boxed{c}\ \boxed{d}$ $n=4,\ r=2$

P_1	P_2
a	b
a	c
a	d
b	c
b	d
c	d

# of combinations $= 6$

$\hookrightarrow \frac{12}{2} = 6$

$$\boxed{\#\text{of combinations} \triangleq C_r^n = \frac{P_r^n}{r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}}$$

Note: $\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{n} = 1$

Example: 5 students. How many 3-members teams could be performed?

Solution: $\#\text{of teams} = C_3^5 = \binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5 * 4 * \cancel{3!}}{\cancel{3!} * 2} = \underline{\underline{10}}$ teams.

* Bernouli Trial:

$\Rightarrow$ Trial or exp. with two possible outcomes.

A "success" $\bar{A}$ "fail" $\Rightarrow$ depend on your decision.

e.g: Flip a Coin:

Two possible outcomes: $A = \{H\}$ $P(A) = p$

$\bar{A} = \{T\}$ $P(\bar{A}) = 1-p$

* If we repeat the Bernouli Trial n times, Then the number of successes $K = 0, 1, 2, \dots, N$.

- What is the probability of K $P(K) = ?$

Example: flip a coin 3 times, $P(K=2) = ?$

Solution: $A = \{H\} \rightarrow P(A) = p$

$\bar{A} = \{T\} \rightarrow P(\bar{A}) = 1-p$

$$S = \{ \underbrace{HHH}_{K=3}, \underbrace{HHT, HTH, THH}_{K=2}, \underbrace{TTH, THT, HTT}_{K=1}, \underbrace{TTT}_{K=0} \}$$

So, $K = 0, 1, 2, 3$

$$P(K=2) = P(\{HHT, HTH, THH\}) = P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$$

$$= 3\big(P(H)\,P(H)\,P(T)\big) = \underline{\underline{3\left(p^2 (1-p)\right)}}$$

Rule: N Bernouli Trials, $P(A) = p$, $P(\bar{A}) = 1-p$, $K = 0, 1, 2, \dots, N$

$\Rightarrow$ use the following furmela: $$P(K) = \binom{N}{K} p^K (1-p)^{N-K}$$

* Now Back to the previous example: find $P(K=0)$?

$$P(K=0) = \binom{3}{0} p^0 (1-p)^3 = (1-p)^3$$

OR: $P(K=0) = P\{TTT\} = P\{T\}\,P\{T\}\,P\{T\} = (1-p)^3$

$\Rightarrow$

⇒ find $P(K=1)$?

$$P(K=1) = \binom{3}{1} p^1 (1-p)^2 = 3p(1-p)^2$$

OR: $P(K=1) = P\{TTH\} + P\{THT\} + P\{HTT\} = 3p(1-p)^2$

Example: Flip a coin 100 times, what is the probability that the head appears at most 2 times, $P(H)=0.4$?

Solution: since he asked about the head ⇒ make it the success event.

$A = \{H\}$, $N=100$ ⇒ $p = P\{H\} = 0.4$

$K = 0, 1, \cdots, 100$ ⇒ $P\{\text{head appears at most 2 times}\} = P\{K \le 2\}$

$= P\{K=0\} + P\{K=1\} + P\{K=2\}$

$$\left.\begin{aligned} \text{Term 1} &= \binom{100}{0}(0.4)^0 (0.6)^{100} \\ \text{Term 2} &= \binom{100}{1}(0.4)^1 (0.6)^{99} \\ \text{Term 3} &= \binom{100}{2}(0.4)^2 (0.6)^{98} \end{aligned}\right\} \Rightarrow \boxed{\text{Answer} = 1.482 * 10^{-19}}$$

*** See the Text Book: Example 1.7-1, 1.7-2, 1.7-3

* * * * *

End of CH1.

※CHAPTER(2): Random Variables (RV):

* RV def: function of sample space.

exp. → $S = \{e_1, e_2, \cdots, e_N\} \Longrightarrow X(S) \triangleq X$ → point on the real Line.

$\longrightarrow \mathbb{R}$

RV → Discrete.
RV → Continuous.

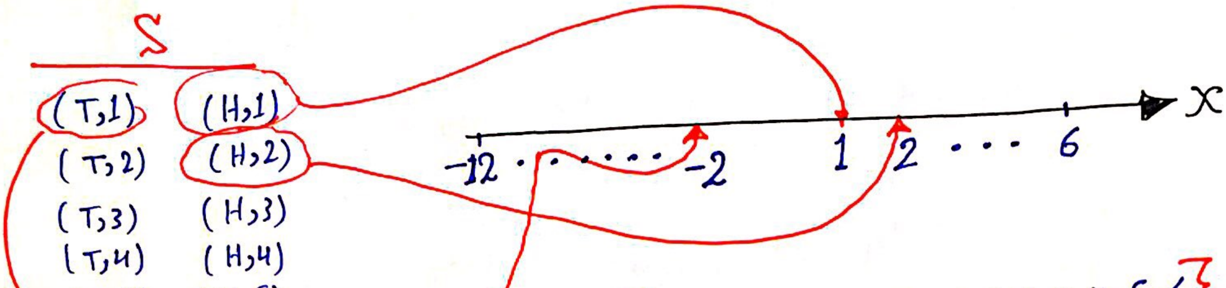
Example: exp. flip a coin & roll a die

ⓐ S ⓑ define R.V.

⟹

Rule: $X(S)$: if H appears: $x \triangleq$ # on the die.
if T appears: $x \triangleq (-2)$ (# on the die).

Solution:

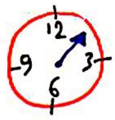


$X = \{-12, -10, -8, -6, -4, -2, 1, 2, 3, 4, 5, 6\}$

Discrete.

$\Rightarrow$ find the following:

- $P(X=-2) = P(\{T,1\}) = \boxed{\frac{1}{12}}$
- $P(X=22) = P\{\emptyset\} = \boxed{\text{Zero.}}$
- $P(X \leq 3) = P(X=-12) + P(X=-10) + \cdots + P(X=3)$
  $= 9 * \frac{1}{12} = \boxed{\frac{3}{4}}$
- $P(X \leq 30) = \boxed{1}$
- $P(X \leq \infty) = \boxed{1}$
- $P(X \leq -\infty) = \boxed{0}$
- $P(0.5 < X \leq 4) = \frac{1}{12} * 4 = \boxed{\frac{1}{3}}$

Example: Wheel of chance  ⓐ S ⓑ R.V if $X(s) = s^2$

Solution: ⓐ $S = \{0 < s \leq 12\}$ $\Rightarrow$ All real #'s between 0 & 12 without including zero.

ⓑ $X = \{0 < x \leq 144\}$

Continuous R.V.

$\Rightarrow$ find $P(0 < X \leq 9) = ?$

$$P(0 < X \leq 9) = P\{0 < s \leq 3\} = \frac{\text{Length of the period}}{\text{Length of the total}} = \frac{3-0}{12-0} = \boxed{\frac{1}{4}}$$

$\Rightarrow$ find $P(X=9) = P(s=3) = 0$

* Always probability for a specific point in Continuous R.V = Zero.

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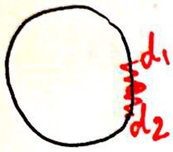
proof :



$\Rightarrow P\{d_1 < s < d_2\} = \dfrac{d_2 - d_1}{12}$ Take the Limit

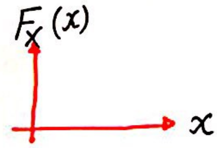
$\lim\limits_{d_2 - d_1 \to 0} P\{d_1 < s < d_2\} = \lim\limits_{d_2 - d_1 \to 0} \dfrac{d_2 - d_1}{12} = \text{Zero}$ #

Note : if it was Discrete R.V $\Rightarrow$ it won't be a Zero.

※ R.V : $\longrightarrow$ ① Distribution Function. (CDF).
$\quad\quad\quad\quad$ ② Density Function. (PDF).

1) CDF :

for any R.V X $\longrightarrow$ $F_X(x) = P(X \leq x)$, $\quad -\infty < x < \infty$



Example: $F_X(x)$ for a Discrete R.V.

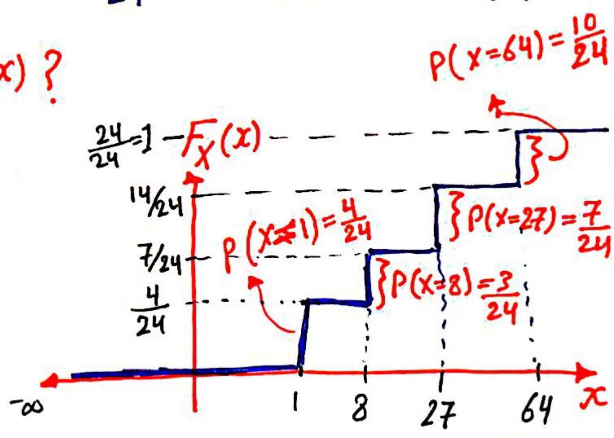
exp. : $S = \{1, 2, 3, 4\}$ $\quad P(1) = \frac{4}{24}$, $P(2) = \frac{3}{24}$, $P(3) = \frac{7}{24}$, $P(4) = \frac{10}{24}$

Define R.V $X = S^3$ find $F_X(x)$?

<u>Solution</u>: $X = \{1, 8, 27, 64\}$

$P(X=1) = \frac{4}{24}$ $\quad\quad P(X=27) = \frac{7}{24}$

$P(X=8) = \frac{3}{24}$ $\quad\quad P(X=64) = \frac{10}{24}$



Take points starting from $-\infty$:

- $F_X(-\infty) = P(X \leq -\infty) = 0$
- $F_X(0) = P(X \leq 0) = 0$

$\Big\} \Rightarrow \bullet\ F_X(1^-) = 0$

- $F_X(1) = P(X \leq 1) = P(X=1) = 4/24$
- $F_X(3) = P(X \leq 3) = P(X=1) = 4/24$
- $F_X(8) = P(X \leq 8) = P(X=1) + P(X=8) = \frac{4}{24} + \frac{3}{24} = \frac{7}{24}$
- $F_X(27) = P(X \leq 27) = P(X=1) + P(X=8) + P(X=27)$

⇒ Writing $F_X(x)$:

$$F_X(x) = \begin{cases} 0 & , x<1 \\ 4/24 & , 1 \leq x < 8 \\ 7/24 & , 8 \leq x < 27 \\ 14/24 & , 27 \leq x < 64 \\ 1 & , 64 \leq x \end{cases}$$

* Always:

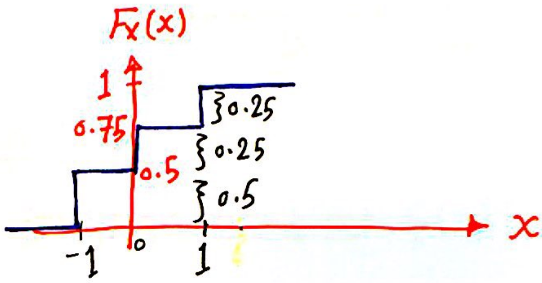
$$\boxed{\begin{aligned} &F_X(-\infty) = 0 \\ &F_X(\infty) = 1 \\ &F_X(x^+) = F_X(x) \end{aligned}}$$

e.g.: $X = \{-1, 0, 1\}$

$P(X=-1) = 0.5$

$P(X=0) = 0.25$

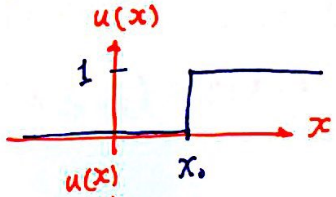
$P(X=1) = 0.25$



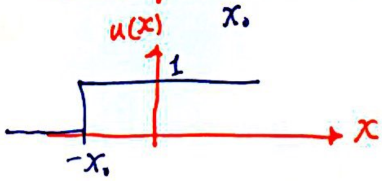
Recall the unit step function:

$$u(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$$u(x-x_0) = \begin{cases} 1 & , x \geq x_0 \\ 0 & , x < x_0 \end{cases}$$



$$u(x+x_0) = \begin{cases} 1 & , x \geq -x_0 \\ 0 & , x \leq -x_0 \end{cases}$$

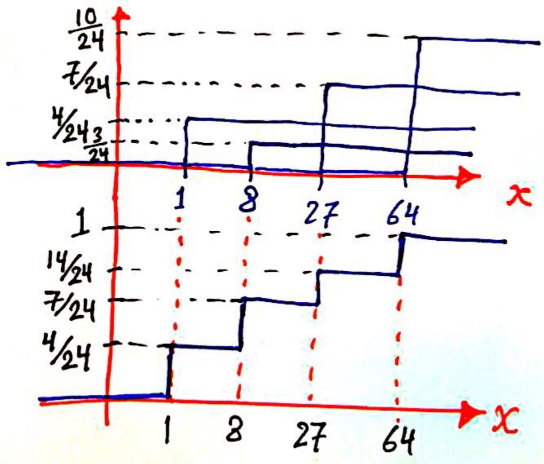


* CDF for Discrete R.V $X \in \{x_1, x_2, \ldots x_N\}$

$$\boxed{F_X(x) = \sum_{i=1}^{N} P(X=x_i)\, u(x-x_i)}$$

⇒ for Example:

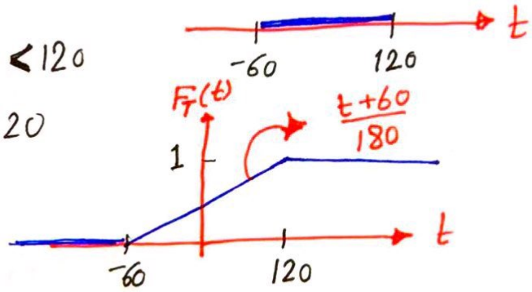
$$F_X(x) = P(X=1)\,u(x-1) + P(X=8)\,u(x-8) + P(X=27)\,u(x-27) + P(X=64)\,u(x-64)$$



Example: Distribution Function Continuous R.V:

R.V T models the temperature of certain location $T=\{-60\leq t\leq 120\}$

$$F_T(t)=P\{T\leq t\}=\begin{cases}0 & ,\ t<-60\\ \dfrac{t+60}{180} & ,\ -60\leq t<120\\ 1 & ,\ t\geq 120\end{cases}$$



※ CDF Properties:

① $F_X(-\infty)=0$ ② $F_X(\infty)=1$ ③ $0\leq F_X(x)\leq 1$

④ for $x_1<x_2 \Rightarrow F_X(x_1)\leq F_X(x_2)$ ⑤ $P(x_1<X\leq x_2)=F_X(x_2)-F_X(x_1)$

↳ CDF is increasing Function.

⑥ $F_X(x^+)=F_X(x)$

* proof for ⑤: $F_X(x_2)=P(X\leq x_2)=P\big((X\leq x_1)\cup(x_1<X\leq x_2)\big)$

$=P(X\leq x_1)+P(x_1<X\leq x_2)$

so, $P(x_1\leq X\leq x_2)=P(X\leq x_2)-P(X\leq x_1)$ #

* for the same example in page ⑯:

Find: a) $P(3<X\leq 27)$

b) $P(1<X<27)$

c) $P(8\leq X<64)$

Solution: a) $P(3<X\leq 27)=P(X=27)-P(X=3)=\frac{14}{24}-\frac{4}{24}=\boxed{\frac{10}{24}}$

OR: $P(3<X\leq 27)=P(X=8)+P(X=27)=\frac{3}{24}+\frac{7}{24}=\boxed{\frac{10}{24}}$

$\frac{7}{24}-\frac{4}{24}$ $\frac{14}{24}-\frac{7}{24}$

b) $P(1<X<27)=P(1<X\leq 27^-)=F_X(27^-)-F_X(1)$

$=\frac{7}{24}-\frac{4}{24}=\boxed{\frac{3}{24}}$

OR: $P(1<X<27)=P(X=8)=\boxed{3/24}$

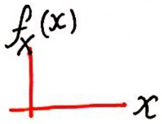
c) $P(8\leq X<64)=P(X=8)+P(8<X\leq 64^-)=\frac{3}{24}+F_X(64^-)-F_X(8)=\frac{3}{24}+\frac{14}{24}-\frac{7}{24}=\boxed{\frac{10}{24}}$

Note: for Contineous R.V:

$P(X=x_1)=0$, $P(x_1<X<x_2)=P(x_1\leq X\leq x_2)=P(x_1\leq X<x_2)$

2] PDF:

for R.V X, the PDF: $f_X(x)=\dfrac{dF_X(x)}{dx}$



Example: PDF for Discrete R.V.

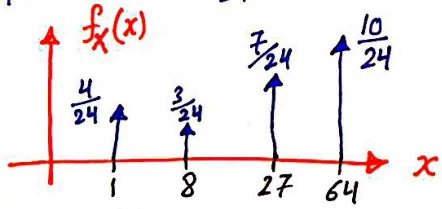
$X=\{1,8,27,64\}$, $P(X=1)=\frac{4}{24}$ $P(X=27)=\frac{7}{24}$

$P(X=8)=\frac{3}{24}$ $P(X=64)=\frac{10}{24}$

Find $f_X(x)=?$

Solution: $f_X(x)=\dfrac{dF_X(x)}{dx}=\dfrac{d}{dx}\left[\frac{4}{24}u(x-1)+\frac{3}{24}u(x-8)+\frac{7}{24}u(x-27)+\frac{10}{24}u(x-64)\right]$

$=\frac{4}{24}\delta(x-1)+\frac{3}{24}\delta(x-8)+\frac{7}{24}\delta(x-27)+\frac{10}{24}\delta(x-64)$



* For Discrete R.V $X\in\{x_1,x_2,x_3\cdots x_N\}$

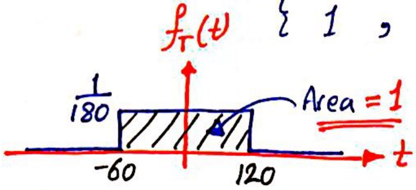
Then, $f_X(x)=\sum_{i=1}^{N}P(X=x_i)\,\delta(x-x_i)$

Example: PDF for Contineous R.V. $T=\{-60\leq t\leq 120\}$

Find $f_X(x)=?$

Solution: $f_T(t)=\dfrac{dF_T(t)}{dt}$; we know $F_T(t)=P(T\leq t)=\begin{cases}0 & , t<-60\\ \frac{t+60}{180} & , -60\leq t<120\\ 1 & , t\geq 120\end{cases}$

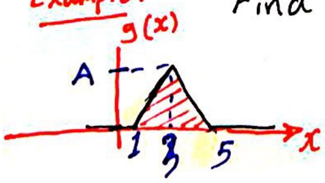
$\Rightarrow$ $f_T(t)=\begin{cases}0 & , t<-60\\ \frac{1}{180} & , -60\leq t<120\\ 0 & , t\geq 120\end{cases}$



* PDF Properties:

① $f_X(x)\geq 0$ ② $\int_{-\infty}^{\infty}f_X(x)\,dx=1$ "if NOT 1 then it is NOT PDF"

Example: Find the Constant A such that $g(x)$ is PDF?



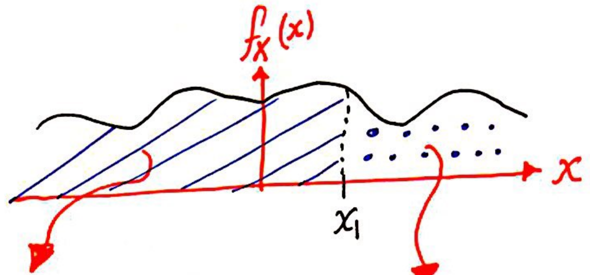
Solution: $\int_{-\infty}^{\infty}g(x)\,dx=1$ Because $g(x)$ is PDF.

$\Rightarrow\left(\frac{1}{2}*2*A\right)*2=1\Rightarrow\boxed{A=\frac{1}{2}}$

③ $F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz$

Example: $F_X(\infty) = \int_{-\infty}^{\infty} f_X(z)\,dz = 1$

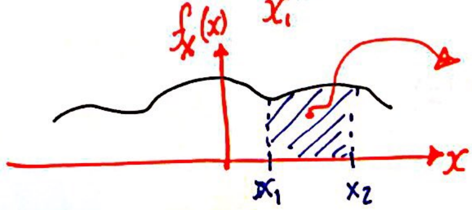
$F_X(x_1) = P(X \le x_1) = \int_{-\infty}^{x_1} f_X(z)\,dz$



$F_X(x_1) = P(X \le x_1)$

$1 - F_X(x_1) = P(X > x_1) = \int_{x_1}^{\infty} f_X(z)\,dz$

④ $P(x_1 < X \le x_2) = \int_{x_1}^{x_2} f_X(z)\,dz$



* **proof:** $P(x_1 < X < x_2) = F_X(x_2) - F_X(x_1) = P(X \le x_2) - P(X \le x_1)$

$$= \int_{-\infty}^{x_2} f_X(z)\,dz - \int_{-\infty}^{x_1} f_X(z)\,dz = \int_{x_1}^{x_2} f_X(z)\,dz \quad \#$$

※ **Important R.V's:**

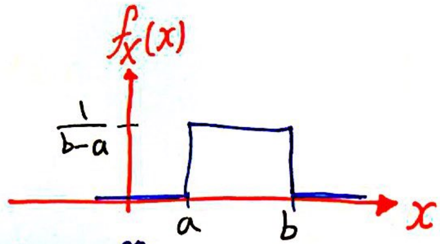
1. Uniform R.V.
2. Exponential R.V.
3. Gaussian R.V.
4. Bernouli R.V.
5. Binomial R.V.

