CH #1

12th. Sep. 2018

* Set definition:-

↳ A collection of objects "elements".

-Ex: Set A, B, C, ..., ..., K

$A = \{2, 4, 6\}$

$B = \{a, e, f\}$

* $a \in A$: element a belongs to set A

$a \notin A$: element a doesn't belong to set A

-Ex: $C = \{-1, 0, 10, 2.7\}$

$-1 \in C$, $13 \notin C$

* How to specify sets:-

[1] Tabular Method

ex: $B = \{1, 3, 5, 7, 9\}$

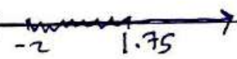
ex: 

$C = \{1.2 \le x \le 9\}$

[2] Rule Method

ex: C = {all real numbers from 2 to 9}

ex: B = {all odd numbers from 1 to 9}

ex: $B = \{-2 < b \le 1.75\}$ 

* Set Classification [1]

→ Countable $D = \{-1, 0, 1.2, 5\}$

$F = \{0, 2, 4, 6, ---\}$

→ Uncountable $K = \{-1.2 \le K \le 3\}$

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Set Classification [2]

Finite
eg: D
Set is empty
OR has elements
that can be counted.

infinite
eg: F countable
K Uncountable.

* Empty Sets: Set of no elements "ϕ", "{ }" (null set)

* $A \subseteq B$: Set A is contained in B
A is subset of B
All elements of A are in B

- Ex $A = \{-1, 0, 1, 3, 7, 11, 12\}$
$B = \{0, 11, 12\}$
$C = \{1, 7, 11, 13\}$
$B \subseteq A$

* Note: ϕ is subset of any set.

* Universal Set "S"
↳ the set that contains all other sets in certain situations.
eg: $A = \{a, b, c\}$
Determine all subsets of A?
$\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{\cancel{b, a}\}, \{a, c\}, \{b, c\}, \{a, b, c\}$
→ A is a universal set for this situation.

* Note: the number of subsets that can be performed of N-elements set is $\boxed{2^N}$

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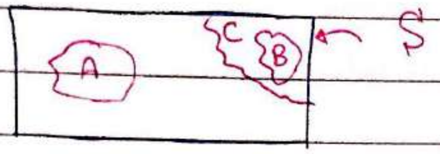
* Venn Diagram



$B \subseteq C$

Ex 1.1-1, 1.1-2

* Set operations:-

1. Set equality

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

ex $A = \{-2, 0, 1, 4\}$

$B = \{0, 4, -2, 1\}$

$C = \{-2, 1, 4, 5\}$

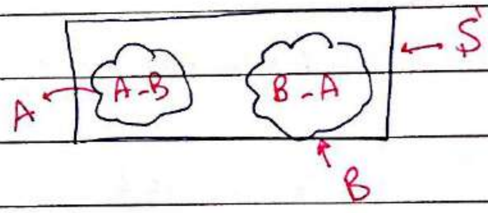
$\left.\begin{array}{l} A \subseteq B \\ B \subseteq A \end{array}\right\} \to A = B$

$C \neq A$

2. Set Difference

$A - B$: All elements in A but not in B

$B - A$: All elements in B but not in A



Ex $A = \{1, 2.2, 4, 5, 9\}$

$B = \{-1, 1, 3.2, 5, 10\}$

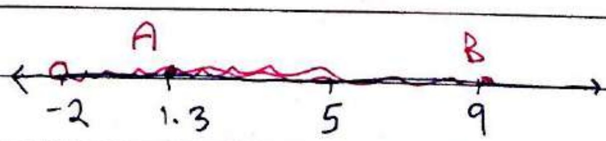
$A - B = \{2.2, 4, 9\}$

$B - A = \{-1, 3.2, 10\}$

Note: $A - B \neq B - A$.

Ex $A = \{-2 \leq a \leq 5\}$

$B = \{1.3 \leq b \leq 9\}$



$A - B = \{-2 < c < 1.3\}$

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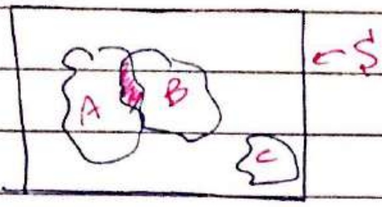
3 Set Intersection "$\cap$" "and" "together" "at the same time"

$A \cap B$: the set of common elements between A and B



Note: if $A \cap B = \emptyset$ then A and B are disjoint (mutually exclusive).

- Ex: $A = \{1, 5, 7, 11, 15\}$

$B = \{5, 11, 20, 25\}$

$A \cap B = \{5, 11\}$

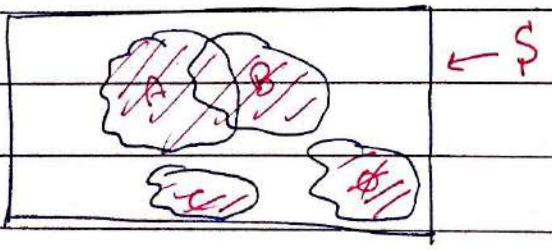
4 Set Union "$\cup$"

$A \cup B$: the set of all elements in A and B

Ex: $A = \{1, 5, 7, 11, 15\}$

$B = \{5, 11, 20, 25\}$

$A \cup B = \{1, 5, 7, 11, 15, 20, 25\}$



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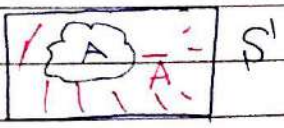
(5) Set Complement "Not"

$\bar{A}$: Set of all elements not in A.



Notes: $A \cap \bar{A} = \phi$

$A \cup \bar{A} = S'$

$\bar{\phi} = S'$

$\bar{S} = \phi$

3. Algebra of Set :-

[1] Commutitave Law

$A \cap B = B \cap A$

$A \cup B = B \cup A$

[2] Distributive Law

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

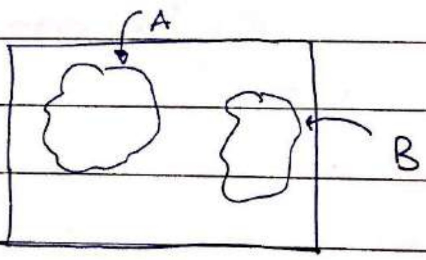
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

[3] Associative Law

$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$

$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$

[4] DeMorgan's Law



$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

↳ Neither A nor B

[5] Principle of Duality ($\cup \to \cap$ / $\cap \to \cup$ / $S \to \phi$ / $\phi \to S$)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$S \cap \phi = \phi$

$\phi \cup S = S$

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* Mathematical Model of Expirements :-

1. Sample Space "S" (Same as Universal Set)
   ↳ the set of all possible outcomes of an expirement.

- Ex exp: flip a coin

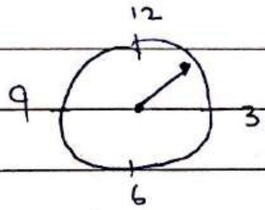
  $S = \{ H, T \}$
  "OR"

- Ex exp: flip a die

  $S = \{1, 2, 3, 4, 5, 6\}$

- Ex exp: wheel of chance

  $S = \{ 0 < S \leq 12 \}$



2. Events
   ↳ Subset of the Sample Space.

- Ex exp: Roll a dice.
  ① find S
  ② define the events (a) A "the appeared number is even"
  (b) B "the appeared number is integer"
  (c) C "the appered number is between 4 & 4.5" 4 is included.
  (d) D: "the appered number is -ve".

Sol ① $S = \{1, 2, 3, 4, 5, 6\}$

② (a) $A = \{2, 4, 6\}$

(b) $B = \{1, 2, 3, 4, 5, 6\} = S$

(c) $C = \{4\}$

(d) $D = \emptyset$ "impossible event"

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3 Probability Axioms: "P"

$P(A)$: the probability of the occurance of A

1) $0 \le P(A) \le 1$

$P(\emptyset)$ "impossible event"

$P(S)$ "certain event"

Ex: Flip a fair coin $S = \{H, T\}$

$P(\{H\}) = 1/2$

$P(\{T\}) = 1/2$

Ex: Roll a fair dice

$S = \{1, 2, 3, 4, 5, 6\}$

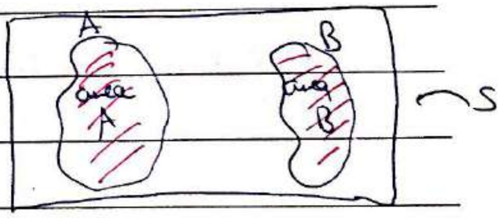
$P(S) = 1$ $\qquad P(\{5\}) = 1/6$

$P(A) = 3/6$ $\qquad P(B) = 1$

$P(C) = 1/6$ $\qquad P(D) = P(\{\}) = 0$

2) If events A and B are disjoint

then $P(A \cup B) = P(A) + P(B)$



Proof: $P(A \cup B) = \dfrac{\text{area } A + \text{area } B}{\text{area } S}$

$= \dfrac{\text{area } A}{\text{area } S} + \dfrac{\text{area } B}{\text{area } S}$

$= P(A) + P(B)$ #

Ex: Roll an unfair dice with

$P(\{1\}) = 0.1$ $\qquad P(\{4\}) = 0.05$

$P(\{2\}) = 0.15$ $\qquad P(\{5\}) = 0.2$

$P(\{3\}) = 0.2$ $\qquad P(\{6\}) = 0.3$

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Define the probability that the appeared number is odd

Sol Let Event A: the appered # is odd.

$A = \{1, 3, 5\}$

$p(A) = p(\{1,3,5\})$

$= p(\{1\} \cup \{3\} \cup \{5\})$

$= p(\{1\}) + p(\{3\}) + p(\{5\})$

$= 0.1 + 0.2 + 0.2 = 0.5$

Ex: Roll two fair dice and observe the appered numbers.

1) find S

2) Determine the events.

$A: \{ Sum = 7 \}$

$B = \{ ~~Sum >~~ \; 8 < Sum \leq 11 \}$

$C = \{ 10 < Sum \}$

3) find $p(A)$, $p(B)$, $p(C)$, $p(A \cap B)$, $p(B \cap C)$, $p(A \cup B)$

$p(B \cup C)$

Sol 1) $S = \left\{ \begin{matrix} (1,1), (1,2), \dots, (1,6) \\ \vdots \\ (6,1), (6,2), \dots, (6,6) \end{matrix} \right\}$

2) $A = \{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}$

$B = \{ (3,6), (4,5), (5,4), (6,3), (4,6), (5,5), (6,5), (6,4)$

$(5,6)$

$C = \{ (6,5), (6,6), (5,6) \}$

$p(A) = 6/36 = 1/6$

$p(B) = 9/36$.

$p(C) = 3/36$.

$p(A \cap B) = p(\emptyset) = \text{Zero}$

$p(B \cap C) = p(\{(5,6), (6,5)\}) = 2/36$

$p(A \cup B) = 15/36$

$p(B \cup C) = 10/36$

$p(A) + p(B) = \frac{6}{36} + \frac{9}{36} = \frac{15}{36}$

(disjoint)

$\neq p(B) + p(C)$.

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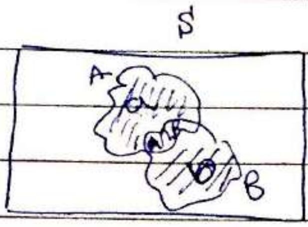
* Joint Probability "$P(A \cap B)$"

↳ the probability of the occurance of A and B at the same time

LHS $\{ P(A \cap B) = P(A) + P(B) - P(A \cup B) \}$ RHS

- Proof:



$$RHS = P(A) + P(B) - P(A \cup B)$$
$$= P(a) + P(A \cap B) + P(b) + P(A \cap B)$$
$$- (P(a) + P(A \cap B) + P(b))$$
$$= P(A \cap B) \quad \#$$

- As a consequence: -

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- As a special case, if $A \cap B = \emptyset$ (disjoint)

$$P(A \cup B) = P(A) + P(B)$$

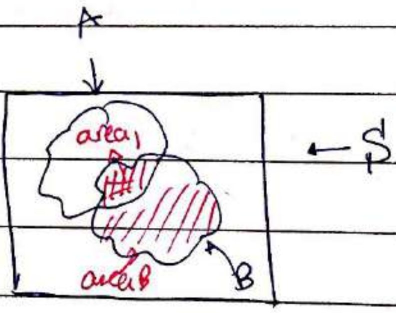
- Note: $P(A \cup B) \leq P(A) + P(B)$

* Conditional Probability: -

"$P(A/B)$": the probability of event A given that event B has occured.

↳ "given" or "condition"

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$



$$P(A/B) = \frac{a_1/S}{a_B/S} = \frac{P(A \cap B)}{P(B)}$$

$$P(B \cap B) = P(A/B) \cdot P(B)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(B/A) \cdot P(A)$$

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* Ex: Given a box of 100 resistors

	5%	10 %	Total
22 Ω	10	14	24
47 Ω	28	16	44
100 Ω	24	8	32
	62	38	100

exp # 1: draw out one resistor

Define the events A: the resistor is 47 Ω

B: the resistor is with 5% tolerance

C: the resistor is 100 Ω

find $p(A)$, $p(B)$, $p(C)$, $p(A/B)$, $p(A/C)$, $p(B/C)$?

Sol $S = \{ \underbrace{(22\Omega, 5\%), ---, (22\Omega, 5\%)}_{10}$

$\underbrace{(22\Omega, 10\%), ---, (22\Omega, 10\%)}_{14} ---- \}$

1) $p(A) = p(\text{the resistor is } 47\Omega) = 44/100$

2) $p(B) = p(\text{the resistor is } 5\%) = 62/100$

3) $p(C) = p(\text{the resistor is } 100\Omega) = 32/100$.

4) $p(A/B) =$ 28 / 62

↳ the resistor is 47 Ω given that 5% tolerance.

$$p(A/B) = \frac{p(A \cap B)}{p(B)} = \frac{28/100}{62/100} = 28/62.$$

5) $p(A/C) =$ zero.

6) $p(B/C) = 24/32$

OR $$p(B/C) = \frac{p(B \cap C)}{p(C)} = \frac{24/100}{32/100} = 24/32$$!!

* Ex: Given a box of 80 resistors.

10Ω	15Ω	20Ω	30Ω
18	12	33	17

exp#1: Draw out one resistor

$p(\text{the resistor is } 10\Omega) = 18/80$

$p(\text{the resistor is } 15\Omega) = 12/80$

$p(\text{the resistor is } 20\Omega) = 33/80$

$p(\text{the resistor is } 30\Omega) = 17/80$

exp#2: Draw out two resistors without replacement

Find $p(\text{the } 2^{nd} \text{ resistor is } 30\Omega \text{ \& the } 1^{st} \text{ resistor is } 15\Omega)$

$p(\text{the } 2^{nd}\ 30\Omega / 1^{st}\ 15\Omega) \cdot p(1^{st}\ 15\Omega)$

$$\frac{17}{79} \cdot \frac{12}{80}$$

or $p(1^{st}\ 15\Omega / \overset{②}{2^{nd}}\ 30\Omega) \cdot p(\overset{①}{2^{nd}}\ 30\Omega)$

↳ Using ① the Law of total probality & ② Bayes ~~Law~~ rule

* Statistical Independence:

↳ let A and B with $p(A) \neq 0$, $p(B) \neq 0$.

If $p(A/B) = p(A)$ and $p(B/A) = p(B)$

then A and B are said to be statistically independent.

- As a consequence

$$p(A/B) = \frac{p(A \cap B)}{p(B)} = p(A)$$

$\rightarrow p(A \cap B) = p(A) \cdot p(B)$

↳ can be used to examine dependency.

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- Ex: the same previous example but with replacement

$p(2^{nd}\ 30\Omega / 1^{st}\ 10\Omega) = 17/80$

$p(2^{nd}\ 10\Omega \cap 1^{st}\ 15\Omega) = \frac{18}{80} \cdot \frac{12}{80}$.

- Note:

If A_1 and A_2 are independent

as a result:

(1) A_1 and $\bar{A}_2$ are independent

(2) $\bar{A}_1$ and A_2 are independent

(3) $\bar{A}_1$ and $\bar{A}_2$ are independent.

- Ex: Given A_1 and A_2 are independent events with

$p(A_1) = 0.6 \qquad p(A_2) = 0.3$.

Find (1) $p(A_1 \cap \bar{A}_2) = p(A_1) \cdot p(\bar{A}_2) = (0.6)(0.7)$

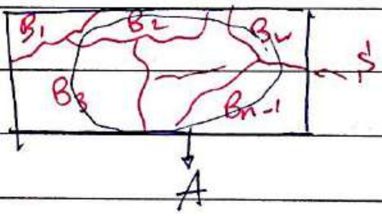
(2) $p(\bar{A}_2 / \bar{A}_1) = p(\bar{A}_2) = 0.7$

* Law of Total Probability:

↳ Let $B_1, B_2, \dots, B_n$ such that

(1) $B_i \cap B_j = \emptyset \quad \forall\ i \neq j = 1, 2, \dots, n$ — ماكو تقاطع بين أي تنين

(2) $\bigcup_{i=1}^{N} B_i = S$ — احتماليه كلهم = S



$$\boxed{p(A) = \sum_{i=1}^{N} p(A/B_i) \cdot p(B_i)}$$

- Proof:- $p(A) = p(A \cap S) = p\left(A \cap \left(\bigcup_{i=1}^{N} B_i\right)\right)$

$= p(A \cap (B_1 \cup B_2 \cup B_3 \dots \cup B_N))$

$= p((A \cap B_1) \cup (A \cap B_2) \dots \cup (A \cap B_N))$ (disjoint to each other)

$= p(A \cap B_1) + p(A \cap B_2) + \dots + p(A \cap B_N)$

$= \sum_{i=1}^{N} p(A \cap B_i) = \sum_{i=1}^{N} p(A/B_i) \cdot p(B_i)$ #

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* Bayes Rule:

$$P(B_i/A) = \frac{P(B_i \cap A)}{P(A)} = \boxed{\frac{P(A/B_i) \cdot P(B_i)}{P(A)}}$$

*Ex 1-

10Ω	15Ω	20Ω	30Ω
18	12	33	17

exp: Draw out two resistors without replacement.

(1) $P(2^{nd}\ 20\Omega)$

(2) $P(1^{st}\ 10\Omega / 2^{nd}\ 20\Omega)$

Sol (1) $P(2^{nd}\ 20\Omega) = P(2^{nd}\ 20\Omega \cap 1^{st}\ 10\Omega) \cup P(2^{nd}\ 20\Omega \cap 1^{st}\ 15\Omega)$
$\cup\, P(2^{nd}\ 20\Omega \cap 1^{st}\ 20\Omega) \cup P(2^{nd}\ 20\Omega \cap 1^{st}\ 30\Omega)$

$$= \overset{33/79}{P(2^{nd}\ 20\Omega / 1^{st}\ 10\Omega)} \cdot P(1^{st}\ 10\Omega) + \overset{33/79}{P(2^{nd}\ 20\Omega / 1^{st}\ 15\Omega)} \cdot P(1^{st}\ 15\Omega)$$
$$+ \underset{32/79}{P(2^{nd}\ 20\Omega / 1^{st}\ 20\Omega)} \cdot P(1^{st}\ 20\Omega) + \underset{33/79}{P(2^{nd}\ 20\Omega / 1^{st}\ 30\Omega)} \cdot P(1^{st}\ 30\Omega).$$

(2) $P(1^{st}\ 10\Omega / 2^{nd}\ 20\Omega) = \dfrac{P(1^{st}\ 10\Omega \cap 2^{nd}\ 20\Omega)}{\underset{\text{from (1)}}{P(2^{nd}\ 20\Omega)}} = \dfrac{P(2^{nd}\ 20\Omega / 1^{st}\ 10\Omega) \cdot P(1^{st}\ 10\Omega)}{P(2^{nd}\ 20\Omega)}$

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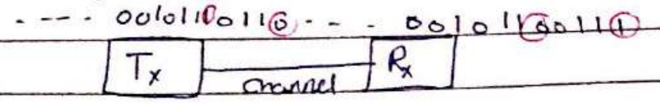
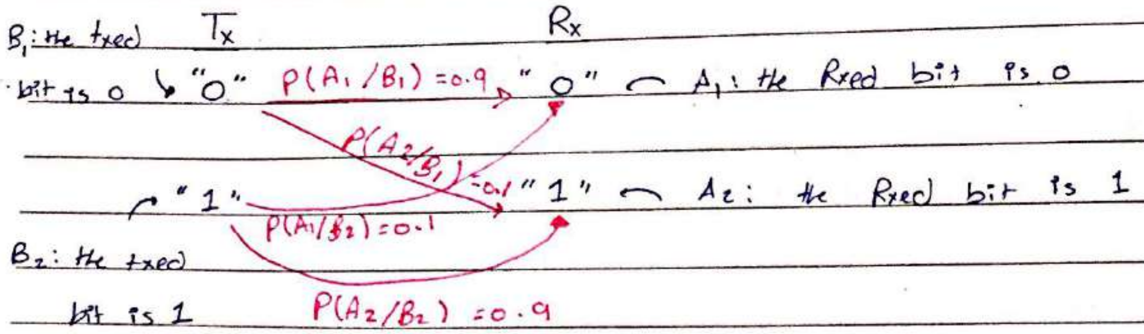
Ex Binary Communication channel (BCC)





$P(B_1)=0.6$, $P(B_2)=0.4$

find : $P(A_1)$, $P(A_2)$, $P(B_1/A_1)$, $P(B_2/A_1)$, $P(B_1/A_2)$, $P(B_2/A_2)$?

Sol $P(A_1) = P(A_1/B_1) \cdot P(B_1) + P(A_1/B_2) \cdot P(B_2)$

$= (0.9)(0.6) + (0.1)(0.4) = 0.58$

$P(A_2) = 1 - P(A_1) = 1 - 0.58 = 0.42$.

$P(B_1/A_1) = \frac{P(B_1 \cap A_1)}{P(A_1)} = \frac{P(A_1/B_1) \cdot P(B_1)}{P(A_1)} = \frac{(0.9)(0.6)}{0.58} = 0.931$

$P(B_2/A_1) = 1 - P(B_1/A_1) = 1 - 0.931 = 0.069$

$P(B_1/A_2) = \frac{P(A_2/B_1) \cdot P(B_1)}{P(A_2)} = \frac{(0.1)(0.6)}{0.42}$

$P(B_2/A_2) = 1 - P(B_1/A_2)$.

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*Combined Experiments :-

↳ experiment consists of multiple sub-experiments.

-Ex exp: flip a coin & roll a dice.

$S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)$

$(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$.

exp → Sub-exp #1 : flip a coin $S_1 = \{H, T\}$.

exp → Sub-exp #2 : Roll a dice. $S_2 = \{1, 2, 3, 4, 5, 6\}$.

$$\boxed{S = S_1 \times S_2}$$

combination. (all possible outcomes).

-Ex Roll 3 dice

$S = S_1 \times S_2 \times S_3$

S_1	S_2	S_3
1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6

$6 \times 6 \times 6 = 6^3$

In exp', define C: " the coin is $\underbrace{H}_{As_1}$ and the $\underbrace{\text{appeared \# is odd}}_{As_2}$ " ?

$C = \{ (H,1), (H,3), (H,5) \}$

OR

C → Sub-event 1 : $As_1 = \{H\}$

C → Sub-event 2 : $As_2 = \{1,3,5\}$.

$C = As_1 \times As_2$

"And", " $\cap$ "

$p(C) = 3/12$

OR $p(C) = p(As_1 \times As_2)$

$= p(As_1 \cap As_2)$

$= p(As_1) \times p(As_2)$

$= \frac{1}{2} * \frac{3}{6} = 3/12$

the and between combinat is always * because they are independent

Ex Roll two dice

$p(\underbrace{1^{st} \text{ odd}}_{As_1} \text{ and } \underbrace{2^{nd} \text{ even}}_{As_2})$.

$p(As_1) * p(As_2)$

$3/6 * 3/6 = 9/36$

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"Counting Rules"

* Permutations and Combinations:

- Permutations:

↳ all possible sequences of ordering (the order is important)

$\boxed{r}$ elements taken from $\boxed{n}$ elements without replacement.

P_r^n : number of permutations.

- Ex Given 4 cards

A	B	C	D

order 2 elements.

$\boxed{n=4}$ $\boxed{r=2}$

place 1	place 2
A	B
A	C
A	D
B	A
B	C
B	D
C	A
C	B
C	D
D	A
D	B
D	C

$P_2^4 = 12 = \underset{\text{place 1}}{4} \times \underset{\text{place 2.}}{3}$

$P_3^4 = 24 = 4 \times 3 \times 2$

$$* \; P_r^n = n(n-1)(n-2) \cdots (n-r+1)$$

$$P_r^n = \frac{n!}{(n-r)!}$$

Five Apple

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- Combinations:

↳ All possible sequences of taking $\boxed{r}$ elements from $\boxed{n}$ elements $\boxed{\text{without replacement}}$ $\boxed{\text{(order is not important)}}$.

C_r^n : number of combinations.

$$C_2^4 = \frac{12}{2} = 6 = \frac{P_2^4}{2!}$$

$$C_n^r = \frac{P_r^n}{r!} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

- Ex Perform a 3 - member team from 6 students.

$$C_3^6 = \frac{6!}{3!\,(6-3)!} = \frac{6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}\ 3!} = \frac{6 \times 5 \times 4}{3 \times 2}$$

$$= 2 \times 5 \times 2 = \boxed{20}$$

* $\binom{n}{1} = n$ $\qquad$ $\binom{n}{0} = 1$

$\binom{n}{n} = 1$ $\qquad$ $0! = 1$

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* Bernoulli Trial :-

↳ Experiment with two possible out comes

$A, \bar{A}$

Success ← → Fail

$p(A) = p$

$p(\bar{A}) = 1-p$

Ex Flip a coin

$S = \{H, T\}$

$A = \{H\}$

$\bar{A} = \{T\}$

Ex Take an exam

$S = \{S, F\}$

$A = \{S\}$

$\bar{A} = \{F\}$

* Repeat Bernoulli trial n times.

(1) what is the number of possible ~~sequences~~ successes K?

$K = 0, 1, 2, ----, n$

(2) what is the probability that the number of successes $K = i$?

Ex Flip a coin 3 times :- given $p(\{H\}) = p$ $p(\{T\}) = 1-p$

1 Flip is Bernoulli trial → Success $= \{H\}$

2 → Fail $= \{T\}$

HHH	$K=3$
HHT HTH THH	$K=2$
HTT THT TTH	$K=1$
TTT	$K=0$

$N = 3$

$K = 0, 1, 2, 3$

$p(K=0) = p(\{T, T, T\}) = p(T) \cdot p(T) \cdot p(T) = (1-p)^3$

$p(K=2) = p(\{HHT\} \{HTH\} \{THH\})$

$= p(HHT) + p(HTH) + p(THH)$

$= \cancel{3(1-p)^3}$ $3p^2(1-p)$

Five Apple

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* Repeat Bernouli N time

$$p(k) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\sum_{k=0}^{N} p(k) = 1$$

- Ex Flip a coin 70 times, $p(\{H\}) = 0.2$, $p(\{T\}) = 0.8$ Compute the probability that the tail appears at most 2 times?

Sol p(tail appears at most 2 times)

$p(k \leq 2) = p(k=0 \cup k=1 \cup k=2)$

$= p(k=0) + p(k=1) + p(k=2)$.

$= \binom{70}{0}(0.8)^0(0.2)^{70} + \binom{70}{1}(0.8)(0.2)^{69} + \binom{70}{2}(0.8)^2(0.2)^{68}$

- Ex 1.7-1

Submarine has 3 torpedos

p(torbedo to hit) $= 0.4$

the carrier will be sunk if two torkedos ore more will hit it.

Sol $N = 3$, $p = 0.4$

p(the carrier will be sunk)

$= p(k \geq 2)$

$= p(k=2) + p(k=3)$

$= \binom{3}{2}(0.4)^2(0.6) + (0.4)^3 = 0.352$.

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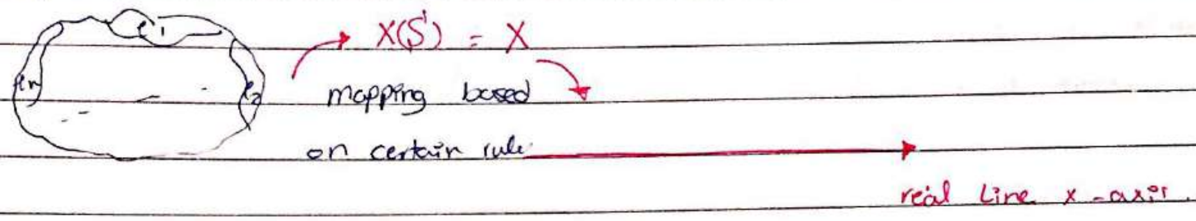
CH#2

* Random Variables : - "R.V"

↳ RV is a function of the sample space S

-Exp:



* R.V
- → Discrete : $X = \{x_1, x_2, x_3, \dots x_n\}$.
- → Continuos : $X = \{ x_1 < x < x_2 \}$
- → Mixed

-Ex exp: Flip a coin and roll a dice

Define R.V X = { If head appears : x = number on the dice
If Tail appears : x = (−2) * number on the dice.