$\boxed{1}$ Uniform R.V:

$X \sim U(a,b)$ $a \,\&\, b \in (-\infty, \infty)$

$X = \{a \le x \le b\}$

• $f_X(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & , o.w. \end{cases}$

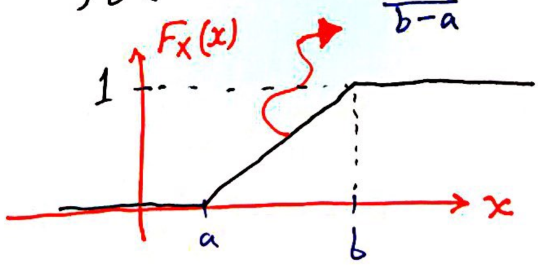


$P\{X \le x\}$

• $F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz = \begin{cases} 0 & , x < a \\ \int_a^x \frac{1}{b-a}\,dz & , a \le x < b \\ 1 & , b \le x \end{cases}$

check: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

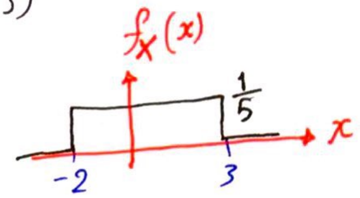
$\Rightarrow F_X(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \le x < b \\ 1 & , b \le x \end{cases}$



Example: $X \sim U(a,b) \Rightarrow X \sim U(-2,+3)$

find a) $P\{-1 \leq X \leq 0\}$

b) $P\{1 < X < 4\}$



Solution:

a) $P\{-1 \leq X \leq 0\} = \int_{-1}^{0} f_X(x)\,dx = \int_{-1}^{0} \frac{1}{5}\,dx = \boxed{\frac{1}{5}}$

b) $P\{1 < X < 4\} = \int_{1}^{4} f_X(x)\,dx = \int_{1}^{3} \frac{1}{5}\,dx = \boxed{\frac{2}{5}}$

2 Exponential R.V:

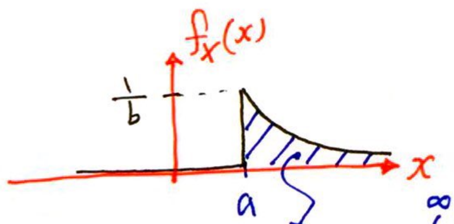
$X \sim exp(a,b)$

$a \,\&\, b \in (-\infty, \infty)$

- $f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & , x \geq a \\ 0 & , x < a \end{cases}$

*practical example:

waiting time to start a phone call.



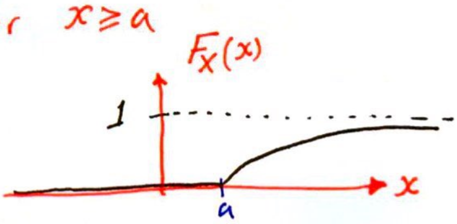
check: Area $= \int_{a}^{\infty} \frac{1}{b} e^{-\frac{(x-a)}{b}} dx = 1$

- CDF of $exp(a,b)$:

$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi = \begin{cases} 0 & , x < a \\ \int_{a}^{x} \frac{1}{b} e^{-\frac{(x'-a)}{b}} dx' & , x \geq a \end{cases}$

$\int_{a}^{x} \frac{1}{b} e^{-\frac{(x'-a)}{b}} dx' = \frac{1}{\cancel{b}} \left. \frac{e^{-(x'-a)/b}}{-1/\cancel{b}} \right|_{a}^{x} = \underline{\underline{1 - e^{-\frac{(x-a)}{b}}}}$ for $x \geq a$

$\Rightarrow F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & , x \geq a \\ 0 & , x < a \end{cases}$

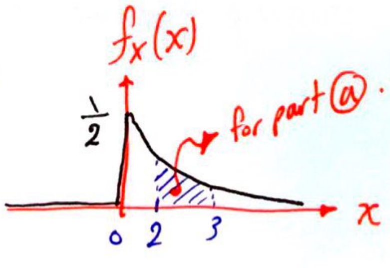


Example: $X \sim exp(0,2)$ Find: a) $P\{2 < X < 3\}$ b) $P\{X < 2 \cap X > 3\}$

c) $P\{X < 2 \cup X > 3\}$

Solution: $\Rightarrow$ To solve find $f_X(x)$ firstly.

$f_X(x) = \begin{cases} \frac{1}{2} e^{-x/2} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$

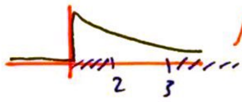


$\Longrightarrow$ Continue.

$\Rightarrow$ a) $P\{2<X<3\} = \int_2^3 \frac{1}{2} e^{-x/2} dx = \frac{1}{2} \left. \frac{e^{-x/2}}{-\frac{1}{2}} \right|_2^3 = e^{-1} - e^{-1.5} = \boxed{0.1447}$

OR: find $F_X(x) \Rightarrow P\{2<X<3\} = F_X(3) - F_X(2)$

b) $P\{x<2 \cap x>3\} = P\{\emptyset\} = 0$

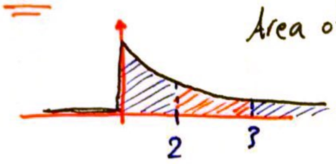


No intersection.

c) $P\{x<2 \cup X>3\} \Rightarrow$ since No intersection $\Rightarrow$ $x<2$, $x>3$ are disjoint.

$= P\{X<2\} + P\{X>3\} = \int_0^2 \frac{1}{2} e^{-x/2} dx + \int_3^\infty \frac{1}{2} e^{-x/2} dx = 1 - e^{-1} + e^{-1.5} - 0 = \boxed{0.8553}$

OR:



Area of $P\{x<2 \cup X>3\} = 1 - P\{2<X<3\}$

$= 1 - \int_2^3 \frac{1}{2} e^{-x/2} dx = \boxed{0.8553}$

3 Gaussian R.V "Normal":

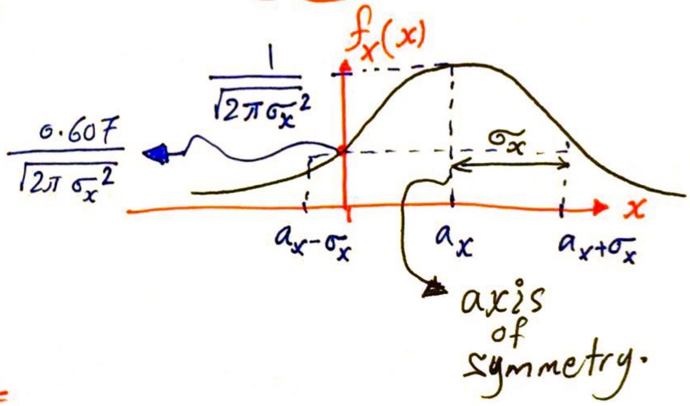
$X \sim N(a_x, \sigma_x^2)$

- $a_x \in (-\infty, \infty)$
- $\sigma_x > 0$
- $x \in (-\infty, \infty)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$$

$a_x \equiv$ Average Value for $X \sim N(a_x, \sigma_x^2)$

$\sigma_x \equiv$ Standard Deviation.



Note:

1) $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx = \underline{\underline{1}}$$

2) $$\int_{a_x}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx = \int_{-\infty}^{a_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx = \underline{\underline{\frac{1}{2}}}$$

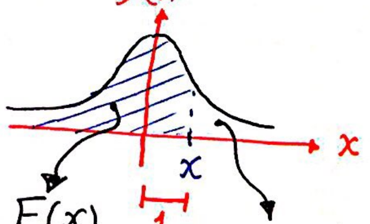
*Special Case:

$\left.\begin{matrix} a_x = 0 \\ \sigma_x = 1 \end{matrix}\right\} \Rightarrow X \sim N(0,1) \Rightarrow$ standard gaussian R.V.

- Distribution Function for $X \sim N(0,1) \Rightarrow F(x)$
- Distribution Function for $X \sim N(a_x, \sigma_x^2) \Rightarrow F_X(x)$

*** $X \sim N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



$$\Rightarrow F(x) = P\{X \leq x\} = \int_{-\infty}^{x} f(\xi)\, d\xi$$

$$\Rightarrow \boxed{F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\acute{x}^2/2}\, d\acute{x}}$$

↳ solved by using Tables.

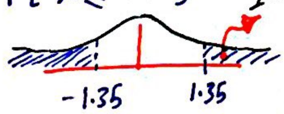
Example: Given $X \sim N(0,1)$ Find: a) $P\{X \leq -1.35\}$ b) $P\{0.1 \leq X \leq 2.3\}$
c) $P\{0.1 < X < 2.3\}$ d) $P\{-2.3 < X < -0.1\}$ e) $P\{-1.5 < X < 1.4\}$ f) $P\{X < 0.23\}$

Solution: since Continuous R.V we don't care for the equal sign.

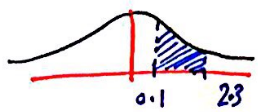
f) $P\{X < 0.23\}$

$\Rightarrow P\{X < 0.23\} = F(0.23) = \boxed{0.5910}$

a) $P\{X \leq -1.35\}$ $\quad 1 - F(1.35)$ $\quad \Rightarrow F(-1.35) = 1 - F(1.35) = 1 - 0.9115 = \boxed{0.0885}$



b) $P\{0.1 \leq X \leq 2.3\} = F(2.3) - F(0.1) = 0.9893 - 0.5398 = \boxed{0.4495}$



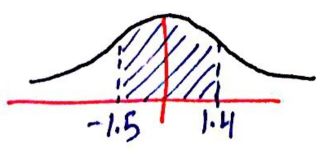
c) Same as (b) since it is Continuous R.V.

d) $P\{-2.3 < X < -0.1\} = F(-0.1) - F(-2.3) = 1 - F(0.1) - 1 + F(2.3)$

$= F(2.3) - F(0.1) = 0.9893 - 0.5398 = \boxed{0.4495}$

e) $P\{-1.5 < X < 1.4\} = F(1.4) - F(-1.5) = F(1.4) - 1 + F(1.5)$

$= 0.9192 - 1 + 0.9332 = \boxed{0.8524}$



** $X \sim N(a_x, \sigma_x^2)$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(z-a_x)^2}{2\sigma_x^2}} dz$$

By substitution: Set $u = \frac{z-a_x}{\sigma_x}$

$z = -\infty \rightarrow u = -\infty$

$z = x \rightarrow u = \frac{x-a_x}{\sigma_x}$

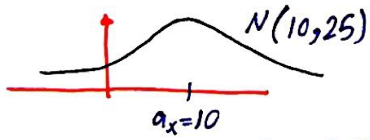
$du = \frac{dz}{\sigma_x}$

$$\Rightarrow F_X(x) = \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \frac{1}{\sqrt{2\pi\cancel{\sigma_x^2}}} e^{-\frac{u^2}{2}} \cdot \cancel{\sigma_x}\, du$$

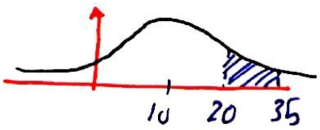
$$\Rightarrow \Rightarrow \quad F_X(x) = \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}}_{N(0,1)} du \Rightarrow F_X(x) = F\left(\frac{x-a_x}{\sigma_x}\right)$$

Example: $X \sim N(10, 25)$ Find: a) $P\{20 < X \leq 35\}$ b) $P\{5 < X \leq 15\}$

Solution: $a_x = 10$, $\sigma_x^2 = 25 \Rightarrow \sigma_x = 5$



a)



$P\{20 < X \leq 35\} = F_X(35) - F_X(20)$

$= F\left(\frac{35-10}{5}\right) - F\left(\frac{20-10}{5}\right) = F(5) - F(2)$ almost 1.

$= 1 - 0.977 = \boxed{0.0228}$

b)

$P\{5 < X \leq 15\} = F_X(15) - F_X(5) = F\left(\frac{15-10}{5}\right) - F\left(\frac{5-10}{5}\right)$

$= F(1) - F(-1) = F(1) - 1 + F(1) = 2F(1) - 1$

$= 2 * 0.8413 - 1 = \boxed{0.6826}$

First Material * * * First Material * * * First Material * * *

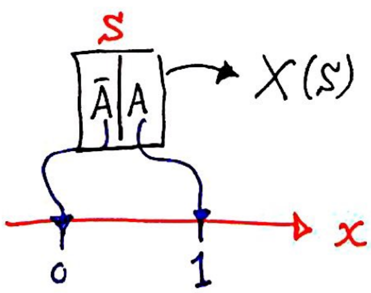
4 Bernouli R.V:

$\Rightarrow$ Discrete R.V, related to the Bernouli Trial.

$S = \{A, \bar{A}\}$ — A: success, $\bar{A}$: fail

$P(A) = p$

$P(\bar{A}) = 1-p$



$$X = \begin{cases} 1, & \text{if } A \text{ occurs.} \\ 0, & \text{if } \bar{A} \text{ occurs.} \end{cases}$$

$X \in \{0, 1\}$ $\quad P\{X=1\} = P\{A\} = p$ & $P\{X=0\} = P\{\bar{A}\} = 1-p$

$\Rightarrow f_X(x)$ & $F_X(x)$:

- $f_X(x) = (1-p)\,\delta(x) + p\,\delta(x-1)$
- $F_X(x) = (1-p)\,u(x) + p\,u(x-1)$

$\boxed{5}$ Binomial R.V: $\Rightarrow$ Discrete.

Def.: X : "Number of success in repeating N Bernouli trials."

$$X \in \{\overset{x_1}{0}, \overset{x_2}{1}, \overset{x_3}{2}, \ldots, \overset{x_{N+1}}{N}\}$$

- $f_X(x) = \sum_{i=1}^{N+1} P\{X=x_i\}\,\delta(x-x_i)$

$$\Rightarrow P\{X=2\} = P\{\underset{x_i}{K=2}\} = \binom{N}{\underset{x_i}{2}} p^{\overset{x_i}{2}} (1-p)^{\overset{x_i}{N-2}}$$

$$\Rightarrow f_X(x) = \sum_{i=1}^{N+1} \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i}\,\delta(x-x_i)$$

- $F_X(x) = \sum_{i=1}^{N+1} \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i}\,u(x-x_i)$

※ Conditional Density & Distribution Functions:

* Recall: for R.V $X \Rightarrow F_X(x) = P\{X \le x\}$ $\Rightarrow$ Distribution Function of X.

$\Rightarrow$ Conditional Distribution Function of X:

$$F_X(x/B) = P\{X \le x / B\} \qquad \Rightarrow \quad F_X(x/B) = \frac{P\{X \le x \cap B\}}{P\{B\}}$$

$\Rightarrow$ Conditional Density Function of X:

$$f_X(x/B) = \frac{d\,F_X(x/B)}{dx}$$

* Note: All properties for PDF & CDF same as proparties of Conditional CDF & PDF.

Example:

R	G	B
5	35	60

BOX1: Total = 100

R	G	B
80	60	10

BOX2: Total = 150

Exp: "Randomly select a box & then draw out a ball from the selected box".

Define R.V $X = \begin{cases} 1 & \text{, the ball is R.} \\ 2 & \text{, the ball is G.} \\ 3 & \text{, the ball is B.} \end{cases}$, $X = \{1, 2, 3\}$

Find: a) $F_X(x/B_1)$ b) $F_X(x|B_2)$ c) $F_X(x)$

event $B_1 \equiv$ "the selected box is BOX1".

event $B_2 \equiv$ "the selected box is BOX2".

Solution:

a) $F_X(x|B_1) = P\{X=1/B_1\}u(x-1) + P\{X=2/B_1\}u(x-2) + P\{X=3/B_1\}u(x-3)$

$$= \frac{5}{100}u(x-1) + \frac{35}{100}u(x-2) + \frac{60}{100}u(x-3)$$

if he asked about $f_X(x/B_1) \Rightarrow$ The same But replace unit by impulse.

b) $F_X(x|B_2) = \frac{80}{150}u(x-1) + \frac{60}{150}u(x-2) + \frac{10}{150}u(x-3)$

c) $F_X(x) = P\{X=1\}u(x-1) + P\{X=2\}u(x-2) + P\{X=3\}u(x-3)$

- $P\{X=1\} = P\{R\} = P\{(R\cap B_1)\cup(R\cap B_2)\} = P\{R\cap B_1\} + P\{R\cap B_2\}$

$= P\{R/B_1\}P\{B_1\} + P\{R/B_2\}P\{B_2\}$ "Total Probability".

$= \frac{5}{100} * (0.5) + \frac{80}{150} * (0.5) = \boxed{0.292}$

if Not given, take it 0.5, 0.5.

- $P\{X=2\} = \frac{35}{100} * (0.5) + \frac{60}{150} * (0.5) = \boxed{0.375}$.

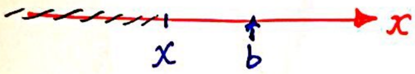
- $P\{X=3\} = \frac{60}{100} * (0.5) + \frac{10}{150} * (0.5) = \boxed{0.333}$.

※ Given R.V X with $F_X(x)$:

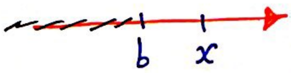
↳ Determine $F_X(x|B)$ such $B = \{X \leq b\}$, b: real number.

$$F_X(x|B) = P\{X \leq x / X \leq b\} = \frac{P\{X \leq x \cap X \leq b\}}{P\{X \leq b\}}$$

for $b > x$:



for $b < x$:



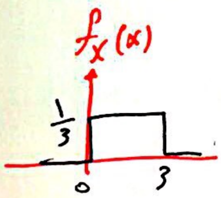
$$\Rightarrow F_X(x|B) = \begin{cases} \dfrac{P\{X \leq x\}}{P\{X \leq b\}} = \dfrac{F_X(x)}{F_X(b)} & , b > x \\ \dfrac{P\{X \leq b\}}{P\{X \leq b\}} = 1 & , b < x \end{cases} = \boxed{\begin{cases} \dfrac{F_X(x)}{F_X(b)} & , b > x \\ 1 & , b < x \end{cases}}$$

if we change the condition $B = \{X \leq b\} \Rightarrow B = \{X \geq b\}$

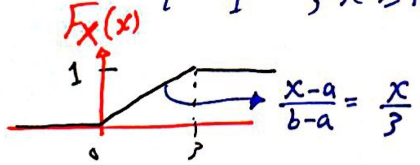
$F_X(x|B)$ will change.

Example: R.V $X \sim U(0,3)$, Find $F_X(x / X \leq 1)$?

solution: $\Rightarrow \underline{b = 1}$, $F_X(x/X \leq 1) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & , x < 1 \\ 1 & , x \geq 1 \end{cases}$

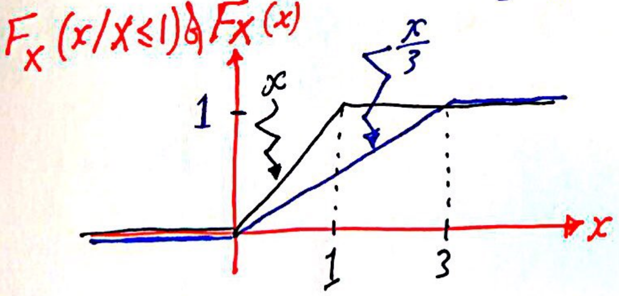


$\Rightarrow$



so, Now we Re-write $F_X(x/X \leq 1)$.

$$\Rightarrow F_X(x/X \leq 1) = \begin{cases} \dfrac{x/3}{1/3} = x & , 0 < x < 1 \\ 1 & , x > 1 \\ 0 & , x < 0 \end{cases}$$



* * * *

End of CH2

※ CHAPTER (3): Operations on one R.V:

- Expectation (or "mean" or "average value") $X \longrightarrow E[X] \triangleq \overline{X}$

↳ (DC value).

Example: grades for 20 students with mark out of 50.

# of students	grade
6	21
5	9
4	45
5	16

$\Rightarrow G = \{21, 9, 45, 16\}$

↳ as R.V.

average grade $\equiv \overline{G}$

$$\overline{G} = \frac{21*6+9*5+45*4+16*5}{20}$$

$$= 21 \cdot \frac{6}{20} + 9 \cdot \frac{5}{20} + 45 \cdot \frac{4}{20} + 16 \cdot \frac{5}{20}$$

$$\Rightarrow \overline{G} = 21\, P\{G=21\} + 9\, P\{G=9\} + 45\, P\{G=45\} + 16\, P\{G=16\}$$

* for Discrete R.V $X = \{x_1, x_2, \dots, x_N\}$

Then, $E[X] \triangleq \overline{X} = \sum_{i=1}^{N} x_i\, P\{X = x_i\}$

Example: exp.: $S = \{1,2,3,4\}$, $P(1) = 0.2$, $P(2) = 0.4$, $P(3) = 0.1$

Define: R.V $Y = S^2 - 1$, Find $\overline{Y}$?

Solution: $Y = \{\overset{y_1}{0}, \overset{y_2}{3}, \overset{y_3}{8}, \overset{y_4}{15}\} \Rightarrow \overline{Y} = \sum_{i=1}^{4} y_i\, P\{Y = y_i\}$

$$\Rightarrow \overline{Y} = (0)(0.2) + (3)(0.4) + (8)(0.1) + (15)(0.3) = \boxed{6.5}$$

$P(4) = 1 - (0.1 + 0.2 + 0.4) = \underline{\underline{0.3}}$

※ For Continuous R.V X with $f_X(x)$:

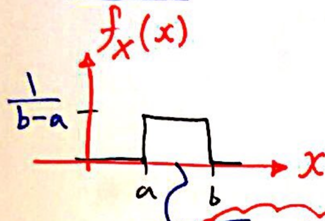
the expectation (average of X) is given by:

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Example: mean of uniform R.V.

$X \sim U(a,b)$, Find $\bar{X}$?

Solution:



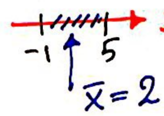
$$\bar{X} = \frac{a+b}{2}$$

$\Rightarrow$ Mathematically:

$$\bar{X} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

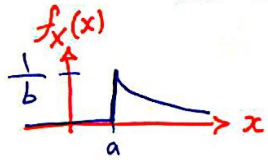
$$= \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{b^2-a^2}{2(b-a)} \Rightarrow \bar{X} = \frac{b+a}{2}$$

$\hookrightarrow$ e.g. $X \sim U(-1,5) \Rightarrow \bar{X} = \frac{-1+5}{2} = \underline{\underline{2}}$



Example: Find mean value for the exponential R.V $X \sim \exp(a,b)$?

Solution: $f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & , x \geq a \\ 0 & , x < a \end{cases}$



$\bar{X}$ can't be determined directly.

$\Rightarrow$ Mathematically: $\bar{X} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^{\infty} x \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}}\,dx.$

$$\Rightarrow \bar{X} = \frac{1}{b} e^{a/b} \int_a^{\infty} x\, e^{-x/b}\,dx$$ "integration by parts"

$u = x \rightarrow du = dx$
$dv = e^{-x/b}dx \rightarrow v = \frac{e^{-x/b}}{-1/b}$

$$\Rightarrow \bar{X} = \frac{1}{b} e^{a/b}\left[\underbrace{-bx\,e^{-x/b}\Big|_a^{\infty}}_{\text{it will result } \frac{\infty}{e^{\infty}} = \frac{\infty}{\infty}} + \int_a^{\infty} b\,e^{-x/b}\,dx\right]$$

By using L'Hopital Rule.

$$\frac{-bx}{e^{x/b}} \Rightarrow \left.\frac{-b}{\frac{1}{b}e^{x}}\right|_{x=\infty} = \text{Zero.}$$

$$\Rightarrow \bar{X} = \frac{e^{a/b}}{b}\left[0 + ab\,e^{-a/b} + \left.\frac{b\,e^{-x/b}}{-1/b}\right|_a^{\infty}\right]$$

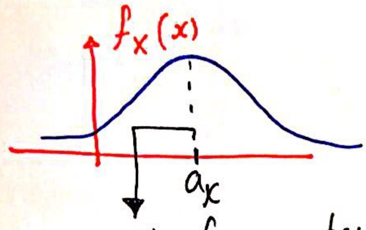
$$= \frac{1}{b} e^{a/b}\left[ab\,e^{-a/b} + 0 + b^2 e^{-a/b}\right]$$

$$\Rightarrow \bar{X} = a + b$$

$\hookrightarrow$ e.g. $X \sim \exp(1,3) \Rightarrow \bar{X} = 1+3 = \underline{\underline{4}}$

e.g. $X \sim \exp(-1,5) \Rightarrow \bar{X} = -1+5 = \underline{\underline{4}}$

Example: If R.V $X \sim N(a_x, \sigma_x^2)$, Find $\bar{X}$?



axis of symmetry.

since a_x is axis of symmetry $\Rightarrow$ $\bar{X} = a_x$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}$$

***Mathematically:** $\bar{X} = \int_{-\infty}^{\infty} x f_X(x)\, dx \Rightarrow$ "By substitution".

let $y = \frac{x - a_x}{\sigma_x} \rightarrow dy = \frac{1}{\sigma_x} dx$ & $y = -\infty \rightarrow \infty$

$\hookrightarrow \sigma_x y + a_x = x$

$$\bar{X} = \int_{-\infty}^{\infty} \frac{(a_x + \sigma_x y)}{\sqrt{2\pi \cancel{\sigma_x^2}}} e^{-y^2/2} \cdot \cancel{\sigma_x} dy = \int_{-\infty}^{\infty} \frac{a_x}{\sqrt{2\pi}} e^{-y^2/2} dy + \int_{-\infty}^{\infty} \frac{\sigma_x}{\sqrt{2\pi}} \underbrace{y}_{\text{odd}} \underbrace{e^{-y^2/2}}_{\text{even}} dy$$

$$= a_x \underbrace{\int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy}_{X \sim N(0,1) \Rightarrow \text{Area}=1} + \int_{-\infty}^{\infty} \text{odd } dy = a_x + 0$$

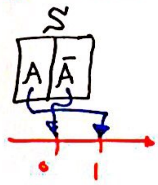
$$\Rightarrow \boxed{\bar{X} = a_x} \ \#$$

Example: $X \sim$ Bernouli(P), Find $\bar{X}$?

Solution: $X = \{0, 1\}$

$P(A) = P$

$P(\bar{A}) = 1 - P$



$f_X(x) = (1-p)\,\delta(x) + p\,\delta(x-1)$

$$\bar{X} = \sum_{i=1}^{2} x_i \, P\{X = x_i\}$$

$\Rightarrow \bar{X} = (0)\,P\{X=0\} + (1)\,P\{X=1\} = 0 + P \Rightarrow \bar{X} = P$

$\hookrightarrow$ e.g. $X \sim$ Bernouli$(0.4) \Rightarrow \underline{P = 0.4}$ $\quad f_X(x) = 0.6\,\delta(x) + 0.4\,\delta(x-1)$

$\Rightarrow \underline{\bar{X} = 0.4}$

Example: $X \sim$ Binomial(p, N), Find $\bar{X}$?

Solution: $X = \{\overset{x_1}{0}, \overset{x_2}{1}, 2, \ldots, \overset{x_{N+1}}{N}\} \Rightarrow f_X(x) = \sum_{i=1}^{N+1} P\{X = x_i\}\,\delta(x - x_i)$

$\Rightarrow P\{X = x_i\} = \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i}$

$$\Rightarrow \bar{X} = \sum_{i=1}^{N+1} x_i \, P\{X = x_i\} = \cancel{(0)\,P\{X=0\}} + (1)\,P\{X=1\} + \cdots + N\,P\{X=N\}$$

$$= \sum_{i=1}^{N} i \, P\{X = i\} = \sum_{i=1}^{N} i \binom{N}{i} p^i (1-p)^{N-i} = \sum_{i=1}^{N} \cancel{i} \frac{N!}{\cancel{i}(i-1)!\,(N-i)!} p^i (1-p)^{N-i}$$

$$= Np \sum_{i=1}^{N} \frac{(N-1)!}{(i-1)!\,(N-i)!} p^{i-1} (1-p)^{N-i}$$

$\Rightarrow \Rightarrow$ Continue $\Rightarrow \Rightarrow$

$\Rightarrow$ Let $K = i-1 \rightarrow i = 1+K$

$$\bar{X} = NP \sum_{K=0}^{N-1} \frac{(N-1)!}{K!\,(N-1-K)!} P^K (1-P)^{N-1-K}$$ Let $N-1 = M$

$$\Rightarrow \bar{X} = NP \sum_{K=0}^{M} \frac{M!}{K!\,(M-K)!} P^K (1-P)^{M-K} = NP \sum_{K=0}^{M} \binom{M}{K} P^K (1-P)^{M-K}$$

$X = \{0, 1, 2, \ldots, M\} \Rightarrow P\{X = i\} = \binom{M}{i} P^i (1-P)^{M-i}$

So $\sum_{K=0}^{M} \binom{M}{K} P^K (1-P)^{M-K}$ represent the summation of all proparties so it gives $\underline{\underline{1}}$.

$$\Rightarrow \bar{X} = NP * 1 \Rightarrow \bar{X} = NP$$

※ Expectation of function of R.V:

X is R.V.

Let $g(X)$ is function of the R.V X.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

* If X is discrete R.V:

Then, $E[g(x)] = \sum_{i=1}^{N} g(x_i) P\{X = x_i\}$

<u>Example:</u> $X = \{-1, 2, 5, 9\}$, $P\{X=-1\} = 0.1$, $P\{X=2\} = 0.6$, $P\{X=5\} = 0.15$

Let $g(X) = X^2 - 1$, Find $E[g(X)]$?

<u>Solution:</u> $g(X) = \{g(x_1), g(x_2), g(x_3), g(x_4)\} = \{0, 3, 24, 80\}$

$$E[g(X)] = \sum_{i=1}^{4} g(x_i) P\{X = x_i\} = g(x_1) P\{X = x_1\} + g(x_2) P\{X = x_2\} + g(x_3) P\{X = x_3\} + g(x_4) P\{X = x_4\}$$

$$= (0)(0.1) + (3)(0.6) + (24)(0.15) + (80)(0.15) = \boxed{17.4}.$$

Example: let X a R.V with $f_X(x)$, and $g(X) = aX + b$.
where a & b real numbers & they are constants.
Determine $E[g(X)]$?

Solution: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx$

$$\Rightarrow E[g(X)] = a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} f_X(x)\,dx$$

$$\hookrightarrow E[g(X)] = a\bar{X} + b * (1)$$

$$\Rightarrow E[g(X)] = a\bar{X} + b$$

$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$f_X(x) = \sum_{i=1}^{N} P\{x_i\}\,\delta(x - x_i)$

$E[X] = \int_{-\infty}^{\infty} x \sum_{i=1}^{N} P\{x_i\}\,\delta(x-x_i)\,dx$

$= \sum_{i=1}^{N} P\{x_i\} \int_{-\infty}^{\infty} x\,\delta(x-x_i)\,dx$

$= \sum_{i=1}^{N} x_i\,P\{x_i\}$ ※

so, No difference in using $E[X]$ in form of integration or summation in this Ex.

* Some proparties:

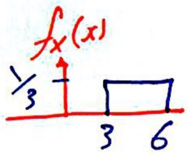
- $E[aX] = aE[X]$
- $E[b] = b$ (average of constant is the constant itself.)

$\hookrightarrow$ e.g. Let X a R.V with $\bar{X} = 2$ find:

a) $E[-X] = -1\,\bar{X} = -2$

b) $E[X-1] = \bar{X} - 1 = 1$

Example: let $X \sim U(3,6)$, find $E[X^2]$?



solution: $g(X) = X^2$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{3}^{6} \frac{1}{3}x^2\,dx = \frac{1}{9}x^3\Big]_3^6 = \frac{6^3 - 3^3}{9} = \boxed{21}.$$

※ **Moments about the orgin:**

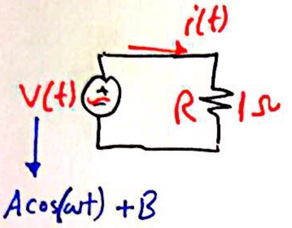
$m_n = E[X^n]$, $X^n \hookrightarrow g(X)$

$n = 0, 1, 2, \ldots$
0 or +ve integer.

$$\Rightarrow m_n = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

- Zeroth order:
  $m_0 = E[X^0] = E[1] = \underline{\underline{1}}$
- first order:
  $m_1 = E[X^1] = E[X] = \underline{\underline{\bar{X}}}$ (average value (DC)) $\rightarrow$ DC average power.
- Second Order:
  $m_2 = E[X^2]$ (Total Average Power in the R.V X) $\rightarrow$ AC power + DC power.

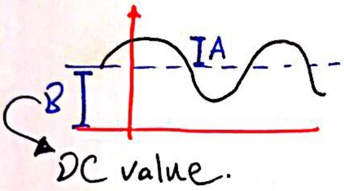
*Recall:



• $P(t) = v(t)\, i(t) = \frac{v^2(t)}{R} = v^2(t)$

• Average Power $(\bar{P}) = \frac{1}{T}\int v^2(t)\, dt$

$$\Rightarrow \bar{P} = \frac{1}{T}\int_T \left(B^2 + A^2\cos^2(\omega t) + 2AB\cos(\omega t)\right) dt .$$



Do the integration & remember that integration of cosine on its period is zero.

$$\Rightarrow \bar{P} = B^2 + \frac{A^2}{2} + 0 \Rightarrow \bar{P} = \underbrace{\frac{A^2}{2}}_{\text{AC Average power.}} + \underbrace{B^2}_{\text{DC Average power.}}$$

* AC average power is:

$$X = E[X^2] - \bar{X}^2 = m_2 - m_1^2 \triangleq \underbrace{Var(X)}_{\text{variance.}} \triangleq \sigma_x^2$$

Example: $X \sim exp(a,b)$, Find m_2?

Solution: $m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^{\infty} x^2 \cdot \frac{1}{b}\, e^{-\frac{(x-a)}{b}}\, dx$.

$\Rightarrow m_2 = \frac{e^{a/b}}{b}\int_a^{\infty} x^2 e^{-x/b}\, dx$ (Integration By Parts 2 times)

$\Rightarrow m_2 = (a+b)^2 + b^2$; if he aske to find $Var(X)$?

$\Rightarrow Var(X) = m_2 - m_1^2 = E[X^2] - E[X]^2 = (a+b)^2 + b^2 - (a+b)^2$

$\Rightarrow Var(X) = b^2$

Example: $X \sim U(a,b)$, Find: a) m_0 b) m_1 c) m_2 d) $\sigma_x^2 \triangleq Var(X)$

Solution: a) $m_0 = E[X^0] = \boxed{1}$

b) $m_1 = \bar{X} = \boxed{\frac{a+b}{2}}$

c) $m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^b \frac{x^2}{b-a}\, dx = \frac{1}{b-a} \cdot \frac{x^3}{3}\Big]_a^b$

$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} = \boxed{\frac{b^2+ab+a^2}{3}}$

d) $Var(X) = E[X^2] - E[X]^2 = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$

$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12} = \frac{b^2-2ab+a^2}{12} = \boxed{\frac{(b-a)^2}{12}}$

※ Moments about the Mean: (Central Moments).

$\mu_n = E[(X-\bar{X})^n]$, $n = 0, 1, 2, \ldots$

- $\mu_0 = E[(X-\bar{X})^0] = 1$
- $\mu_1 = E[X-\bar{X}] = E[X] - E[\bar{X}] = \text{Zero}$. ($E[\bar{X}] = \bar{X}$)
- $\mu_2 = E[(X-\bar{X})^2] \triangleq Var(X) \triangleq \sigma_x^2$

$\Rightarrow$ for μ_2:

$$\mu_2 = E[X^2 - 2\bar{X}X + \bar{X}^2] = E[X^2] - 2\bar{X}E[X] + \bar{X}^2 = \underline{\underline{E[X^2] - \bar{X}^2}}$$

$$\Rightarrow \mu_2 = m_2 - m_1^2$$

Example: $X \sim N(a_x, \sigma_x^2)$.

- $\bar{X} = a_x$
- $Var(X) = \sigma_x^2$ (prove?)

$Var(X) = E[X^2] - \bar{X}^2 = E[X^2] - a_x^2 \triangleq \sigma_x^2$

$$\Rightarrow E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-a_x)^2}{2\sigma_x^2}} dx \quad \text{(Do the integration)} = \sigma_x^2 + a_x^2$$

※ Characteristic Function $\phi_X(\omega)$:

DEF.: R.V X with $f_X(x)$, Then: $\phi_X(\omega) = E[\underbrace{e^{j\omega X}}_{g(X)}]$

$$\Rightarrow \boxed{\begin{aligned} \phi_X(\omega) &= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx \\ f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x}\phi_X(\omega)\,d\omega \end{aligned}}$$

$m_n = E[X^n] = \int x^n f_X(x)\,dx$ (Hard to solve)

$\Rightarrow$ we use: $m_n = (-j)^n \left.\dfrac{d^n \phi_X(\omega)}{d\omega^n}\right|_{@\omega=0}$

Example: $X \sim \exp(a, b)$, Find: a) $\phi_X(\omega)$ b) m_1 & m_2 using part $\underline{\underline{a}}$.

Solution: $\phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx = \int_a^{\infty} e^{j\omega x} \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}} dx$.

$$\Rightarrow \phi_X(\omega) = \frac{e^{a/b}}{b}\int_a^{\infty} e^{j\omega x} \cdot e^{-x/b} dx = \frac{e^{a/b}}{b}\int_a^{\infty} e^{-(\frac{1}{b}-j\omega)x} = \frac{e^{a/b}}{b}\left[\frac{e^{-(\frac{1}{b}-j\omega)x}}{-(\frac{1}{b}-j\omega)}\right]_a^{\infty}$$

$$\phi_X(\omega) = \frac{e^{a/b}}{b}\left[\frac{e^{-(\frac{1}{b}-j\omega)a}}{1/b - j\omega}\right] \Rightarrow \phi_X(\omega) = \frac{e^{j\omega a}}{1 - j\omega b}$$

(b) $m_1 = (-j)^1 \dfrac{d\phi_X(\omega)}{d\omega}\Big|_{@\omega=0} = -j\left[\dfrac{(1-j\omega b)\, ja\, e^{j\omega a} - e^{j\omega a}\,(-jb)}{(1-j\omega b)^2}\right]$

$\Rightarrow m_1\Big|_{@\omega=0} = -j\;\dfrac{ja+jb}{1} = -jj\,(a+b) \Rightarrow m_1 = a+b$

$m_2 = (-j)^2 \dfrac{d^2\phi_X(\omega)}{d\omega^2}\Big|_{@\omega=0} = \dots \Rightarrow m_2 = (a+b)^2+b^2$

Homework: for $X \sim U(a,b)$ **Find** $\phi_X(\omega)$? **Answer:** $\boxed{\phi_X(\omega) = \dfrac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}}$

※ Moments Generating Function:

$$M_X(\nu) = E[e^{\nu X}] = \int_{-\infty}^{\infty} e^{\nu x} f_X(x)\,dx \qquad m_n = \frac{d^n M_X(\nu)}{d\nu^n}\Big|_{@\nu=0}$$

Example: $X \sim \exp(a,b)$

- $M_X(\nu) = \dfrac{e^{\nu a}}{1-b\nu}$
- $m_1 = \bar{x} = \dfrac{(1-\nu b)(a e^{\nu a}) - e^{\nu a}(-b)}{(1-b\nu)^2}\Big|_{@\nu=0} = \underline{\underline{a+b}}$

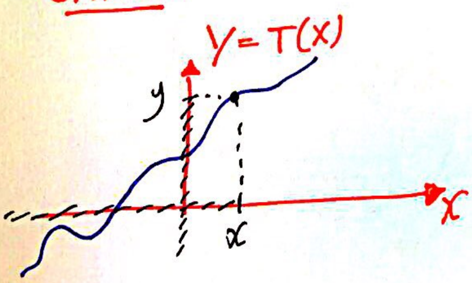
※ Transformation of R.V:

R.V X —— $\boxed{T(\cdot)}$ —— $Y = T(X)$ ← Transformation.

$\left.\begin{array}{l} F_X(x) \\ f_X(x) \end{array}\right\}$ are Known. $\Rightarrow$ find $\begin{array}{l} F_Y(y) \\ f_Y(y) \end{array}$?