Sol
$$S = \left\{ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (H,1), & (H,2), & (H,3), & (H,4), & (H,5), & (H,6) \\ (T,1), & (T,2), & (T,3), & (T,4), & (T,5), & (T,6). \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ -2 & -4 & -6 & -8 & -10 & -12. \end{array} \right.$$

$X = \{-12, -10, -8, -6, -4, -2, 6, 5, 4, 3, 2, 1\}$

*Notes: $\{X \leq x\}$, $\{x_1 < x < x_2\}$, $\{X = x_1\}$ → events
S as events r.v's

*Note : $P(X = -\infty) = 0$, $P(X = \infty) = 0$

$P(X \leq -\infty) = 0$

for previous example

$P(X = -4) = P(\{T,2\}) = 1/12$, $P(X = 16) = \text{zero}$, $P(X \leq 20) = 1$

~~P(X ≥ 1.5~~ $P(1.5 \leq X < 5) = P(X \in \{2,3,4\}) = P(\{H,2\}, \{H,3\}, \{H,4\}) = 3/12$

$P(X \leq \infty) = 1$

Five Apple

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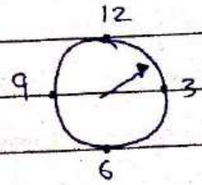
* For Discrete R.V

$p(X=x)$ exists!

- Ex Continuos R.V : wheel of chance

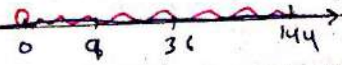
Find (1) S

(2) Define R.V $X = S^2$



Sol $S = \{0 \le S \le 12\}$

$X = \{0 \le X \le 144\}$



$p(0 \le X \le 9) = p(0 \le S \le 3) = \frac{3-0}{12-0} = \frac{3}{12}$

$p(36 \le X \le 144) = p(6 \le S \le 12) = 1/2$

$p(d_1^2 < X < d_2^2) = p(d_1 \le S \le d_2) = \frac{d_2 - d_1}{12} = \frac{\Delta d}{12}$

$$\lim_{\Delta d \to 0} \frac{\Delta d}{12} = p(S = \text{point}) = 0$$

* For Continuos R.V

$p(X=x) = 0$.

$\Rightarrow p(x_1 < X < x_2) = p(x_1 \le X < x_2) = p(x_1 < X \le x_2) = p(x_1 \le X \le x_2)$

$p(X \le -\infty) = 0$

$p(X \le \infty) = 1$

$p(-2 \le X \le 9) = p(0 \le X \le 9) = 3/12$.

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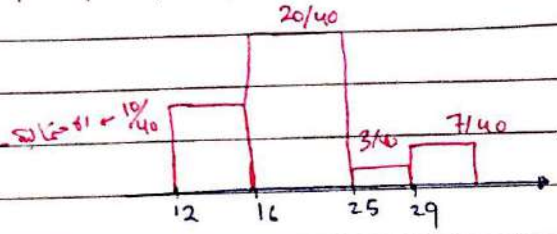
* Distribution and Density Function:

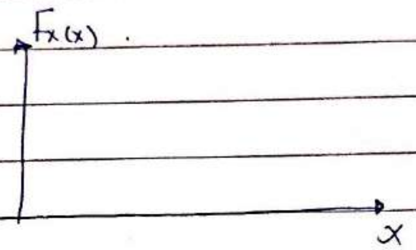
* R.V $G = \{25, 16, 12, 29\}$
  $\quad\;\; 3 \quad 20 \quad 10 \quad 7$

40 students.

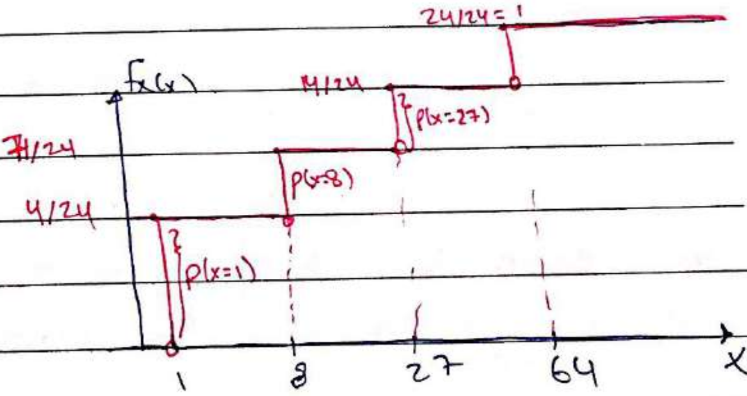


* X → Commulative Distribution Function ($F_X(x)$) "cdf"
  $F_X(x) = P(X \le x)$
  → Probability Density Function ($f_X(x)$) "pdf"

- CDF:
  $F_X(x) = P(X \le x)$
  $(-\infty, \infty)$



- Ex $S = \{1, 2, 3, 4\}$
  $P(\{1\}) = 4/24$, $P(\{2\}) = 3/24$, $P(\{3\}) = 7/24$, $P(\{4\}) = 10/24$
  Define R.V $X = S^3$
  Find (1) X (2) $F_X(x)$



Sol $X = \{1, 8, 27, 64\}$

$F_X(x) = P(X \le x)$

$F_X(-\infty) = P(X \le -\infty) = 0.$

$F_X(-2) = P(X \le -2) = 0$

$F_X(1^-) = P(X \le 1^-) = \text{Zero}$

$F_X(1) = P(X \le 1) = P(X = 1) = 4/24$

$F_X(2) = P(X \le 2) = P(X = 1) = 4/24$

$F_X(8) = P(X \le 8) = P(X=1) + P(X=8) = 4/24 + 3/24 = 7/24$

$F_X(100) = 1$

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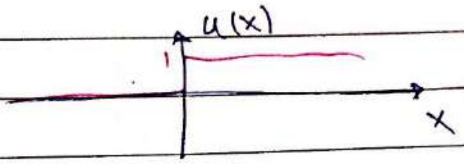
* Notes: $F_X(-\infty) = 0$

$F_X(\infty) = 1$

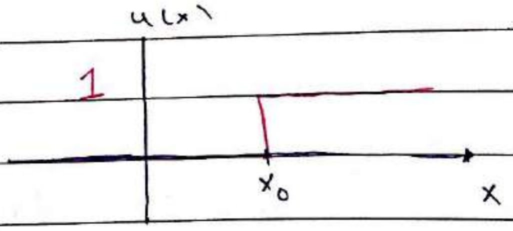
$$F_X(x) = \begin{cases} 0, & x < 1 \\ 4/24, & 1 \le x < 8 \\ 7/24, & 8 \le x < 27 \\ 14/24, & 27 \le x < 64 \\ 1, & x \ge 64 \end{cases}$$

* Unit Step function:-

$$u(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$$



$$\rightarrow u(x - x_0) = \begin{cases} 1, & x \ge x_0 \\ 0, & x < x_0 \end{cases}$$



↳ for the previous example

$$F_X(x) = \frac{4}{24} u(x-1) + \frac{3}{24} u(x-8) + \frac{7}{24} u(x-27) + \frac{10}{24} (x-64)$$

ارتفاع كل step مضروبة ب u للvalue.

* For Discrete R.V $X = \{x_1, x_2, x_3, \dots, x_n\}$

with $p(X=x_1), P(X=x_2) \dots, p(X=x_n)$

Then

$$F_X(x) = p(X=x_1)u(x-x_1) + p(X=x_2)u(x-x_2) + \dots + p(X=x_n)u(x-x_n)$$

$$F_X(x) = \sum_{i=1}^{N} p(X=x_i)\,u(x-x_i)$$

- practice $X = \{-1, 0.1, 2, 3, 4.5\}$

$p(-1) = 0.1$ $\quad p(2) = 0.05$ $\quad p(4.5) = 0.5$

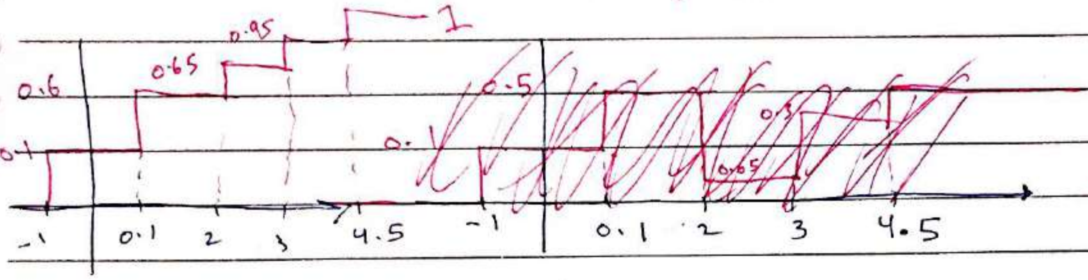
$p(0.1) = 0.5$ $\quad p(3) = 0.3$

Find and plot $F_X(x)$?

$F_X(x) = \sum_{i=1}^{N} p(X = x_i)\, u(x - x_i)$

$= p(X=-1)u(x+1) + p(X=0.1)u(x-0.1) + p(X=2)u(x-2) + p(X=3)u(x-3) + p(X=4.5)u(x-4.5)$

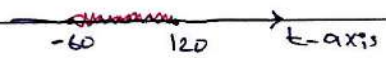
$= 0.1u(x+1) + 0.5u(x-0.1) + 0.05u(x-2) + 0.3u(x-3) + 0.5u(x-4.5).$



- Ex CDF for continuous R.V

Let a R.V T represents the temperature in F° for certain location.

$T = \{-60° \le t \le 120°\}$



Find CDF?

Sol $F_T(t) = p(T \le t)$

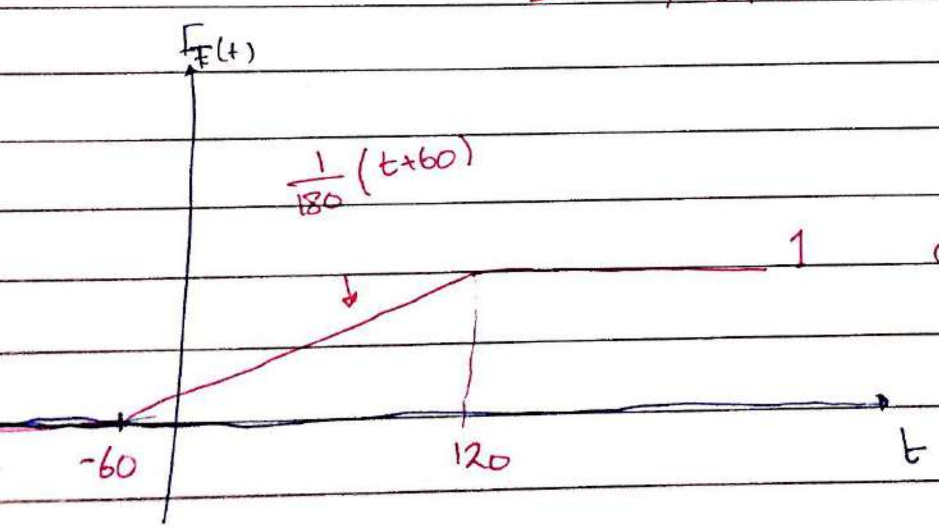
$$F_T(t) = p(T \le t) = \begin{cases} 0, & t < -60 \\ \frac{t+60}{180}, & -60 \le t < 120 \\ 1, & 120 \le t \end{cases}$$

$\rightarrow 180 = 120 - -60.$

$p(T \le 10) = p(-60 \le T \le 10)$

$= \frac{10 - (-60)}{120 - (-60)}$ interval.

$p(T \le 140) = 1$



continuous

$p(T=5) = 0$

$F_T(5) = 65/180$

$p(T \le 5)$

$F_T(-\infty) = 0$ $\quad F_T(\infty) = 1$

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* CDF properties :-

1. $F_X(-\infty) = 0$
2. $F_X(\infty) = 1$
3. $0 \leq F_X(x) \leq 1$
4. $F_X(x) = F_X(x^+)$ (i.e $F_X(x)$ is continuous from the right).
5. If $x_1 \leq x_2$ then $F_X(x_2) \geq F_X(x_1)$

   (i.e $F_X(x)$ is non-decreasing function)

6. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

→ proof :-

$$\xrightarrow[\quad x_1 \qquad x_2 \quad]{}$$

$$F_X(x_2) = P(X \leq x_2) = P(X \leq x_1 \ \cup \ x_1 < X \leq x_2)$$

$$= P(X \leq x_1) + P(x_1 < X \leq x_2)$$

$$F_X(x_2) = F_X(x_1) + P(x_1 < X \leq x_2)$$

then $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$ #!

- Ex For previous example (Temprature)

$$P(-80 \leq T \leq 25) = F_T(25) - F_T(-80)$$

$$= \frac{25+60}{180} - 0$$

لو شلت ال equality

أو خليتها ما بتفرق

مشان هيك فترات

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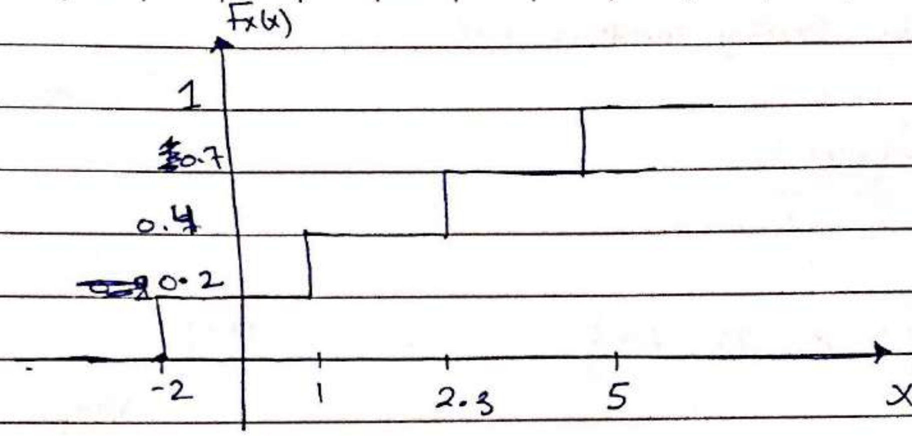
$*F_x$



Find:-

1) $P(X = 2.3) = 0.7 - 0.4 = \boxed{0.3}$

2) $P(2 < X \leq 7) = F_x(7) - F_x(2) = 1 - 0.4 = \boxed{0.6}$

3) $P(0 \leq X \leq 4) = \underline{P(X=0)} + P(0 < X \leq 4)$

مش موجودة عندي
عشان $F_x(0)$ ثابت!

$= 0 + F_x(4) - F_x(0)$

$= 0.7 - 0.2 = \boxed{0.5}$

4) $P(1 \leq X < 3) = P(X=1) + P(1 < X \leq 3^-)$

$= (0.4 - 0.2) + F_x(3^-) - F_x(1)$

$= (0.2) + 0.7 - 0.4$

$= \boxed{0.5}$

5) $P(2.3 < X < 5) = P(2.3 < X \leq 5^-)$

$= F_x(5^-) - F_x(2.3)$

$= 0.7 - 0.7 = \boxed{0}$

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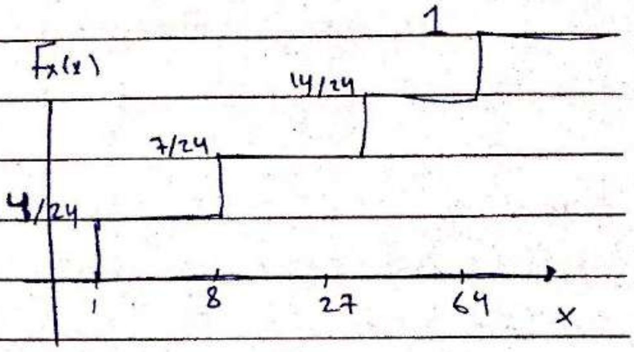
* Probability Density Function "pdf"

$$f_X(x) = \frac{dF_X(x)}{dx}$$

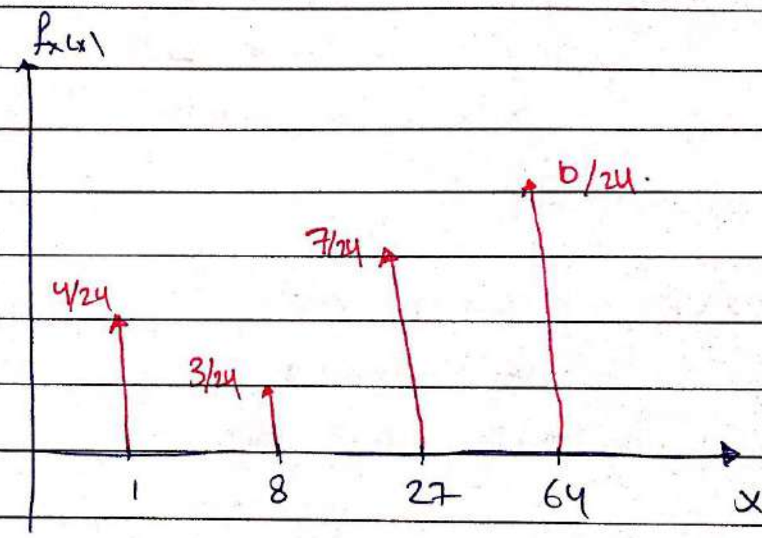
- Ex $X = \{1, 8, 27, 64\}$

Find $f_X(x)$?



Sol $F_X(x) = \frac{4}{24} u(x-1) + \frac{3}{24} u(x-8) + \frac{7}{24} u(x-27) + \frac{10}{24} u(x-64)$

$f_X(x) = \frac{dF_X(x)}{dx} = \frac{4}{24} \delta(x-1) + \frac{3}{24} \delta(x-8) + \frac{7}{24} \delta(x-27) + \frac{10}{24} \delta(x-64)$



* for discrete R.V $X = \{x_1, x_2, \ldots, x_N\}$

$$F_X(x) = \sum_{i=1}^{N} P(X = x_i) \, u(x - x_i)$$

$$f_X(x) = \sum_{i=1}^{N} P(X = x_i) \, \delta(x - x_i)$$

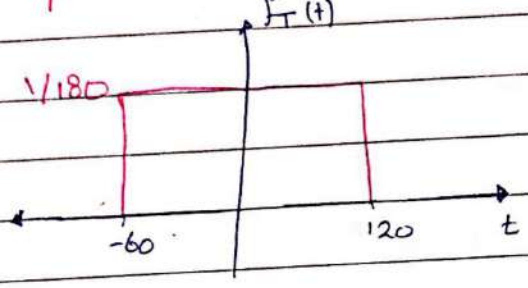
- Temprature Example:-

$T = \{-60^\circ \leq t \leq 120\}$.

$$F_T(t) = \begin{cases} 0 & , \ t < -60 \\ \frac{t+60}{180} & , \ -60 \leq t \leq 120 \\ 1 & , \ t \geq 120 \end{cases}$$

Find $f_T(t)$?

Sol $f_T(t) = \frac{d F_T(t)}{dt} = \begin{cases} 0 & , \ t < -60 \\ 1/180 & , \ -60 \leq t \leq 120 \\ 0 & , \ t \geq 120 \end{cases}$



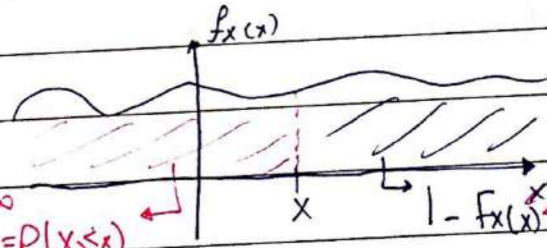
Uniform R.V

* PDF properties :-

[1] $f_X(x) \geq 0$

[2] $\int_{-\infty}^{\infty} f_X(x)\, dx = \boxed{1} = P(S)$

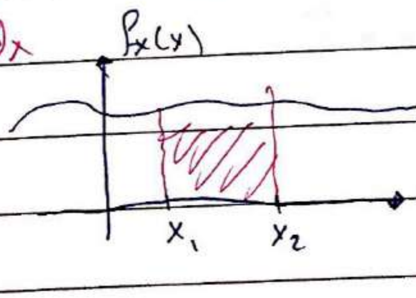
[3] $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$



$F_X(x) = P(X \leq x)$

$1 - F_X(x) = P(X \geq x) = \int_x^{\infty} f_X(y)\, dy.$

[4] $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx$



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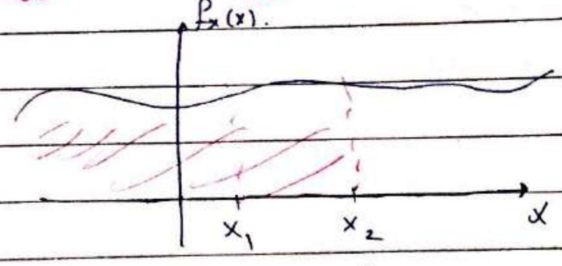
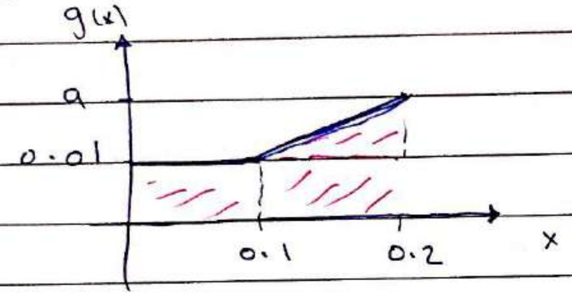
- proof for $\boxed{4}$:-

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

$$= \int_{-\infty}^{x_2} f_X(x)\,dx - \int_{-\infty}^{x_1} f_X(x)\,dx.$$



* given g(x) pdf, find the value of a?



Sol $\int_{-\infty}^{\infty} g(x)\,dx = 1$

$= (0.1)(0.01) + (0.1)(0.01) + (0.5)(0.1)(a - 0.01) = 1$

* Note Discrete R.V $X = \{x_1, x_2, \dots, x_n\}$

$$f_X(x) = \sum_{i=1}^{N} P(X = x_i)\,\delta(x - x_i).$$

check $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$??

$$\Rightarrow \int_{-\infty}^{\infty} \sum_{i=1}^{N} \underbrace{P(X = x_i)}_{\text{constant}}\,\delta(x - x_i)\,dx.$$

1 (تكامل δ على فترة كلها = 1)

$$\sum_{i=1}^{N} P(X = x_i) \int_{-\infty}^{\infty} \delta(x - x_i)\,dx$$

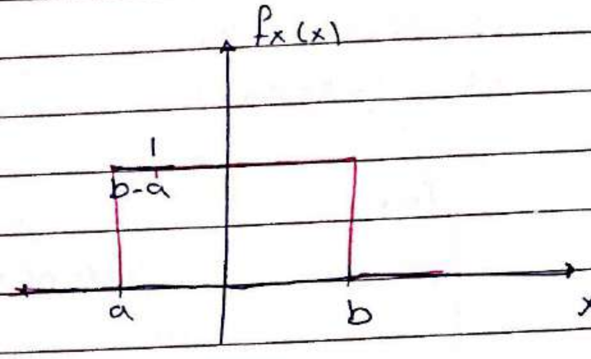
$$\therefore \sum_{i=1}^{N} P(X = x_i) = 1$$

* Common R.V Types:-

1) Uniform R.V

$X \sim U(a,b)$

- $b > a$
- a & b real numbers
- $X = \{a \leq x \leq b\}$.



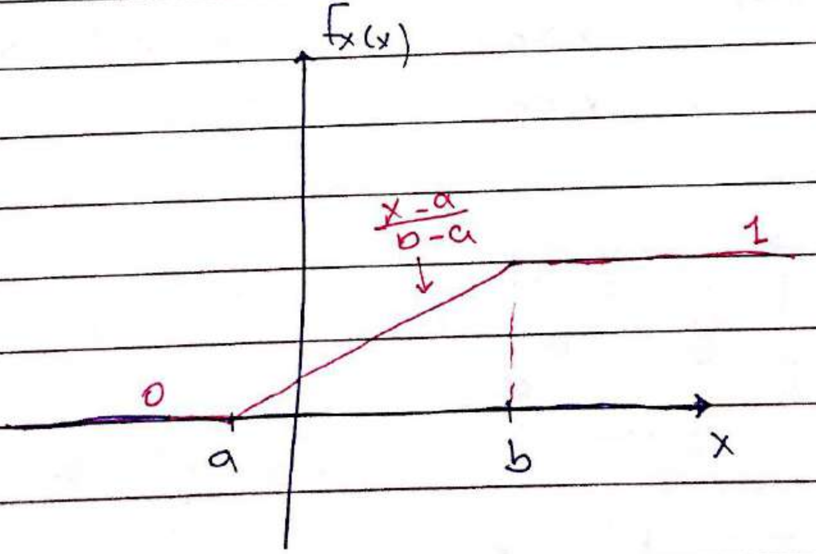
$$f_X(x) = \begin{cases} \frac{1}{b-a} & , \; a \leq x \leq b \\ 0 & , \; o.w \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \begin{cases} 0 & , \; x < a \\ \int_a^x \frac{1}{b-a} & , \; a < x < b \\ 1 & , \; x > b \end{cases}$$

بنشوف حسب قيمة x وين موجودة.

كأنه كاملة على كل الفترة.

$$= \begin{cases} 0 & , \; x < a \\ \frac{x-a}{b-a} & , \; a \leq x < b \\ 1 & , \; x \geq b \end{cases}$$



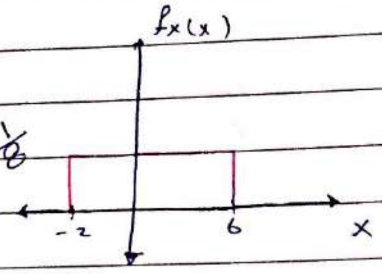
Ex $X \sim U(-2,6)$

Find (1) $P(1 \le X \le 9)$

(2) $P(0 \le X < 3)$

(3) $P(-3 < X < 1)$

Sol



(1) $P(1 < X \le 9) = \int_1^9 f_X(x)\,dx = \int_1^6 \frac{1}{8}\,dx = \boxed{\frac{5}{8}}$

(2) $P(0 \le X < 3) = \boxed{\frac{3}{8}}$

(3) $P(-3 < X < 1) = \boxed{\frac{3}{8}}$

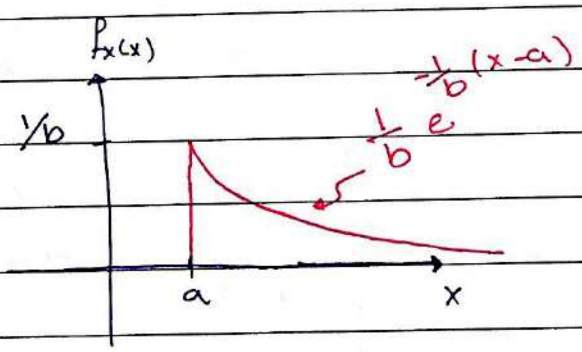
* practical example on Uniform R.V :-

Wireless communication.

$\boxed{2}$ Exponential R.V :-

$X \sim (a, b)$

- $b > 0$
- a is real number
- $X = \{ x \ge a \}$



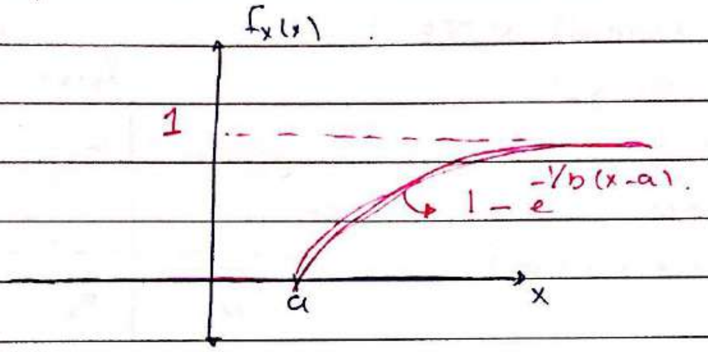
$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{1}{b}(x-a)} & , \ x \ge a \\ 0 & , \ x < a \end{cases}$$

Check: $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^{\infty} \frac{1}{b} e^{-\frac{1}{b}(x-a)}\,dx$

$= \frac{\cancel{1}}{\cancel{b}} \frac{e^{-\frac{1}{b}(x-a)}}{-\cancel{1/b}} \Big|_{a}^{\infty} = 1 - 0 = \boxed{1}$ #

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \begin{cases} 0 & , x < a \\ \int_a^x \frac{1}{b} e^{-\frac{1}{b}(x-a)} & , x > a \end{cases}$$

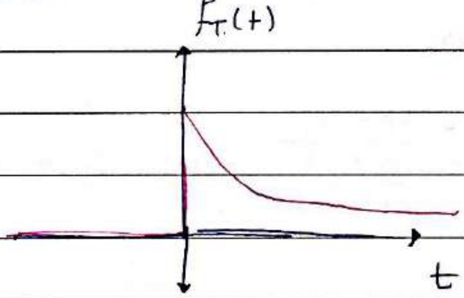
$$= \begin{cases} 0 & , x < a \\ 1 - e^{-1/b(x-a)} & x > a \end{cases}$$



*practical example on exp (a, b)

waiting time in phone calls!