$T(\cdot)$ → monotonic → Increasing. / Decreasing.
$T(\cdot)$ → Non-monotonic.

CASE1: Monotonic - Increasing:



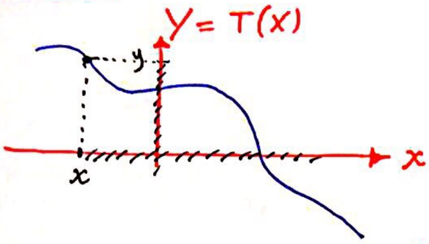
$Y = T(X) \Rightarrow X = T^{-1}(Y)$

$F_Y(y) = P\{Y \le y\} = P\{X \le x\} = P\{X \le T^{-1}(y)\}$

$\Rightarrow \boxed{F_Y(y) = F_X\left(T^{-1}(y)\right)}$ *

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(T^{-1}(y))}{dy} \underset{\text{chain Rule.}}{=} \frac{dF_X(T^{-1})}{dy}\cdot\frac{dT^{-1}(y)}{dy} \Rightarrow \boxed{f_Y(y) = f_X(T^{-1}(y))\cdot\frac{dT^{-1}(y)}{dy}} \;**$$

CASE 2: Monotonic - Decreasing:



$$F_Y(y) = P\{Y \leq y\} = P\{X \geq x\} = 1 - P\{X \leq x\}$$
$$= 1 - P\{X \leq T^{-1}(y)\}$$

$$\Rightarrow \boxed{F_Y(y) = 1 - F_X(T^{-1}(y))} \; *$$

$$\Rightarrow f_Y(y) = \frac{d F_Y(y)}{dy} = \boxed{-f_X(T^{-1}(y)) \cdot \frac{d T^{-1}(y)}{dy}} \; **$$

$*$ $\frac{d T^{-1}(y)}{dy} \equiv$ Represent the slope., Notice in Case2: (-ve) slope $\Rightarrow$ (-ve) $f_Y(y)$.

$*$ So, we could write Monotonic - T as follows:

$$f_Y(y) = f_X(T^{-1}(y)) \left| \frac{d T^{-1}(y)}{dy} \right| \; ***$$

Example: $X \sim N(a_x, \sigma_x^2)$, Define $Y = T(X) = aX + b$, a & b are real constants.

$\Rightarrow$ Find $f_Y(y)$?

Solution:
- $\bar{Y} = a\bar{X} + b = a\,a_x + b$
- $\sigma_Y^2 = a^2 \sigma_x^2$ (variance for the constant is zero).

$$f_X(x) = \frac{1}{\sqrt{2\pi \sigma_x^2}} \cdot e^{\frac{-(x-a_x)^2}{2\sigma_x^2}} \Rightarrow f_Y(y) = f_X(T^{-1}(y)) \cdot \left| \frac{d T^{-1}(y)}{dy} \right|$$

$$Y = aX + b \Rightarrow X = T^{-1}(y) = \frac{y-b}{a} \Rightarrow \frac{d T^{-1}(y)}{dy} = \frac{1}{a}$$

So, $f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$ (Now replace each x by $\frac{y-b}{a}$ in $f_X(x)$)

$$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi \sigma_x^2}} \cdot e^{-\frac{\left(\frac{y-b}{a} - a_x\right)^2}{2\sigma_x^2}} \cdot \frac{1}{|a|}$$

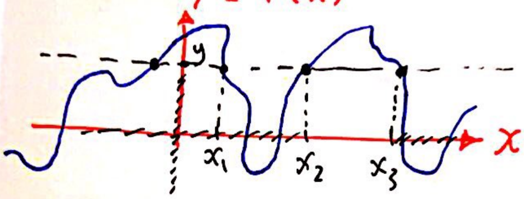
put it inside the root.

$$= \frac{1}{\sqrt{2\pi \underbrace{\sigma_x^2 a^2}_{\sigma_Y^2}}} \, e^{\frac{-(y - \overbrace{(a_x a + b)}^{a_y})^2}{2 \underbrace{a^2 \sigma_x^2}_{\sigma_Y^2}}} \Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \, e^{\frac{-(y-a_y)^2}{2\sigma_y^2}}$$

$*$ Notice that the change just in the mean & the variance.

CASE 3: Non-Monotonic:

$Y = T(X)$



$F_Y(y) = P\{Y \le y\}$ (Hard to find)

we use:

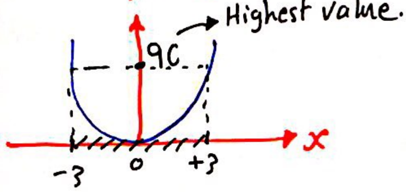
$$\boxed{f_Y(y) = \sum_n \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx}\Big|_{x=x_n} \right|}}$$

$x_1, x_2, x_3 \dots x_n$ are the roots for $Y - T(x) = 0$

Example: $X \sim U(-3,3)$, $Y = T(X) = CX^2$, C is constant, Find $f_Y(y)$?

Solution:

$Y = T(X)$



$\Rightarrow -3 \le X \le +3$

$0 \le Y \le 9C$

$Y = T(x) \Rightarrow Y - T(X) = 0$

$Y - CX^2 = 0 \Rightarrow X^2 = Y/C \Rightarrow X = \pm\sqrt{Y/C}$

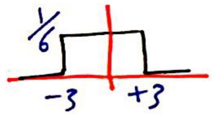
$x_1 = \sqrt{y/c}$

$x_2 = -\sqrt{y/c}$

- $\frac{dT(x)}{dx} = 2CX$

$$\Rightarrow f_Y(y) = \frac{f_X(x_1)}{\left|\frac{dT(x)}{dx}\Big|_{x=x_1}\right|} + \frac{f_X(x_2)}{\left|\frac{dT(x)}{dx}\Big|_{x=x_2}\right|}$$

$f_X(x)$



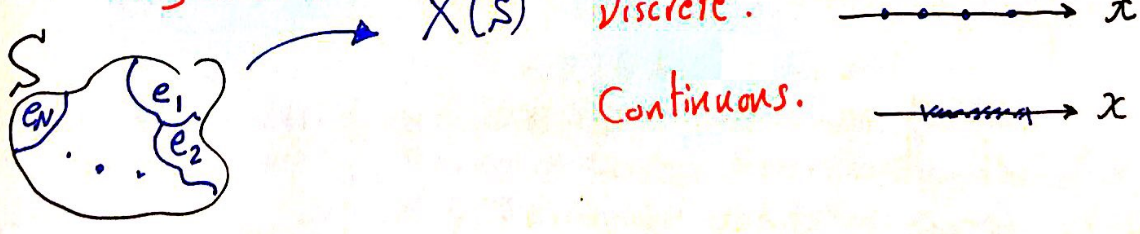
$$= \frac{1/6}{|2Cx_1|} + \frac{1/6}{|2Cx_2|} = \frac{1/6}{2C\sqrt{y/c}} + \frac{1/6}{2C\sqrt{y/c}}$$

$$\boxed{f_Y(y) = \frac{1}{6\sqrt{Cy}}} \quad ; \quad 0 \le y \le 9C$$

End of CH3 * * * * End of CH3

※ CHAPTER (4): Multiple R.V:

single RV

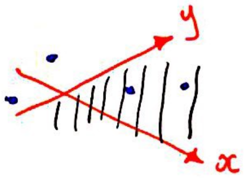


$X(S)$ Discrete.

Continuous.

* two-dimensional R.V:

S: outcomes $\to X(s)$, $Y(s)$



(X, Y).

⇒ N-Dimensional R.V:

S: $X_1(s)$, $X_2(s)$, $\vdots$, $X_N(s)$ $\Rightarrow (X_1, X_2, \dots, X_N)$

But we will focuse on 2-Dim.

* $(X,Y) \longrightarrow \boxed{F_{X,Y}(x,y) = P\{\underbrace{X \le x}_{A \subseteq S} \underbrace{,}_{\text{"and"}, \cap} \underbrace{Y \le y}_{B \subseteq S}\}}$

Called: "Joint Distribution Function".

$\Downarrow$

$A \cap B$.

* Proparties for $F_{X,Y}(x,y)$:

① $F_{X,Y}(-\infty,-\infty) = 0$ ⇒ proof: $F_{X,Y}(-\infty,-\infty) = P\{X \le -\infty, Y \le -\infty\} = P\{\phi \cap \phi\} = 0$ #

$F_{X,Y}(-\infty,y) = 0$ ⇒ proof: $F_{X,Y}(-\infty,y) = P\{X \le -\infty, Y \le y\} = P\{\phi \cap B\} = 0$ #

$F_{X,Y}(x,-\infty) = 0$ ⇒ Proof: $F_{X,Y}(x,-\infty) = P\{X \le x, Y \le -\infty\} = P\{B \cap \phi\} = 0$ #

② $F_{X,Y}(\infty,\infty) = 1$ ⇒ proof: $F_{X,Y}(\infty,\infty) = P\{X \le \infty, Y \le \infty\} = P\{S \cap S\} = P\{S\} = 1$ #

③ $0 \le F_{X,Y}(x,y) \le 1$

④ $F_{X,Y}(x,y)$ is Non-decreasing function.

⑤ $P\{x_1 < X \le x_2, y_1 < Y \le y_2\} = F_{X,Y}(x_2,y_2) + F_{X,Y}(x_1,y_1) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2)$

Exercise: prove proparty #5.

⑥ $F_{X,Y}(x,\infty) = F_X(x)$ ⇒ proof: $F_{X,Y}(x,\infty) = P\{X \le x, Y \le \infty\} = P\{\underbrace{X \le x}_{A} \cap S\}$

$= P\{X \le x\} = F_X(x)$

$F_{X,Y}(\infty,y) = F_Y(y)$

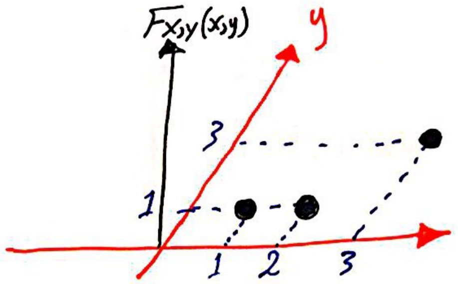
- $F_{X,Y}(x,y) \equiv$ Joint distribution function.
- $F_X(x) \equiv$ Marginal distribution function of X.
- $F_Y(y) \equiv$ Marginal distribution function of Y.

Example: Discrete 2-Dim. R.V (X,Y).

$(X,Y) \in \{(1,1),(2,1),(3,3)\}$

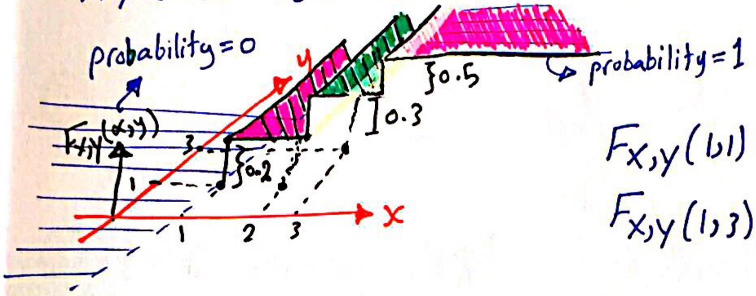
Given that: $P\{(1,1)\} = 0.2$, $P\{(2,1)\} = 0.3$, $P\{(3,3)\} = 0.5$

Find $F_{X,Y}(x,y)$?



Solution:

$$F_{X,Y}(x,y) = P\{X \le \overset{(-\infty,\infty)}{x}, Y \le \overset{(-\infty,\infty)}{y}\}$$



$F_{X,Y}(1,1) = P\{X \le 1, Y \le 1\} = P\{X=1, Y=1\} = P(1,1) = 0.2$

$F_{X,Y}(1,3) = P\{X \le 1, Y \le 3\} = P(1,1) = 0.2$

$\Rightarrow$ Now, writing $F_{X,Y}(x,y)$ in form of unit step:

$$F_{X,Y}(x,y) = 0.2\,u(x-1)\,u(y-1) + 0.3\,u(x-2)\,u(y-1) + 0.5\,u(x-3)\,u(y-3)$$

$F_X(x) = F_{X,Y}(x,\infty) = 0.2\,u(x-1) + 0.3\,u(x-2) + 0.5\,u(x-3) \Rightarrow X \in \{1,2,3\}$

$F_Y(y) = F_{X,Y}(\infty, x) = 0.2\,u(y-1) + 0.3\,u(y-1) + 0.5\,u(y-3) = 0.5\,u(y-1) + 0.5\,u(y-3)$

$\Rightarrow Y \in \{1,3\}$

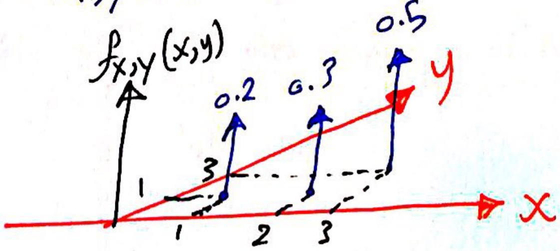
※ Joint Density Function:

$(X,Y) \longrightarrow F_{X,Y}(x,y)$

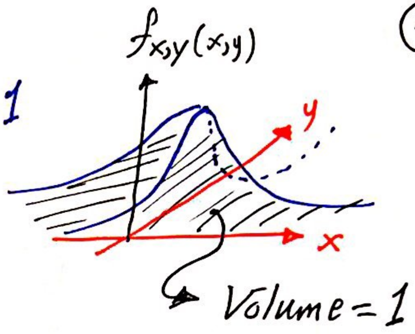
$\hookrightarrow f_{X,Y}(x,y)$:
$$f_{X,Y}(x,y) = \frac{d^2 F_{X,Y}(x,y)}{dx\,dy}$$

Example: for the previous example find $f_{X,Y}(x,y)$?

Solution: $f_{X,Y}(x,y) = 0.2\,\delta(x-1)\delta(y-1) + 0.3\,\delta(x-2)\delta(y-1) + 0.5\,\delta(x-3)\delta(y-3)$



* Properties of $f_{X,Y}(x,y)$:



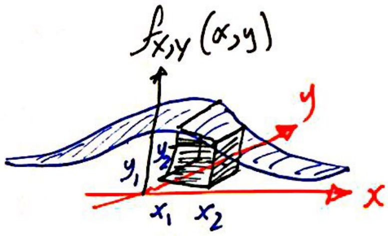
① $f_{X,Y}(x,y) \geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

③ $F_{X,Y}(x,y) = \int_{\xi_2=-\infty}^{y}\int_{\xi_1=-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$

④ $F_X(x) = \int_{\xi_1=-\infty}^{x}\int_{\xi_2=-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)\,d\xi_2\,d\xi_1$

⑤ $F_Y(y) = \int_{\xi_2=-\infty}^{y}\int_{\xi_1=-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$



⑥ $P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy$

$P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\}$

⑦ $$\boxed{\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \end{aligned}}$$

Example: Given $g(x,y) = \begin{cases} b e^{-x}\cos(y), & 0 \leq x \leq 2,\ 0 \leq y \leq \pi/2 \\ 0, & \text{o.w} \end{cases}$

find the value of $\underline{\underline{b}}$ such that $g(x,y)$ is density function?

Solution: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy = 1 = \int_0^{\pi/2}\int_0^{2} b\, e^{-x}\cos(y)\,dx\,dy$

$\Rightarrow\ b\int_0^{\pi/2} \cos(y)\left(\int_0^2 e^{-x}\,dx\right)dy = b\int_0^{\pi/2}\cos y\,(1-e^{-2})\,dy$

$= b(1-e^{-2})\,\sin y\,\Big|_0^{\pi/2} = b(1-e^{-2}) * (1-0) = 1$

$\Rightarrow\ b = \dfrac{1}{1-e^{-2}}\ \Rightarrow\ \boxed{b = 1.1565}$

Example: Given $f_{X,Y}(x,y) = x\, e^{-x(y+1)}\, u(x)\, u(y)$

Find the marginal density functions?

Solution: Need to find $f_X(x)$ & $f_Y(y)$!?

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = x\, u(x)\, e^{-x} \int_0^{\infty} e^{-xy}\, dy$$

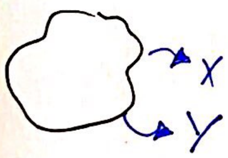
$$= \cancel{x} e^{-x} u(x) \left. \frac{e^{-xy}}{-\cancel{x}} \right|_0^{\infty} = e^{-x} u(x) [1-0] \Rightarrow \boxed{f_X(x) = e^{-x} u(x)}$$

$$\Rightarrow f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = u(y) \underbrace{\int_0^{\infty} x e^{-x(y+1)}\, dx}_{\text{By Parts.}}$$

$u = x \rightarrow du = dx$

$dv = e^{-x(y+1)} dx \rightarrow v = \dfrac{e^{-x(y+1)}}{-(y+1)}$ solving: .... $\boxed{f_Y(y) = \dfrac{u(y)}{(1+y)^2}}$

* Satistical Independence:



(X, Y) → Joint Distribution Function $F_{X,Y}(x,y)$.
→ Joint Density Function $f_{X,Y}(x,y)$.

If X & Y are two independent, Then:

$$F_{X,Y}(x,y) = P\{X \leq x \cap Y \leq y\} = P\{X \leq x\} \cdot P\{Y \leq y\}$$

$$\Rightarrow \boxed{F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)}$$

$$* \quad f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y} = \frac{dF_Y(y)}{dy} \cdot \frac{dF_X(x)}{dx} \Rightarrow \boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

*** For the N independent R.V's $X_1, X_2, \cdots, X_N$:**

$$f_{X_1 X_2 \cdots X_N}(x_1, x_2, \cdots, x_N) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot \cdots \cdot f_{X_N}(x_N)$$

Example: $f_{X,Y}(x,y) = x e^{-x(y+1)} u(x)\, u(y)$ Are X & Y independent?

Solution: as found before:

$f_X(x) = e^{-x} u(x) \Rightarrow f_X(x) \cdot f_Y(y) = \dfrac{1}{(y+1)^2} e^{-x} u(x)\, u(y) \neq f_{X,Y}(x,y)$

$f_Y(y) = \dfrac{u(y)}{(1+y)^2}$

so, X & Y are NOT independent.

Example: $f_{X,Y}(x,y) = \frac{1}{12} u(x) u(y) e^{-\frac{x}{4} - y/3}$

Are X&Y independent ?

Solution: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{1}{12} u(x) e^{-\frac{x}{4}} \int_{0}^{\infty} e^{-y/3} dy$

$$= \frac{1}{12} u(x)\, e^{-x/4} \left(\frac{e^{-y/3}}{\frac{1}{3}} \Big|_{\infty}^{0} \right) \Rightarrow \boxed{f_X(x) = \frac{1}{4} u(x)\, e^{-x/4}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{1}{12} u(y)\, e^{-y/3} \int_{0}^{\infty} e^{-x/4} dx \Rightarrow \boxed{f_Y(y) = \frac{1}{3} u(y)\, e^{-y/3}}$$

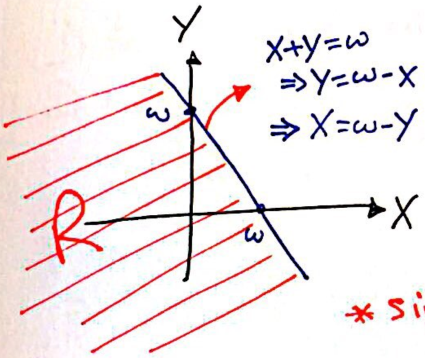
$\Rightarrow f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y) \Rightarrow$ so, X&Y are Independent.

※ Density Function of Sum of Two Independent R.V's:

IF X&Y are two independent R.V's with $f_{X,Y}(x,y)$.

Define: $W = X+Y$, Need to find $f_W(w)$?!

$$\Rightarrow F_W(w) = P\{W \leq w\} = P\{X+Y \leq w\}$$



$$\Rightarrow F_W(w) = \iint_R f_{X,Y}(x,y)\,dx\,dy$$

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_{X,Y}(x,y)\,dx\,dy$$

* since two independent:

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_X(x) \cdot f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\,dx\,dy$$

for $f_W(w) = \frac{d F_W(w)}{dw} = \int_{-\infty}^{\infty} f_Y(y) \left[\frac{d \int_{-\infty}^{w-y} f_X(x)\,dx}{dw} \right] dy$

using: Appendix G "Leibniz's Theorem".

$$\frac{d \int_{-\infty}^{w-y} f_X(x)\,dx}{dw} = f_X(w-y) \cdot \frac{d(w-y)}{dw} - \cancelto{0}{f_X(-\infty) \cdot \frac{d(-\infty)}{dw}} + \int_{-\infty}^{w-y} \cancelto{0}{\frac{d f_X(x)}{dw}}\,dx = f_X(w-y)$$

$$\Rightarrow f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\,dy \Rrightarrow \boxed{f_W(w) = f_Y(y) * f_X(x)}$$

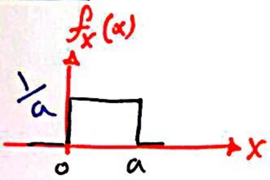
Convolution.

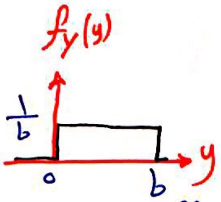
OR: $f_W(\omega) = f_X(x) \star f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(\omega - x)\, dx.$

Example: $X \sim U(0, a)$ & $Y \sim U(0, b)$, $b > a > 0$, X & Y are independent.

Find the density function for $W = X + Y$?

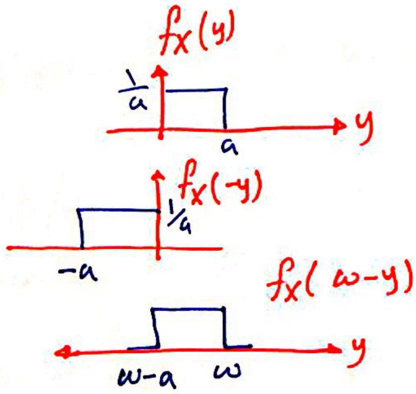
Solution:

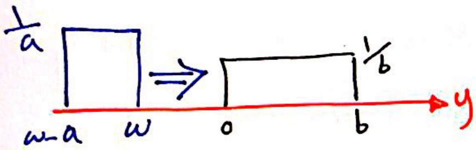




* Most of the time we move the one with the smallest width.

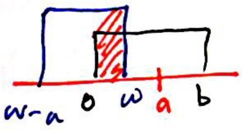
$$\Rightarrow f_W(\omega) = f_Y(y) \star f_X(x) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(\omega - y)\, dy$$



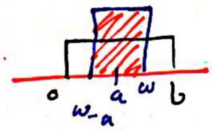


1) for $\omega < 0$: $f_W(\omega) = 0$

2) for $0 < \omega < a$: $f_W(\omega) = \int_0^{\omega} \frac{1}{ab}\, dy = \frac{\omega}{ab}$

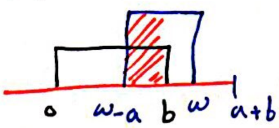


3) for $a < \omega < b$:



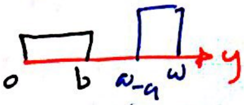
$f_W(\omega) = \int_{\omega-a}^{\omega} \frac{1}{ab}\, dy = \frac{\omega - \omega + a}{ab} = \frac{1}{b}$

4) for $b+a > \omega > b$:



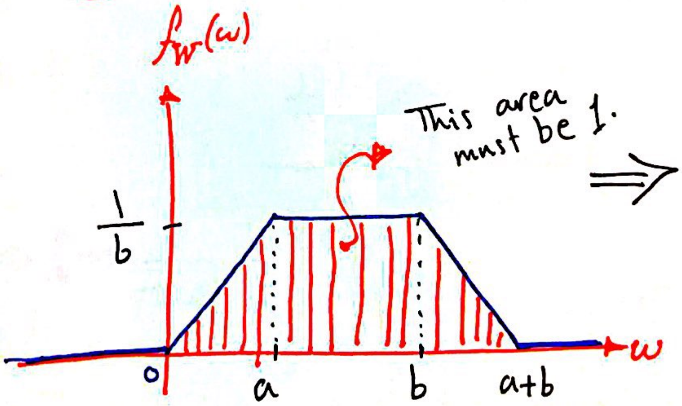
$f_W(\omega) = \int_{\omega-a}^{b} \frac{1}{ab}\, dy = \frac{b - \omega + a}{ab} = \frac{(a+b) - \omega}{ab}$

5) $a+b < \omega$:



$f_W(\omega) = 0$

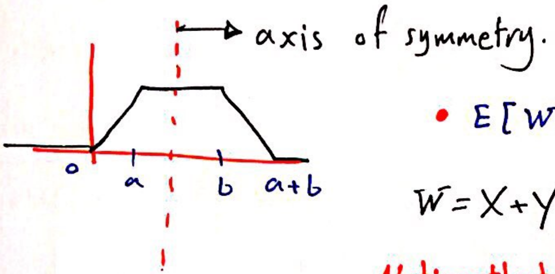
$$f_W(\omega) = \begin{cases} 0 & , \omega < 0 \\ \frac{1}{ab}\,\omega & , 0 < \omega < a \\ 1/b & , a < \omega < b \\ \frac{(a+b) - \omega}{ab} & , b < \omega < a+b \\ 0 & , \omega > a+b \end{cases}$$



$\Rightarrow$

⇒ check for the area:

$$\int_{-\infty}^{\infty} f_W(w)\,dw = \left(\frac{1}{2}a\ \frac{1}{b}\right) * 2 + \frac{b-a}{b} = 1 \ \checkmark$$



• $E[W] = \overline{W} = \int_{-\infty}^{\infty} w\, f_W(w)\,dw = \frac{a+b}{2}$

$W = X + Y \Rightarrow \overline{W} = \frac{a+b}{2}$, $\overline{X} = \frac{a}{2}$, $\overline{Y} = \frac{b}{2}$

Notice that: $\overline{W} = \overline{X} + \overline{Y}$

* More general: for $X_1, X_2, \ldots, X_N$ are independent R.V's (joint).

If $W = X_1 + X_2 + \cdots + X_N$

Then: $f_W(w) = f_{X_1}(x_1) \star f_{X_2}(x_2) \star \cdots \star f_{X_N}(x_N)$

↳ This is the exact density function for W.

Hard to find so we use the following Theorem:

* Central-Limit-Theorem (CLT):

If $X_1, X_2, \ldots X_N$ are indep. & $W = X_1 + \cdots + X_N$.

Then, the density function for W can be approximated as:

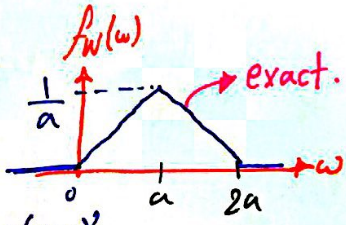
$W \sim N(a_w, \sigma_w^2)$; $a_w = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$

$\sigma_w^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$

Example: $X \sim U(0,a)$, $Y \sim U(0,a)$, $W = X + Y$.

Find: a) the exact $f_W(w)$. b) Find the approximate $f_W(w)$.

Solution: a) $f_W(w) = f_X(x) \star f_Y(y)$ as done before:



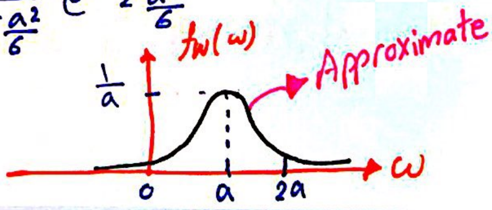
b) $W \sim N(a_w, \sigma_w^2)$

$a_w = \overline{X} + \overline{Y} = \frac{a}{2} + \frac{a}{2} = a$

$\sigma_w^2 = \sigma_x^2 + \sigma_y^2 = \frac{a^2}{12} + \frac{a^2}{12} = \frac{a^2}{6}$

$X \sim U(a,b) \rightarrow \sigma_x^2 = \frac{(b-a)^2}{12}$

$$\Rightarrow f_W(w) = \frac{1}{\sqrt{2\pi \frac{a^2}{6}}}\, e^{-\frac{(x-a)^2}{2\frac{a^2}{6}}}$$



Example: Given X_1, X_2, X_3 are indep. R.V's

Let $X = X_1 + X_2 + X_3$

Find: Approximate $f_X(x)$?

	mean	variance
X_1	-1	2
X_2	0.6	1.5
X_3	1.8	0.8

Solution:

$X \sim N(a_X, \sigma_X^2) \Rightarrow a_X = \bar{X}_1 + \bar{X}_2 + \bar{X}_3 = -1 + 0.6 + 1.8 = \underline{\underline{1.4}}$

$$\sigma_X^2 = \overset{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2 + 1.5 + 0.8} = \underline{\underline{4.3}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi(4.3)}} e^{-\frac{(x-1.4)^2}{2 * 4.3}}$$

END of CH4 * * * END of CH4.

※ CHAPTER (5): Operation on Multiple R.V's:

- Expectation of function of multiple R.V's:

Let X & Y are two joint R.V's with $f_{X,Y}(x,y)$, and $g(X,Y)$ is function of X & Y, Then:

$$\boxed{E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy.}$$

- Assume $g(X,Y) = g(X)$:

$$\Rightarrow E[g(X,Y)] = \iint g(x)\, f_{X,Y}(x,y)\, dx\, dy = \int_{-\infty}^{\infty} g(x) \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy}_{\text{represents } f_X(x).}\, dx$$

$$\Rightarrow \boxed{E[g(X,Y)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.}$$

practice: $f_{X,Y}(x,y) = u(x)\,u(y)\, e^{-x(y+1)}\, x$, $g^1_{X,Y}(x,y) = x e^{-(2+y)}$, $g^2_{X,Y}(x,y) = \frac{2}{x}$

Find a) $E[g^1_{X,Y}(x,y)]$

b) $E[g^2_{X,Y}(x,y)]$

Answers: for (a) use $E = \iint g(x,y)\, f_{X,Y}(x,y)\, dx\, dy.$

for (b) use $E = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx.$

Example: $X_1, X_2 \longrightarrow f_{X_1,X_2}(x_1,x_2)$

$X = \alpha_1 X_1 + \alpha_2 X_2$, α_1 & α_2 are Constants.

show that: $\overline{X} = \alpha_1 \overline{X_1} + \alpha_2 \overline{X_2}$

Solution: $X = \alpha_1 X_1 + \alpha_2 X_2 = g(X_1, X_2)$

$$\Rightarrow E[X] = \overline{X} = E[g(X_1,X_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\alpha_1 x_1 + \alpha_2 x_2)\, f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \alpha_1 \iint x_1 f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2 + \alpha_2 \iint x_2 f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2$$

$$= \alpha_1 \int x_1 \underbrace{\int f_{X_1,X_2}(x_1,x_2)\, dx_2}_{\text{Marginal of } X_1}\, dx_1 + \alpha_2 \int x_2 \underbrace{\int f_{X_1,X_2}(x_1,x_2)\, dx_1}_{\text{Marginal of } X_2}\, dx_2.$$

$$= \alpha_1 \int x_1 f_{X_1}(x_1)\, dx_1 + \alpha_2 \int x_2 f_{X_2}(x_2)\, dx_2$$

$$\overline{X} = \alpha_1 \overline{X_1} + \alpha_2 \overline{X_2} \quad \#$$

* **In general:** for $X = \sum_{i=1}^{N} \alpha_i X_i = \alpha_1 X_1 + \cdots + \alpha_N X_N$

$$\Rightarrow \overline{X} = \sum_{i=1}^{N} \alpha_i \overline{X_i} = \alpha_1 \overline{X_1} + \cdots + \alpha_N \overline{X_N}$$

Example: X_1, X_2, X_3 with $\overline{X_1} = -1$, $\overline{X_2} = 2$, $\overline{X_3} = 3$

Let $X = 2X_1 - X_2 + \frac{1}{2} X_3$, Find $\overline{X}$?

Solution: $\overline{X} = 2*(-1) - 2 + \frac{1}{2}*3 = -4 + \frac{3}{2} = \underline{\underline{-2.5}}$

Example: $X = \alpha_1 g_1(x_1) + \alpha_2 g_2(x_2)$ show that: $\overline{X} = \alpha_1 \overline{g_1(x_1)} + \alpha_2 \overline{g_2(x_2)}$

Solution: it would be as the previous proof just replace each X_1 by $g_1(X_1)$ & X_2 by $g_2(X_2)$.

※ Joint Moments:

For two R.V's $X \& Y \rightarrow f_{X,Y}(x,y)$

$$m_{nK} = E[X^n Y^K] = \iint x^n y^K f_{X,Y}(x,y)\, dx\, dy$$

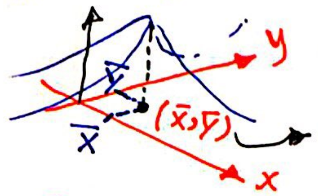
$n, K = 0, 1, 2, \ldots$ * $(n+K)$ is the order.

- Zeroth joint moment:

$m_{00} = E[1] = 1$

- First joint moment:

$m_{10} = E[X]$
$m_{01} = E[Y]$ → center of gravity.



- Second joint moment:

$m_{20} = E[X^2] \triangleq m_2$ for X.

$m_{02} = E[Y^2] = m_2$ for Y.

$m_{11} = E[XY] = R_{XY}$ → Correlation of $X \& Y$.

Notes:

* If $R_{XY} = 0 \Rightarrow X \& Y$ are orthogonal.
* If $R_{XY} = E[XY] \triangleq E[X] \cdot E[Y] \Rightarrow X \& Y$ are Uncorrelated.
* If $X \& Y$ are Independent $\Rightarrow$ Then they must be uncorrelated.

proof: $R_{XY} = E[XY] = \iint xy\, f_{X,Y}(x,y)\, dx\, dy = \iint xy\, \underbrace{f_X(x) f_Y(y)}_{\text{since indep}}\, dx\, dy$

$= \int_{-\infty}^{\infty} y f_Y(y) \underbrace{\int_{-\infty}^{\infty} x f_X(x)\, dx}_{\bar{X}}\, dy = \bar{X} \underbrace{\int_{-\infty}^{\infty} y f_Y(y)\, dy}_{\bar{Y}}$

$= \bar{X} \cdot \bar{Y}$ (so, uncorrelated).

* If $X \& Y$ are Uncorrelated $\Rightarrow$ Not necessary to be independent. (We don't Know).
* Independent $\Rightarrow f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$.
* Uncorrelated $\Rightarrow E[XY] = R_{XY} = \bar{X} \cdot \bar{Y}$

Example: $X \begin{cases} \bar{X} = 3 \\ \sigma_X^2 = 2 \end{cases}$, $Y = -6X + 22$.

Find: a) R_{XY}? b) Are $X \& Y$ orthogonal? c) Are $X \& Y$ Uncorrelated?

Solution:

a) $R_{xy} = E[XY] = E[X(-6X+22)] = E[-6X^2+22X]$

$= -6\,E[X^2] + 22\,E[X] = -6(\sigma_x^2+\bar{X}^2) + 22\,\bar{X}$

$= -6(2+3^2) + (22)(3) = \underline{\underline{\text{Zero}}}.$

b) since $R_{xy}=0 \Rightarrow$ They are Orthogonal.

c) We check if $R_{xy} \overset{?}{=} \bar{X}\bar{Y}$

$\bar{Y} = E[-6X+22] = -6\bar{X}+22 = \underline{\underline{4}}$ $4*3=12 \neq 0$

since $R_{xy} \neq \bar{X}\bar{Y} \Rightarrow$ X & Y <u>aren't</u> uncorrelated.

※ Joint Central Moment:

<u>Recall</u>: for R.V X: $\mu_n = E[(X-\bar{X})^n]$

* for two R.V's X & Y: $\mu_{nk} = E[(X-\bar{X})^n (Y-\bar{Y})^k]$

The order is $n+k$.

- Zeroth order: $\mu_{00} = 1$
- First Order: $\mu_{10} = E[(X-\bar{X})] = 0$
  $\mu_{01} = E[(Y-\bar{Y})] = 0$
- Second Order: $\mu_{20} = E[(X-\bar{X})^2] = E[X^2] - (\bar{X})^2 = \sigma_x^2$
  $\mu_{02} = E[(Y-\bar{Y})^2] = \sigma_y^2$
  $\mu_{11} = E[(X-\bar{X})(Y-\bar{Y})] = \underline{\underline{C_{xy}}}$ "Covariance"

*<u>Note</u>: $C_{xx} = \sigma_x^2$, $C_{yy} = \sigma_y^2$

$C_{xy} \to 0$, $C_{xy} \to +ve$, $C_{xy} \to -ve$.

* $C_{xy} = E[(X-\bar{X})(Y-\bar{Y})] = E[XY - \bar{Y}X - \bar{X}Y + \bar{X}\bar{Y}]$

$= R_{xy} - \bar{Y}\bar{X} - \cancel{\bar{X}\bar{Y}} + \cancel{\bar{X}\bar{Y}} \Rightarrow C_{xy} = R_{xy} - \bar{X}\bar{Y}$

$\Rightarrow R_{xy} = C_{xy} + \bar{X}\bar{Y}$

The Relation between Correlation & Covariance.

*If X&Y are orthogonal: $\underline{\underline{C_{xy} = -\bar{X}\bar{Y}}}$

*If X&Y are uncorrelated: $\underline{\underline{C_{xy} = 0}}$
↓
or independent.

*if $C_{xy}=0 \Rightarrow$ Uncorrelated for sure but Not necessary To be indep.

※ Correlation Parameter (Coefficient): ρ_{xy}

$$\rho_{xy} = \frac{\mu_{11}}{\sqrt{\mu_{20}\,\mu_{02}}} = \frac{C_{xy}}{\sigma_x \sigma_y}$$

- If x & y are uncorrelated: $R_{xy} = \bar{x}\bar{y} \rightarrow C_{xy} = 0 \rightarrow \rho_{xy} = 0$

- If $X = Y$: $C_{xy} = C_{xx} = \sigma_x^2 \rightarrow \rho_{xy} = 1$
  (High Correlation) $\sigma_y = \sigma_x$

$$-1 \leq \rho_{xy} \leq 1$$

"Uncorrelated" ← $0 \leq |\rho_{xy}| \leq 1$ → "Highly Correlated"

Example: for $X = \alpha_1 X_1 + \alpha_2 X_2$, find σ_x^2?

Solution: $\sigma_x^2 = E[(x-\bar{x})^2]$

$$x - \bar{x} = (\alpha_1 x_1 + \alpha_2 x_2) - (\alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2) = \alpha_1 (x_1 - \bar{x}_1) + \alpha_2 (x_2 - \bar{x}_2)$$

$$\sigma_x^2 = E[(x-\bar{x})^2] = E\left[\left(\alpha_1 (x_1 - \bar{x}_1) + \alpha_2 (x_2 - \bar{x}_2)\right)^2\right]$$

$$= E\left[\alpha_1^2 (x_1 - \bar{x}_1)^2 + \alpha_2^2 (x_2 - \bar{x}_2)^2 + 2\alpha_1\alpha_2 (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\right]$$

$$= \alpha_1^2 \sigma_{x_1}^2 + \alpha_2^2 \sigma_{x_2}^2 + 2\alpha_1\alpha_2 C_{x_1 x_2}$$

$C_{xy} = C_{yx}$.

$$\Rightarrow \sigma_x^2 = \alpha_1^2 \sigma_{x_1}^2 + \alpha_2^2 \sigma_{x_2}^2 + \alpha_1\alpha_2 C_{x_1 x_2} + \alpha_2 \alpha_1 C_{x_2 x_1}$$

*** General Case:** $X = \sum_{i=1}^{N} \alpha_i X_i = \alpha_1 X_1 + \dots + \alpha_N X_N$

$$\sigma_x^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j C_{x_i x_j} \quad ; \quad i \neq j$$

*** Note:** if the R.V's Uncorrelated "special Case".
$x_1, x_2, \dots, x_N$

$$X = \alpha_1 X_1 + \dots + \alpha_N X_N \Rightarrow \sigma_x^2 = \alpha_1^2 \sigma_{x_1}^2 + \dots + \alpha_N^2 \sigma_{x_N}^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2$$

Since $C_{x_i x_j} = 0 \quad \forall i \& j$

※ Joint Gaussian R.V's:

Recall: $X \sim N(a_x, \sigma_x^2) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$

- X_1 & X_2 are said to be jointly gaussian, if their joint density function $f_{X_1,X_2}(x_1,x_2)$ is given by:

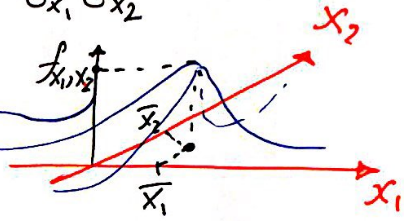
$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_{x_1}^2} - \frac{2\rho(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{x_2}^2}\right]}$$

Given in the exam.