$T \sim \exp(0, \lambda)$

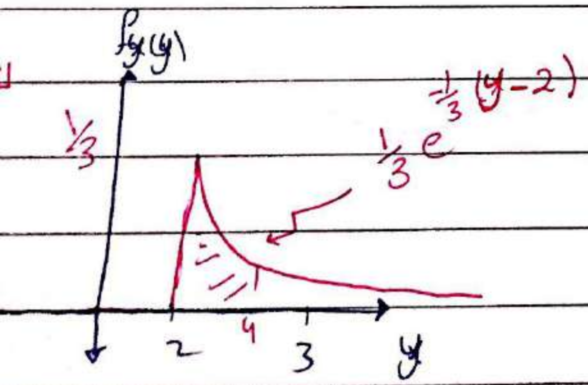


Ex $Y \sim \exp(2, 3)$

Find (1) $P(2 < Y < 4)$

(2) $P(Y < 2 \cap Y > 4)$

(3) $P(Y < 2 \cup Y > 4)$



$$(1)\ P(2 < Y < 4) = \int_2^4 \frac{1}{3} e^{-\frac{1}{3}(y-2)}\,dy.$$

$$= e^{\frac{-(2-2)}{3}} - e^{\frac{-(4-2)}{3}} = 1 - e^{-2/3}$$

(2) $p(Y<2 \cap Y>4) = p(\phi) = \text{zero}.$

2 4

(3) $p(Y<2 \cup Y>4) = p(Y<2) + p(Y>4)$

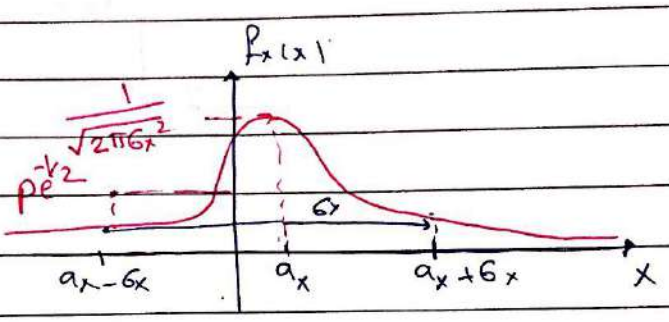
$$= 0 + \int_4^{\infty} \frac{1}{3} e^{-\frac{1}{3}(y-2)}\, dy$$

$$= 1-(1-e^{-\frac{2}{3}}) = e^{-2/3}$$

[3] Gaussian (Normal) R.V's.

$X \sim N(a_x, \sigma_x^2)$

- $a_x \in (-\infty, \infty)$
- $\sigma_x > 0$
- $X = \{-\infty < x < \infty\}$



a_x: R.V mean (Average value)

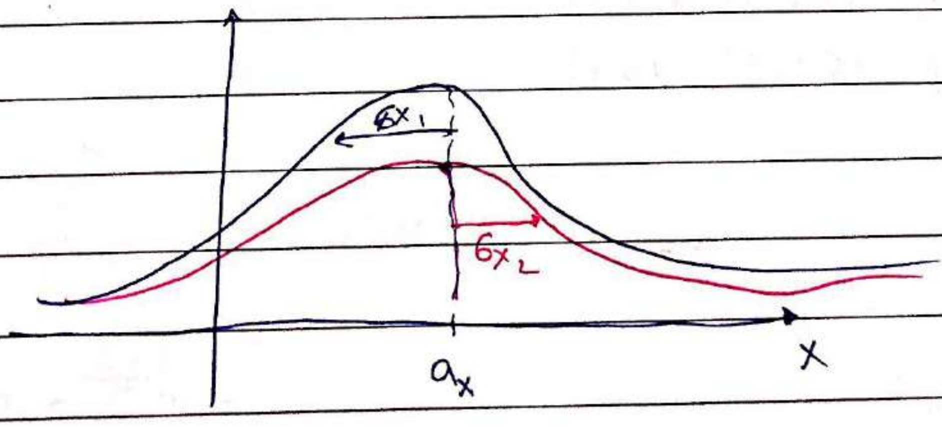
σ_x: Standard deviation.

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$$

- $X_1 \sim N(a_x, \sigma_{x_1}^2)$ $X_2 \sim N(a_x, \sigma_{x_2}^2)$
- $\sigma_{x_2} > \sigma_{x_1}$

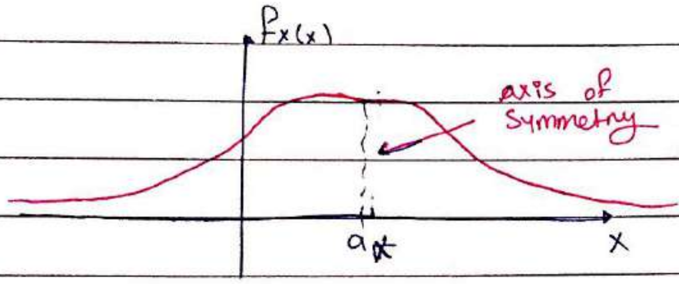
كل ما كبرت σ_x

بينزل ال peak.



* $$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x}} dx = \boxed{1}$$

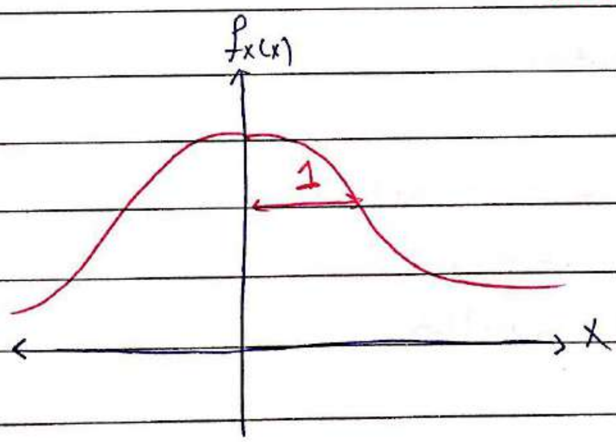
*



$$\int_{-\infty}^{a_x} N\sim(a_x, \sigma_x^2)\,dx = 0.5 = p(x \leq a_x).$$

$$\int_{a_x}^{\infty} N\sim(a_x, \sigma_x^2)\,dx = 0.5 = p(x \geq a_x).$$

- If $a_x = 0$, $\sigma_x = 1$
then $X \sim N(0,1)$ is called
the standard gaussian R.V $\qquad f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$



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* CDF for gaussian R.V :-

- Two cases :

Case I : $X \sim N(0,1)$, cdf $F_x(x)$ ← رموز بس

case II : $X \sim N(a_x, \sigma_x^2)$, cdf $F_x(x)$ ← للتعويض .

CASE I :- $X \sim N(0,1)$

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$F(x) = \int_{-\infty}^{x} f_x(y)\,dy = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$$

↳ this integral has no analytical solution!!

- Ex $X \sim N(0,1)$

find [1] $p(X < 0.23)$ [2] $p(X < -1.35)$

[3] $p(0.1 < X < 2.3)$ [4] $p(-1.5 < x < 1.4)$

[5] $p(-2.3 < x < -0.1)$.

Sol [1] $p(X < 0.23) = F_x(0.23)$

$= 0.591$

[2] $p(X < -1.35) = F(-1.35)$

$= 1 - F(1.35)$

$= 1 - 0.9115$

[3] $p(0.1 < x < 2.3) = F_x(2.3) - F(0.1)$

$= 0.9893 - 0.5398$

$= 0.4495.$

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$\boxed{4}$ $P(-1.5 < X < 1.4) = F(1.4) - F(-1.5)$

$= F(1.4) - (1 - F(1.5))$

$= F(1.4) - 1 + F(1.5)$

$= 0.9192 - 1 + 0.9332$

$= 0.8524$

$\boxed{5}$ $P(-2.3 < X < -0.1) = F(-0.1) - F(-2.3)$

$= 1 - F(0.1) - (1 - F(2.3))$

$= 1 - F(0.1) - 1 + F(2.3) = F(2.3) - F(0.1)$

$= 1 - 0.5398 - 1 + 0.9893$

$= 0.9893 - 0.5398$

$= 0.4495$

*Case II

$X \sim N(a_x, \sigma_x^2)$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-a_y)^2}{2\sigma_y^2}}\,dy$$

Using substitution integration.

$$\boxed{F_X(x) = F\left(\frac{x - a_x}{\sigma_x}\right)}$$

Ex $X \sim N(7, \frac{1}{4})$

find $\boxed{1}$ $P(X \leq 7.3)$

$$P(X \leq 7.3) = F_X(7.3) = F\left(\frac{x - a_x}{\sigma_x}\right)$$

$$= F\left(\frac{7.3 - 7}{1/2}\right) = F(0.6)$$

$= 0.7257$

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$\boxed{2}$ $P(-1.2 < x < 6)$

$$P(-1.2<x<6) = F_X(6) - F_X(-1.2)$$
$$= F\left(\frac{6-7}{1/2}\right) - F\left(\frac{-1.2-7}{1/2}\right)$$
$$= F(-2) - F(-16.4)$$
$$= 1-F(2) - (1-F(16.4))$$
$$= \cancel{1} - F(2) \cancel{-1} + F(16.4)$$
$$= F(16.4) - F(2)$$
$$= 1 - F(2)$$
$$= 1 - 0.9772$$
$$= 0.0228.$$

* Q(x) function.

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$Q(x) = 1 - F(x)$$

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[4] Bernolli R. V (Discrete):~

$\cdot\ X \sim B(p)$

- Bernolli Trial

$S = \{A, \bar{A}\} \rightsquigarrow P(A) = p$

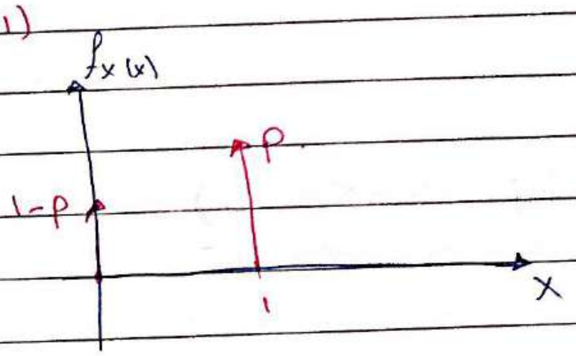
$P(\bar{A}) = 1 - p.$

$$X = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{if } \bar{A} \text{ occurs.} \end{cases}$$

$X = \{0, 1\} \rightsquigarrow P(X=0) = 1-p \ /\ P(X=1) = p$

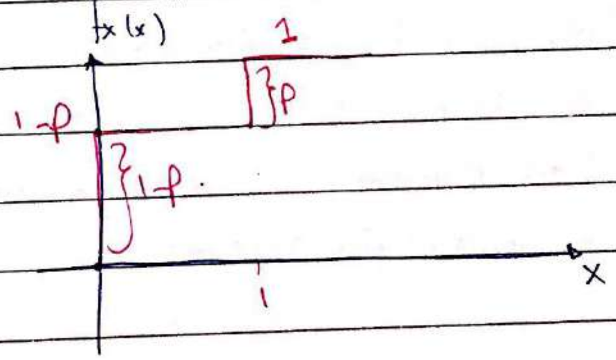
$\hookrightarrow f_X(x) = P(X=0)\,\delta(x) + P(X=1)\,\delta(x-1)$

$= (1-p)\,\delta(x) + p\,\delta(x-1)$



$F_X(x) = P(X=0)\,u(x) + P(X=1)\,u(x-1)$

$= (1-p)\,u(x) + p\,u(x-1).$



- Ex 5 $Y \sim B(0.15)$

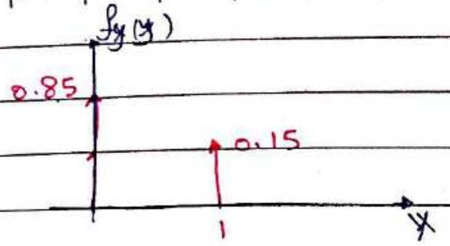
Find 1) $F_Y(y)$ & $f_Y(y)$?

2) $P(Y < 3)$

3) $P(-1 < Y < 0.75)$

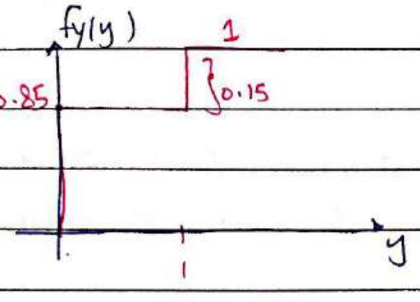
Sol 1) $f_Y(y) = (1-p)\delta(y) + p\,\delta(y-1)$

$= 0.85\,\delta(y) + 0.15\,\delta(y-1)$



$F_Y(y) = (1-p)u(y) + p\,u(y-1)$

$= 0.85\,u(y) + 0.15\,u(y-1)$



2) $P(Y<3)$

$F_X(3) = 1$

3) $P(-1 < y < 0.75)$

$= P(-1 < y \leq 0.75^-)$

$= F_X(0.75^-) - F_X(-1)$

$= 0.85 - 0 = \boxed{0.85}$

$\boxed{5}$ Binomial R.V: (Discrete)

$X \sim b(P, N)$

X: the number of success in repeating a Bernolli trial with $P(\text{success}) = P \sim N$ times.

- $X = \{0, 1, 2, \ldots, N\}$.

$P(X=K) = \binom{N}{K} P^N (1-P)^{N-K}$.

$$f_X(x) = \sum_{K=0}^{N} \binom{N}{K} P^K (1-P)^{N-K} \delta(x-K).$$

$$F_X(x) = \sum_{K=0}^{N} \binom{N}{K} P^K (1-P)^{N-K} u(x-K).$$

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*Conditional CDF & PDF:

- Recalls for R.V X $\Rightarrow F_X(x) = P(X \le x)$ → Distribution function of X

- Conditional distribution function of X:-

$$F_X(x/B) = P(X \le x / B) \Rightarrow F_X(x/B) = \frac{P(X \le x \cap B)}{P(B)}$$

- Conditional density function of X:-

$$f_X(x/B) = \frac{d F_X(x/B)}{dx}$$

- Note: All properties of pdf & cdf are the same for conditional cdf & pdf

- Ex:

R	G	B
5	35	60 ~~85~~

Box 1

R	G	B
80	60	100

Box 2

exp: Randomly select a box then draw out a ball from the selected box.

Define R.V $X = \begin{cases} 1, & \text{the ball is Red} \\ 2, & \text{the ball is Green} \\ 3, & \text{the ball is Blue.} \end{cases}$

Find: a) $F_X(x/B_1)$ b) $F_X(x/B_2)$ c) $F_X(x)$

B_1 = the selected box is box 1

B_2 = the selected box is box 2

Five Appl

- Solution:

a) $F_X(x/B_1) = P(X=1/B_1)\,u(x-1) + P(X=2/B_1)\,u(x-2) + P(X=3/B_1)\,u(x-3)$

$= \frac{5}{100}\,u(x-1) + \frac{35}{100}\,u(x-2) + \frac{60}{100}\,u(x-3)$

b) $F_X(x/B_2) = \frac{80}{150}\,u(x-1) + \frac{60}{150}\,u(x-2) + \frac{10}{150}\,u(x-3)$

c) $F_X(x) = P(X=1)\,u(x-1) + P(X=2)\,u(x-2) + P(X=3)\,u(x-3)$

$P(X=1) = P(R) = P\{(R\cap B_1) \cup (R\cap B_2)\} = P(R\cap B_1) + P(R\cap B_2)$

$= P(R/B_1)P(B_1) + P(R/B_2)P(B_2)$

$= \frac{5}{100}(0.5) + \frac{80}{150}(0.5) = 0.292$

$P(X=2) = \frac{35}{100}(0.5) + \frac{60}{150}(0.5) = 0.375$

$P(X=3) = \frac{60}{100}(0.5) + \frac{10}{150}(0.5) = 0.333$

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$F_X(x) = P(X \le x)$

*** Conditional CDF & PDF:**

$X \sim F_X(x) = P(X \le x)$

$\hookrightarrow F_X(x/B) = P(X \le x / B)$

- Ex: Let X a R.V with $F_X(x)$

If $B = \{X \le b\}$, where b is constant. Determine $F_X(x/B)$

Sol: $F_X(x/B) = P(X \le x / X \le b) = \dfrac{P(X \le x \cap X \le b)}{P(X \le b)}$

$$= \begin{cases} \dfrac{P(X \le b)}{P(X \le b)} = 1 & , x > b \\[2ex] \dfrac{P(X \le x)}{P(X \le b)} = \dfrac{F_X(x)}{F_X(b)} & , x > b \end{cases}$$

b x

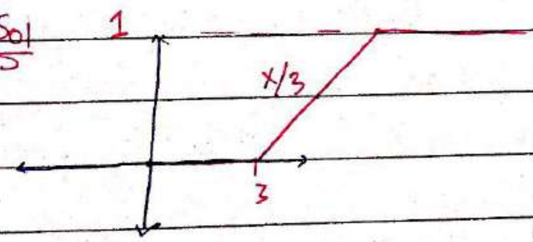
بدي منطقة التقاطع

x b

$$\boxed{F_X(x / X \le b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & , x \le b \\[2ex] 1 & , x > b \end{cases}}$$

- Ex: $X \sim U(0, 3)$. Find [1] $F_X(x)$ [2] $F_X(x / x \le 1)$

Sol:



$$F_X(x / x \le 1) = \begin{cases} \dfrac{F_X(x)}{F_X(1)} & , x \le 1 \\[2ex] 1 & , x > 1 \end{cases}$$

$$\begin{cases} \dfrac{x/3}{1/3} = x & , 0 \le x \le 1 \\[2ex] 0/1/3 = 0 & , x < 0 \end{cases}$$

$$= \begin{cases} 0 & , x < 0 \\ x & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases}$$

Five Apple

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*CH 3

Operation on one R.V "Statistical"

[I] Expectation:

↳ "Average value"

↳ "mean value"

↳ "DC value"

$$\boxed{X \rightarrow E[X] = \overline{X}}$$

- Ex The expectation of Discrete R.V

20 students.

grades		# of sts
g_1	3	2
g_2	17	6
g_3	25	8
g_4	29	4

Find the average grade? $(\overline{G})$

Sol $\overline{G} = \dfrac{3(2) + 17(6) + 25(8) + 29(4)}{20}$

$= 3 * \frac{2}{20} + 17 \cdot \frac{6}{20} + 25 \cdot \frac{8}{20} + 29 \cdot \frac{4}{20}$

Let R.V $G = \{3, 17, 25, 29\}$

$p(G=3) = 2/20$ $\quad$ $p(G=25) = 8/20$

$p(G=17) = 6/20$ $\quad$ $p(G=29) = 4/20$.

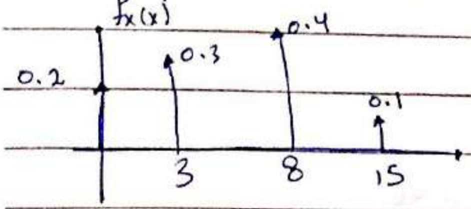
* $E[G] = \overline{G} = g_1 \cdot p(G=g_1) + g_2\, p(G=g_2) + \text{---} + g_4\, P(G=g_4)$

Five An

* For discrete R.V $X: \{x_1, x_2, x_3, ---, x_n\}$

$$\bar{X} = \sum_{i=1}^{N} x_i \, p(X = x_i)$$

- Ex $X = \{0, 3, 8, 15\}$



Find $\bar{X}$?

Sol $\bar{X} = 0(0.2) + 3(0.3) + 8(0.4) + 15(0.1)$

* In general, for any R.V X with $f_X(x)$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) \cdot dx \qquad (*)$$

- Using (*) for discrete R.V

$$\bar{X} = \int x \cdot \sum p(x = x_i) \, \delta(x - x_i) \, dx$$

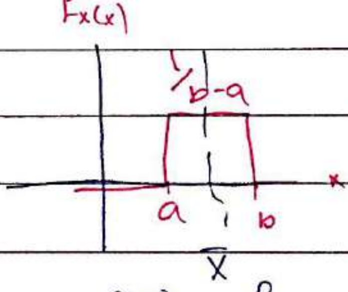
$$= x_i \sum p(x = x_i) \underbrace{\int \delta(x - x_i) \, dx}_{}^{1}$$

* $Y \sim u(-5, 4) \rightarrow \bar{Y} = -0.5$
$Z \sim u(0, 3) \rightarrow \bar{Z} = 1.5$

- Ex $X \sim u(a, b)$ find $\bar{X}$

Sol
$$\bar{X} = \int_{-\infty}^{\infty} x f_X(x) \, dx$$

$$= \int_{a}^{b} \frac{x}{b-a} \, dx = \left. \frac{x^2}{2(b-a)} \right|_a^b = \frac{b^2 - a^2}{2(b-a)}$$



axis of symmetry!

$$\boxed{\bar{X} = \frac{a+b}{2}}$$

Five Apple

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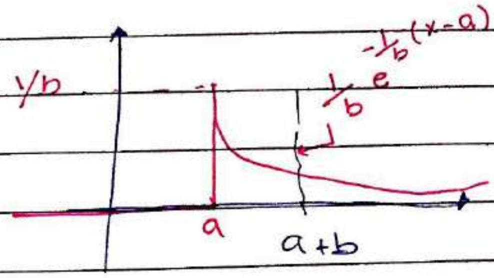
* Ex $X \sim \exp(a,b)$. Show that $\bar{X} = a+b$.



Sol

$$\bar{X} = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{a}^{\infty} x \cdot \frac{1}{b} e^{-\frac{1}{b}(x-a)}\, dx$$

$$= \frac{e^{a/b}}{b} \int_{a}^{\infty} \underbrace{x}_{u} \underbrace{e^{-x/b}\, dx}_{dv}.$$

$u = x \longrightarrow du = 1$ $\qquad\qquad u \cdot v - \int v\, du$

$dv = e^{-x/b} \longrightarrow v = \frac{e^{-x/b}}{-1/b}$

$$\bar{X} = \frac{e^{a/b}}{b}\left[+bxe^{-x/b} \Big|_{\cancel{\infty}}^{\cancel{a}} + b\int_{a}^{\infty} e^{-x/b}\, dx \right]$$

$$= \frac{e^{a/b}}{b}\left[abe^{-a/b} - 0 + \frac{b^2 e^{-x/b}}{\cancel{+1/b}} \Big|_{\cancel{a}}^{\cancel{\infty}a} \right]$$

$$= \frac{e^{a/b}}{b}\left[abe^{-a/b} + b^2 e^{-a/b} - 0 \right]$$

$$= \frac{\cancel{e^{a/b}}}{\cancel{b}}\left[a\cancel{b}\,\cancel{e^{-a/b}} + {}^{b}\cancel{b^2}\,\cancel{e^{-a/b}} \right]$$

$$= \boxed{a+b}$$

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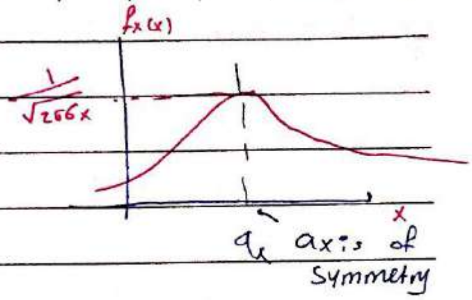
Ex $X \sim N(a_x, \sigma_x^2)$

Show that $\bar{X} = a_x$?



Sol $\bar{X} = \int_{-\infty}^{\infty} x f_{x(x)}\, dx$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx.$$

By subtitution. let $u = \frac{x - a_x}{\sigma_x}$. $\rightarrow x = \sigma_x \cdot u + a_x$

$x : -\infty \rightarrow \infty$

$u : -\infty \rightarrow \infty$ $\qquad du = \frac{1}{\sigma_x} dx$

$$= \int_{-\infty}^{\infty} \frac{(\sigma_x u + a_x)}{\sqrt{2\pi\cancel{\sigma_x^2}}} \cdot e^{-u^2/2} \cdot \cancel{\sigma_x}\, du$$

$$= \frac{\sigma_x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u}_{odd}\, \underbrace{e^{-u^2/2}}_{even}\, du + \underbrace{\int_{-\infty}^{\infty} \frac{a_x}{\sqrt{2\pi}} e^{-u^2/2} du}_{1}$$

standard pdf

Zero $+ a_x (1)$

$= \boxed{a_x}$

- Ex $X \sim \beta(p)$, Find $\bar{X}$

Sol $X = \{0, 1\}$

$p(x=0) = 1-p$

$p(x=1) = p$

$\bar{X} = 0(1-p) + 1(p) = \boxed{p}$

Ex $X \sim$ binomial (p, N).

Show that $\bar{X} = Np$.

Sol $X = \{0, 1, 2, \ldots, N\}$

$$\bar{X} = \sum_{i=0}^{N} i\, p(x=i)$$

$$= \sum_{i=\cancel{0}}^{N} i\, p(x=i) = \sum_{i=1}^{N} i \cdot \binom{N}{i} p^{i} (1-p)^{N-i}$$

$$= \sum_{i=1}^{N} i \cdot \frac{\overset{N(N-1)!}{N!}}{\underset{(i-1)!}{i!}\,(N-i)!} \underset{p\,p^{i-1}}{p^{i}} (1-p)^{N-i}$$

$$= Np \sum_{i=1}^{N} \frac{(N-1)!}{(i-1)!\,(N-i)!} p^{(i-1)} (1-p)^{N-i}$$

Let $k = i-1$, $k+1 = i$

$$= Np \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} p^{\cancel{k+1-1}} (1-p)^{N-1-k}$$

Let $N-1 = M$

$$= Np \sum_{k=0}^{M} \frac{M!}{k!\,(M-k)!} p^{k} (1-p)^{M-k}$$

$$= Np \sum_{k=0}^{M} \binom{M}{k} p^{k} (1-p)^{M-k}$$

$$= Np(1)$$

$$= \boxed{Np}$$

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* mean of function of R.V

$X \sim f_X(x)$

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$\boxed{E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx.}$$

• for discrete R.V : $X = \{x_1, x_2, ---, x_n\}$

$$g(x) = \{g(x_1), g(x_2), ---, g(x_N)\}$$

$$\text{mean}[X] = \sum_i x_i\, P(X = x_i)$$

$$\boxed{E[g(x)] = \sum_{i=1}^{N} g(x_i)\, P(X = x_i)}$$

- Ex $X = \{-1, 2, 5, 9\}$

$P\{0.1, 0.6, 0.15, 0.15\}$

Let $g(x) = x^2 - 1$

find $[g(x)]$?

Sol $E[g(x)] = 0(0.1) + 3(0.6) + 24(0.15) + 80(0.15)$ $\quad g(x) = \{0, 3, 24, 80\}$

$= 17.4$

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$\bar{X} = \int_{-\infty}^{\infty} x f_X(x)\, dx$

for discrete R.V

$\bar{X} = \sum_{i=1}^{N} x_i\, p(x = x_i)$

Expectation of function R.V

$X \sim f_X(x)$

If $g(x)$ is function of x

$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$

Ex $Z \sim u(6, -2)$

Find 1) $E[Z]$ 2) $g(z) = -2z^2 + z$

Sol 1) $E[Z] = \frac{6-2}{2} = 2$

2) $E[g(z)] = \int_{-\infty}^{\infty} g(z) \cdot f_Z(z)\, dz$

$= \int_{-2}^{6} \frac{(-2z^2 + z)}{8}\, dz$ ---

Ex Show that $E[g_1(x) + g_2(x) + \cdots + g_n(x)] = E[g_1(x)] + E[g_2(x)] + \cdots + E[g_n(x)]$

Sol $E[g_1(x) + g_2(x) + \cdots + g_n(x)] = \int_{-\infty}^{\infty} (g_1(x) + g_2(x) + \cdots + g_n(x))\, f_X(x)\, dx$

$= \int_{-\infty}^{\infty} g_1(x) f_X\, dx + \cdots + \int_{-\infty}^{\infty} g_n(x) f_X(x)\, dx$

$= E[g_1(x)] + \cdots + E[g_n(x)]$

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* $E\left[\sum_{i=1}^{N} g_i(x)\right] = \sum_{i=1}^{N} E\left[g_i(x)\right]$

* $E\left[a\,g(x)\right] = a\,E\left[g(x)\right]$

* $E[a] = a$

- Ex Let $Y \sim \exp(1, 5)$

  Find the mean of $g(y) = \frac{-1}{2} Y^2 + 2Y - 6$.

Sol $E[g(y)] = E\left[\frac{-1}{2} Y^2 + 2y - 6\right]$

$= \frac{-1}{2} E[y^2] + 2E[y] - 6$

$E[y] = 1 + 5 = 6$

$E[Y^2] = \int_{-\infty}^{\infty} y^2 . f_y(y)\, dy$

$= \int_{1}^{\infty} y^2 . \frac{1}{5} e^{\frac{-1}{5}(y-1)}$

$= \frac{1}{5} e^{\frac{+1}{5}} \int_{1}^{\infty} y^2 e^{-y/5}\, dy$ (solved "by parts" technique)

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+ Moments about the origin:-

$m_n = E[\underbrace{X^n}_{g(x)}]$, $n = 0, 1, 2, \dots$

$= \int_{-\infty}^{\infty} x^n f_X(x)\, dx$

- Zeroth order moment

$m_0 = E[X^0] = E[1] = 1$

- 1st moment "DC value", "mean value"

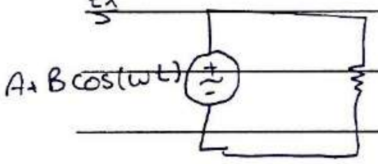
$m_1 = E[X] = \bar{X}$

$m_1^2 = \bar{X}^2$ = "DC average Power".