$\rho = \frac{C_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}}$

⁂ Special Case:

when X_1 & X_2 are uncorrelated: $\frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}}$



$$\Rightarrow \rho = 0 \rightarrow f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}} e^{-\left[\frac{(x_1-\bar{x}_1)^2}{2\sigma_{x_1}^2} + \frac{(x_2-\bar{x}_2)^2}{2\sigma_{x_2}^2}\right]}$$

$$\Rightarrow f_{X_1,X_2}(x_1,x_2) = \underbrace{\frac{1}{\sqrt{2\pi\sigma_{x_1}^2}} e^{-\frac{(x_1-\bar{x}_1)^2}{2\sigma_{x_1}^2}}}_{f_{X_1}(x_1)} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma_{x_2}^2}} e^{-\frac{(x_2-\bar{x}_2)^2}{2\sigma_{x_2}^2}}}_{f_{X_2}(x_2)}$$

$\therefore$ X_1 & X_2 Are Independent.

for any R.V's.

$*$ Independent $\rightleftarrows$ Uncorrelated. $*$

only for jointly gaussian R.V's.

✱ For N jointly gaussian R.V's:

$X_1, X_2, \ldots, X_N$ are said to be jointly gaussian if their density function is given by:

$$f_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \frac{|[C_x]^{-1}|^{1/2}}{(2\pi)^{N/2}} \cdot e^{-\frac{[X-\bar{X}]^T[C_X]^{-1}[X-\bar{X}]}{2}}$$

Given in the Exam.

Continue.

⇒ where:
$$[C_X]_{N\times N} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} & \cdots & C_{X_1X_N} \\ C_{X_2X_1} & C_{X_2X_2} & \cdots & C_{X_2X_N} \\ \vdots & & & \\ C_{X_NX_1} & \cdots & \cdots & C_{X_NX_N} \end{bmatrix}$$

$\sigma_{X_1}^2$, $\sigma_{X_N}^2$

Covariance Matrix.

Equal To each other.

$$\underset{N\times 1}{[X-\bar{X}]} = \begin{bmatrix} X_1-\bar{X}_1 \\ X_2-\bar{X}_2 \\ \vdots \\ X_N-\bar{X}_N \end{bmatrix}, \quad \underset{1\times N}{[X-\bar{X}]^T} = [X_1-\bar{X}_1 \quad X_2-\bar{X}_2 \quad \cdots \quad X_N-\bar{X}_N]$$

↳ For 2 R.V's: X_1 & X_2

$$[C_X] = \begin{bmatrix} \sigma_{X_1}^2 & C_{X_1X_2} \\ C_{X_2X_1} & \sigma_{X_2}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix}$$

$$|[C_X]| = \sigma_{X_1}^2\sigma_{X_2}^2 - \rho^2\sigma_{X_1}^2\sigma_{X_2}^2 = (1-\rho^2)\,\sigma_{X_1}^2\sigma_{X_2}^2$$

$$[C_X]^{-1} = \frac{1}{(1-\rho^2)\sigma_{X_1}^2\sigma_{X_2}^2}\begin{bmatrix} \sigma_{X_2}^2 & -\rho\sigma_{X_1}\sigma_{X_2} \\ -\rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_1}^2 \end{bmatrix} = \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_{X_1}^2} & \frac{-\rho}{\sigma_{X_1}\sigma_{X_2}} \\ \frac{-\rho}{\sigma_{X_1}\sigma_{X_2}} & \frac{1}{\sigma_{X_2}^2} \end{bmatrix} \cdots ①$$

$$|[C_X]^{-1}| = \left(\frac{1}{\sigma_{X_1}^2\sigma_{X_2}^2} - \frac{\rho^2}{\sigma_{X_1}^2\sigma_{X_2}^2}\right)\cdot\frac{1}{(1-\rho^2)^2} = \frac{1}{\sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2)} \cdots ②$$

$$[X-\bar{X}]^T = [X_1-\bar{X}_1 \quad X_2-\bar{X}_2] \cdots ③$$

$$[X-\bar{X}] = \begin{bmatrix} X_1-\bar{X}_1 \\ X_2-\bar{X}_2 \end{bmatrix} \cdots ④$$

* multiply ③ * ① * ④ , Then substitute them with ② into the main equation of $f_{X_1,\ldots,X_N}(x_1,\ldots,x_N)$

⇒ You will get:

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}(1-\rho^2)}\, e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_{X_1}^2} - \frac{2\rho(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_{X_1}\sigma_{X_2}} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{X_2}^2}\right]}$$

*** Second Material *** Second Material ***

* Transformation of Multiple R.V's:

Recall: $X \xrightarrow{} \boxed{T(\cdot)} \rightarrow Y = T(X)$ $\qquad X = T^{-1}(Y)$

$f_X(x)$ $\qquad f_Y(y) = ?$

* for $X_1, \dots, X_N \rightarrow f_{X_1,\dots,X_N}(x_1,\dots,x_N)$ is Known.

$$
\left.\begin{aligned}
Y_1 &= T_1(X_1,\dots,X_N)\\
Y_2 &= T_2(X_1,\dots,X_N)\\
&\vdots\\
Y_N &= T_N(X_1,\dots,X_N)
\end{aligned}\right]
$$

⇒ Conditions:

1) all T_i's are Continuous Functions.
2) all T_i's have derivatives.
3) $X_1 = V_1(Y_1,\dots,Y_N)$
$\vdots$
$X_N = V_N(Y_1,\dots,Y_N)$

Then; $f_{Y_1,\dots,Y_N}(y_1,\dots,y_N) = f_{X_1,\dots,X_N}\big(V_1(y_1,\dots,y_N),\dots,V_N(y_1,\dots,y_N)\big)\cdot |J|$

Absolute Value.

$$
J = \left|\begin{bmatrix}
\frac{dV_1}{dy_1} & \frac{dV_1}{dy_2} & \cdots & \frac{dV_1}{dy_N}\\
\frac{dV_2}{dy_1} & \frac{dV_2}{dy_2} & \cdots & \frac{dV_2}{dy_N}\\
\vdots & & & \\
\frac{dV_N}{dy_1} & \frac{dV_N}{dy_2} & \cdots & \frac{dV_N}{dy_N}
\end{bmatrix}\right| \rightarrow \text{Determinent.}
$$

Jacobian Matrix.

Example: $X_1, X_2 \rightarrow f_{X_1,X_2}(x_1,x_2)$

$Y_1 = aX_1 + bX_2$; a,b,c,d are constants & $ad - bc \neq 0$
$Y_2 = cX_1 + dX_2$

Find $f_{Y_1,Y_2}(y_1,y_2)$?

Solution:

$$
\begin{aligned}
&\quad aX_1 + bX_2 = Y_2\\
&+(cX_1 + dX_2 = Y_2) * \tfrac{-b}{d}\\
\hline
&ax_1 - \frac{bc}{d}x_1 = Y_1 - \frac{b}{d}Y_2
\end{aligned}
$$

$\Rightarrow \left[x_1\left(a - \frac{bc}{d}\right) = Y_1 - \frac{b}{d}Y_2\right] * d$

$\Rightarrow x_1(ad - bc) = dY_1 - bY_2$

$\Rightarrow \boxed{X_1 = \dfrac{dY_1 - bY_2}{ad - bc}} \rightarrow V_1(Y_1, Y_2)$

$$
\begin{aligned}
&\quad aX_1 + bX_2 = Y_1\\
&+(cX_1 + dX_2 = Y_2) * \tfrac{-a}{c}\\
\hline
&bx_2 - \frac{ad}{c}x_2 = Y_1 - \frac{a}{c}Y_2
\end{aligned}
$$

$\Rightarrow X_2 = \dfrac{Y_1 - \frac{a}{c}Y_2}{b - \frac{ad}{c}} * \dfrac{c}{c}$

$\Rightarrow \boxed{X_2 = \dfrac{cY_1 - aY_2}{bc - ad}} \rightarrow V_2(Y_1, Y_2)$

⇒

⇒ Now we find J :

$$J = \left| \begin{bmatrix} \frac{dV_1}{dy_1} & \frac{dV_1}{dy_2} \\ \frac{dV_2}{dy_1} & \frac{dV_2}{dy_2} \end{bmatrix} \right| = \left| \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{c}{bc-ad} & \frac{-a}{bc-ad} \end{bmatrix} \right| = \frac{ad}{(ad-cb)^2} - \frac{bc}{(ad-bc)^2}$$

$$\Rightarrow \quad J = \frac{ad-bc}{(ad-bc)^2} \quad \Rightarrow \quad J = \frac{1}{ad-bc}$$

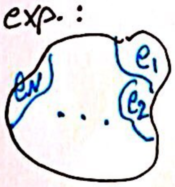
$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(\frac{dy_1-by_2}{ad-bc}, \frac{cy_1-ay_2}{bc-ad}\right) \cdot \frac{1}{|ad-bc|} \quad \#.$$

END of CH5 * * * END of CH5

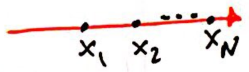
※ CHAPTER(6): Random Process : (R.P) → stochastic.

– Temporal Characteristic:

exp.:



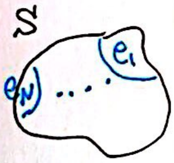
R.V
$X(S) = X$ ⇒



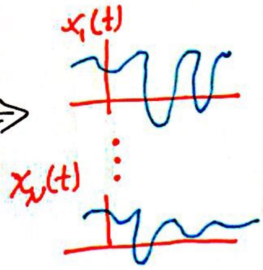
real-line axis.

$X = \{x_1, x_2, \dots, x_N\}$

S

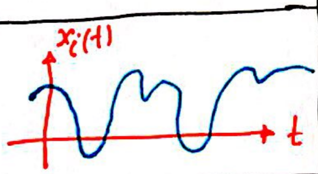
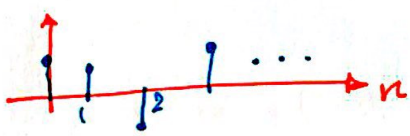
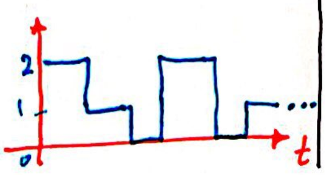
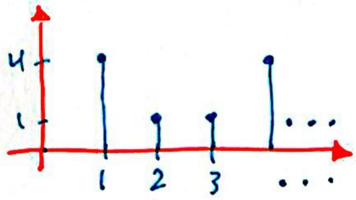


$X(t,S) = X(t)$ ⇒



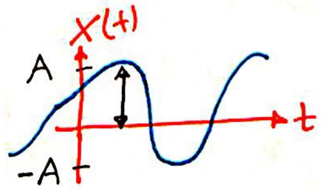
This family of time signals perform the R.P $X(t)$.
each one is called sample function.
or Realization.

※ R.P Classifications:

	(CT) Continuous-time.	(DT) Discrete-time.
Continuous Amplitude		
Discrete Amplitude.		

* R.P → Deterministic
  - ↳ Can be Represented Mathematically.
  - e.g: $X(t) = A\cos(\omega_0 t + \theta)$
  - $\Rightarrow$ at least one of A, ω_0, θ should be R.V.
* R.P → Non-Deterministic → Can't be represented mathematically.
  - e.g: Noise.

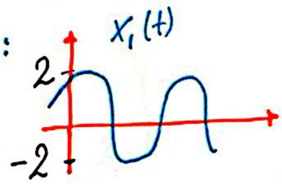
* For $X(t) = A\cos(\omega_0 t + \theta)$:



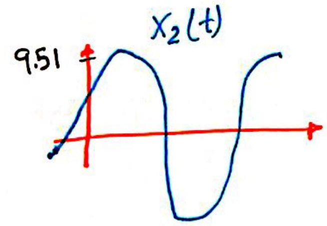
- ω_0, θ constants.

  $A \sim U(0,10)$

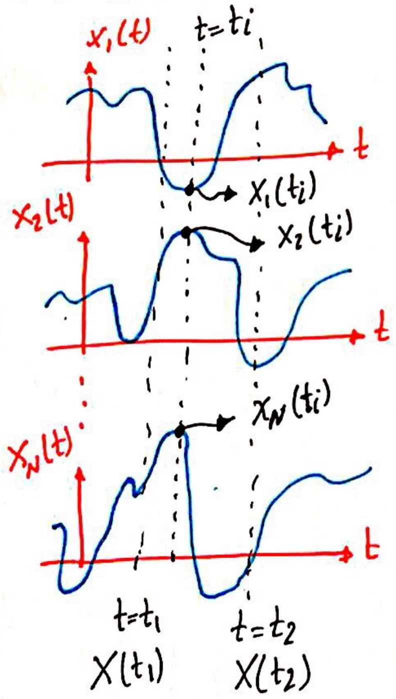
  could give:



OR



* R.P $X(t)$:



$X(t_i) = \{x_1(t_i), x_2(t_i), \dots, x_N(t_i)\}$

↳ $= X_i$

$X(t_1) \longrightarrow f_X(x; t_1)$

$X(t_2) \longrightarrow f_X(x; t_2)$

"First Order Distribution"

$X \rightsquigarrow f_X(x)$

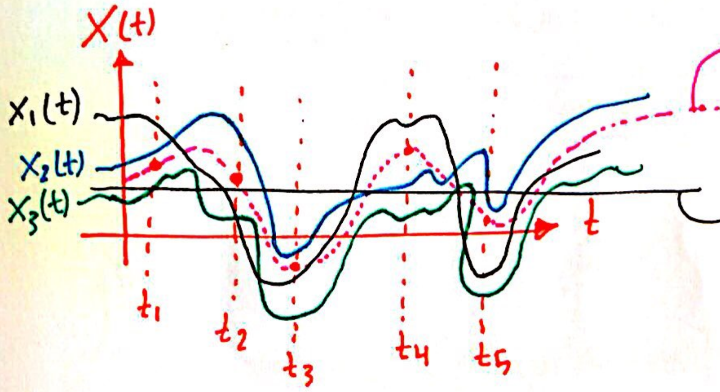
$f_X(x_1, x_2; t_1, t_2)$

"Second Order Distribution"

※ R.P Mean:

$E[X(t)] = m_X(t)$ → The Mean of R.P is Function of Time.

$$\Rightarrow \boxed{E[X(t)] = \int_{-\infty}^{\infty} x\, f_X(x;t)\, dx}$$



$m_X(t)$ "one signal"

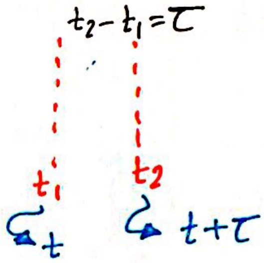
DC Value for $X(t)$: $= A\left[E[X(t)]\right]$

A → Time Average.

※ Auto-Correlation Function:

$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$

$$\boxed{R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)]}$$

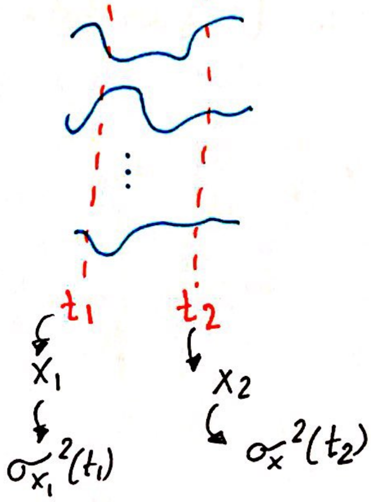


- when $\tau = 0$: $\boxed{R_{XX}(t,t) = E[X^2(t)]}$

※ R.P Variance:

$$\boxed{\sigma_X^2(t) = E[X^2(t)] - m_X^2(t)}$$

$\sigma_X^2(t)$ → function of time.



※ Auto-Covariance Function:

$C_{XX}(t, t+\tau) = E\left[(X(t) - m_X(t))(X(t+\tau) - m_X(t+\tau))\right]$

$$\Rightarrow \boxed{C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - m_X(t)\, m_X(t+\tau)}$$

- When $\tau = 0$: $\Rightarrow C_{XX}(t,t) = R_{XX}(t,t) - m_X^2(t) = \boxed{\sigma_X^2(t)}$

Example: $X(t) = A\cos(\omega_0 t + \theta)$

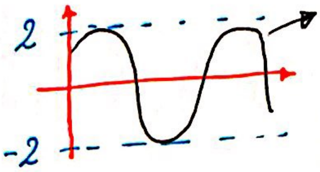
$A \sim N(2, 9)$ $\quad$ ω_0 & θ are constants.

Find: a) $m_X(t)$? $\qquad$ b) $\sigma_X^2(t)$?

Solution:

a) $m_X(t) = E[X(t)] = E[\underbrace{A}_{R.V}\underbrace{\cos(\omega_0 t+\theta)}_{\text{Not } R.V}] = \cos(\omega_0 t+\theta)\; E[A]$

$$\Rightarrow \boxed{m_X(t) = 2\cos(\omega_0 t+\theta)}$$



b) $\sigma_X^2(t) = E[X^2(t)] - m_X^2(t)$

- $E[X^2(t)] = E[A^2\cos^2(\omega_0 t+\theta)] = \cos^2(\omega_0 t+\theta)\; E[A^2]$

$= \cos^2(\omega_0 t+\theta)\;(9+4) = 13\cos^2(\omega_0 t+\theta)$

$$\Rightarrow \sigma_X^2(t) = 13\cos^2(\omega_0 t+\theta) - 4\cos^2(\omega_0 t+\theta) = \boxed{9\cos^2(\omega_0 t+\theta)}$$

* let A & ω_0 R.V's, $\theta = 0$

$$E[X(t)] = E[\underbrace{A\cos(\omega_0 t)}_{2\ R.V's.}] = E[g(A,\omega_0)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(A,\omega_0)\, f_{A,\omega_0}(A,\omega_0)\, dA\, d\omega_0$$

* Stationary:

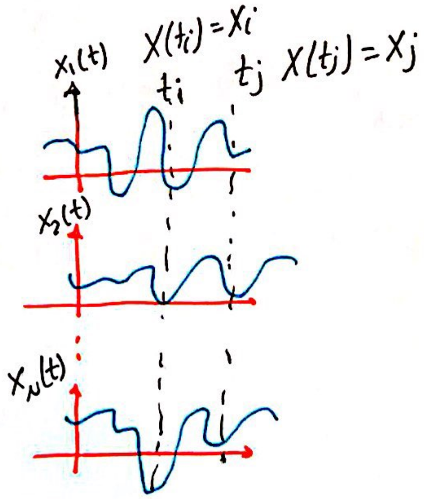
⇒ **In general:** The R.P is said to be stationary if all its statistical Properties do NOT change with time.

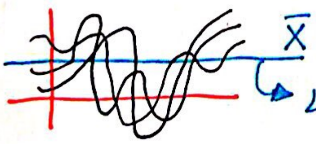
- **First Order Stationary:**

$f_X(x, t_i) = f_X(x, t_j) \quad \forall\, t_i, t_j$

Result: $m_X(t_i) = m_X(t_j) = \bar{X} \quad \forall\, t_i, t_j$

$\bar{X}$ → Constant Mean.





↳ DC value for the R.P.

$E[X(t)] = \bar{X}$

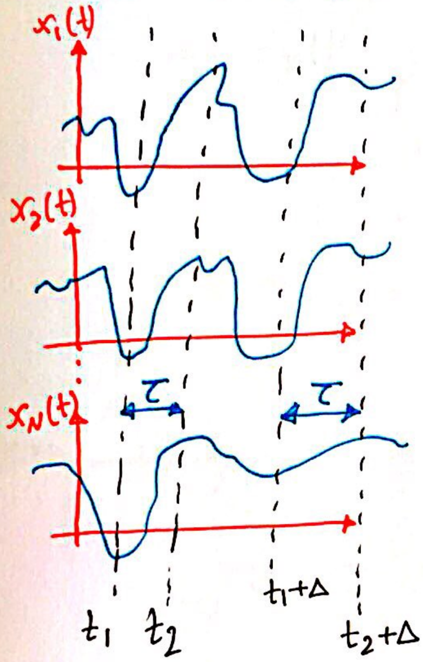
* To find DC Value:

1) $m_X(t) = E[X(t)]$

2) $A[m_X(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} m_X(t)\,dt = DC \text{ value}.$

if stationary $= \bar{X}$

• Second Order Stationary:



$\boxed{\tau = t_2 - t_1}$

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\Delta, t_2+\Delta)$$

$\forall\ t_1, t_2, \Delta$

* Autocorrelation:

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

τ

$$= \iint x_1 x_2 f_X(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

τ

$$R_X(t_1+\Delta, t_2+\Delta) = E[X(t_1+\Delta)X(t_2+\Delta)]$$

$$= \iint x_1 x_2 f_X(x_1, x_2; t_1+\Delta, t_2+\Delta)\, dx_1 dx_2$$

$\Rightarrow \boxed{R_{XX}(t, t+\tau) = R_{XX}(\tau)}$; Autocorrelation between two times equal to the Autocorrelation of the difference between these two times.

e.g: $R_{XX}(1, 4) = R_{XX}(9, 12)$

* Wide Sense Stationary R.P: "WSS"

Def: A R.P $X(t)$ is said to be WSS if:

$\boxed{1}$ $E[X(t)] = m_X(t) = \bar{X} \longrightarrow$ R.P has Constant Mean. "DC value for the R.P"

$\boxed{2}$ $R_{XX}(t, t+\tau) = R_{XX}(\tau)$

As result: $\sigma_X^2(t) = \sigma_X^2 \longrightarrow$ R.P has Constant Variance. "AC power in $X(t)$"

$\bar{X}$: DC value. $\bar{X}^2$: DC power for $X(t)$. σ_X^2: AC power.