- 2nd moment

$m_2 = E[X^2]$ = " Total average power".

Ex



Determine the total average power dissipated by the resistor (1 Ω value)

Sol $p(t) = \frac{v^2(t)}{R} = v^2(t)$

$P = \frac{1}{T} \int_0^T p(t)\, dt = \frac{1}{T} \int_0^T v^2(t)\, dt$

$= A^2 + \frac{B^2}{2}$

DC avg Power — AC avg Power.

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- $m_2 = E[x^2] =$ Total avg power
$= $ Dc avg power + Ac avg power.

$\Rightarrow$ $\boxed{\text{Ac avg power of } X = E[x^2] - E^2[x]}$ "$Var(x)$" = "σ_x^2"

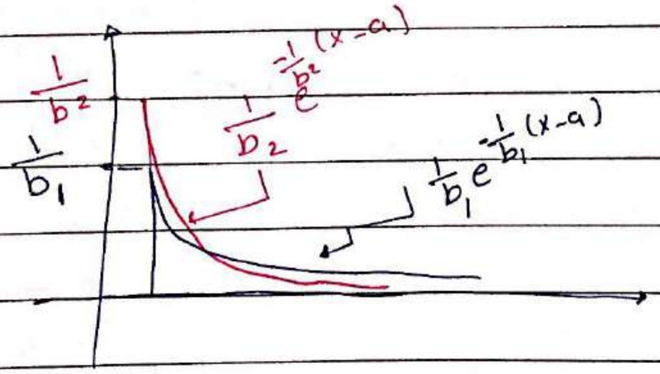
- Ex $X \sim exp(a, b)$
find (1) DC Avg power
(2) Total avg power
(3) Ac Total power.

Sol (1) DC avg power $= \bar{X}^2 = \boxed{(a+b)^2}$
(2) Total avg power $= m_2 = E[x^2]$

$$= \int_a^\infty x^2 \cdot \frac{1}{b} e^{-1/b\,(x-a)} = \boxed{(a+b)^2 + b^2}$$

(3) Ac avg power $= E[x^2] - E^2[x]$
$= \boxed{b^2}$

- $X_1 \sim exp(a, b_1)$
$X_2 \sim exp(a, b_2)$
$b_1 > b_2$



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Ex $X \sim u(a,b)$

(1) $\bar{X}$ (2) $\bar{X}^2$ (3) m_2 (4) AC power.

Sol (1) $\bar{X} = \boxed{\frac{a+b}{2}}$ (DC value)

(2) $\bar{X}^2 = \boxed{\frac{(a+b)^2}{4}}$ (DC power).

(3) $m_2 = E[x^2] = \int_a^b x^2 \cdot \frac{1}{b-a} \, dx$

$$= \frac{1}{b-a} \frac{x^3}{3} \Big|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2 + ab + a^2)}{3\cancel{(b-a)}}$$

$$= \boxed{\frac{b^2 + ab + a^2}{3}}$$ total power.

(4) AC power "Var(x)" $= m_2 - \bar{X}^2$.

$$= \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{b^2 + a^2 - 2ab}{12}$$

$$= \boxed{\frac{(b - a)^2}{12}}$$

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- Ex $X \sim N(a_x, \sigma_x^2)$

(1) $\overline{X} = a_x$

(2) $m_2 = \overline{X^2} = \int x^2 \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx = \boxed{a_x^2 + \sigma_x^2}$

(3) AC power $= m_2 - \overline{X}^2 = a_x^2 + \sigma_x^2 - a_x^2$
$= \boxed{\sigma_x^2}$

* Moments about the mean
"Central moments"

$$\mu_n = E[(x-\overline{x})^n] = \int (x-\overline{x})^n f_x(x)\, dx \quad , n=0,1,\ldots$$

- $\boxed{\mu_0} = \boxed{1}$

* $\boxed{\mu_1} = E[(x-\overline{x})^1] = E[x] - E[\overline{x}]$
$= E[x] - \overline{X} = \boxed{\text{zero}}$

- $\boxed{\mu_2} = E[(x-\overline{x})^2] \longrightarrow$ AC power
$\quad \hookrightarrow Var(x)$

- $\boxed{\mu_2} = E[(x-\overline{x})^2] = E[x^2 - 2x\overline{x} + \overline{x}^2]$

$= E[x^2] - 2\overline{x}E[x] + E[\overline{x}^2]$
$= E[x^2] - 2\overline{X}^2 + \overline{X}^2$
$= E[x^2] - \overline{X}^2$
$= \boxed{m_2 - m_1^2}$

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- $X = \{0, 3, 4.5, 7\}$

$P = \{0.1, 0.2, 0.1, 0.6\}$

Find (1) $\overline{X}$ (2) $\overline{X}^2$ (3) m_2 (4) μ_2.

Sol (1) $\overline{X} = \sum_{i=1}^{N} x_i P(x = x_i)$

$= 0(0.1) + 3(0.2) + 4.5(0.1) + 7(0.6)$

$= 5.25$

(2) $\overline{X}^2 = (5.25)^2$

$= 27.5625$

(3) $m_2 = E[x^2]$

$x^2 = \{0, 9, 20.25, 49\}$.

$E[x^2] = 0(0.1) + 9(0.2) + 20.25(0.1) + 49(0.6)$

$= 33.225$.

(4) $\mu_2 = m_2 - m_1^2$

$= 33.225 - 27.5625$

$= 5.6625$.

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* Characteristic Function:

$X \sim f_X(x)$

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

$$E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$f_X(x) \longleftrightarrow \phi_X(\omega)$.

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$\boxed{m_n = (-j)^n \left. \frac{d^n \phi_X(\omega)}{d\omega^n} \right|_{\omega=0}}$$

* Ex $X \sim \exp(a, b)$

Find $\phi_X(\omega)$

using $\phi_X(\omega)$ find m_1 and m_2.

$\frac{e^{j\omega a}}{1 - j\omega b}$.

Sol

$$\phi_X(\omega) = \int_{a}^{\infty} e^{j\omega x} \frac{1}{b} e^{-\frac{1}{b}(x-a)}\, dx$$

$$= \frac{e^{a/b}}{b} \int_{a}^{\infty} e^{-\left(\frac{1}{b} - j\omega\right) x}\, dx$$

$$= \frac{e^{a/b}}{b} \cdot \left. \frac{e^{-\left(\frac{1}{b} - j\omega\right) x}}{-\left(\frac{1}{b} - j\omega\right)} \right]_{a}^{\infty} = \frac{e^{a/b}}{b} \cdot \left[0 - \frac{e^{-\left(\frac{1}{b} - j\omega\right) a}}{-\left(\frac{1}{b} - j\omega\right)} \right]$$

$$= \boxed{\frac{e^{j\omega a}}{1 - j\omega b}}$$

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2) $\bar{X} = m_1 = -j \dfrac{d\phi_x(\omega)}{d\omega} \Big|_{\omega=0}$

$$= -j\left[\frac{(1-j\omega b)(ja\,e^{j\omega a}) - e^{j\omega a}(-jb)}{(1-j\omega b)^2}\right]\Bigg|_{\omega=0}.$$

$$= -j\left[\frac{ja + jb}{1}\right] = -j(j)(a+b)$$
$$= \boxed{a+b}$$

$m_2 = (-j)^2 \dfrac{d^2\phi_x(\omega)}{d^2\omega} \Big|_{\omega=0}.$ $(a+b)^2+b^2$

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- Ex $X \sim u(a,b)$

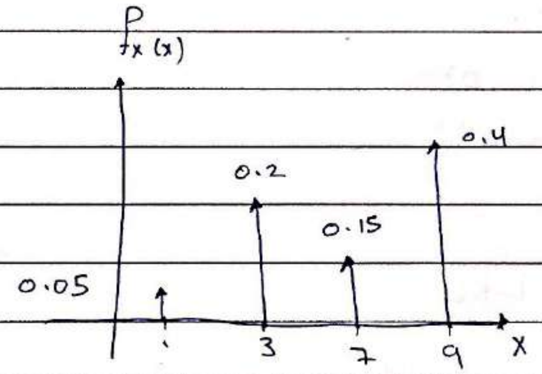
Show that $\phi_x(\omega) = \dfrac{e^{j\omega b} - e^{j\omega a}}{j(b-a)\omega}$

- Sol $\phi_x(\omega) = \int_a^b \dfrac{e^{j\omega x}}{b-a} dx = \dfrac{e^{j\omega x}}{j\omega(b-a)} \Big|_a^b =$ #!

$m_1 = -j \dfrac{d\phi_x(\omega)}{d\omega} \Big|_{\omega=0}$ $m_1 = \frac{a+b}{2}$

- Ex $\phi_x(\omega)$ for Discrete R.V :-

$X = \{1, 3, 7, 9\}$



- Sol $\phi_x(\omega) = E[e^{j\omega x}] = \sum_{i=1}^{4} e^{j\omega x} P(X = x_i)$

$= 0.05 e^{j\omega} + 0.2 e^{3j\omega} + 0.15 e^{7j\omega} + 0.4 e^{9j\omega}$

$\overline{X} = 1(0.05) + 3(0.2) + 7(0.15) + 9(0.4)$

OR

$\overline{X} = -j \dfrac{d\phi_x(\omega)}{d\omega} = -j\left[0.05 j e^{j\omega} + 0.2(3)j e^{j\omega 3} + 0.15(j7) e^{j\omega 7} + 0.4(j9) e^{j9\omega}\right]\Big|_{\omega=0}$

$= (-j)(j)\left[0.05(1) + 0.2(3) + 0.15(7) + 0.4(9)\right]$

$=$

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* The Moment Generating Function (MGF):-

$$X \sim f_X(x)$$
$$M_X(u) = E[e^{ux}] = \int_{-\infty}^{\infty} e^{ux} f_X(x)\, dx$$

$$m_n = \left. \frac{d^n M_X(u)}{du^n} \right|_{u=0}$$

- Ex $X \sim \exp(a, b)$
Find $M_X(u)$?

مكان كل س بحط u.

Sol $M_X(u) = \frac{e^{au}}{1-bu}$

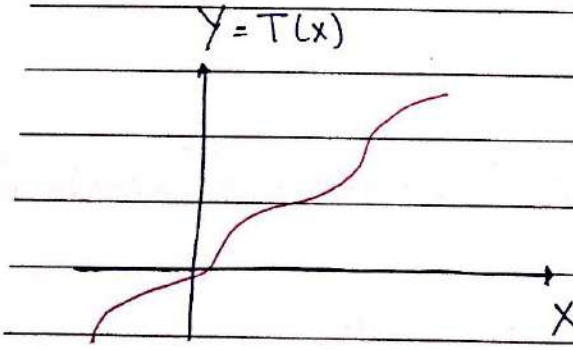
* Transformation of one R.V:

$f_X(x)$ ✓
$F_X(x)$ ✓

$$X \longrightarrow \boxed{T(\cdot)} \longrightarrow Y = T(X)$$

input R.V — Transformation — output R.V

بكون $f_Y(y)$ بعوض عن $f_X(x)$
ولازم نوجده .

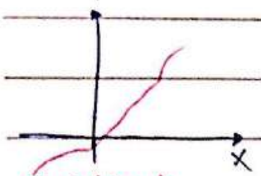


Continuos Transformation

$T(x)$

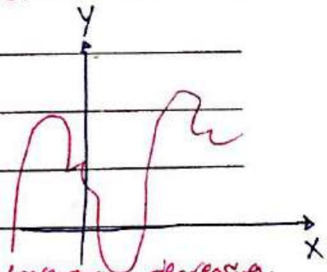
Monotonic | non-monotonic

Increasing | Decreasing



all intervals increasing

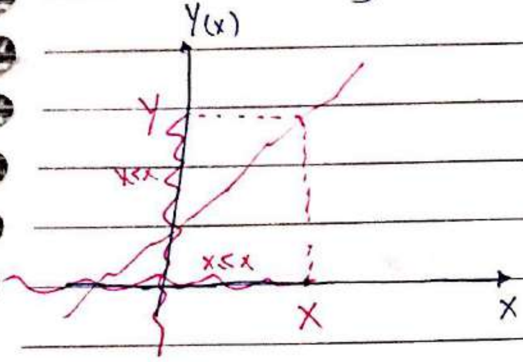
all intervals decreasing



increasing + decreasing.

Case #1

Monotonic increasing Transformation:-



$$F_Y(y) = P(Y \le y) \quad T(x)$$
$$= P(X \le x)$$
$$= F_X(x)$$
$$= F_X(T^{-1}(y))$$

$Y = T(x) = x^2$

$P(x=2) = P(y=4)$

$Y = T(x)$

$x = T^{-1}(y)$

$$\boxed{F_Y(y) = F_X(T^{-1}(y))}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(T^{-1}(y))}{dy}$$ Using chain Rule

$$= \frac{dF_X(T^{-1})}{dT^{-1}} \cdot \frac{dT^{-1}(y)}{dy}$$

$$\boxed{f_Y(y) = f_X(T^{-1}(y)) \cdot \frac{dT^{-1}(y)}{dy}}$$ → +ve slope.

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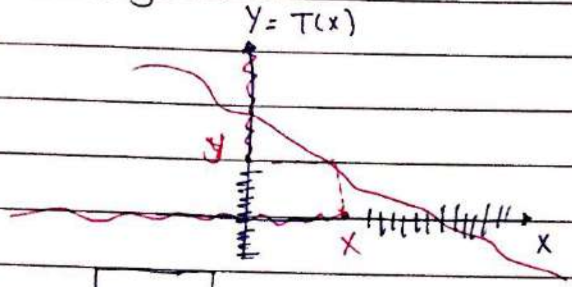
* Case #2:

Monotonic decreasing:-



$$X \longrightarrow \boxed{T(\cdot)} \longrightarrow Y = T(X)$$

$$F_Y(y) = P(Y \le y) = P(X \ge x) = 1 - F_X(x)$$

$$\boxed{F_Y(y) = 1 - F_X(T^{-1}(y))}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d\left[1 - F_X(T^{-1}(y))\right]}{dy}$$

$$= -\frac{d\left[F_X(T^{-1}(y))\right]}{dy}$$

$$\boxed{f_Y(y) = -f_X(T^{-1}(y)) \frac{dT^{-1}(y)}{dy}}$$ -ve slope

* For case #1 & case #2 (monotonic transformation).

$$f_Y(y) = f_X(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|$$

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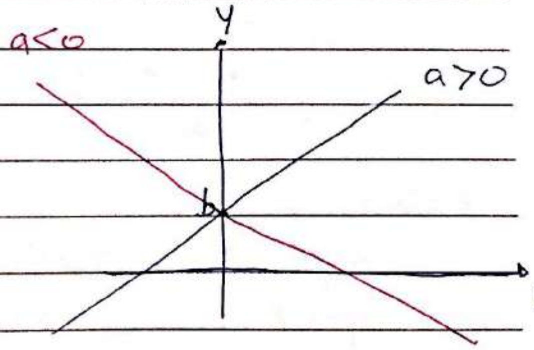
- Ex $X \sim N(a_x, \sigma_x^2)$

$$X \longrightarrow \boxed{a(\cdot)+b} \longrightarrow Y = T(x)$$

$Y = aX + b$

a, b real constants.

Find $f_Y(y)$??



Sol $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}$

→ monotonic transformation.

$$f_Y(y) = f_X\left(T^{-1}(y)\right) \left| \frac{dT^{-1}(y)}{dy} \right|$$

$Y = aX + b$

$X = \dfrac{Y-b}{a} = T^{-1}(y)$.

$\dfrac{dT^{-1}(y)}{dy} = \dfrac{1}{a}$.

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{\frac{-\left(\frac{y-b}{a} - a_x\right)^2}{2\sigma_x^2}} \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi \underbrace{a^2\sigma_x^2}_{\sigma_y^2}}} \cdot e^{\frac{-\left(y - \overbrace{(aa_x+b)}^{a_y}\right)^2}{2\underbrace{a^2\sigma_x^2}_{\sigma_y^2}}}$$

$\dfrac{y-b}{a} - a_x$

$= \dfrac{y-b-a\,a_x}{a}$

$= \dfrac{y-(aa_x+b)}{b}$.

$\sim N\left(\underbrace{aa_x+b}_{E[Y]}, \underbrace{a^2\sigma_x^2}_{Var(Y)}\right)$

* Linear Transformation of Gaussian R.V is a Gaussian R.V.

$Y = aX + b$

$E[Y] = E[aX+b]$

$= aE[X] + b = \boxed{aa_x + b}$

$Var(Y) = Var(aX+b)$

$= a^2\, Var(X) + \text{zero}$

$= \boxed{a^2\sigma_x^2}$

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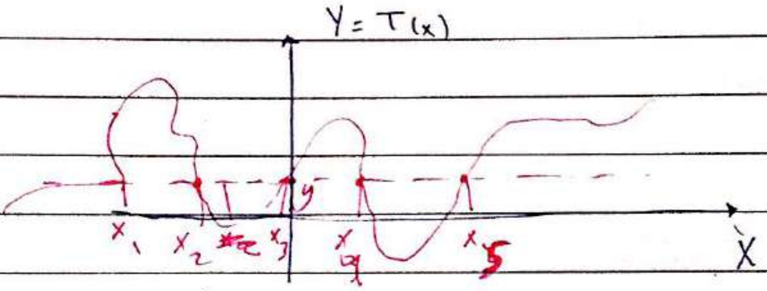
* Case #3

Non-monotonic Transformation.



$$F_Y(y) = P(Y \le y)$$

$$= P(X \le x_1 \cup x_2 < X < x_3 \cup x_4 < X < x_5)$$

$$= P(X \le x_1) + P(x_2 < X < x_3) + P(x_4 < X < x_5)$$

$$= F_X(x_1) + F_X(x_3) - F_X(x_2) + F_X(x_5) - F_X(x_4)$$

* For non-monotonic Transformation.

$$f_Y(y) = \sum_{n=1}^{N} \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx} \right|_{x = x_n}}$$

where $x_1, x_2, x_3, \ldots, x_N$ are the roots of $T(x) - y = 0$.

- Ex: $X \sim U(-3, +3)$

$$X \longrightarrow \boxed{C(\cdot)^2} \longrightarrow Y = T(x) = CX^2$$

$C > 0$

Find $f_Y(y)$?

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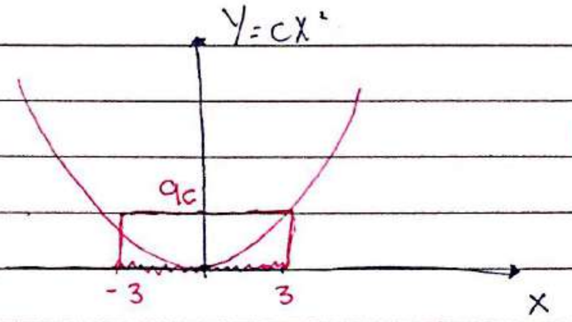
Sol $Y = T(x) = cX^2$



Non-monotonic

$Y = \{0 < y < 9c\}$

Solve for $T(x) - y = 0$

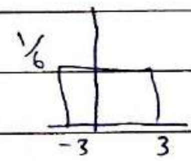
$cX^2 - y = 0$

$X = \pm\sqrt{\frac{y}{c}}$, $X_1 = \sqrt{y/c}$ $\quad X_2 = -\sqrt{y/c}$

$$f_Y(y) = \frac{f_X(x_1)}{\left|\frac{dT(x)}{dx}\right|_{x=x_1}} + \frac{f_X(x_2)}{\left|\frac{dT(x)}{dx}\right|_{x=x_2}}$$

$\frac{dT(x)}{dx} = 2cX$

$$f_X(x) = \begin{cases} 1/6 & , -3 < X < 3 \\ 0 & , \text{o.w} \end{cases}$$



$$\Rightarrow \frac{f_X(\sqrt{y/c})}{|2c\sqrt{y/c}|} + \frac{f_X(-\sqrt{y/c})}{|-2c\sqrt{y/c}|}$$

$$= \frac{1/6}{2c\sqrt{y/c}} + \frac{1/6}{2c\sqrt{y/c}}$$

$$= \frac{1/6}{2\sqrt{cy}} + \frac{1/6}{2\sqrt{cy}}$$

$$\boxed{f_Y(y) = \frac{1}{6\sqrt{cy}} \quad , \quad 0 < y < 9c}$$

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b) $X \sim N(0,1) \longrightarrow \boxed{c(\cdot)^2} \longrightarrow Y = T(X) = cX^2.$

$c > 0$

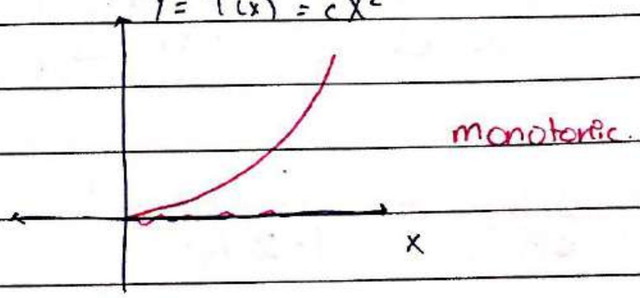
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f_Y(y) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y/c)}{2}}}{2\sqrt{cy}} + \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y/c)}{2}}}{2\sqrt{cy}}$$

$$f_Y(y) = \frac{e^{-y/2c}}{\sqrt{2\pi}\sqrt{cy}}, \quad 0 \le y < \infty \leftarrow \text{chi-squared with degree one R.V}$$

- Ex $X \sim exp(0,1)$

$$f_Y(y) = f_X(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|$$



- H.w: $X \longrightarrow \boxed{2e^{-(\cdot)}} \longrightarrow Y = 2e^{-X}$

decreasing.

$X \sim exp(0,2)$ Find $f_Y(y)$?

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* Chapter 48

Multiple Random Variable:

2 RVs with different pdfs but from same sample space.

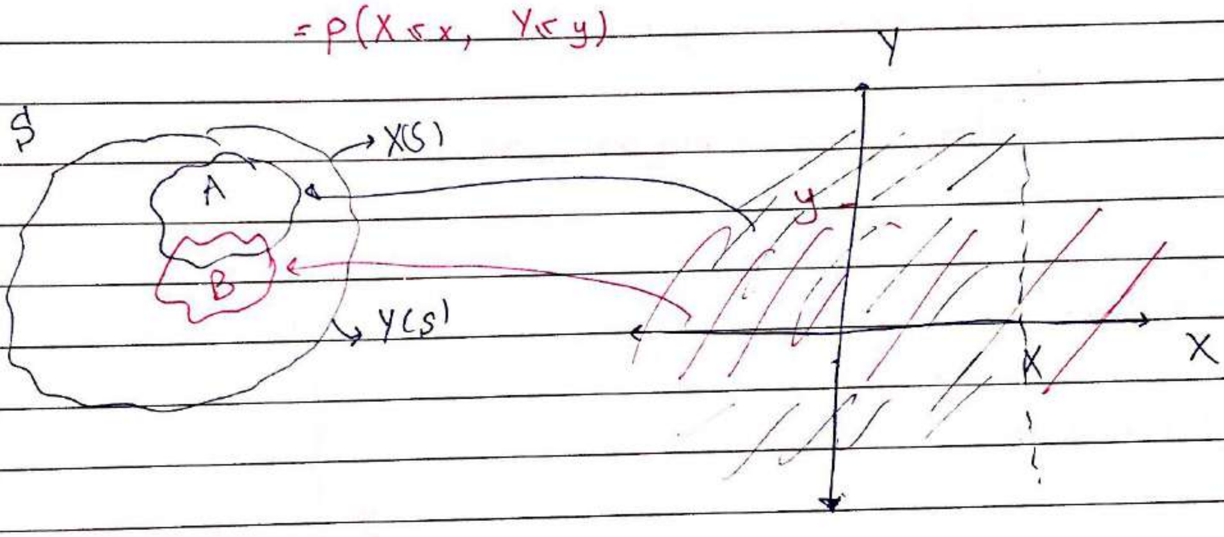
- Joint CDF (X, Y)

$F_{X,Y}(x,y)$.

Recall: $X \sim F_X(x) = P(X \le x)$

$F_{X,Y}(x,y) = P(\underbrace{X \le x}_{\text{event } A} \cap \underbrace{Y \le y}_{\text{event } B})$

$= P(X \le x, Y \le y)$



* Joint cdf properties:-

$\boxed{1}$ ~~$F_{X,Y}(x,y)$~~. $F_{X,Y}(-\infty,-\infty) = P(X \le -\infty, Y \le -\infty)$

$= P(\emptyset \cap \emptyset) = P(\emptyset) = \text{Zero}.$

• $F_{X,Y}(-\infty, y) = P(X \le -\infty, Y \le y)$

$= P(\emptyset \cap B) = P(\emptyset) = \text{Zero}.$

• $F_{X,Y}(x, -\infty) = \text{Zero}.$

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[2] $F_{X,Y}(\infty,\infty) = P(X \le \infty, Y \le \infty) = P(S \cap S) = 1$

[3] $0 \le F_{X,Y}(x,y) \le 1$

[4] $F_{X,Y}(x,y)$ is non-decreasing function.

[5] $\boxed{F_{X,Y}(x,\infty)} = P(X \le x, Y \le \infty)$

$= P(X \le x \cap S)$

$= P(X \le x) = \boxed{F_X(x)}$

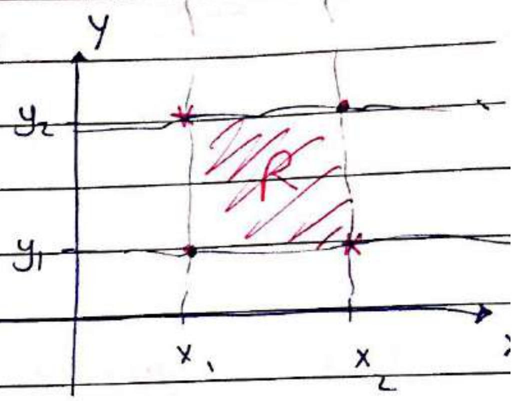
$\boxed{F_{X,Y}(\infty,y)} = \boxed{F_Y(y)}$ marginal cdfs.

[6] $P(x_1 < X \le x_2, y_1 < Y \le y_2)$

$= P((x,y) \in R)$

$= F_{X,Y}(x_2,y_2) + F_{X,Y}(x_1,y_1)$

$- F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)$



$$\left.\begin{aligned} * F_X(x) &= F_{X,Y}(x,\infty) \\ F_Y(y) &= F_{X,Y}(\infty,y) \end{aligned}\right\} \text{marginal cdfs.}$$

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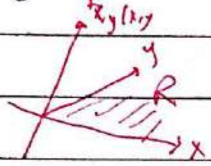
- Problem 4.2-4

X & Y with

$F_{X,Y}(x,y) = \left[1 - e^{-\frac{1}{2}x} - e^{-\frac{1}{2}y} + e^{-\frac{1}{2}(x+y)}\right] u(x)\, u(y)$



$x > 0 \quad , \quad y > 0$

$(x,y) \in R$

Find a) $P(X \leq 1, \ Y \leq 2) = F_{X,Y}(1,2)$

$= 1 - e^{-\frac{1}{2}(1)} - e^{-\frac{1}{2}(2)} + e^{-\frac{1}{2}(3)}$

b) $P(\frac{1}{2} < X \leq 1.5) = F_X(1.5) - F_X(\frac{1}{2})$

$= F_{X,Y}(1.5, \infty) - F_{X,Y}(\frac{1}{2}, \infty)$

$= (1 - e^{-\frac{1.5}{2}}) - (1 - e^{-\frac{1}{4}})$

c) $P(-1.5 < X \leq 2.1, \ 1 < Y \leq 3)$

$= F_{X,Y}(2.1, 3) + F_{X,Y}(-1.5, 1) - F_{X,Y}(2.1, 1) - F_{X,Y}(-1.5, 3)$

$= ---$

- Ex cdf for ~~both~~ discrete 2 R.Vs.

$(X,Y) = \{(1,1), (2,1), (3,3)\}$

$P((1,1)) = 0.2$

$P((2,1)) = 0.3$

$P((3,3)) = 0.5$

Find $F_{X,Y}(x,y)$??