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Example: $X(t) = A\cos(\omega_0 t+\theta)$, A & ω_0 are Constants.
$\theta \sim U(0, 2\pi)$

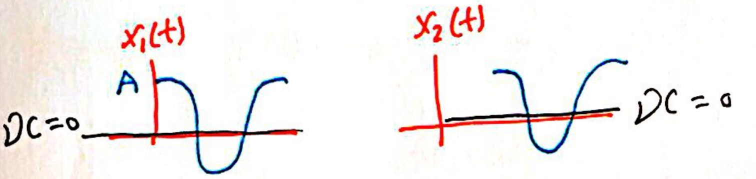
Show that: $X(t)$ is WSS ?

Solution: • $m_X(t) = E[X(t)] = E[\underbrace{A\cos(\omega_0 t+\theta)}_{g(\theta)}]$

$$\Rightarrow m_X(t) = \int_\theta g(\theta) f_\Theta(\theta)\, d\theta = \int_0^{2\pi} A\cos(\omega_0 t+\theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi}\int_0^{2\pi} \cos(\theta+\omega_0 t)\, d\theta$$ "integration of a cosine on its period $=0$"

$\Rightarrow \boxed{m_X(t)=0}$ ...① ; which is constant.



here Always the DC value will be zero as evaluated.

• $R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)] = E[A\cos(\omega_0 t+\theta) \cdot A\cos(\omega_0 t+\omega_0\tau+\theta)]$

remember: $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$\Rightarrow R_{XX}(t, t+\tau) = \frac{A^2}{2} E[\cos(\omega_0\tau) + \cos(2\omega_0 t+\omega_0\tau+2\theta)]$$

$$= \frac{A^2}{2}\cos(\omega_0\tau) + \frac{A^2}{2} E[\cos(2\theta+\omega_0\tau+2\omega_0 t)]$$

↳ it will give zero.

$$= \frac{A^2}{2}\cos(\omega_0\tau)$$ ...②

$\Rightarrow\Rightarrow \boxed{R_{XX}(t, t+\tau) = R_{XX}(\tau)}$

from ① & ② : $X(t)$ is WSS.

※ Proparties of $R_{XX}(\tau)$:

{1} $|R_{XX}(\tau)| \leq R_{XX}(0)$

{2} $R_{XX}(-\tau) = R_{XX}(\tau) \rightarrow$ "even function".

{3} $R_{XX}(0) = E[X^2(t)] \rightarrow$ "Total average power", "mean-squared-value".

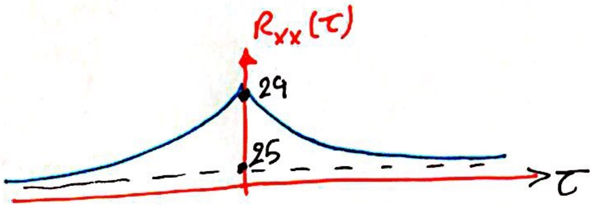
{4} $\lim_{\tau\to\infty} R_{XX}(\tau) = \overline{X}^2 \rightarrow$ "DC Average Power"

Example: Given WSS R.P $X(t)$, with $R_{XX}(\tau) = 25 + \dfrac{4}{1+6\tau^2}$

Find: a) $E[X^2(t)]$?

b) $\bar{X}$?

c) σ_X^2 ?



Solution:

a) $E[X^2(t)] = R_{XX}(0) = \boxed{29}$.

b) $\lim\limits_{\tau \to \infty} R_{XX}(\tau) = \bar{X}^2 = 25 \Rightarrow \boxed{\bar{X} = +5 \text{ or } -5}$.

c) $\sigma_X^2 = E[X^2(t)] - \bar{X}^2 = 29 - 25 = \boxed{4}$.

Example: $X(t) \longrightarrow \otimes \longrightarrow Y(t)$, check if $Y(t)$ is WSS ?

$X(t)$: WSS, $R_{XX}(\tau) = e^{-a|\tau|}$

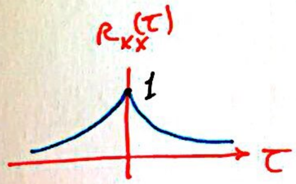
$\cos(\omega_0 t + \theta)$, $\theta \sim U(-\pi, +\pi)$

Solution: $Y(t) = X(t)\cos(\omega_0 t + \theta)$

- $E[Y(t)] = E[X(t) \cdot \cos(\omega_0 t + \theta)]$

These two R.Ps are statistically independent.

$= E[X(t)] \cdot E[\cos(\omega_0 t + \theta)]$



$\Rightarrow \lim\limits_{\tau \to \infty} R_{XX}(\tau) = 0 = \bar{X}^2$; so $\bar{X} = 0$

Also $E[\cos(\theta + \omega_0 t)] = 0$; $\boxed{E[Y(t)] = \text{Zero}}$ ...①

- $R_{XX}(t, t+\tau) = E[Y(t) \cdot Y(t+\tau)] = E[X(t)\cos(\omega_0 t + \theta) \cdot X(t+\tau)\cos(\omega_0 t + \omega_0 \tau + \theta)]$

$= E[\underbrace{X(t) \cdot X(t+\tau)}_{R_{XX}(\tau)}] \cdot E[\underbrace{\cos(\omega_0 t + \theta) \cdot \cos(\omega_0 t + \omega_0 \tau + \theta)}_{\text{As done Before it will give } \frac{1}{2}\cos(\omega_0 \tau)}]$

$= \frac{1}{2} R_{XX}(\tau)\cos(\omega_0 \tau) = \frac{1}{2} e^{-a|\tau|} \cos(\omega_0 \tau) = R_{YY}(\tau)$ $\quad \boxed{R_{XX}(t, t+\tau) = R_{YY}(\tau)}$ ...②

from ① & ②: $Y(t)$ is WSS.

※ Auto-Covariance Function :

$$\boxed{C_{XX}(t,t+\tau) = R_{XX}(t,t+\tau) - m_X(t)\, m_X(t+\tau)}$$

- If $X(t)$ is WSS, Then: $\boxed{\underbrace{C_{XX}(t,t+\tau)}_{C_{XX}(\tau)} = R_{XX}(\tau) - \bar{X}^2}$

※ Cross-Correlation Function :

$$\boxed{R_{XY}(t,t+\tau) = E[X(t)\, Y(t+\tau)]}$$

- If $R_{XY}(t,t+\tau) = 0$, Then: $X(t) \perp Y(t)$.
- If $R_{XY}(t,t+\tau) = E[X(t)] \cdot E[Y(t+\tau)]$, Then:
  $X(t)$ & $Y(t)$ Are Statistically Independent.

※ Cross-Covariance Function :

$$\boxed{C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)\, m_Y(t+\tau)}$$

※※ A R.P's $X(t)$ & $Y(t)$ Are said to be joint WSS if:

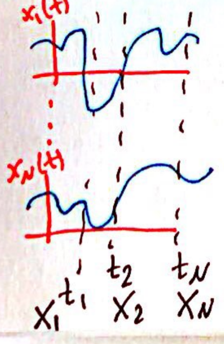
1) $X(t)$ is WSS $\rightarrow$ $m_X(t) = \bar{X}$, $R_{XX}(t,t+\tau) = R_{XX}(\tau)$.

2) $Y(t)$ is WSS $\rightarrow$ $m_Y(t) = \bar{Y}$, $R_{YY}(t,t+\tau) = R_{YY}(\tau)$

3) $R_{XY}(t,t+\tau) = R_{XY}(\tau)$

$\hookrightarrow$ As a Result: $C_{XY}(t,t+\tau) \triangleq C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$

※ Gaussian R.P :



Def: A R.P is said to be gaussian if the N R.V's $X_1, \dots, X_N$ Corresponding to the time instents $t_1, \dots, t_N$ are jointly gaussian. with density function

$$f_X(x_1, \dots, x_N) = \frac{|[C_X]^{-1}|^{1/2}}{(2\pi)^{N/2}}\, e^{-\frac{1}{2}[x-\bar{x}]^T [C_X]^{-1} [x-\bar{x}]}$$

$\Rightarrow$

- $[X-\bar{X}] = \begin{bmatrix} X_1 - m_X(t_1) \\ X_2 - m_X(t_2) \\ \vdots \\ X_N - m_X(t_N) \end{bmatrix}$ • $[C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$

- $C_{ij} = C_{XX}(t_i, t_j) = R_{XX}(t_i, t_j) - m_X(t_i)\, m_X(t_j)$

$i = 1, \dots, N$
$j = 1, \dots, N$

- If $X(t)$ is WSS gaussian R.P : $[X-\bar{X}] = \begin{bmatrix} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \\ \vdots \\ X_N - \bar{X}_N \end{bmatrix}$, $\bar{X} = m_X(t)$

$\Rightarrow\Rightarrow C_{ij} = C_{XX}(t_i, t_j) = C_{XX}(t_j - t_i) = \underline{\underline{R_{XX}(t_j - t_i) - \bar{X}^2}}$

Example: A continuous-time WSS gaussian R.P with mean $\bar{X} = 4$ & Auto-correlation function $R_{XX}(\tau) = 25\, e^{-3|\tau|} + 16$.

* Determine the Covariance matrix for three R.V's $X(t_1), X(t_2), X(t_3)$ Defined as $t_j = t_0 + \frac{j-1}{2}$; $j = 1, 2, 3$?

Solution: $\underset{3\times 3}{[C_X]} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$t_1 = t_0$
$t_2 = t_0 + \frac{1}{2}$
$t_3 = t_0 + 1$

$\Rightarrow C_{ij} = R_{XX}(\underset{\tau}{t_j - t_i}) - \bar{X}^2 = 25\, e^{-3|t_j - t_i|} + 16 - 16$

$\hookrightarrow \lim_{\tau \to \infty} R_{XX}(\tau) = \bar{X}^2 = 16$

$$\boxed{C_{ij} = 25\, e^{-3|t_j - t_i|}}$$

- $C_{11} = 25\, e^{-3(0)} = 25$
- $C_{12} = C_{21} = 25\, e^{-3/2}$
- $C_{22} = 25\, e^{-3(0)} = 25$
- $C_{13} = C_{31} = 25\, e^{-3}$
- $C_{23} = C_{32} = 25\, e^{-3/2}$
- $C_{33} = 25$

$$\Rightarrow [C_X] = \begin{bmatrix} 25 & 25\, e^{-3/2} & 25\, e^{-3} \\ 25\, e^{-3/2} & 25 & 25\, e^{-3/2} \\ 25\, e^{-3} & 25\, e^{-3/2} & 25 \end{bmatrix} \quad \#$$

$\hookrightarrow$ This result just for WSS.

* Time-Average & Ergodicity:

$X(t)$ is R.P:

- $m_X(t) = E[X(t)] \triangleq \bar{X}$ "DC value" — If WSS
- $R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)] \triangleq R_{XX}(\tau)$. — If WSS.

$\Rightarrow$ mathematically: $m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f_X(x;t)\,dx$. → We Cannot use it if we don't Know the R.P density function(S).

$$R_{XX}(t, t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2; t, t+\tau)\,dx_1 dx_2$$

* Estimation for $m_X(t)$ & $R_{XX}(t, t+\tau)$:

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \quad , \quad \hat{R}_{XX}(t, t+\tau) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)\, x_i(t+\tau)$$

- For the result to be accurate: we need $N \to \infty$

$\Rightarrow$ Which is practically somehow hard.

↳ This is solved by using "Ergodicity".

* Ergodicity: A WSS R.P is said to be ergodic if:

$m_X(t) = \bar{X}$, $R_{XX}(t, t+\tau) = R_{XX}(\tau)$

1) $\bar{X} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$ for any sample function. $x(t)$.

$\bar{X}$ ↳ Statistical Average $E[X(t)]$

$\int_{-T}^{T} x(t)\,dt$ → Time-Average.

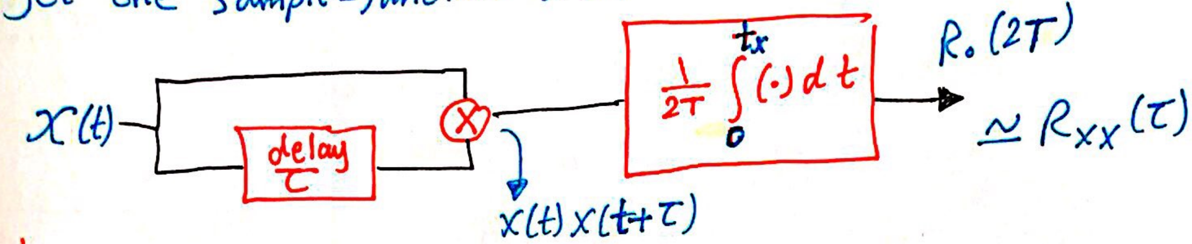
2) $R_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt$ for any sample function $x(t)$.

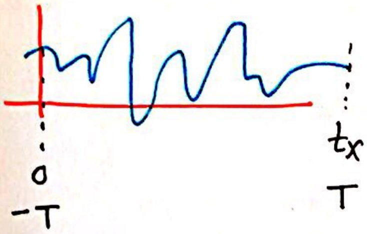
$R_{XX}(\tau)$ ↳ statistical Auto-correlation function.

$\int_{-T}^{T} x(t)\,x(t+\tau)\,dt$ ↳ Time-Average Auto-Correlation.

※ How to measure $R_{xx}(\tau)$:

$\Rightarrow$ get one sample-function $x(t)$.





* To take $\lim_{T\to\infty}(\cdot)$ make T relatively Large.

Example: $X(t) = A\cos(\omega_0 t + \theta)$, $\theta \sim U(0, 2\pi)$

Find: a) $R_{xx}(\tau)$. b) measure $R_{xx}(\tau)$.

Solution: a) $R_{xx}(\tau) = E[X(t)\, X(t+\tau)] = \underset{\text{obtained before}}{\cdots\cdots} = \boxed{\frac{A^2}{2}\cos(\omega_0 \tau)}$

b) Take: $x(t) = A\cos(\omega_0 t + \theta)$ → Constant (NOT R.V).

$$R_0(2T) = \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T} A^2 \cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0 \tau + \theta)\, dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T}\left[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + \theta)\right] dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T}\cos(\omega_0 \tau)\, dt + \frac{A^2}{4T}\int_{-T}^{T}\cos(2\omega_0 t + \omega_0 \tau + \theta)\, dt$$

↳ solve this integral.

$$= \frac{A^2}{4T} * \cos(\omega_0 \tau) * 2T + \frac{A^2}{2}\cos(\omega_0 \tau + 2\theta)\,\frac{\sin(2\omega_0 T)}{2\omega_0 T}$$

$$\Rightarrow \boxed{R_{xx}(\tau) \simeq R_0(2T) = \frac{A^2}{2}\cos(\omega_0 \tau) + \underbrace{\frac{A^2}{2}\cos(\omega_0 \tau + 2\theta)\,\frac{\sin(2\omega_0 T)}{2\omega_0 T}}_{\text{Error}}}$$

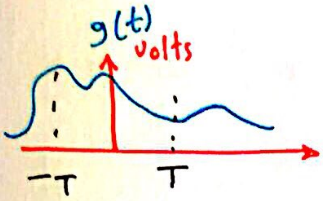
Note: $\frac{\sin(2\omega_0 T)}{2\omega_0 T}$ represents "Sinc"

so, if we take the Limit as $T \to \infty$: we will get the answer in part $\boxed{a}$.

END of CH6 * * * END of CH6

※ CHAPTER (7): R. Processes – Spectral Characteristics:

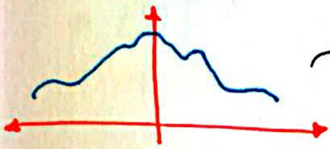
Recall: let $g(t)$ is a deterministic signal.
↳ it is time-limited or Bounded.



→ To study the spectral proparties.

F.T → $G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt.$

complex function.



→ shows how the signal voltage distributed over the frequancy.

• Average Power in $g(t)$:

Time Domain

$$P_{gg} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g^2(t)\, dt \longrightarrow$$

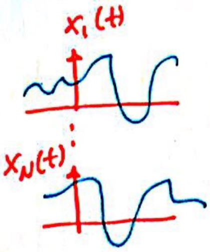
Frequancy Domain

$$P_{gg} = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|G(\omega)|^2}{2T}\, d\omega$$

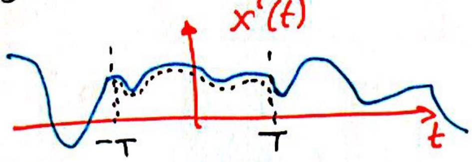
* parsevals' Theorem. *

* How we can study the R.P in the freq. domain?

$X(t)$



• get one sample-function $x^i(t)$:



⇒ Take from $x^i(t)$ the version $x_T^i(t)$ "Truncated":

$$x_T^i(t) = \begin{cases} x^i(t) & , -T<t<T \\ 0 & , \text{o.w} \end{cases}$$

$$x_T^i(t) \xleftrightarrow{F.T} X_T^i(\omega) = \int_{-\infty}^{\infty} x_T^i(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x^i(t)\, e^{-j\omega t}\, dt.$$

* for the power:

$$x_T^i(t) \xleftrightarrow{\quad F.T \quad} X_T^i(\omega)$$

$$P_T^i = \frac{1}{2T}\int_{-T}^{T} x_T^{i\,2}(t)\, dt = \frac{1}{2T}\int_{-T}^{T} x^{i\,2}(t)\, dt.$$

$$P_T^i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T^i(\omega)|^2}{2T}\, d\omega$$

* average power in $X_T^i(t)$:

$$P_T^i = \frac{1}{2T}\int_{-T}^{T} x_T^{i\,2}(t)\,dt = \frac{1}{2T}\int_{-T}^{T} x^{i\,2}(t)\,dt. \quad \xleftrightarrow{F.T} \quad P_T^i = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|X_T^i(\omega)|^2}{2T}\,d\omega$$

* average power in $x^i(t)$:

$$P^i = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^{i\,2}(t)\,dt. \quad \xleftrightarrow{F.T} \quad P^i = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|X_T^i(\omega)|^2}{2T}\,d\omega$$

* $P^i = \{P^1, P^2, \dots, P^N\}$ where: $P_1 \equiv$ Average Power for $x^1(t)$.

$\vdots$

$P_N \equiv$ Average Power for $X^N(t)$.

*Average Power of $X(t)$:

$$\Rightarrow \quad P_{XX} = E[P^i] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\,dt.$$

*** "Time Domain".

* Time Average for any signal: $A[\cdot] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\cdot)\,dt.$

$$\Rightarrow \quad P_{XX} = E[P^i] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\,dt = A\left[E[X^2(t)]\right]$$

* $$P_{XX} = E[P^i] = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{E[|X_T(\omega)|^2]}{2T}\,d\omega$$

$$\Rightarrow \quad P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{E[|X_T(\omega)|^2]}{2T}\,d\omega.$$

*** "Frequancy Domain"

$$\underbrace{P_{XX}}_{\text{Watt}} = \underbrace{\frac{1}{2\pi}}_{\cancel{\frac{1}{rad}}}\int_{-\infty}^{\infty}\underbrace{\frac{E[|X_T(\omega)|^2]}{2T}}_{\frac{\text{Watt}}{\text{HZ}}}\cdot\underbrace{d\omega}_{\cancel{rad}\cdot HZ}$$

→ Power - Density - Spectrum. ($\mathcal{S}_{XX}(\omega)$)

R.P: $X(t) \longrightarrow \mathcal{S}_{XX}(\omega)$

$$\mathcal{S}_{XX}(\omega) = \lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T}$$

where: $X_T(\omega) = \int_{-T}^{T} X(t)\,e^{-j\omega t}\,dt.$

sub. the R.P expression.

$$\boxed{\begin{array}{l} P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{S}_{XX}(\omega)\,d\omega \\ \text{OR} \\ P_{XX} = A\left[E[X^2(t)]\right]\end{array}}$$

*if $x(t)$ is WSS:

$$P_{XX} = E[X^2(t)]$$

Example: Given $X(t) = A_o \cos(\omega_o t + \theta)$; $\theta \sim U(0, \frac{\pi}{2})$

Find: a) P_{XX} using Time-Domain.

b) $\rho_{XX}(\omega)$.

c) use part (b) To find P_{XX}.

Solution: a) $P_{XX} = A\left[E[X^2(t)]\right] \Rightarrow E[X^2(t)] = E[A_o^2 \cos^2(\omega_o t + \theta)]$

$$\Rightarrow E[X^2(t)] = E\left[\frac{A_o^2}{2} + \frac{A_o^2}{2}\cos(2\theta + 2\omega_o t)\right]$$

$$= \frac{A_o^2}{2} + \frac{A_o^2}{2}\int_0^{\pi/2} \cos(2\theta + 2\omega_o t)\cdot\frac{2}{\pi}\,d\theta = \underline{\underline{\frac{A_o^2}{2} - \frac{A_o^2}{2}\sin(2\omega_o t)}}$$

it is function of time so Not a WSS.

$$P_{XX} = A\left[\frac{A_o^2}{2} - \frac{A_o^2}{2}\sin(2\omega_o t)\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\Big(\frac{A_o^2}{2} - \underbrace{\frac{A_o^2}{2}\sin(2\omega_o t)\Big)dt}_{\text{zero.}} = \frac{A_o^2}{2} - 0 = \frac{A_o^2}{2}$$

b) $\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} = \rho_{XX}(\omega)$

- $X_T(\omega) = \int_{-T}^{T} X(t)\, e^{-j\omega t}\,dt = \int_{-T}^{T} A_o\cos(\omega_o t + \theta)\, e^{-j\omega t}\,dt$

$\hookrightarrow = \frac{e^{j(\omega_o t + \theta)} + e^{-j(\omega_o t + \theta)}}{2}$

$$= \int_{-T}^{T} \frac{A_o e^{j(\omega_o t + \theta)} + A_o e^{-j(\omega_o t + \theta)}}{2}\, e^{-j\omega t}\,dt.$$

$$= \frac{A_o}{2} e^{j\theta}\int_{-T}^{T} e^{j(\omega_o - \omega)t}\,dt + \frac{A_o}{2} e^{-j\theta}\int_{-T}^{T} e^{-j(\omega_o - \omega)t}\,dt \quad \ldots \text{Do the integration.}$$

$$= A_o T e^{j\theta}\,\frac{\sin((\omega - \omega_o)T)}{(\omega - \omega_o)T} + A_o T e^{-j\theta}\,\frac{\sin((\omega + \omega_o)T)}{(\omega + \omega_o)T}$$

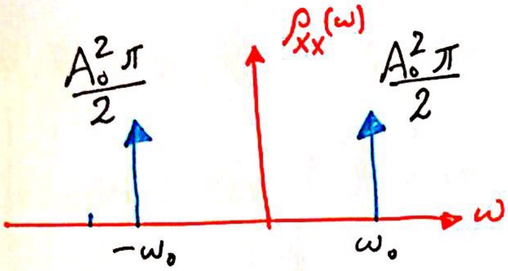
$$|X_T(\omega)|^2 = X_T(\omega)\cdot X_T^*(\omega)$$

$$\frac{1}{2T}E\left[|X_T(\omega)|^2\right] = \frac{1}{2T}E\left[X_T(\omega)\cdot X_T^*(\omega)\right] \quad \ldots \text{Do the expectation.}$$

$$= \frac{A_o^2\pi}{2}\left[\frac{T}{\pi}\,\frac{\sin^2((\omega - \omega_o)T)}{((\omega - \omega_o)T)^2} + \frac{T}{\pi}\,\frac{\sin^2((\omega + \omega_o)T)}{((\omega + \omega_o)T)^2}\right] \quad \ldots\ldots (1)$$

$$\rho_{XX}(\omega) = \lim_{T\to\infty} \text{equ.}(1) = \frac{A_o^2}{2}\pi\left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right]$$

using $\lim_{T\to\infty}\frac{T}{\pi}\left[\frac{\sin(\alpha T)}{\alpha T}\right]^2 = \delta(\alpha)$



c) $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A_0^2\pi}{2}\delta(\omega-\omega_0) + \frac{A_0^2\pi}{2}\delta(\omega+\omega_0)\right)d\omega$

$= \frac{1}{2\pi}\left[\frac{A_0^2\pi}{2} + \frac{A_0^2\pi}{2}\right] \Rightarrow P_{XX} = \frac{A_0^2}{2}$

※ $S_{XX}(\omega)$ Proparties: → so you could integrate from $0 \to \infty$ & multiplyby 2.

$\boxed{1}$ $S_{XX}(\omega) \geq 0$ $\boxed{2}$ $S_{XX}(-\omega) = S_{XX}(\omega)$ even function. $\boxed{3}$ $S_{XX}(\omega)$ is Real.

$\boxed{4}$ $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$. $\boxed{5}$ $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$.

$\boxed{6}$ $S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]\, e^{-j\omega\tau}\,d\tau$

$\boxed{7}$ $A[R_{XX}(t,t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{+j\omega\tau}\,d\omega$.

* If $X(t)$ is WSS for $\boxed{6}$ & $\boxed{7}$:

$R_{XX}(t,t,\tau) = R_{XX}(\tau)$. $\Rightarrow$ $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\,d\tau$

$A[R_{XX}(\tau)] = R_{XX}(\tau)$ $\quad R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{+j\omega\tau}\,d\omega = FT^{-1}\{S_{XX}(\omega)\}$

* For WSS $X(t)$: $R_{XX}(\tau) \xleftrightarrow{FT} S_{XX}(\omega)$

Example: Given $X(t) = A_0\cos(\omega_0 t+\theta)$; $\theta \sim U(0, 2\pi)$, Find $S_{XX}(\omega)$?

Solution: As found before $R_{XX}(t,t+\tau) = E[X(t)\,X(t+\tau)] = \frac{A_0^2}{2}\cos(\omega_0\tau)$

$\therefore$ $X(t)$ is WSS.

$\Rightarrow S_{XX}(\omega) = FT\{R_{XX}(\tau)\} = FT\{\frac{A_0^2}{2}\cos(\omega_0\tau)\} = \underline{\frac{A_0^2}{2}\pi\,\delta(\omega-\omega_0) + \frac{A_0^2}{2}\pi\,\delta(\omega+\omega_0)}$

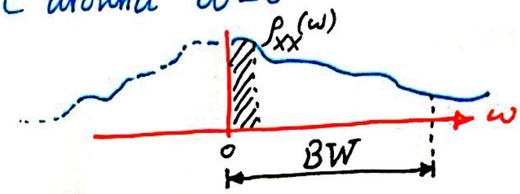
same result as we got before.

※ R. Process Classification:

→ **Baseband Process:**

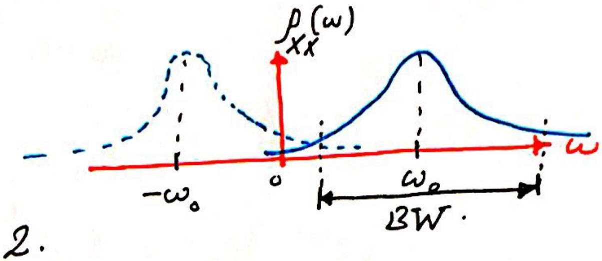
Its spectral components are around $\omega = 0$.

e.g: "Voice".



→ **Bandpass Process:**

Its spectral components are clustered near some frequancy ω_0.

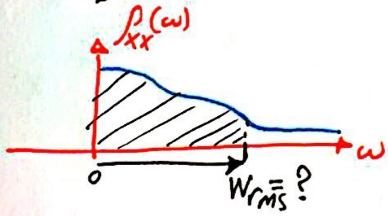


⇒ here we could also integrate for the +ve part, Then multiply by 2.

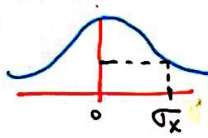
※ Bandwidth:

For R.P's ⇒ we will focuse on Root-Mean-Square BW. (W_{rms})

- W_{rms} for Baseband R.P:



same as $N(0, \sigma_x^2)$ ⇒



$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\int x^2 f_x(x)\,dx}$$

Area of pdf $= 1$

⇒ Area of $\rho_{XX}(\omega) \neq 1$

So we find the Normalized $\rho_{XX}(\omega)$:

$$\rho_{XX}^{norm}(\omega) = \frac{\rho_{XX}(\omega)}{\int_{-\infty}^{\infty} \rho_{XX}(\omega)\,d\omega}$$

⇒ proof (that Area=1):

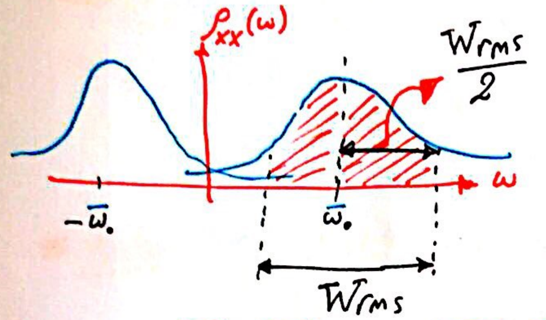
$$\int_{-\infty}^{\infty} \rho_{XX}^{norm}(\omega)\,d\omega = \int_{-\infty}^{\infty} \frac{\rho_{XX}(\omega)}{\underbrace{\int_{-\infty}^{\infty} \rho_{XX}(\omega)\,d\omega}_{\text{constant}}}\,d\omega = \frac{1}{\int_{-\infty}^{\infty} \rho_{XX}(\omega)\,d\omega} \cdot \int_{-\infty}^{\infty} \rho_{XX}(\omega)\,d\omega$$

$$= 1 \quad \#$$

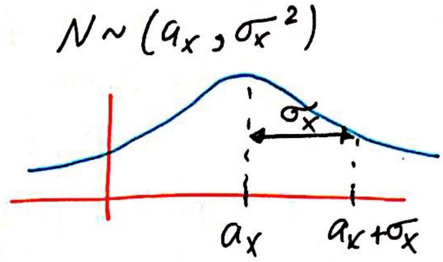
$$W_{rms}^{bb} = \sqrt{\int_{-\infty}^{\infty} \omega^2 \rho_{XX}^{norm}(\omega)\,d\omega}$$

$$\Rightarrow \quad W_{rms}^{bb} = \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 \rho_{XX}(\omega)\,d\omega}{\int_{-\infty}^{\infty} \rho_{XX}(\omega)\,d\omega}} \quad ***$$

- W_{rms} for Bandpass R.P:



same as $\Longrightarrow$



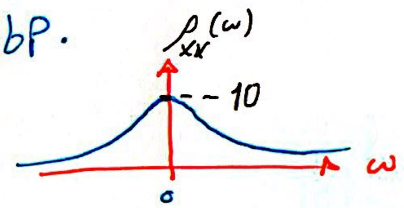
$$\boxed{W_{rms}^{bP} = 2\sqrt{\frac{\int_{-\infty}^{\infty}(\omega-\bar{\omega}_0)^2 \rho_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty}\rho_{XX}(\omega)\, d\omega}}} \quad ***$$

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{E[(X-a_X)^2]} = \sqrt{\int_{-\infty}^{\infty}(x-a_X)^2 f_X(x)\, dx}.$$

Example: $X(t)$ with $\rho_{XX}(\omega) = \frac{10}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2}$, Find W_{rms}?

Solution: Draw $\rho_{XX}(\omega)$ To Know the type bb or bP.

Take 3 values: $\omega = 0$, $\omega = 1$, $\omega = \infty$



so it is Baseband process.

- $\int_{-\infty}^{\infty}\rho_{XX}(\omega)\, d\omega = \int_{-\infty}^{\infty}\frac{10}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2} d\omega = 10^5 \int_{-\infty}^{\infty}\frac{1}{(100+\omega^2)^2} d\omega$

$\Rightarrow$ using the formula in App. C: $\Rightarrow \int_{-\infty}^{\infty}\rho_{XX}(\omega)\, d\omega = \boxed{50\pi}$

- $\int_{-\infty}^{\infty}\omega^2\rho_{XX}(\omega)\, d\omega = \int_{-\infty}^{\infty}\frac{10\,\omega^2}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2} d\omega = 10^5 \int_{-\infty}^{\infty}\frac{\omega^2}{(100+\omega^2)^2} d\omega$

$\Rightarrow$ using the formula in App. C: $\Rightarrow \int_{-\infty}^{\infty}\rho_{XX}(\omega)\,\omega^2 d\omega = \boxed{5000\pi}$

$W_{rms}^{bb} = \sqrt{\frac{5000\pi}{50\pi}} \Longrightarrow W_{rms}^{bb} = 10$ rad/sec.

※ Cross-Power Density Spectrum:

Let $X(t)$ & $Y(t)$ are two Real R.P's. ; & $W(t) = X(t) + Y(t)$.

- $R_{WW}(t, t+\tau) = E[W(t)\, W(t+\tau)] = E[(X(t)+Y(t))\,(X(t+\tau)+Y(t+\tau))]$

$$= \underbrace{E[X(t)X(t+\tau)]}_{R_{XX}(t,t+\tau)} + \underbrace{E[X(t)Y(t+\tau)]}_{R_{XY}(t,t+\tau)} + \underbrace{E[Y(t)X(t+\tau)]}_{R_{YX}(t,t+\tau)} + \underbrace{E[Y(t)Y(t+\tau)]}_{R_{YY}(t,t+\tau)} \Longrightarrow$$

$$\Rightarrow \boxed{R_{WW}(t,t+\tau) = \underbrace{R_{XX}(t,t+\tau) + R_{YY}(t,t+\tau)}_{\text{Auto-Correlation functions.}} + \underbrace{R_{XY}(t,t+\tau) + R_{YX}(t,t+\tau)}_{\text{Cross-Correlation functions.}}}$$

*** Note:** If $X(t)$ & $Y(t)$ are orthognal; cross-correlation $= 0$

$$\Rightarrow \boxed{R_{WW}(t,t+\tau) = R_{XX}(t,t+\tau) + R_{YY}(t,t+\tau)}$$

- $$S_{WW}(\omega) = \int_{-\infty}^{\infty} A[R_{WW}(t,t+\tau)]\, e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]\, e^{-j\omega\tau} d\tau + \int_{-\infty}^{\infty} A[R_{YY}(t,t+\tau)]\, e^{-j\omega\tau} d\tau$$

$$+ \int_{-\infty}^{\infty} A[R_{XY}(t,t+\tau)]\, e^{-j\omega\tau} d\tau + \int_{-\infty}^{\infty} A[R_{YX}(t,t+\tau)]\, e^{-j\omega\tau} d\tau$$

$$\Rightarrow \boxed{S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + \underbrace{S_{XY}(\omega) + S_{YX}(\omega)}_{\text{Cross-PDS.}}}$$

***Note:** If $X(t)$ & $Y(t)$ are orthogonal; Cross-PDS $= 0$

$$\Rightarrow \boxed{S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega)}$$

- $$P_{WW} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{WW}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)\, d\omega$$

$$\Rightarrow \boxed{P_{WW} = P_{XX} + P_{YY} + \underbrace{P_{XY} + P_{YX}}_{\text{Cross-Average Power.}}}$$

***Note:** If $X(t)$ & $Y(t)$ are orthogonal: $\Rightarrow \boxed{P_{WW} = P_{XX} + P_{YY}}$

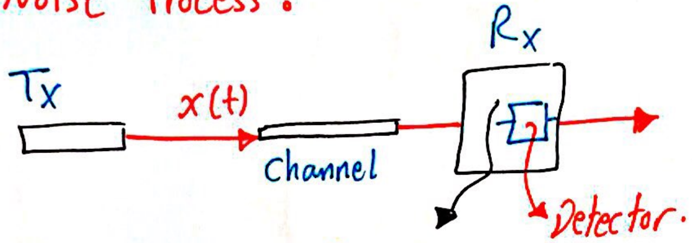
***Note:**

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[\overbrace{|X_T(\omega)|^2}^{X_T(\omega)\cdot X_T^*(\omega)}]}{2T}, \quad S_{XY} = \lim_{T\to\infty} \frac{E[X_T(\omega)\, Y_T^*(\omega)]}{2T} \;\ldots$$

Note that:

$$\boxed{S_{XY}^* = S_{YX}}$$

※ Noise Process :

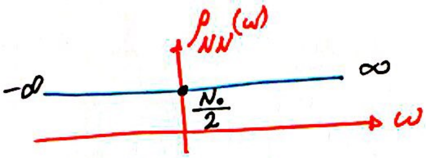


$r(t) = x(t) + N(t)$

received signal ← $r(t)$; Transmitted signal ← $x(t)$; Noise ← $N(t)$.

⁂ we study noise characteristics in the frequancy domain : $\rho_{NN}(\omega)$?

* White Noise :



Def.: $N(t)$ is called white Noise if its PDS is given by :

$$\rho_{NN}(\omega) = \frac{N_0}{2}$$ for all values of ω.

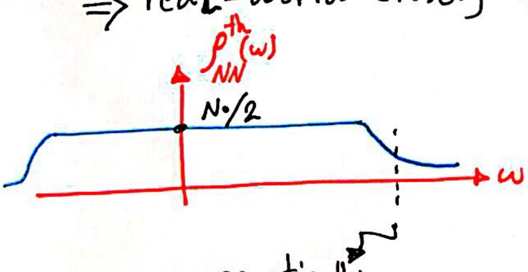
⇒ Always $N(t)$ is assumed to be "WSS".

$$R_{NN}(\tau) = FT^{-1}\left\{\frac{N_0}{2}\right\} = \frac{N_0}{2}\,\delta(\tau)$$

* for the power : $P_{NN} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2}\, d\omega = \infty$ "White Noise Unrealizable".

* Thermal Noise :

⇒ real-world closely approximates white noise.



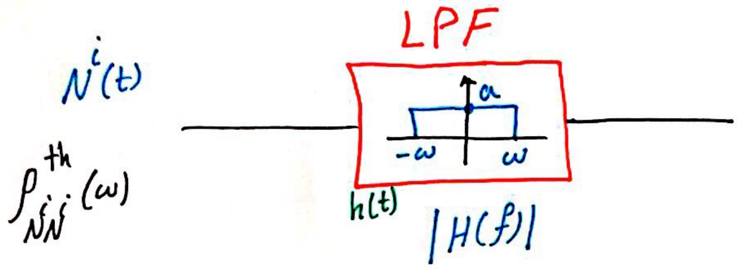
practically it reaches 1000GHZ.

⇒ flat (constant) over a wide range of frequancies.

"AWGN"

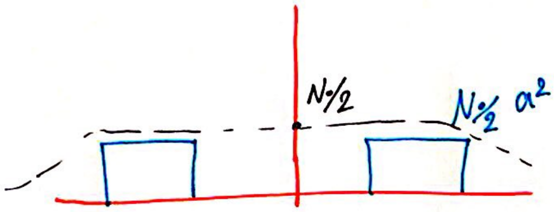
A → Addition. W → White. G → Gaussian. N → Noise.

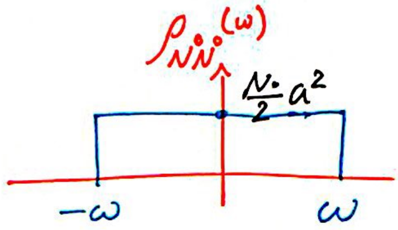
$r(t) = x(t) + N(t)$ Low-Pass-Filter. Convolution.



$$N^o(t) = N^i(t) * h(t)$$

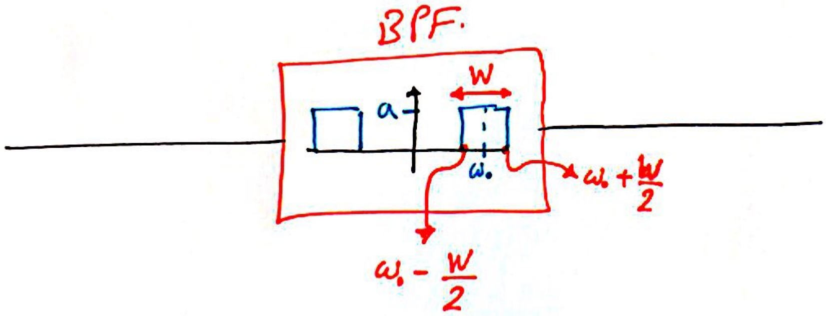
$$\rho^{th}_{N^oN^o}(\omega) = \rho^{th}_{N^iN^i}(\omega)\,|H(f)|^2$$





Home Work:

Do the same for BPF. "Band-Pass-Filter".



END of CH7 * * * END of CH7

* End of Material *

*Good Luck. *