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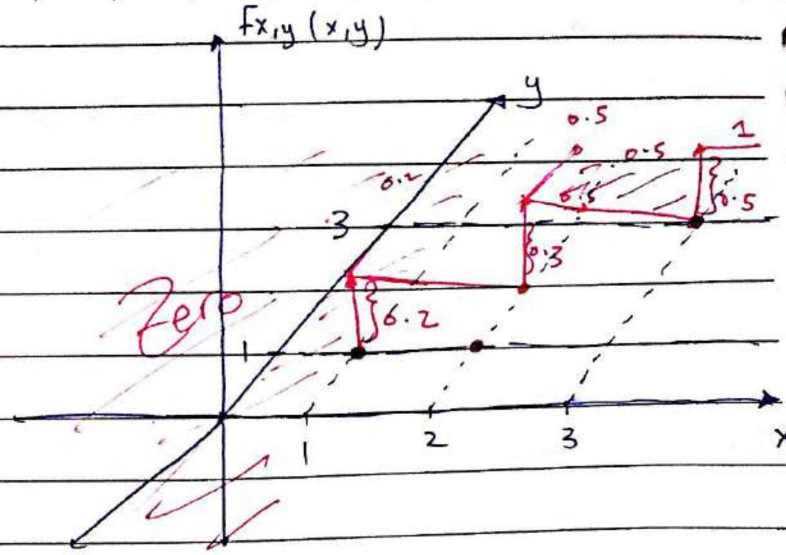
<u>Sol</u>

$F_{x,y}(x,y) = P(X \le x,\ Y \le y)$

$F_{x,y}(1,1) = P(X \le 1,\ Y \le 1)$

$= P((1,1)) = 0.2$



$$F_{x,y}(x,y) = 0.2\,u(x-1)\,u(y-1) + 0.3\,u(x-2)\,u(y-1) + 0.5\,u(x-3)\,u(y-3)$$

$\rightarrow F_X(x) = F_{x,y}(x, \infty)$ عند Y لـ ∞ بتكون 1 ف u(y-1) بعد 1 بتعطي قيمة 1

$= 0.2\,u(x-1) + 0.3\,u(x-2) + 0.5\,u(x-3)$

plot?

$\rightarrow F_Y(y) = F_{x,y}(\infty, y)$

$= 0.2\,u(y-1) + 0.3\,u(y-1) + 0.5\,u(y-3)$

$= 0.5\,u(y-1) + 0.5\,u(y-3)$

plot?

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* Joint pdf

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

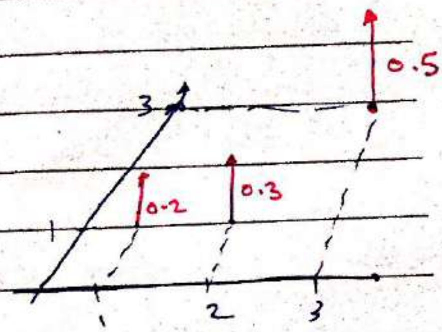
partially, once with respect to x & once with y.

~~[illegible]~~

pdf for discrete 2.D R.Vs

for the joint cdf in the last example find $f_{X,Y}(x,y)$?

Sol $f_{X,Y}(x,y) = 0.2\,\delta(x-1)\,\delta(y-1)$
$+ 0.3\,\delta(x-2)\,\delta(y-1)$
$+ 0.5\,\delta(x-3)\,\delta(y-3)$



* Joint density function properties:-

[1] $f_{X,Y}(x,y) \geq 0$

[2] $\int_{-\infty}^{\infty}\int f_{X,Y}(x,y)\, dx\, dy = 1$

[3] $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(z_1, z_2)\, dz_1\, dz_2 \rightarrow$ " $F_{X,Y}(\infty,\infty) = 1$ "

[4] $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(z_1, \cancel{y}\, z_2)\, dz_2\, dz_1$

↳ just adds const. no y's

[5] $F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(z_1, z_2)\, dz_1\, dz_2$

Five

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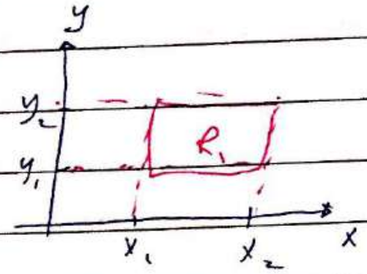
$\boxed{5}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx$$

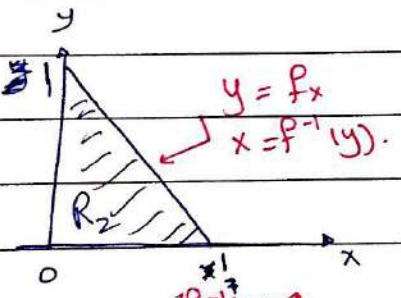
$\boxed{6}$ $p(x_1 < X \leq x_2 \,,\, y_1 < Y \leq y_2)$



$$= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{x,y}(x,y)\, dx\, dy$$

$$= \iint_{R_1} f_{x,y}(x,y)\, dx\, dy$$

Ex



$$p((x,y) \in R_2) = \cancel{\int_0^{1} \int_0^{f^{-1}(y)} f_{x,y}(x,y)\, dy\, dx}$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{f^{-1}(y)} f_{x,y}(x,y)\, dx\, dy.$$

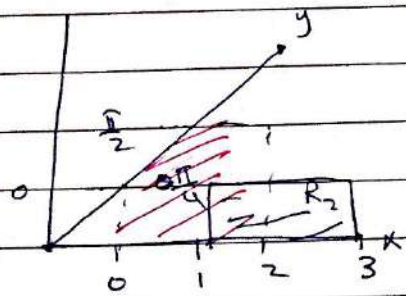
- Ex let $g(x,y) = \begin{cases} b\,e^{-x}\cos(y), & 0\le x\le 2, \quad 0\le y\le \pi/2 \\ 0, & o.w. \end{cases}$

$b>0$

Find b such that $g(x,y)$ is joint pdf ??

$$\int_{-\infty}^{\infty}\int b\,e^{-x}\cos(y)\,dx\,dy = 1$$

$$\int_0^{\pi/2}\int_0^{2} b\,e^{-x}\cos(y)\,dx\,dy = 1$$



$$= b\int_0^{\frac{\pi}{2}}\cos(y)\left(\int_0^{2} e^{-x}dx\right)dy = 1$$

$$\int_0^{2} e^{-x}dx = -e^{-x}\Big|_2^0 = 1-e^{-2}$$

$$= b(1-e^{-2})\int_0^{\frac{\pi}{2}}\cos(y)\,dy$$

$$= b(1-e^{-2})\,\sin y\Big|_0^{\frac{\pi}{2}} = 1$$

$$= b(1-e^{-2}) = 1$$

$$\boxed{b = \frac{1}{1-e^{-2}}}$$

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Find ($P(x,y) \in R$)　　　　　　　　　　　　　　　　　　من الرسمة السابقة

$$= \iint_R \cancel{f_{x,y}}\, g(x,y)\, dx\, dy$$

$$= \int_{y=0}^{\frac{\pi}{4}} \int_{x=1}^{2} b\, e^{-x} \cos(y)\, dx\, dy$$

x بتاخد بس من 1 - 2 ده
عند ال 3 مش معرف الفنكشن
بتكون صفر

- Ex $f_{x,y}(x,y) = x\, e^{-x(y+1)}\, u(x)\, u(y)$.

$$= x\, e^{-x(y+1)} \quad , \quad x > 0 \quad , \quad y > 0$$

Find $f_X(x)$ $f_Y(y)$?

Sol $f_X(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy$

$$= \int_0^{\infty} x\, e^{-x(y+1)}\, dy$$

$$= x\, e^{-x} \int_0^{\infty} e^{-xy}\, dy = x\, e^{-x} \left(\frac{e^{-xy}}{-x} \Big|_0^{\infty} \right)$$

$$= x\, e^{-x} \cdot \frac{1}{x}$$

$$= \boxed{e^{-x}} \quad , \quad \boxed{x > 0}$$

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$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{0}^{\infty} x\, e^{-x(y+1)}\, dx \quad \Rightarrow \text{Using parts technique.}$$

$$= \frac{1}{(1+y)^2}, \quad y > 0.$$

* Statistical Independence :

Given (X,Y) with $F_{X,Y}(x,y) = P(X \le x,\ Y \le y)$

- Recal two event A, B are independent iff.

$P(A|B) = P(A)$

$P(B|A) = P(B)$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$.

↳ when X and Y are independent, then

$F_{X,Y}(x,y) = P(X \le x) \cdot P(Y \le y)$.

$F_{X,Y}(x,y) = \underline{F_X}(x) \cdot \underline{F_Y}(y)$. marginal

$$f_{X,Y}(x,y) = \underline{f_X}(x) \cdot \underline{f_Y}(y).$$

- Ex $f_{X,Y}(x,y) = X e^{-x(y+1)} u(x)\, u(y)$.

$f_X(x) = e^{-x} u(x)$

$f_Y(y) = \frac{1}{(1+y)^2} u(y)$.

check if X, Y are ind?

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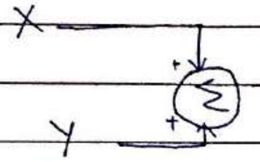
Sol. $f_X(x) \cdot f_Y(y)$

$e^{-x}u(x) \cdot \frac{1}{(1+y)^2}\, u(y) \neq f_{X,Y}(x,y)$

⇒ X, Y are not independent.

* Sum of two independent Random Variables:-

- Let X, Y be independent R.Vs with joint pdf $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$.



$\rightarrow W = X + Y$

find $f_W(w)$??

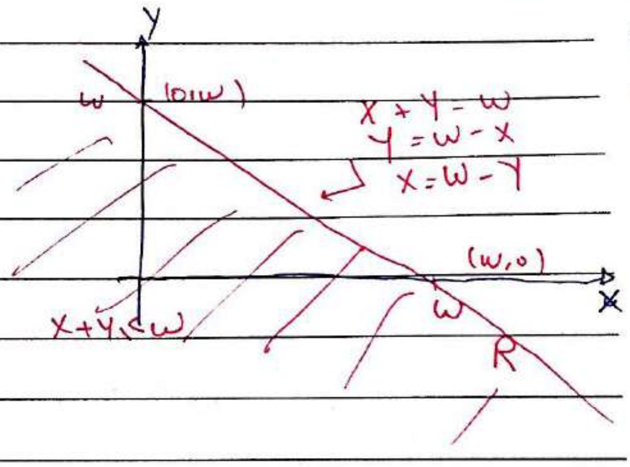
→ $F_W(w) = P(W \le w)$

$= P(X + Y \le w)$

• $X + Y = w$

$Y = w - X$

$X = w - Y$



↳ $P((X,Y) \in R) = \iint_R f_{X,Y}(x,y)\,dx\,dy$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_X(x) \cdot f_Y(y)\,dx\,dy \quad (*)$$

$$= \int_{-\infty}^{w-y}\int_{-\infty}^{\infty} f_X(x) \cdot f_Y(y)\,dy\,dx \quad (**)$$

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→ Start from (*):-

$$F_W(w) = \int_{-\infty}^{\infty} f_Y(y) \left(\int_{-\infty}^{w-y} f_X(x)\,dx \right) dy.$$

$$f_W(w) = \frac{dF_W(w)}{dw} = \int_{-\infty}^{\infty} f_Y(y) \left(\frac{d \int_{-\infty}^{w-y} f_X(x)\,dx}{dw} \right) dy.$$

· Using Leibni's rule (Appendix G-1, G-2).

$$\hookrightarrow f_W(w) = \int_{-\infty}^{\infty} f_Y(y) \left[f_X(w-y) \overbrace{\frac{d(w-y)}{dw}}^{1} - f_X(-\infty) \cdot \overbrace{\frac{d(-\infty)}{dw}}^{\text{zero.}} + \int_{-\infty}^{w-y} \underbrace{\frac{d f_X(x)}{dw}}_{\text{zero.}} dx \right] dy.$$

$$\Rightarrow f_W(w) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(w-y)\,dy$$
$$= f_Y(y) * f_X(x)$$

→ if we start from (**)

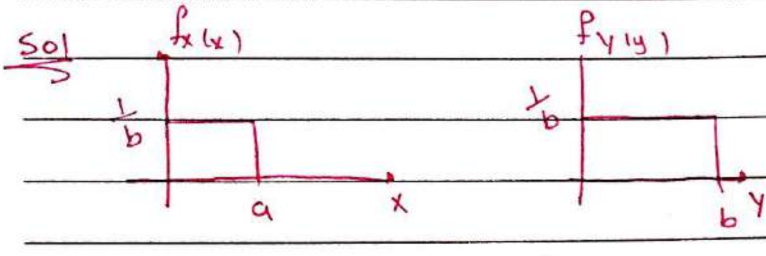
$$f_W(w) = f_X(x) * f_Y(y)$$
$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx$$

$W = X - Y$

X, Y independent.

Ex: let $X \sim u(0,a)$ and $Y \sim u(0,b)$
where $b > a > 0$. If X & Y are independent, find $f_W(w)$??

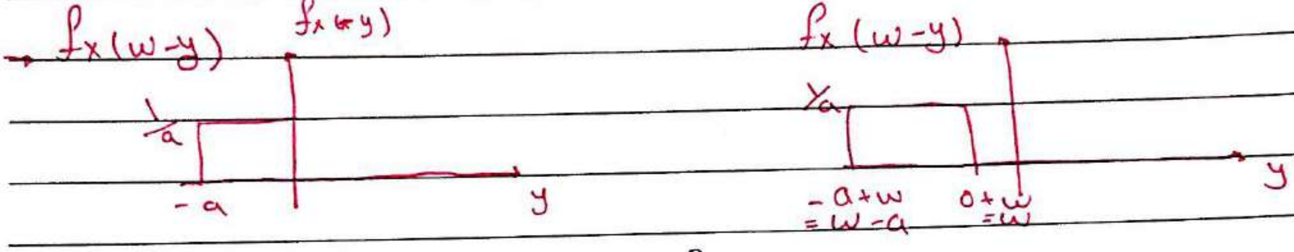
Sol

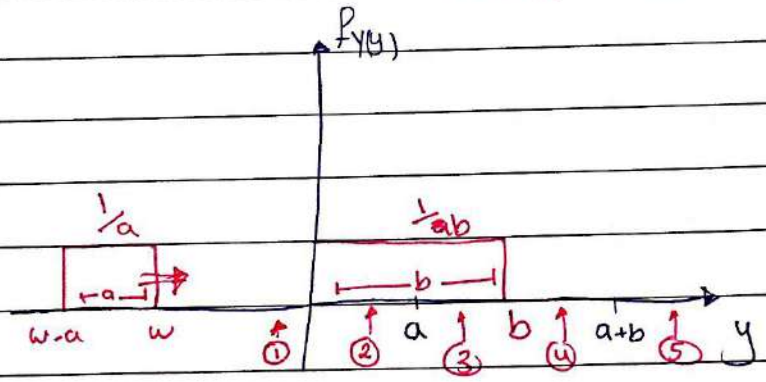


بحيث اذا كبير بحرك الاصغر مشان نعمله convolution

بحركه

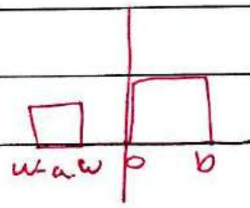
$$f_W(w) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_Y(y) \cdot \underline{f_X(w-y)}\, dy$$





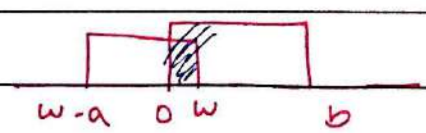
Case ①

$$w < 0 \rightarrow \int_{-\infty}^{\infty} (0)\, dy = \boxed{0}$$



Case ②

$0 < w < a$



$$\int_0^w \frac{1}{ab}\, dy = \frac{y}{ab}\Bigg|_0^w = \boxed{\frac{w}{ab}}$$

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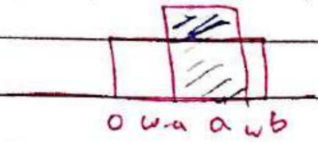
Case ③

$a < w < b$

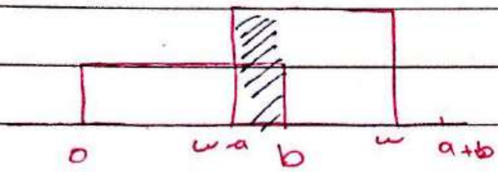
$$f_W(w) = \int_{w-a}^{w} \frac{1}{ab}\,dy$$



$$= \frac{y}{ab}\Big|_{w-a}^{w} = \frac{w-w+a}{ab} = \boxed{\frac{1}{b}}$$

Case ④

$b < w < a+b$

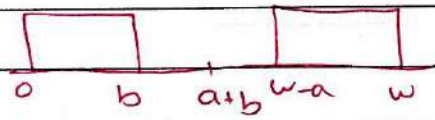


$$\int_{w-a}^{b} \frac{1}{ab}\,dy = \frac{y}{ab}\Big|_{w-a}^{b}$$

$$= \frac{b-w+a}{ab} = \boxed{\frac{(a+b)-w}{ab}}$$

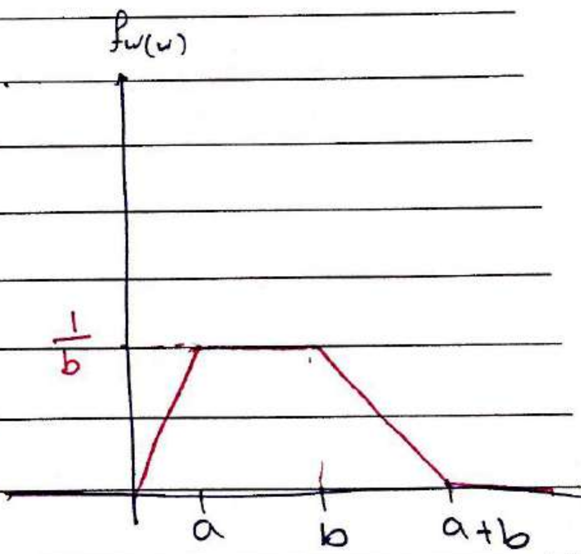
Case ⑤

$w > a+b$



$$f_W(w) = \int 0\,dy = \boxed{0}$$

$$f_W(w) = \begin{cases} 0 & , \quad w < 0 \\ \frac{w}{ab} & , \quad 0 < w < a \\ \frac{1}{b} & , \quad a < w < b \\ \frac{(a+b)-w}{ab} & , \quad b < w < a+b \\ 0 & , \quad w > a+b \end{cases}$$



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* **Notes:**

$$X \sim f_X(x) \longleftrightarrow \phi_X(\overset{\text{omega}}{\omega}) \quad (\text{characteristic function})$$

$$Y \sim f_Y(y) \longleftrightarrow \phi_Y(\omega)$$

$$W = X + Y \sim f_W(w) = f_X(x) * f_Y(y)$$

$$\phi_W(\omega) = \phi_X(\omega) \cdot \phi_Y(\omega)$$

* If $W = X_1 + X_2 + \dots + X_N$ independent

$$f_W(w) = f_{X_1}(x_1) * f_{X_2}(x_2) \; \dots \; * f_{X_n}(x_n)$$

$\hookrightarrow$ exact pdf of W

كتير صعب
خصوصاً إذا عندي بس كل RVs
بستخدم الثيورم اللي تحت.

* **Central Limit theorem (CLT):**

Let $X_1, X_2, \dots, X_N$ be N independent R.V's then the density of $W = X_1 + X_2 + \dots + X_N = \sum_{i=1}^{N} X_i$ can be approximated as.

$$W \sim N(a_w, \sigma_w^2)$$

$$a_w = \overline{X}_1 + \overline{X}_2 + \dots + \overline{X_N}$$

$$\sigma_w^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \dots + \sigma_{x_N}^2$$

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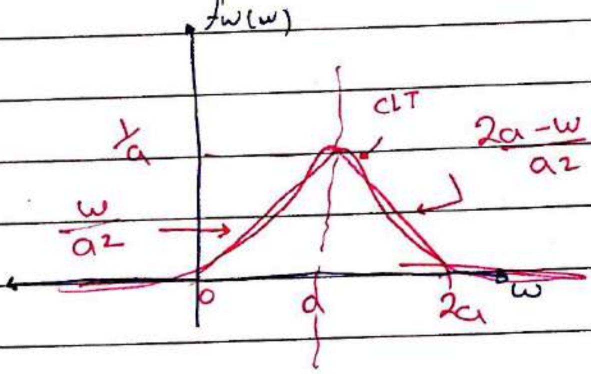
- Ex $X_1 \sim u(0,a)$, $X_2 \sim u(0,a)$

Let $W = X_1 + X_2$

Find 1) the exact $f_W(w)$ (convolution) 2) the approximate $f_W(w)$: (CLT)

Sol (1) $f_W(w) = f_{X_1}(x_1) * f_{X_2}(x_2)$



(2) approximate using CLT

$W \sim N(a_W, \sigma_W^2)$

$a_W = \overline{X_1} + \overline{X_2}$

$= \frac{a}{2} + \frac{a}{2} = a$

$\sigma_W^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2$

$= \frac{a^2}{12} + \frac{a^2}{12} = \frac{a^2}{6}$

$\Rightarrow W \sim N(a, \frac{a^2}{6})$

$f_W(w) = \frac{1}{\sqrt{\pi \frac{a^2}{3}}} e^{\frac{-(w-a)^2}{a^2/3}}$, $-\infty < w < \infty$.

Problem 4.7-3.

4.6-14

4.7-4 (b+c)

CH # 5:

Operation on Multiple RV's:-

- Expectation (mean) of multiple R.V's:-

Let X & Y be two R.V's with joint pdf $f_{x,y}(x,y)$ and $g(x,y)$ is function of X & Y.

then:

$$\boxed{E[g(x,y)] = \iint_{-\infty}^{\infty} g(x,y) \cdot f_{x,y}(x,y)\, dx\, dy}$$

- As a special case, if $g(x,y) = g(x)$

then:

$$E[g(x)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) \cdot f_{x,y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} g(x) \cdot \left(\int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy \right) dx$$

$$\boxed{E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx}$$

- As a special case, if $g(x,y) = g(y)$

then:

$$\boxed{E[g(y)] = \int_{-\infty}^{\infty} g(y)\, f_Y(y)\, dy}$$

where $f_Y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx$

Five A

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Ex * Given X_1 & X_2 with $f_{X_1,X_2}(x_1,x_2)$

Show that

$$E[\alpha_1 X_1 + \alpha_2 X_2] = \alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2$$

where α_1, α_2 are real constants.

Sol Let $g(x_1, x_2) = \alpha_1 X_1 + \alpha_2 X_2$

$$E[g(x_1,x_2)] = \iint_{-\infty}^{\infty} g(x_1,x_2) f_{x_1,x_2}(x_1,x_2)\,dx_1\,dx_2$$

$$= \iint (\alpha_1 x_1 + \alpha_2 x_2) f_{x_1,x_2}(x_1,x_2)\,dx_1\,dx_2$$

$$= \iint \alpha_1 x_1 f_{x_1,x_2}(x_1,x_2)\,\cancel{dx_1\,dx_2}^{\,dx_2 dx_1} + \iint \alpha_2 x_2 f_{x_1,x_2}(x_1,x_2)\,\cancel{dx_1\,dx_2}^{\,dx_2 dx_1}$$

$$= \alpha_1 \int x_1 f_{x_1}(x_1)\,dx_1 + \alpha_2 \int x_2 f_{x_2}(x_2)\,dx_2$$

$$= \alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2 \quad \#$$

* In general

$$E\left[\sum_{i=1}^{N} \alpha_i X_i\right] = \sum_{i=1}^{N} \alpha_i \bar{X}_i$$

* Ex: given X_1, X_2 and X_3 with

	$E[\cdot]$	$\sigma^2(\cdot)$
X_1	-1	1
X_2	$1/2$	2
X_3	0	3

Find the mean of $Z = -2X_1 + \frac{1}{2}X_2^2 + X_3^2 - 4$

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Sol

$\bar{Z} = -2\bar{X}_1 + 1/2\, E[X_2^2] + E[X_3^2] - 4$ $\qquad \sigma^2(\cdot) = E[(\cdot)^2] - E^2[\cdot]$

$= -2(-1) + 1/2\left(32 + (\tfrac{-1}{2})^2\right) + (3 + (0)^2) - 4$

* Joint Moments:

Let X & Y be two R.Vs with joint pdf $f_{X,Y}(x,y)$, then:

$m_{n,k} = E[X^n Y^k]$ is the joint moment with order $\boxed{n+K}$

where $n = 0, 1, 2, ---$

$K = 0, 1, 2, ---$

$$m_{nk} = E[\underbrace{X^n Y^k}_{g(x,y)}] = \int_{-\infty}^{\infty}\!\!\int x^n y^k \cdot f_{XY}(x,y)\, dx\, dy.$$

* Zeroth order

$m_{00} = E[X^0 Y^0] = E[1] = 1$

* axis of Symmetry for two-dimensional is $(\bar{X}, \bar{Y})$

. 1^{st} order

$\left.\begin{array}{l} m_{10} = E[X] \\ m_{01} = E[Y] \end{array}\right\}$ Center of gravity.

. 2^{nd} order

$m_{20} = E[X^2]$ "total power in X"

$m_{02} = E[Y^2]$ "total power in Y"

$m_{11} = E[XY] = R_{XY}$ "The correlation"

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- Recall $g_1(t)$ & $g_2(t)$

Correlation $= \int_{-\infty}^{\infty} g_1(t)\, g_2(t+\tau)\, d\tau$

- Correlation measures the relationship between X & Y

* Notes - If $R_{xy} = E[XY] = 0$, then X & Y are said to be orthogonal

- If $R_{xy} = E[XY] = E[X] \cdot E[Y]$, then X & Y are uncorrelated.
  ↳ X & Y are generated from different sources.

- Ex Show that if X and Y are independent, they must be uncorrelated.

Independent → Uncorrelated.

$f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$ ← $\times$ — $R_{xy} = \bar{X}\,\bar{Y}$

↳ only for gaussian distribution.

proof $R_{xy} = E[XY]$

$= \iint xy\, f_{x,y}(x,y)\, dx\, dy$

$= \iint xy\, f_x(x)\, f_y(y)\, dx\, dy$

$= \int y f_y(y) \left(\underbrace{\int x\, f_x(x)\, dx}_{\bar{X}} \right) dy$

$= \bar{X} \cdot \bar{Y}$

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- If X and Y are uncorrelated $\longrightarrow$ Independent
  $\hookrightarrow$ not independent.

- If X and Y are uncorrelated gaussian $\longrightarrow$ Independent.

- Note If X & Y are independent, then
$$E[g(x)\,g(y)] = E[g(x)] \cdot E[g(y)].$$

- Ex Let X be a R.V with mean $\overline{X} = 3$ and variance $\sigma_x^2 = 2$
Let another R.V $Y = -6X + 22$
Find (1) R_{xy}?
(2) ~~[illegible]~~ Are X and Y uncorrelated?

Sol (1)
$$R_{xy} = E[XY]$$
$$= E[X(-6X+22)]$$
$$= E[-6X^2 + 22X]$$
$$= -6\,E[X^2] + 22\,E[X]$$
$$= -6(2+3^2) + 22(3) = 0$$

(2) Check if $R_{xy} = \overline{X} \cdot \overline{Y}$
$$\overline{X} = 3$$
$$\overline{Y} = E[-6X + 22]$$
$$= -6\overline{X} + 22 = -6(3) + 22 = 4$$

$$R_{xy} = 0 \neq \overline{X} \cdot \overline{Y} = 12$$

$\Rightarrow$ X and Y are not uncorrelated.

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- Independent $\xrightarrow{\text{must be}}$ Uncorrelated

or

not independent $\xrightarrow{\cancel{\text{must}}}$ Correlated.

$\xleftarrow{\text{must}}$

- **Ex**

$X \sim u(-1, +1)$

$Y = X^2$

Find (1) R_{xy} (2) $E[X]\,E[Y]$

Sol (1) $R_{xy} = E[XY] = E[X^3] = \int_{-1}^{1} x^3 \cdot \frac{1}{2}\, dx$

$$= \left.\frac{x^4}{8}\right|_{-1}^{1} = \frac{1}{8} - \left(\frac{1}{8}\right) = 0$$

(2) $E[X]\,E[Y] = 0$

$R_{xy} = E[X]\,E[Y]$, so X & Y are uncorrelated.

* Joint Central moments:-

$$\mu_{nk} = E\left[(x-\bar{x})^n (y-\bar{y})^k\right]$$

$$= \iint_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^k f_{x,y}(x,y)\, dx\, dy.$$

order $n+k$.

$n = 0, 1, 2, \ldots$

$k = 0, 1, 2, \ldots$

- zeroth order:

$\mu_{00} = 1$

- 1st order:

$\mu_{10} = E[x-\bar{x}] = E[x] - E[\bar{x}] = 0$

$\mu_{01} = E[y-\bar{y}] = 0$

- 2nd order:

$\mu_{20} = E[(x-\bar{x})^2] = \sigma_x^2$ "Ac power"

$\mu_{02} = E[(y-\bar{y})^2] = \sigma_y^2$

$\mu_{11} = E[(x-\bar{x})(y-\bar{y})] = C_{xy}$ "Covariance of x & y"

- $C_{xy} = E\left[(x-\bar{x})(y-\bar{y})\right]$

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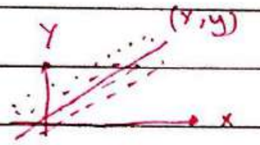
$$C_{xy} = E[xy - \bar{y}x - \bar{x}y + \bar{x}\bar{y}]$$

$$= E[xy] - \bar{y}\bar{x} \cancel{- \bar{x}\bar{y}} \cancel{+ \bar{x}\bar{y}}$$

$$C_{xy} = R_{xy} - \bar{y}\bar{x} \quad , \quad R_{xy} = E[xy] = \bar{x}\bar{y}$$

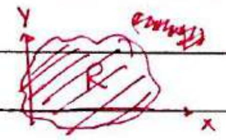
* The covariance C_{xy} measures the correlation between x & y

↳ means the linear relationship between x & y

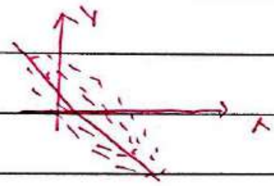


- C_{xy} ~ +ve

↳ 0 ~ Uncorrelated ~



↳ -ve



- C_{xy} bigger → stronger linear rs ((قريبة على بعض)).

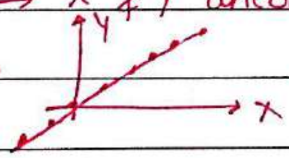
- The correlation Coefficient: ρ_{xy} — لقياس العلاقة الخطية بينهم

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y}$$

$$-1 \le \rho_{xy} \le 1 \Rightarrow 0 \le |\rho_{xy}| \le 1$$

- $\rho_{xy} = 0 \rightarrow x$ & y uncorrelated

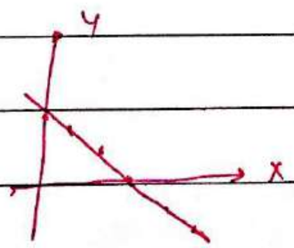
$\rho_{xy} = +1 \rightarrow$



$$Y = ax + b \quad , \quad a > 0$$

لو كان عندي نقطة وحدة مش عالخط ما بكون +1

$\rho_{xy} = -1$



$$Y = ax + b$$

$$a < 0$$

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* Independent $\longrightarrow$ Uncorrelated $\longrightarrow C_{xy} = 0$

$f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$

$E[g(x)\, g(y)] = \bar{g}(x) \cdot \bar{g}(y)$

$\rho_{xy} = 0$

$R_{xy} = \bar{X}\,\bar{Y}$

. $C_{xy} = 0 \rightarrow$ independent

$\hookrightarrow$ not independt (dependent) " $y = f(x)$ "

* Dependent $\longrightarrow$ Correlated " $C_{xy} \neq 0$, $\rho_{xy} \neq 0$, $R_{xy} \neq \bar{X}\,\bar{Y}$ "

" $f_{x,y}(x,y) \neq f_x(x) \cdot f_y(y)$ " $\hookrightarrow$ Uncorrelated

$E[g(x)\, g(y)] \neq \bar{g}(x)\, \bar{g}(y)$

* Correlated $\longrightarrow$ dependent

$$\begin{tikzcd}
\text{Independent} \arrow[r, shift left] & \text{Uncorrelated} \arrow[l, shift left] \arrow[dl, shift left] \\
\text{Dependent} \arrow[r, shift left] \arrow[ur, shift left] & \text{Correlated} \arrow[l, shift left]
\end{tikzcd}$$

* Ex $\quad X \rightarrow \bar{X} = 3 \;,\; \sigma_x^2 = 2$

$Y = -6X + 22.$

Sol $\quad \rho_{xy} = -1 \rightarrow$ مو شكله بدنا حل

dependent $\rightarrow$ Uncorrelated.

* $C_{x_1 x_2} = C_{x_2 x_1}$

$C_{x_1 x_1} = \sigma_{x_1}^2$

$C_{x_2 x_2} = \sigma_{x_2}^2$

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* Ex Let X_1 & X_2 be two joint R.V's

$X = \alpha_1 X_1 + \alpha_2 X_2$

where α_1 & α_2 are real constants

Determine the variance of X?

Sol we know $E[X] = E[\alpha_1 X_1 + \alpha_2 X_2]$

$= \alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2$

$Var(X) = E[(X-\bar{X})^2]$

$X - \bar{X} = (\alpha_1 X_1 + \alpha_2 X_2) - (\alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2)$

$= \alpha_1 (X_1 - \bar{X}_1) + \alpha_2 (X_2 - \bar{X}_2)$

$Var(X) = E[(X-\bar{X})^2]$

$= E[\alpha_1^2 (X_1 - \bar{X}_1)^2 + \alpha_2^2 (X_2 - \bar{X}_2)^2 + 2\alpha_1 \alpha_2 (X_1 - \bar{X}_1)(X_2 - \bar{X}_2)]$

$$Var(X) = \alpha_1^2 \sigma_{x_1}^2 + \alpha_2^2 \sigma_{x_2}^2 + 2\alpha_1\alpha_2 C_{x_1 x_2}$$

* $Var(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1^2 \sigma_{x_1}^2 + \alpha_2^2 \sigma_{x_2}^2 + \alpha_1\alpha_2 C_{x_1 x_2} + \alpha_2 \alpha_1 C_{x_2} C_{x_1}$

*

$Var(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3) = \alpha_1^2 \sigma_{x_1}^2 + \alpha_2^2 \sigma_{x_2}^2 + \alpha_3 \sigma_{x_3}^2 + 2\alpha_1 \alpha_2 C_{x_1 x_2}$
$+ 2\alpha_1 \alpha_3 C_{x_1 x_3} + 2\alpha_2 \alpha_3 C_{x_2 x_3}$

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$$* \ Var\left(\sum_{i=1}^{N} \alpha_i x_i\right) = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2 + \sum_{\substack{i=1}}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} \alpha_i \alpha_j C_{x_i x_j}$$

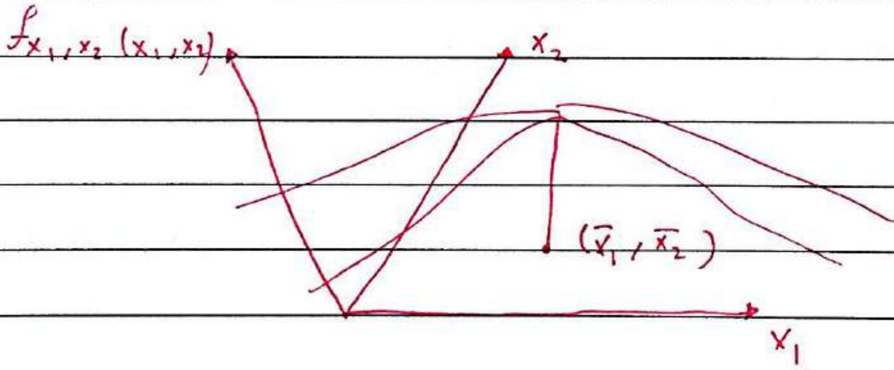
if $X_1, X_2, \dots X_n$ are independent
then $Var\left(\sum_{i=1}^{N} \alpha_i x_i\right) = \alpha_i^2 \sigma_{x_i}^2$

* Jointly Gaussian R.Vs:

- X_1 and X_2 are said to be jointly gaussian "bivirate" if their joint density function is given by :-

$$f_{x_1,x_2}(x_1,x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho^2}} \cdot \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x_1-\bar{x}_1)^2}{\sigma_{x_1}^2} - \frac{2\rho(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-\bar{x}_2)^2}{\sigma_{x_2}^2}\right]\right)$$

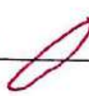
$$\rho = \rho_{x_1 x_2} = \frac{C_{x_1 x_2}}{\sigma_{x_1}\sigma_{x_2}}$$

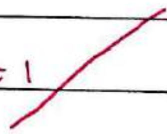


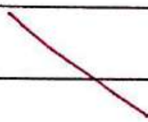
$-\infty < X_1 < \infty$

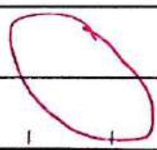
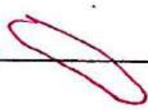
$-\infty < X_2 < \infty$

$\rho = 0$ circle

$\rho > 0$  اذا كبرت [illegible] 

$\rho = 1$ 

$\rho = -1$ 

$\rho < 0$  اذا كبر $\rho < 0$ 

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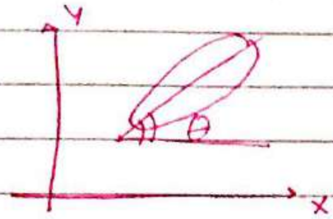

$\theta = \frac{1}{2} \tan^{-1}\left[\frac{2\rho \sigma_x \sigma_y}{\sigma_x^2 - \sigma_y^2}\right]$

- Case 1:

$\sigma_1 > \sigma_2 \longrightarrow \quad 0 < \theta < 45^\circ$

- Case 2:

$\sigma_1 = \sigma_2 \longrightarrow \quad \theta = 45^\circ$

- Case 3

$\sigma_1 < \sigma_2 \longrightarrow \quad 0 < \theta < 90^\circ$

* Find the marginals pdf:-

$f_{x_1}(x_1) = \int_{-\infty}^{\infty} f_{x_1 x_2}(x_1, x_2)\, dx_2 = \frac{1}{\sqrt{2\pi \sigma_{x_1}^2}} e^{\frac{-(x_1 - \bar{x}_1)^2}{2\sigma_{x_1}^2}}$:gaussian.

$X_1 \sim N(\bar{x}_1, \sigma_{x_1}^2)$

$f_{x_2}(x_2) = \int_{-\infty}^{\infty} f_{x_1 x_2}(x_1, x_2)\, dx_1 = \frac{1}{\sqrt{2\pi \sigma_{x_2}^2}} \cdot e^{\frac{-(x_2 - \bar{x}_2)^2}{2\sigma_{x_2}^2}}$

$X_2 \sim N(\bar{x}_2, \sigma_{x_2}^2)$

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* Special case

if X_1 & X_2 are uncorrelated.

$$\rho = 0 \rightarrow f_{X_1 X_2}(x_1, x_2) = \underbrace{\frac{1}{\sqrt{2\pi\sigma_{x_1}^2}} e^{\frac{-(x_1-\bar{x}_1)^2}{2\sigma_{x_1}^2}}}_{f_X(x_1)} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma_{x_2}^2}} e^{\frac{-(x_2-\bar{x}_2)^2}{2\sigma_{x_2}^2}}}_{f_X(x_2)}.$$

* For N jointly Gaussian R.V's:-

$X_1, X_2, \ldots, X_N$ are said to be jointly gaussian if their density function is given by:-

$$f_{X_1, \ldots, X_N}(x_1, \ldots, x_N) = \frac{\left|[C_X]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} \cdot e^{-\frac{[X-\bar{X}]^T [C_X]^{-1} [X-\bar{X}]}{2}}$$

where $[C_X]_{N\times N} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} & \cdots & C_{X_1X_N} \\ C_{X_2X_1} & C_{X_2X_2} & & C_{X_2X_N} \\ \vdots & \vdots & & \\ C_{X_NX_1} & \sigma & \cdots & C_{X_NX_N} \end{bmatrix}$

Covariance Matrix

$\sigma_{x_1}^2$

$\sigma_{x_N}^2$

Equal to each other.

* $$[X-\bar{X}]_{N\times 1} = \begin{bmatrix} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \\ \vdots \\ X_N - \bar{X}_N \end{bmatrix}$$

$$[X-\bar{X}]^T_{1\times N} = \begin{bmatrix} X_1 - \bar{X} & X_2 - \bar{X}_2 & \cdots & X_N - \bar{X}_N \end{bmatrix}.$$

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- For 2 R.V's : X_1 & X_2.

$$[C_x] = \begin{bmatrix} \sigma_{x_1}^2 & C_{x_1 x_2} \\ C_{x_2 x_1} & \sigma_{x_2}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{x_1}^2 & \rho \sigma_{x_1} \sigma_{x_2} \\ \rho \sigma_{x_1} \sigma_{x_2} & \sigma_{x_2}^2 \end{bmatrix}$$

$$\left| [C_x] \right| = \sigma_{x_1}^2 \sigma_{x_2}^2 - \rho^2 \sigma_{x_1}^2 \sigma_{x_2}^2 = (1-\rho^2)(\sigma_{x_1}^2 \sigma_{x_2}^2)$$

$$[C_x]^{-1} = \left(\frac{1}{\sigma_{x_1}^2 \sigma_{x_2}^2} - \frac{\rho^2}{\sigma_{x_1}^2 \sigma_{x_2}^2} \right) \cdot \frac{1}{(1-\rho^2)^2} = \frac{1}{\sigma_{x_1}^2 \sigma_{x_2}^2 (1-\rho^2)}$$

$$[X - \bar{X}]^T = \begin{bmatrix} X_1 - \bar{X}_1 & X_2 - \bar{X}_2 \end{bmatrix}$$

$$[X - \bar{X}] = \begin{bmatrix} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \end{bmatrix}$$

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* Transformation of Multiple R.V's :-

- Recall:

$$X \longrightarrow \boxed{T(\cdot)} \longrightarrow Y = T(x) \qquad x = T^{-1}(y)$$

$\hookrightarrow f_X(x)$ $\qquad\qquad$ $\hookrightarrow f_Y(y) = ?$

- for $x_1, \ldots, x_N \longrightarrow f_{X_1,\ldots X_N}(x_1, \ldots, x_N)$ is known.

* $Y_1 = T_1(x_1, \ldots, x_N)$
  $Y_2 = T_2(x_1, \ldots, x_N)$
  $Y_N = T_N(x_1, \ldots, x_N)$

Conditions:-

1) all T_i's are continuos functions.
2) all T_i's have derivatives.
3) $x_1 = V_1(Y_1, \ldots, Y_N)$
   $\vdots$
   $x_N = V_N(Y_1, \ldots, Y_N)$

- Then;

$$f_{Y_1,\ldots Y_N}(y_1, \ldots, y_N) = f_{X_1,\ldots X_N}\big(V_1(y_1, \ldots, y_N), \ldots, V_N(y_1, \ldots, y_N)\big) \cdot |J|$$

* $J = \begin{vmatrix} \frac{dV_1}{dy_1} & \frac{dV_1}{dy_2} & \cdots & \frac{dV_1}{dy_N} \\ \frac{dV_2}{dy_1} & \frac{dV_2}{dy_2} & \cdots & \frac{dV_2}{dy_N} \\ \vdots & \vdots & & \vdots \\ \frac{dV_N}{dy_1} & \frac{dV_N}{dy_2} & \cdots & \frac{dV_N}{dy_N} \end{vmatrix}$ $\rightsquigarrow$ Determinent

"Jacobian Matrix"

Ex $X_1, X_2 \longrightarrow f_{X_1,X_2}(x_1,x_2)$.

$Y_1 = aX_1 + bX_2$

$Y_2 = cX_1 + dX_2$

a, b, c, d are constants and $ad - bc \neq 0$

Find $f_{Y_1,Y_2}(y_1,y_2)$?

Sol $ax_1 + bx_2 = y_1$

$+ (cx_1 + dx_2 = y_2) * \frac{-b}{d}$

$ax_1 - \frac{b}{d} c x_1 = y_1 - \frac{b}{d} y_2$

$\Rightarrow \left[x_1 (a - \frac{bc}{d}) = y_1 - \frac{b}{d} y_2 \right] * d$

$x_1 (ad - bc) = dy_1 - by_2$

$\rightarrow \boxed{x_1 = \frac{dy_1 - by_2}{ad - bc}} \rightarrow v_1(y_1, y_2)$

$* \quad ax_1 + bx_2 = y_1$

$+ (cx_1 + dx_2 = y_2) * \frac{-a}{c}$

$bx_2 - \frac{ad}{c} x_2 = y_1 - \frac{a}{c} y_2$

$\Rightarrow x_2 = \frac{y_1 - \frac{a}{c} y_2}{b - \frac{ad}{c}} * \frac{c}{c}$

$\boxed{x_2 = \frac{cy_1 - ay_2}{cb - ad}} \rightarrow v_2(y_1, y_2)$

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- Now, we find J :-

$$J = \left| \begin{matrix} \frac{dv_1}{dy_1} & \frac{dv_1}{dy_2} \\ \frac{dv_2}{dy_1} & \frac{dv_2}{dy_2} \end{matrix} \right| = \left| \begin{matrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{c}{bc-ad} & \frac{-a}{bc-ad} \end{matrix} \right|$$

$$= \frac{ad}{(ad-cb)^2} - \frac{bc}{(ad-bc)^2} = \frac{ad-bc}{(ad-bc)^2}$$

$$\boxed{J = \frac{1}{ad-bc}}$$

$$\Rightarrow f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(\frac{dy_1 - by_2}{ad-bc}, \frac{cy_1 - ay_2}{bc-ad}\right) \cdot \frac{1}{|ad-bc|}$$

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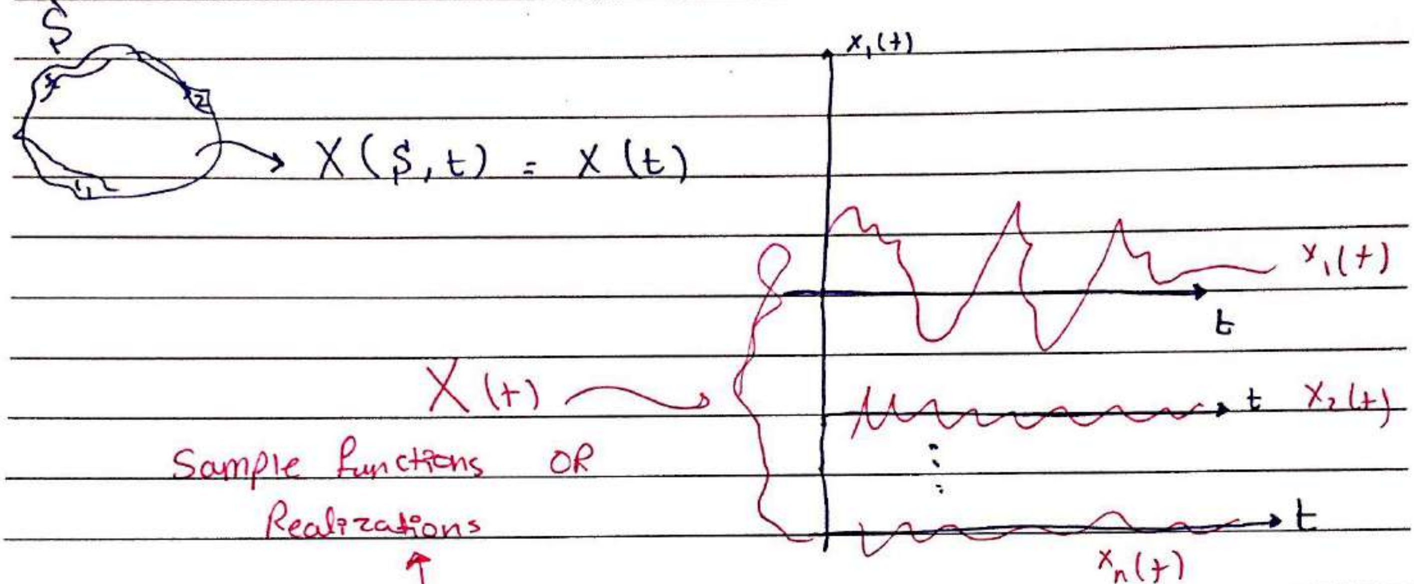
* CH #6 8

Random processes – Temprol characteristics: – (R.P)
"Stachastic"

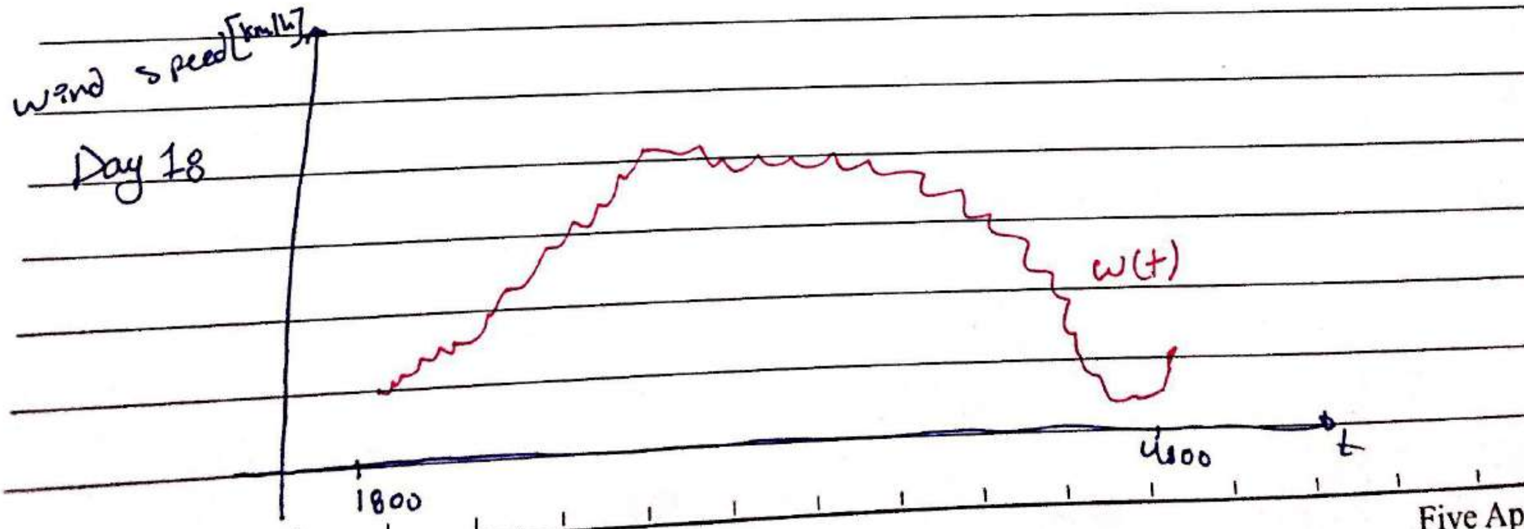
– R.P:



$X(t) = \{ x_1(t), x_2(t), \ldots, x_n(t) \}$.

family of time – waveforms.

– Ex: wind speed at certain location.
from 1:00 am to 4:00 am –



Five Appl

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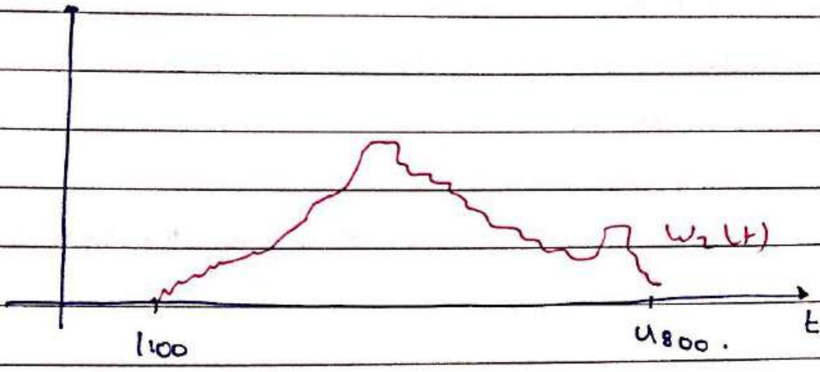
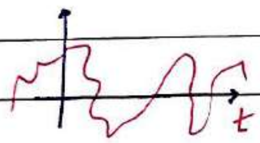
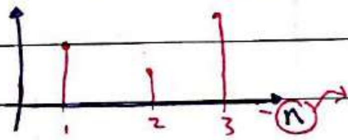
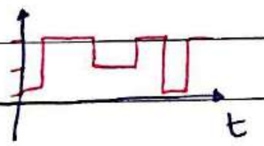
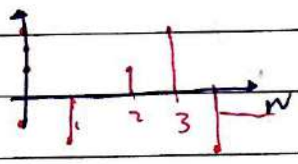
- Day 28 -



Same behaviours but not exactly the same!

Ex : Noise in communication system. is R.p
$\{N(t)$.

* R.p Classifications :-

	$X(t)$ Continuos - Time	$X[n]$ Discrete - Time
Continuos Amplitude		
Discrete Amplitude		

ints معرفة على

R.p

Deterministic
↳ has a mathematical description
$X(t) = \underline{A}\cos(\underset{w_c}{\underline{\theta t + \phi}})$
↳ at least one must be R.V

Non-deterministic.
Ex: Wind speed
Noise

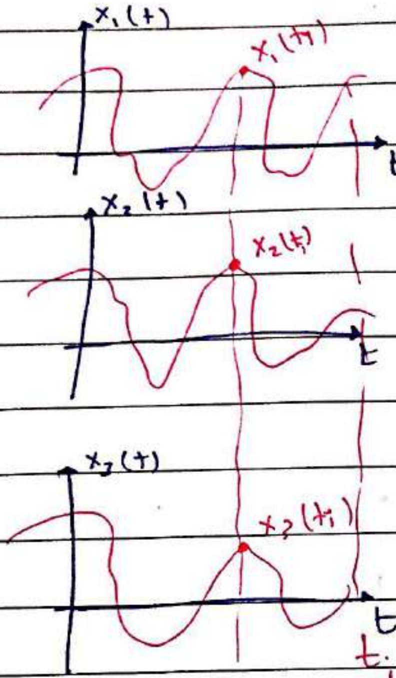
- $X(t) = A\cos(\omega_c t + \theta)$

ω_c, θ constants.

$A \sim u(0,1) \Rightarrow$ R.P

إذا باخذ تقطيعه من R.P بطلع R.V

- Def: R.P is a family of sample time-waveforms



$X(t_i) = X_i = \{x_1(t_i), x_2(t_i), \ldots\}$

is a R.V

$X(t_j)$ is another R.V

t_j ↳ $f_X(x_j, t_j)$.

t_i → $f_X(x_i, t_i)$

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21st. Nov. 2018

• R.P 1st order distributions -

- at $t = t_i \longrightarrow X(t_i) = X_i \overset{pdf}{\sim} f_x(x_i ; t_i)$ — Apply CH2 + CH3.

- at $t = t_j \longrightarrow X(t_j) = X_j \sim f_x(x ; t_j)$

So, R.P pdf: $X(t) \overset{pdf}{\sim} f_x(x ; t)$

cdf $\hookrightarrow F_x(x ; t)$

• R.P 2nd order distributions: — Apply CH4 + CH5.

$(X_i, X_j) \sim f_x(x_i, x_j ; t_i, t_j)$

$\hookrightarrow (X(t), X(t+\tau)) \sim f_x(x_1, x_2 ; t, t+\tau)$

$\hookrightarrow F_x(x_1, x_2 ; t, t+\tau)$

* R.P mean: $E[X(t)]$

$E[X(t)] = \int x . f_x(x;t)\, dx = m_x(t)$

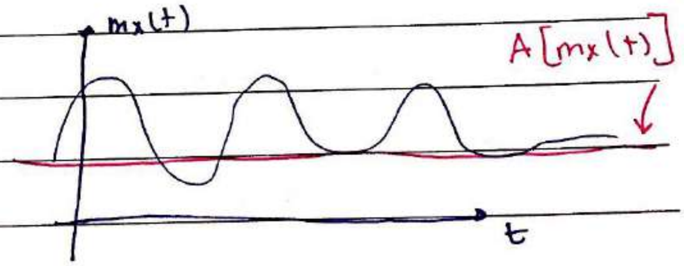
بنعطيه ماد لرمزة بالأخير ع
يكون بحسبان.

Statistical Average.

$\hookrightarrow$ In general is a time-waveform.

* DC-value of R.P $X(t)$

$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} m_x(t)\, dt.$



$= A[m_x(t)]$

Time avg.

$\rightarrow A[g(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, dt$

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* R.P variance : $Var(X(t)) = \sigma_x^2(t)$

$$\sigma_x^2(t) = E[x^2(t)] - m_x^2(t)$$

instantaneous total power.

↳ Avg total power $= A[E[x^2(t)]]$.

instantaneous AC-power.

Avg AC-power $= A[\sigma_x^2(t)]$

instantaneous DC-power

Avg DC-power $= A^2[m_x(t)]$

. R.P auto correlation functions -

- Recall : x_1 & x_2 : Correlation $R_{x_1x_2} = E[x_1 x_2]$

$$= \int\int_{-\infty}^{\infty} x_1 x_2 f_{x_1,x_2}(x_1,x_2)\, dx_1 dx_2$$

. For R.P:

Auto: from the same process.

$$R_{xx}(t, t+\tau)$$

$$= E[x(t)\, x(t+\tau)]$$

$$= \int\int x_1 x_2 f_x(x_1, x_2 ; t, t+\tau)\, dx_1 dx_2$$

- Notes -

If τ = zero -

$$R_{xx}(t,t) = E[x^2(t)]$$

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R.P auto covariance function:

- Recall: X_1 & X_2 $\rightarrow$ $C_{x_1 x_2} = R_{x_1 x_2} - \bar{x}_1 \bar{x}_2$

For R.P

$$C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - m_x(t) \cdot m_x(t+\tau).$$

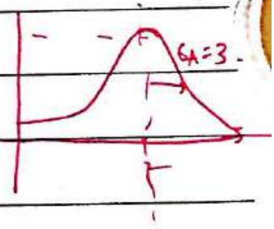
- Notes:

If $\tau = 0$.

$$C_{xx}(t,t) = R_{xx}(t,t) - m_x^2(t)$$
$$= E[x^2(t)] - m_x^2(t)$$
$$= \sigma_x^2(t).$$

- Ex: Given a R.P $X(t) = A\cos(w_0 t + \theta)$ where $A \sim N(2, 9)$, w_0 and θ are constants

Find ① $f_x(x;t)$ ② $m_x(t)$ ③ $\sigma_x^2(t)$ ④ DC-Value.



Sol ①

$$X(t) = \underset{\text{R.V}}{A} \; \underbrace{\cos(w_0 t + \theta)}_{\text{not R.V}}.$$

$$X(t) \sim N(m_x(t), \sigma_x^2(t))$$

هاي بتطلع برا

لأنها مش R.V

$$m_x(t) = E[x(t)] = E[A\cos(w_0 t + \theta)]$$
$$= 2\cos(w_0 t + \theta)$$

$\downarrow$

mean for A.

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③ $\sigma_x^2(t) = Var(X(t)) = Var(A\cos(\omega_0 t + \theta))$

$$= \cos^2(\omega_0 t + \theta)\,Var(A)$$

$$= 9\cos^2(\omega_0 t + \theta)$$

$\boxed{1}$ $X(t) \sim N(2\cos(\omega_0 t + \theta),\ 9\cos^2(\omega_0 t + \theta))$

$$\rightarrow f_x(x,t) = \frac{1}{\sqrt{18\pi\cos^2(\omega_0 t + \theta)}}\ exp - \frac{(x - 2\cos(\omega_0 t + \theta))^2}{18\cos^2(\omega_0 t + \theta)}$$

④ DC value $= A[m_x(t)] = A[2\cos(\omega_0 t + \theta)]$

$$= \frac{1}{\cancel{2}T}\int_0^T 2\cos(\omega_0 t + \theta)\,dt.$$

$$= \boxed{Zero}.$$

*Ex 2: Given a R.P $X(t) = A\cos(\omega_0 t + \theta)$ where A & θ are two R.Vs $f_{A,\theta}(a,\theta)$, ω_0 is const.

$$\rightarrow m_x(t) = E[x(t)] = E[\underset{R.V}{A}\cos(\omega_0 t + \underset{R.V}{\theta})]$$

CH5.

$$= \iint a\cos(\omega_0 t + \theta)\, f_{A,\theta}(a,\theta)\, da\, d\theta$$

HW for previous ex

$R_{xx}(t, t+\tau) = E[x(t)x(t+\tau)]$

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26th. Nov. 2018

*Stationarity:

- Def: In general a R.P $x(t)$ is said to be stationary if it doesn't change it's statistics with time.

- 1st order stationarity:

$f_x(x;t_i) = f_x(x;t_j)$ for any t_i, t_j — doesn't change with time.

which means

$f_x(x;t) = f_x(x)$

- as a result:

① $m_x(t) = E[x(t)] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \bar{X}$

$m_x(t) = \bar{X}$ "constant"

- DC-value for $X(t) = \underbrace{A}_{\text{time Avg.}}[\bar{x}] = \bar{X}$

② $\sigma_x^2(t) = E[x^2(t)] - m_x^2(t)$

$= \int x^2 f_x(x)\, dx - \bar{x}^2$

$=$ constant $- \bar{x}^2 = \sigma_x^2$

$\sigma_x^2(t) = \sigma_x^2$ "constant variance.

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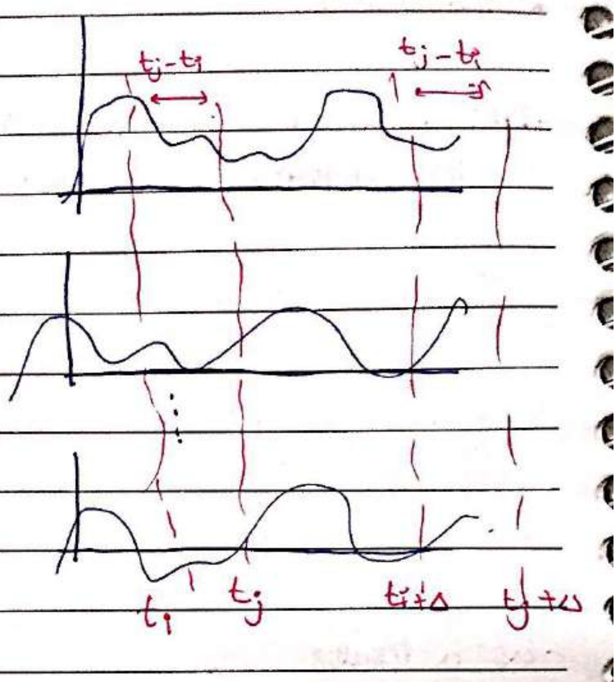
- 2^{nd} order stationarity:

$$f_X(x_1, x_2; t_i, t_j) = f_X(x_1, x_2, t_i+\Delta, t_j+\Delta)$$
for any t_i, t_j, Δ



the separation must be equal.
if I have tj, ti with separation
$2(t_j-t_i)$ I can't apply 2^{nd} order stat.

$$(X(t), X(t+\tau)) \sim f_X(x_1, x_2; t, t+\tau)$$
$$= f_X(x_1, x_2; \tau)$$
↳ separation

- as a result:-

① The R.P is 1^{st} order stationary.

② $R_{XX}(t, t+\tau) = E[X(t) \cdot X(t+\tau)]$

$$= \iint_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t, t+\tau)$$

$$= \iint_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; \tau) dx_1 dx_2 = R_X(\tau)$$

- $R_{XX}(1, 6)$, $R_{XX}(3, 8)$
↳ they have the same joint density function because they have the same time difference "separation = 5".

③ $C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - m_X(t)\, m_X(t+\tau)$
$C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$

$C_{XX}(3, 4.5) = C_{XX}(\cancel{4}, 7)$.
5.5

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* **Wide-Sense Stationary (WSS) R.Ps.**

- Def: A R.P $X(t)$ is said to be WSS if
  1. $m_X(t) = \overline{X}$
  2. $R_{XX}(t, t+\tau) = R_{XX}(\tau)$

- as a result:-

(1) $\sigma_X^2(t) = E[X^2(t)] - m_X^2(t)$

$= R_{XX}(0) - \overline{X}^2$

$= \sigma_X^2$ "Constant"

(2) $C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$

* 2nd-Order stationary $\longrightarrow$ WSS

$\not\longleftarrow$

- Ex: Given a R.P $X(t) = A\cos(\omega_0 t + \theta)$
  where A and ω_0 are constants and
  $\theta \sim u(0, 2\pi)$
  Is $X(t)$ WSS or not??

Sol $m_X(t) = E[X(t)] = E[\underbrace{A\cos(\omega_0 t + \theta)}_{g(\theta)}]$

$= A E[\cos(\omega_0 t + \theta)]$

$= A \int g(\theta) f_\theta(\theta)\, d\theta$

$= A \int_0^{2\pi} \cos(\omega_0 t + \theta) \cdot \frac{1}{2\pi}\, d\theta$ zero

"zero ← one period على cos تكامل"

$= A(0) = \boxed{0} \rightarrow \overline{X}$ is constant ✓

* $R_{XX}(t, t+\tau) = E[X(t) \cdot X(t+\tau)]$

$= E[A\cos(\omega_0 t + \theta) \cdot A\cos(\omega_0 t + \omega_0 \tau + \theta)]$

$= A^2 E\left[\frac{1}{2}\cos(\omega_0 \tau) + \frac{1}{2}\cos(2\omega_0 t + \omega_0 \tau + 2\theta)\right]$

$= \frac{A^2}{2}\left[\cos(\omega_0 \tau) + E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta)]\right]$

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$= \frac{A^2}{2}\left[\cos(\omega_0\tau) + \frac{1}{2\pi}\int_0^{2\pi} \cos(2\theta + \cancel{2\omega_0 t + \omega_0\tau})\, d\theta\right]$

$= \frac{A^2}{2}\cos(\omega_0\tau) = R_{xx}(\tau)$

So Process is WSS!

$R_x(2,4) = \frac{A^2}{2}\cos(\omega_0(2))$

$\tau = 2$

* Properties of $R_{xx}(\tau)$:-

$\boxed{1}$ $|R_{xx}(\tau)| \leq R_{xx}(0)$

$\boxed{2}$ $R_{xx}(-\tau) = R_{xx}(\tau)$ "even Function"

$\boxed{3}$ $R_{xx}(0) = E[x^2(t)]$ "Total avg power" "mean squared value"

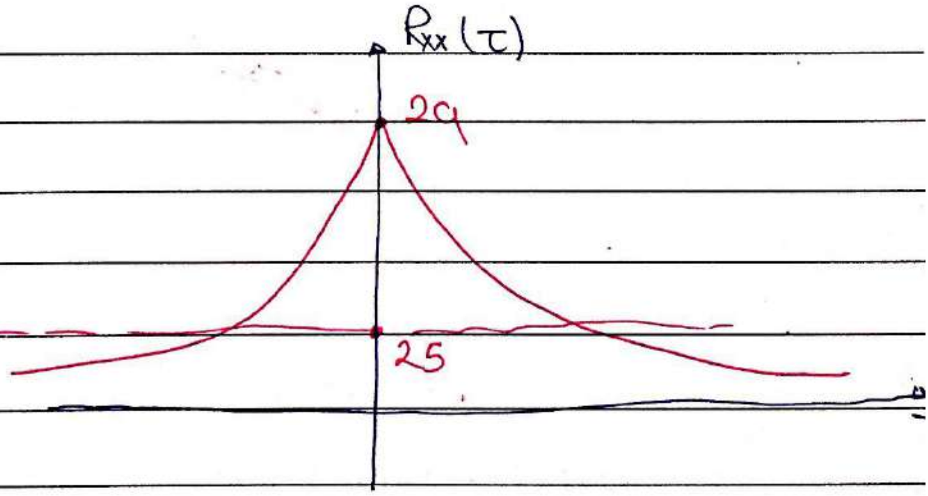
$\boxed{4}$ $\lim_{\tau\to\infty} R_{xx}(\tau) = \bar{X}^2$ "DC Avg power"

- Ex Given WSS R.P $x(t)$ with $R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$

Find: a) $E[x^2(t)]$

b) $\bar{X}$

c) σ_x^2



Sol a) $R_{xx}(0) = 29$

b) $\bar{X}^2 = 25 \rightarrow \bar{X} = \pm 5$

c) $29 - 25 = 4$

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- Ex

$X(t) \longrightarrow \otimes \longrightarrow Y(t)$

$\uparrow$ $\cos(\omega_0 t+\theta)$

$\theta \sim u(-\pi, +\pi)$

WSS

$R_{xx}(\tau) = e^{-a|\tau|}$

check if $Y(t)$ is WSS ??

Sol $Y(t) = x(t) \cos(\omega_0 t+\theta)$

$E[Y(t)] = E[\underbrace{x(t) \cdot \cos(\omega_0 t+\theta)}_{\text{Statistically independent.}}]$

$= E[x(t)] \cdot E[\cos(\omega_0 t+\theta)]$

$\rightarrow \lim_{\tau \to \infty} R_{xx}(-\tau) = 0 = \bar{x}^2 \rightarrow \bar{x} = 0.$

$E[\cos(\theta+\omega_0 t)] = 0$, So $\boxed{E[Y(t)] = \text{zero}}$ --- ①

$R_{xx}(t, t+\tau) = E[Y(t) \cdot Y(t+\tau)] =$

$= E[x(t) \cos(\omega_0 t+\theta) \cdot X(t+\tau) \cos(\omega_0 t + \omega_0 \tau + \theta)]$

$= \underbrace{E[x(t) \cdot x(t+\tau)]}_{R_{xx}(\tau)} \cdot \underbrace{E[\cos(\omega_0 t+\theta) \cdot \cos(\omega_0 t + \omega_0 \tau+\theta)]}_{\frac{1}{2}\cos(\omega_0 \tau)}$

$= \frac{1}{2} R_{xx}(\tau) \cos(\omega_0 \tau) = \frac{1}{2} e^{-a|\tau|} \cos(\omega_0 \tau) = R_{yy}(\tau)$

$\boxed{R_{xx}(t, t+\tau) = R_{yy}(\tau)}$ — ②

From ① & ② it's WSS!

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* Auto - Covariance Functions -

$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - m_X(t)\, m_X(t+\tau)$$

- if $X(t)$ is WSS, then

$$C_{XX}(t, t+\tau) = R_{XX}(\tau) - \bar{X}^2$$

* Cross - Correlation Functions:-

$$R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)]$$

- if $R_{XY}(t, t+\tau) = 0$, then $X(t) \perp Y(t)$

- if $R_{XY}(t, t+\tau) = E[X(t)] \cdot E[Y(t+\tau)]$, then $X(t)$ & $Y(t)$ are statistically independent.

* Cross - Covariance Functions:-

$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - m_X(t)\, m_Y(t+\tau)$$

* A R.P's $X(t)$ & $Y(t)$ are said to be joint WSS if:-

1) $X(t)$ is WSS $\rightarrow m_X(t) = \bar{X}$
$\qquad \rightarrow R_{XX}(t, t+\tau) = R_{XX}(\tau)$

2) $Y(t)$ is WSS $\rightarrow m_Y(t) = \bar{Y}$
$\qquad \rightarrow R_{YY}(t, t+\tau) = R_{YY}(\tau)$

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3) $R_{xy}(t, t+\tau) = R_{xy}(\tau)$

↳ As a result, $C_{xy}(t, t+\tau) = C_{xy}(\tau) = R_{xy}(\tau) - \bar{x}\,\bar{y}$

* Gaussian R.P:

- Def: A R.P is said to be gaussian if the N R.V's $x_1, \dots, x_N$ corresponding to the time instants $t_1, \dots, t_N$ are jointly gaussian with density function

$$f_X(x_1, \dots, x_N) = \frac{\left| [C_x]^{-1} \right|^{1/2}}{(2\pi)^{N/2}} \, e^{-1/2 \, [X - \bar{X}]^T [C_x]^{-1} [X - \bar{X}]}$$

* $[X - \bar{X}] = \begin{bmatrix} x_1 - m_x(t_1) \\ x_2 - m_x(t_2) \\ \vdots \\ x_N - m_x(t_N) \end{bmatrix}$

* $[C_x] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$

* $C_{ij} = C_{xx}(t_i, t_j) = R_{xx}(t_i, t_j) - m_x(t_i)\, m_x(t_j)$

$i = 1, \dots, N$

$j = 1, \dots, N$

if $x(t)$ is WSS gaussian RP:

$[X - \bar{X}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{y}_N \end{bmatrix}$ $\bar{X} = m_x(t)$

$C_{ij} = C_{xx}(t_i, t_j) = C_{xx}(t_i - t_j) = R_{xx}(t_j - t_i) - \bar{x}^2$

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- Ex: A continuos-time WSS gaussian R.P with mean $\bar{X} = 4$ and auto-correlation function $R_{xx}(\tau) = 25e^{-3|\tau|} + 16$

- Determine the covariance matrix for three R.V's $x(t_1), x(t_2), x(t_3)$ defined as $t_j = t_0 + \frac{j-1}{2}$, $j = 1, 2, 3$?

Sol

$$[C_x]_{3\times3} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$t_1 = t_0$
$t_2 = t_0 + 1/2$
$t_3 = t_0 + 1$

$$C_{ij} = R_{xx}(\underbrace{t_j - t_i}_{\tau}) - \bar{X}^2 = 25e^{-3|t_j - t_i|} + 16 - 16.$$

$\hookrightarrow \lim_{\tau\to\infty} R_{xx}(\tau) = \bar{X}^2 = 16$

$$C_{ij} = 25e^{-3|t_j - t_i|}$$

$C_{11} = 25e^{-3(0)} = 25$
$C_{12} = C_{21} = 25e^{-3/2}$
$C_{22} = 25e^{-3(0)} = 25$
$C_{13} = C_{31} = 25e^{-3}$
$C_{23} = C_{32} = 25e^{-3/2}$
$C_{33} = 25.$

$$\Rightarrow [C_x] = \begin{bmatrix} 25 & 25e^{-3/2} & 25e^{-3} \\ 25e^{-3/2} & 25 & 25e^{-3/2} \\ 25e^{-3} & 25e^{-3/2} & 25 \end{bmatrix}$$

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* Time - Average & Ergodicity :

$X(t)$ is R.P

$m_X(t) = E[X(t)] = \bar{X}$ " DC Value "

$R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)] = R_{XX}(\tau)$ if WSS !

$m_X(t)$ mathmetically $= \int_{-\infty}^{\infty} x\, f_X(x;t)\, dx$

$R_{XX}(t, t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2;\, t, t+\tau)\, dx_1 dx_2$

we can't use them if we don't know the R.P density function

* Estimation for $m_X(t)$ & $R_{XX}(t, t+\tau)$:-

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)$$

$$R_{XX}(t, t+\tau) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)\, X_i(t+\tau)$$

- For the result to be accurate : we need $N \to \infty$

→ which is practically hard

↳ This is solved by using "Ergodicity".

Five Apple

* **Ergodicity :** A WSS R.P is said to be ergodic if :-

1) $\overline{X} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, dt$ for any sample function $x(t)$

$\overline{X}$ → Statistical Average $E[x(t)]$

$\frac{1}{2T}\int_{-T}^{+T} x(t)\,dt$ → time-avg

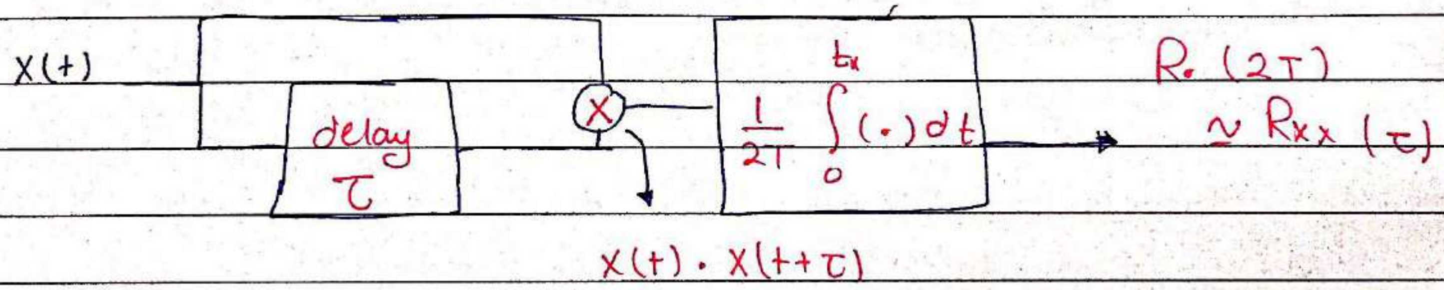
2) $R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, x(t+\tau)\, dt$. for any sample function $x(t)$.

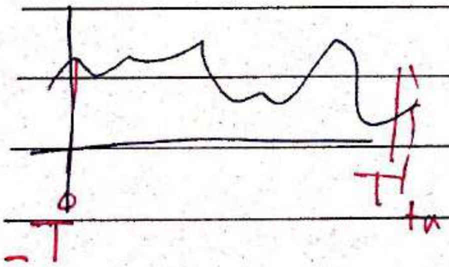
$R_{xx}(\tau)$ → Statistical auto-correlation function.

$\frac{1}{2T}\int_{-T}^{+T} x(t)\,x(t+\tau)\,dt$ → Time-Avg Auto-correlation.

* **How to measure $R_{xx}(\tau)$:-**

get one sample-function $x(t)$





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- Ex $X(t) = A\cos(\omega_0 t + \theta)$, $\theta \sim u(0, 2\pi)$

Find a) $R_{xx}(\tau)$ b) measure $R_{xx}(\tau)$

Sol a) $R_{xx}(\tau) = E[x(t)\, x(t+\tau)] \ldots = \frac{A^2}{2}\cos(\omega_0 \tau)$.

b) $x(t) = A\cos(\omega_0 t + \theta)$

$$R_{xx}(\tau) = \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T} A^2 \cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0 \tau + \theta)\, dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T} \left[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + \theta)\right] dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T} \cos(\omega_0 \tau)\, dt + \frac{A^2}{4T}\int_{-T}^{T} \cos(2\omega_0 t + \omega_0 \tau + \theta)\, dt$$

$$= \frac{A^2}{4T} * \cos(\omega_0 \tau) * 2T + \frac{A^2}{2}\cos(\omega_0 \tau + 2\theta)\frac{\sin(2\omega_0 T)}{2\omega_0 T}$$

$$R_{xx}(\tau) = \frac{A^2}{2}\cos(\omega_0 \tau) + \frac{A^2}{2}\cos(\omega_0 \tau + 2\theta)\frac{\sin(2\omega_0 T)}{2\omega_0 T}$$

Error.

if we take $\lim_{T \to \infty}$ → same answer as part $\boxed{1}$

Five A

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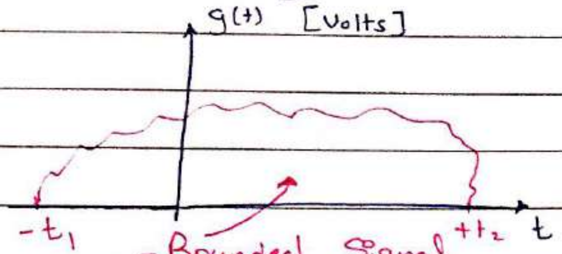
CH #7: Random processes – Spectral characteristics.

– Recall:

Deterministic signal "not R.P."



– Bounded signal

– time – limited signal $\quad \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$

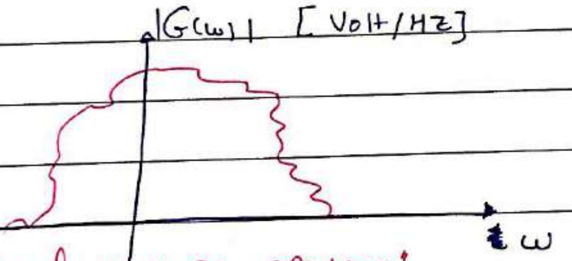
– Energy signal

$$- \, g(t) \xrightarrow[\text{transform}]{\text{Fourier}} G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t} \, dt.$$

only for bounded signals!

Signal spectrum.

$$= |G(\omega)| \, \angle \phi(\omega)$$



"Signal voltage spectrum"

↳ it shows how the signal voltage is distributed over the frequency.

– Total avg – power in g(t):–

$$P_{gg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} g^2(t) \, dt.$$

"for bounded & unbounded".

"computing power in the time domain"

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OR

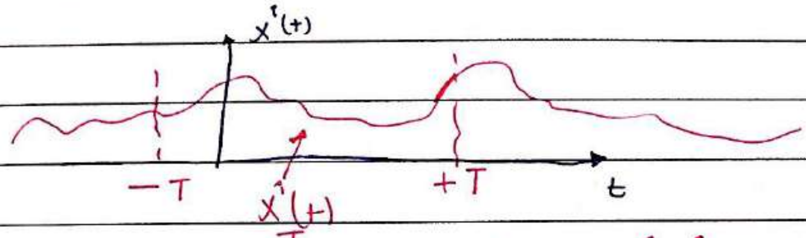
$$P_{gg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \frac{|G(\omega)|^2}{2T} \, d\omega$$ "Parseval's theorem"

"computing power in the frequency domain"

* For Random processes → we concern with the "power-spectral density"

$$X(t) \overset{F.T}{\not\longleftrightarrow} X(\omega)$$

- Take one sample function $x^i(t)$ and assume it's unbounded



→ Truncated Version of $x^i(t)$

- $x^i_T(t) = \begin{cases} x^i(t), & -T \leq t \leq T \\ 0, & \text{o.w} \end{cases}$ "bounded"

$$\to x^i_T(t) \overset{F.T}{\longleftrightarrow} X^i_T(\omega) = \int_{-\infty}^{\infty} x^i_T(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{+T} x^i(t)\, e^{-j\omega t}\, dt.$$

$$P^i(T) = \frac{1}{2T} \int_{-T}^{T} x^{i\,2}_T(t)\, dt$$

$$P^i(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X^i_T(\omega)|^2}{2T}\, d\omega$$

- avg - power in $x^i(t)$ "time domain"

$$P^i = \lim_{T\to\infty} P^i(T)$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^{i^2}(t)\, dt.$$

- avg - power in $x^i(t)$ "Frequency domain"

$$P^i = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X^i_T(\omega)|^2}{2T}\, d\omega$$

"time domain"

$\star P^i \longrightarrow P = \{P^1, P^2, --- P^N\}$

↳ Random Variable.

- avg - Power in the Process $X(t)$:-

$$P_{XX} = E[P] = E\left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^2(t)\, dt\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]\, dt = A\left[E[X^2(t)]\right]$$

"Frequency domain"

$$- P_{XX} = E[P] = E\left[\lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega\right]$$

. where $X_T(\omega) = \int_{-T}^{T} X(t)\, e^{-j\omega t}\, dt$

R.P. R.P

$$\hookrightarrow P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} E\left[\frac{|X_T(\omega)|^2}{2T}\right] d\omega.$$

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$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E\left[\,|X_T(\omega)|^2\right]}{2T} \, d\omega$$

$[\text{watt}]$ $\left[\frac{1}{\text{rad}}\right]$ $\left[\frac{\text{watt}}{\text{Hz}}\right]$ $\left[\frac{\text{rad}}{\text{sec}}\right] = [\text{rad}\cdot\text{Hz}]$

$S_{xx}(\omega)$ = R.P Power spectral density (PSD)

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega.$$

- Summary:

- R.P $X(t)$ $\xrightarrow{\text{To Frequency (spectram) domain}}$ $S_{xx}(\omega)$ "PSD" $= \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$

where $X_T(\omega) = \int_{-T}^{T} X(t)\, e^{-j\omega t}\, dt.$

- Power in time domain

$P_{xx} = A\left[E[X^2(t)]\right]$

- power in frequency domain.

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$$

area under $S_{xx}(\omega)$

if $X(t)$ is WSS

$P_{xx} = E[X^2(t)]$

which is constant.

Five Ap

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- Ex Let $X(t) = \underbrace{A_o \cos(\omega_o t + \theta)}_{\text{real constants}}$, $\theta \sim u(0, \pi/2)$

Find (a) P_{xx} using time domain

(b) $S_{xx}(\omega)$

(c) use part b to find P_{xx}.

- Sol (a) $P_{xx} = A\left[E[X^2(t)]\right]$

$$E[X^2(t)] = E\left[A_o^2 \cos^2(\omega_o t + \theta)\right]$$

$$= E\left[\frac{A_o^2}{2} + \frac{A_o^2}{2}\cos(2\theta + \omega_o t)\right]$$

$$= \frac{A_o^2}{2} + \frac{A_o^2}{2} E\left[\cos(2\theta + \omega_o t)\right]$$

$$= \frac{A_o^2}{2} + \frac{A_o^2}{2}\int_0^{\pi/2} \cos(2\theta + \omega_o t) \cdot \frac{2}{\pi}\, d\theta$$

$$= \frac{A_o^2}{2} - \frac{A_o^2}{2}\sin(2\omega_o t)$$

So, $P_{xx} = A\left[\frac{A_o^2}{2} - \frac{A_o^2}{2}\sin(2\omega_o t)\right]$

$$= \frac{A_o^2}{2} - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\frac{A_o^2}{2}\sin(2\omega_o t)\, dt.$$

$$= \frac{A_o^2}{2} - 0 = \boxed{\frac{A_o^2}{2}}$$

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(b) S_{xx}

First, $X_T(\omega) = \int_{-T}^{+T} x(t)\, e^{-j\omega t}\, dt$

$$= \int_{-T}^{T} A_0 \cos(\omega_0 t + \theta)\, e^{-j\omega t}\, dt.$$

$$= \int_{-T}^{T} \frac{A_0}{2} \cdot \left[e^{j(\omega_0 t + \theta)} + e^{-j(\omega_0 t + \theta)} \right] e^{-j\omega t}\, dt.$$

$$= \frac{A_0}{2} \int_{-T}^{T} e^{j\omega_0 t} \cdot e^{j\theta} \cdot e^{-j\omega t} + e^{-j\omega_0 t} \cdot e^{-j\theta} \cdot e^{-j\omega t}\, dt.$$

$$= \frac{A_0 e^{j\theta}}{2} \int_{-T}^{T} e^{-j(\omega - \omega_0)t}\, dt + \frac{A_0 e^{-j\theta}}{2} \int_{-T}^{+T} e^{-j(\omega + \omega_0)t}\, dt.$$

$$= \frac{A_0 e^{j\theta}}{2} \cdot \left. \frac{e^{-j(\omega - \omega_0)t}}{-j(\omega - \omega_0)} \right|_{-T}^{+T} + \frac{A_0 e^{-j\theta}}{2} \left. \frac{e^{-j(\omega + \omega_0)t}}{-j(\omega + \omega_0)} \right|_{-T}^{+T}$$

$$= A_0 T e^{j\theta} \frac{\sin((\omega - \omega_0)T)}{(\omega - \omega_0)T} + A_0 T e^{-j\theta} \frac{\sin((\omega + \omega_0)T)}{(\omega + \omega_0)T}.$$

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10th. Dec. 2018

* R.P PSD: "PDS" "Power Density Spectrum"

$$X(t) \longrightarrow S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[\,|X_T(\omega)|^2\right]}{2T}$$

where

$$X_T(\omega) = \int_{-T}^{+T} x(t)\, e^{-j\omega t}\, dt.$$

R.P $\nearrow$ $\nwarrow$ R.P.

- Continue the example before.

$X(t) = A\cos(\omega_0 t + \theta)$, $\theta \sim u(0, \pi/2)$

(a) $P_{xx} = \frac{A_0^2}{2}$

(b) find $S_{xx}(\omega)$ using (*)

$$X_T(\omega) = A_0 T e^{j\theta} \frac{\sin\left((\omega-\omega_0)T\right)}{(\omega-\omega_0)T} + A_0 T e^{-j\theta} \frac{\sin\left((\omega+\omega_0)T\right)}{(\omega+\omega_0)T}$$

R.V $\nearrow$

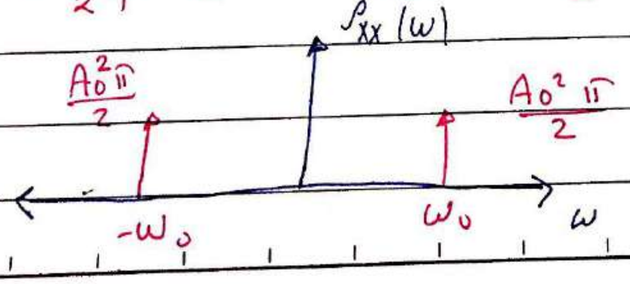
$$|X_T(\omega)| = X_T(\omega) \cdot X_T^*(\omega)$$ E[ ] بجيبوه

$\vdots$

$$\frac{1}{2T} E\left[|X_T(\omega)|^2\right] = \frac{A_0 \pi}{2}\left[\frac{T}{\pi} \frac{\sin^2\left((\omega-\omega_0)T\right)}{\left((\omega-\omega_0)T\right)^2} + \frac{T}{\pi} \frac{\sin^2\left((\omega+\omega_0)T\right)}{\left((\omega+\omega_0)T\right)^2}\right]$$

Using $\lim_{T\to\infty} \frac{T}{\pi} \frac{\sin(\alpha T)}{\alpha T} = \delta(\alpha)$

So; $$\lim \frac{1}{2T} E\left[|X_T(\omega)|^2\right] = \frac{A_0^2 \pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$



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ⓒ $P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{xx}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A_0^2 \pi}{2}\delta(\omega-\omega_0)\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A_0^2\pi}{2}\delta(\omega+\omega_0)\, d\omega$

$= \frac{A_0^2}{4} + \frac{A_0^2}{4} = \boxed{\frac{A_0^2}{2}}$

* $\mathcal{S}_{xx}(\omega)$ properties :-

① $\mathcal{S}_{xx}(\omega) \geq 0$

② $\mathcal{S}_{xx}(\omega) = \mathcal{S}_{xx}(-\omega)$ " Even Function • for real $x(t)$ "

③ $\mathcal{S}_{xx}(\omega)$ is real

④ $P_{xx} = A\left[E\left[x^2(t)\right]\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{xx}(\omega)\, d\omega$

⑤ If $\dot{x}(t) = \frac{dx(t)}{dt}$, then $\mathcal{S}_{\dot{x}\dot{x}}(\omega) = \omega^2 \mathcal{S}_{xx}(\omega)$.

⑥ $\mathcal{S}_{xx}(\omega) = FT\left\{A\left[R_{xx}(t,t+\tau)\right]\right\} = \int_{-\infty}^{\infty} A\left[R_{xx}(t,t+\tau)\right] e^{-j\omega\tau}\, d\tau$

↳ Fourier transform.

⑦ $A\left[R_{xx}(t,t+\tau)\right] = FT^{-1}\left\{\mathcal{S}_{xx}(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{xx}(\omega)\, e^{+j\omega\tau}\, d\omega$

↳ Fourier transform inverse

$$A\left[R_{xx}(t,t+\tau)\right] \xleftrightarrow{FT} \mathcal{S}_{xx}(\omega)$$

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if $X(t)$ is WSS :-

$R_{xx}(t, t+\tau) = R_{xx}(\tau)$

$A[R_{xx}(\tau)] = R_{xx}(\tau)$

So; $R_{xx}(\tau) \xleftrightarrow{FT} \mathcal{S}_{xx}(\omega)$ "FT-pair".

$$\mathcal{S}_{xx}(\omega) = FT\{R_{xx}(\tau)\} = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$R_{xx}(\tau) = FT^{-1}\{\mathcal{S}_{xx}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{xx}(\omega)\, e^{+j\omega\tau}\, d\omega.$$

- Ex $X(t) = A_0 \cos(\omega_0 t + \theta)$

consider 2 cases.

Case I : $\theta \sim u(0, \pi/2) \rightarrow \frac{A_0^2 \pi}{2}\delta(\omega-\omega_0) + \frac{A_0^2 \pi}{2}\delta(\omega+\omega_0)$

Case II : $\theta \sim u(0, 2\pi)$

find $\mathcal{S}_{xx}(\omega)$ using ⑥??

Sol Case II :-

$R_{xx}(t, t+\tau) = E[X(t) \cdot X(t+\tau)] = \text{---} = \frac{A_0^2}{2}\cos(\omega_0 \tau)$

$= R_{xx}(\tau).$

So; $\mathcal{S}_{xx}(\omega) = FT\{R_{xx}(\tau)\} = FT\{\frac{A_0^2}{2}\cos(\omega_0\tau)\}$

$= \frac{A_0^2}{2} FT\{\cos(\omega_0\tau)\}$ found from table.

$= \frac{A_0^2}{2}\left[\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)\right].$

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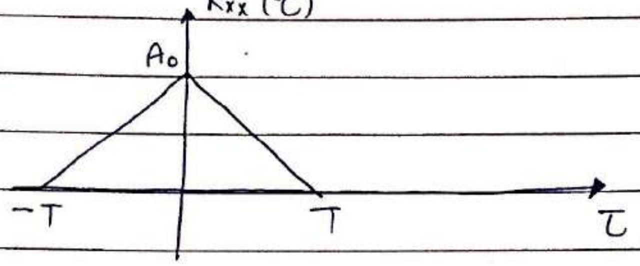
- Ex Given a WSS R.P $x(t)$ with



Find (a) P_{xx} (b) R.P mean (c) $\mathcal{S}_{xx}(\omega)$.

Sol (a) $P_{xx} = E[x^2(t)] = R_{xx}(0) = A_0$

(b) $m_x(t) = \overline{X} = \sqrt{\lim_{T\to\infty} R_{xx}(\tau)} = 0$.

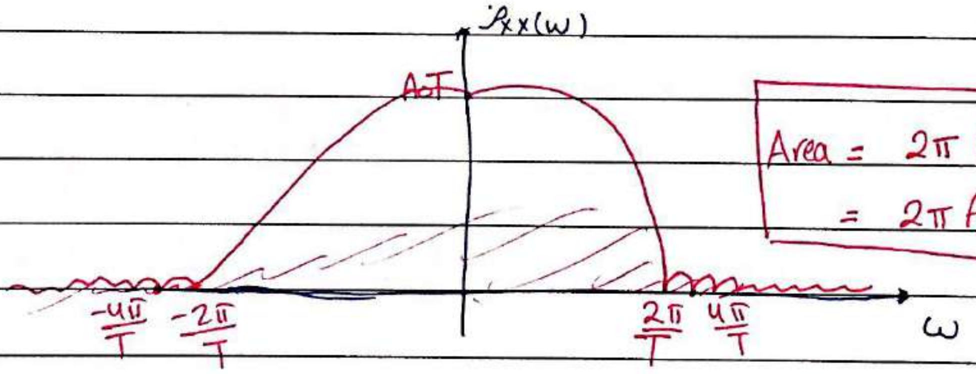
(c) $\mathcal{S}_{xx}(\omega) = FT\{R_{xx}(\tau)\}$

$= FT\{A_0\, tri(\frac{\tau}{T})\}$

$= A_0 \left[T\, \text{Sinc}^2(\frac{\omega T}{2})\right]$

- Note that $R_{xx}(\tau) = \begin{cases} A_0(1 - \frac{|\tau|}{T}) & , -T \le \tau \le T \\ 0 & , o.w \end{cases}$

$= A_0\, tri(\frac{\tau}{T})$



$$\boxed{\begin{aligned} \text{Area} &= 2\pi P_{xx} \\ &= 2\pi A_0. \end{aligned}}$$

$\text{Sinc}(\frac{\omega T}{2}) = \dfrac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}} = 0$

when $\sin(\frac{\omega T}{2}) = 0$

When $\frac{\omega T}{2} = \pm n\pi$

$$\boxed{\omega_{nulls} = \pm \frac{2n\pi}{T}}$$

- Ex Given two WSS R.Ps $X_1(t)$ & $X_2(t)$ with.

$$R_{X_1X_1}(\tau) = a\, e^{-\beta_1|\tau|}$$

$$R_{X_2X_2}(\tau) = a\, e^{-\beta_2|\tau|}, \quad \beta_2 > \beta_1$$

Find
ⓐ $P_{X_1X_1}$ and $P_{X_2X_2}$
ⓑ $S_{X_1X_1}(\omega)$ and $S_{X_2X_2}(\omega)$
ⓒ Determine which one has higher frequency components.

Sol ⓐ $P_{X_1X_1} = R_{X_1X_1}(0) = a$

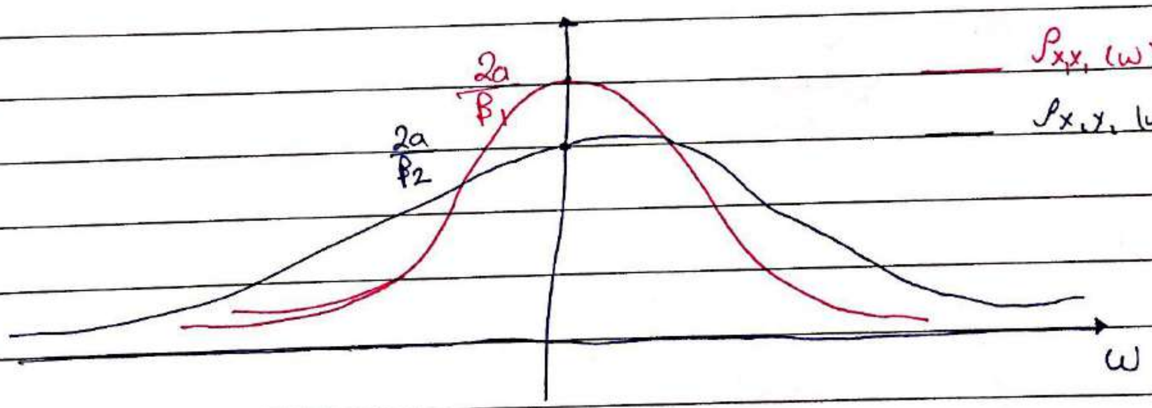
$P_{X_2X_2} = R_{X_2X_2}(0) = a$

both processes have same amount of total avg. power.

ⓑ $S_{X_1X_1}(\omega) = FT\{R_{X_1X_1}(\tau)\} = FT\{a\, e^{-\beta_1|\tau|}\}$ ← from table.

$$= \frac{2a\beta_1}{\beta_1^2 + \omega^2}$$

$$S_{X_2X_2}(\omega) = \frac{2a\beta_2}{\beta_2^2 + \omega^2}$$



$X_2(t)$ has higher frequency component . تغيرات سريعة

$X_2(t)$ $X_1(t)$

check: $R_{X_1X_1}(\tau) > R_{X_2X_2}(\tau)$

Five Appl

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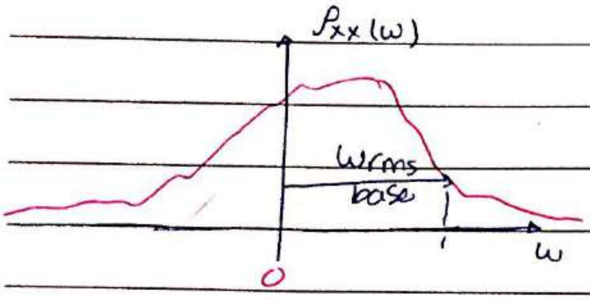
* R.P Bandwidth:

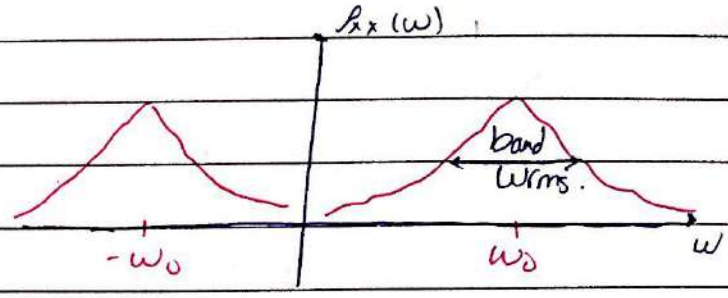
- R.P can be classified as.

1. Baseband process
   - It's frequency components are clustered about $\omega = 0$.



- Voice signal
- Intellegance data (message) in communication system

2. Bandpass process.
   - It's frequency components are clustered about $\omega = \omega_0$



* Root-mean-Squared Bandwidth (Wrms)

Determine Wrms base:-

$$W_{rms} = \sqrt{\int \omega \, S_{xx}^{norm}(\omega) \, d\omega} \quad \text{— (1)}$$

$$\text{where } S_{xx}^{norm}(\omega) = \frac{S_{xx}(\omega)}{\int_{-\infty}^{\infty} S_{xx}(\omega) \, d\omega} \quad \text{— (2)}$$

Sub (1) in (2)

$$W_{rms\ base} = \sqrt{\frac{\int_{-\infty}^{\infty} \omega \, S_{xx}(\omega) \, d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega) \, d\omega}}$$

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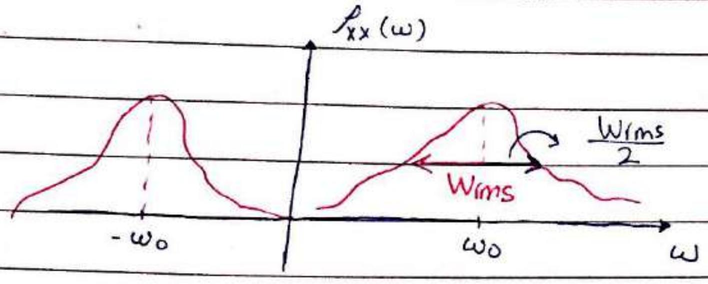
12th . Dec

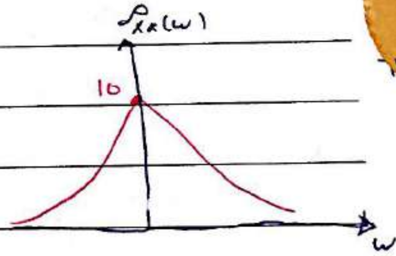
- For Bandpass process:-



$$\frac{W_{rms}}{2} = \sqrt{\int_{-\infty}^{\infty} (\omega-\omega_0)^2 S_{xx}^{norm}(\omega)\,d\omega}$$

$$W_{rms\ band} = 2\sqrt{\frac{\int_{-\infty}^{\infty} (\omega-\omega_0)^2 S_{xx}(\omega)\,d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega}}$$

- Ex 7.1-3: Given $S_{xx}(\omega) = \frac{10}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2}$, Find W_{rms}?



Sol Base band process.

$$\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega = \int_{-\infty}^{\infty} \frac{10}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2}\,d\omega$$

Using C-28 appendix "C"

$$= 10^5 \int_{-\infty}^{\infty} \frac{1}{[100+\omega^2]^2}\,d\omega = \boxed{50\pi}$$

برتب المعادلة حسب الشكل العام.

$$\int_{-\infty}^{\infty} \omega\, S_{xx}(\omega)\,d\omega = \int_{-\infty}^{\infty} \frac{10\,\omega}{\left[1+\left(\frac{\omega}{10}\right)^2\right]^2}\,d\omega = 10^5 \int \frac{\omega^2}{100+\omega^2}\,d\omega = \boxed{5000\pi}$$

So; $W_{rms} = \sqrt{\frac{5000\pi}{50\pi}} = \boxed{10 \text{ rad/sec}}$

Five App

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Cross- PDS & Cross powers:-

R.P $X(t) \rightarrow \mathcal{S}_{XX}(\omega) = \lim_{T\to\infty} \dfrac{E\left[|X_T(\omega)|^2\right]}{2T}$ $\quad \rightarrow X_T(\omega) \cdot X_T^*(\omega)$

R.P $Y(t) \rightarrow \mathcal{S}_{YY}(\omega) = \lim_{T\to\infty} \dfrac{E\left[|Y_T(\omega)|^2\right]}{2T}$ $\quad Y_T(\omega) \cdot Y_T^*(\omega)$

- Define:

$$\mathcal{S}_{XY}(\omega) = \lim_{T\to\infty} \frac{E\left[X_T^*(\omega)\, Y_T(\omega)\right]}{2T}$$

$$\mathcal{S}_{YX}(\omega) = \lim_{T\to\infty} \frac{E\left[Y_T^*(\omega)\, X_T(\omega)\right]}{2T}$$

$\rightarrow$ Cross-PDS.

- $$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XX}(\omega)\, d\omega$$

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{YY}(\omega)\, d\omega$$

$$P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{XY}(\omega)\, d\omega$$

$$P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_{YX}(\omega)\, d\omega$$

$\rightarrow$ Cross-power.

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- Notes:

$$\mathcal{S}^{*}_{xy}(\omega) = \mathcal{S}_{yx}(\omega)$$

$$P^{*}_{xy}(\omega) = P_{yx}(\omega)$$

* Sum of two - processes :-

- Let $W(t) = X(t) \pm Y(t)$

Determine :-

① $R_{ww}(t, t+\tau)$

② $\mathcal{S}_{ww}(t, t+\tau)$

③ P_{ww}

[1] $R_{ww}(t, t+\tau) = E\left[W(t)\ \omega(t+\tau)\right]$

$$= E\left[\underbrace{(X(t)+Y(t))}_{\omega(t)}\ \underbrace{(X(t+\tau)+Y(t+\tau))}_{\omega(t+\tau)}\right]$$

$$= E\left[X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau)\right]$$

$$R_{ww}(t,t+\tau) = \underbrace{R_{xx}(t,t+\tau) + R_{yy}(t,t+\tau)}_{\text{Auto-correlation functions}} + \underbrace{R_{xy}(t,t+\tau) + R_{yx}(t,t+\tau)}_{\text{Cross-correlation functions.}}$$

* If $\boxed{X(t) \perp Y(t)}$: $R_{xy}(t, t+\tau) = R_{yx}(t, t+\tau) = 0$

then

$$R_{ww}(t, t+\tau) = R_{xx}(t, t+\tau) + R_{yy}(t, t+\tau).$$

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② $S_{ww}(\omega) = FT\left\{A\left[R_{ww}(t,t+\tau)\right]\right\}$

$= FT\left\{A\left[R_{xx}(t,t+\tau) + R_{yy}(t,t+\tau) + R_{xy}(t,t+\tau) + R_{yx}(t,t+\tau)\right]\right\}$

$= FT\left\{A\left[R_{xx}(t,t+\tau)\right]\right\} + \text{-----} + FT\left\{A\left[R_{yx}(t,t+\tau)\right]\right\}$

$$S_{ww}(\omega) = S_{xx}(\omega) + S_{yy}(\omega) + S_{xy}(\omega) + S_{yx}(\omega)$$

* If $x(t) \perp y(t)$: $S_{xy}(\omega) = S_{yx}(\omega) = \text{zero}$.

then

$$S_{ww}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$$

③ $P_{ww} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{ww}(\omega)\, d\omega$

$$P_{ww} = P_{xx} + P_{yy} + P_{xy} + P_{yx}.$$

If $x(t) \perp y(t)$

then

$$P_{ww} = P_{xx} + P_{yy}$$

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* Cross - Power density properties :-

1) $S_{XY}(\omega) = S^{*}_{YX}(\omega) = S_{YX}(-\omega)$

2) $Re\{S_{XY}(\omega)\}$ and $Re\{S_{YX}(\omega)\}$ are even functions.

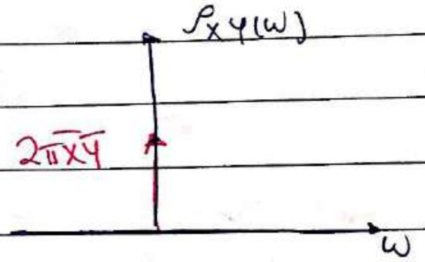
3) $Im\{S_{XY}(\omega)\}$ and $Im\{S_{YX}(\omega)\}$ are odd functions.

4) $S_{XY}(\omega) = 0$, $S_{YX}(\omega) = 0$ when $x(t)$ and $y(t)$ are orthogonal.

5) If $X(t)$ and $Y(t)$ are uncorrelated with means $\bar{X}$ and $\bar{Y}$, then:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t, t+\tau)]\, e^{-j\omega\tau}\, d\tau$$

$$S_{XY}(\omega) = FT\{\bar{X}\,\bar{Y}\} = 2\pi \bar{X}\,\bar{Y}\,\delta(\omega)$$



* If $X(t)$ & $Y(t)$ are joint WSS

$$R_{XX}(\tau) \xleftrightarrow{FT} S_{XX}(\omega)$$
$$R_{YY}(\tau) \xleftrightarrow{FT} S_{YY}(\omega)$$
$$R_{XY}(\tau) \xleftrightarrow{FT} S_{XY}(\omega)$$
$$R_{YX}(\tau) \xleftrightarrow{FT} S_{YX}(\omega)$$

Ex 7.3-1

Ex 7.4-1

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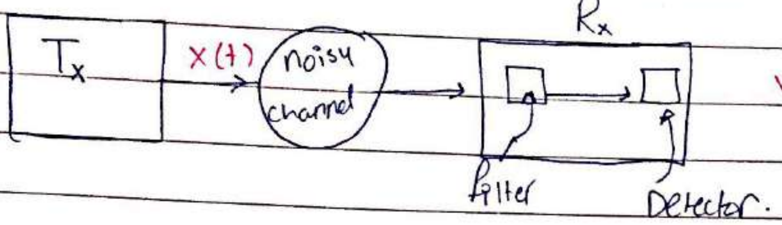
* Noise Process :-

- Communication system :-



$Y(t) = \tilde{x}(t) + n(t)$

↳ noise signal.

* $n(t)$: AWGN process.

A: Additive

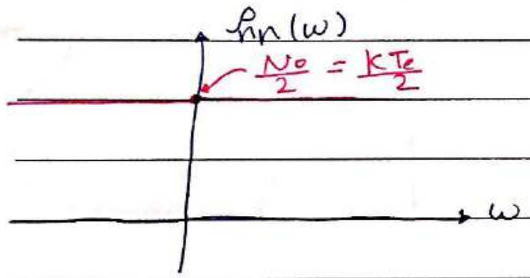
W: White " Kull el frequencies "

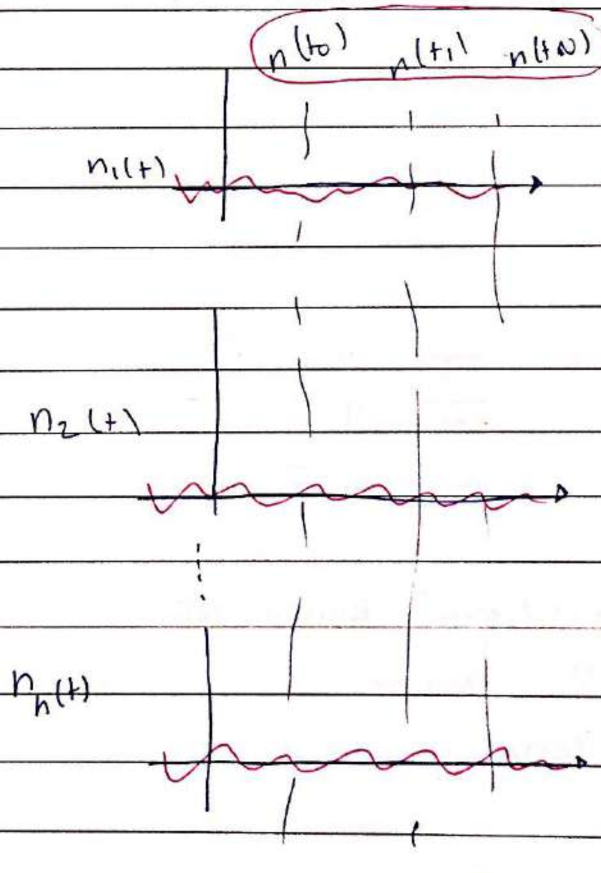
G: Gaussian " $n(t) \sim N(0, \sigma_n^2)$ "

N: Noise

* $n(t) \longrightarrow S_{nn}(\omega)$

$n(t)$ WSS





gaussian

Same $\bar{n} = 0$

Same σ_n^2.

$$S_{nn}(\omega) = \frac{N_0}{2} \quad \forall \omega.$$

* $R_{nn}(\tau) = FT^{-1}\{ S_{nn}(\omega) \} = \frac{N_o}{2} \delta(\tau)$

$P_{nn} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{nn}(\omega)\, d\omega \simeq \infty$

white noise is unrealistic.

* Practically, we deal with "Thermal noise"

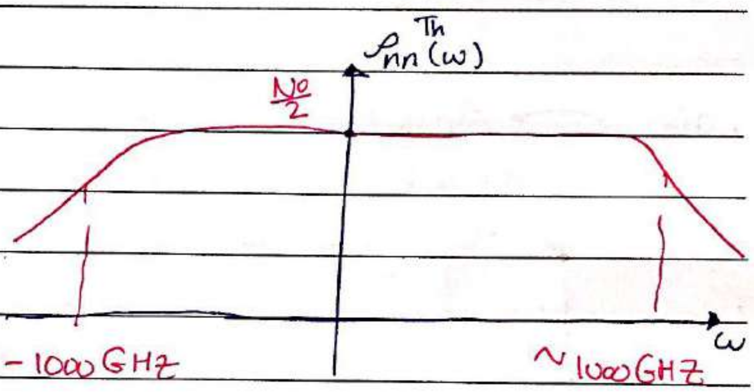
closest noise to represent white noise.

* Because of heat in the material

Random motion

Random current in

Random voltage $n(t)$



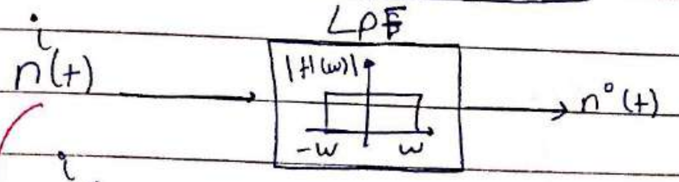
$$S^{Th}_{nn}(\omega) = \frac{\frac{N_o}{2}\, \alpha |\omega| / T}{e^{\alpha |\omega| T} - 1}$$

$\alpha = 7.64 * 10^{-12}$ kelvin-sec

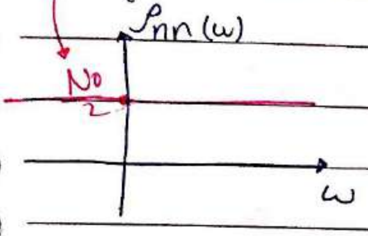
$T = 290°$ kelvin.

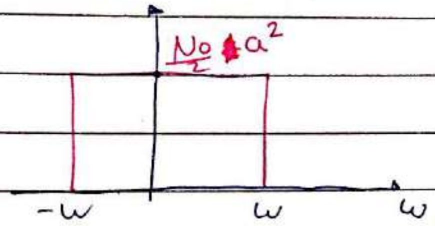
room temprature

* Band Limited Low-pass noise :-



$S^o_{nn}(\omega) = S^i_{nn}(\omega)\,|H(\omega)|^2$





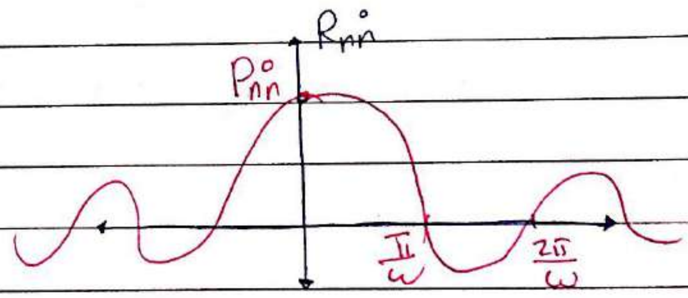
$$P^o_{nn} = \frac{1}{2\pi}\left[\cancel{2}W \cdot \frac{N_o}{\cancel{2}}\, a^2\right]$$

$$P^o_{nn} = \frac{W N_o a^2}{2\pi}$$

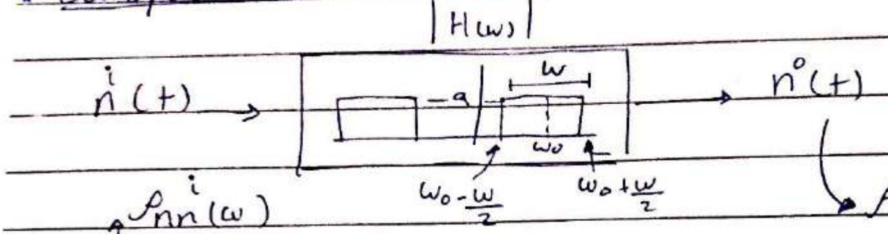
* $$S^o_{nn}(\omega) = \frac{N_o a^2}{2}\, \mathrm{rect}\left(\frac{\omega}{2W}\right)$$

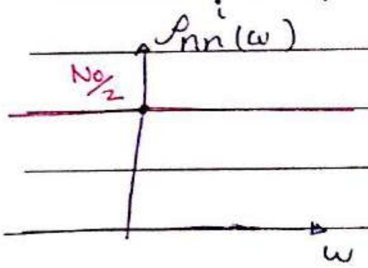
$$R^o_{nn}(\tau) = FT^{-1}\{S^o_{nn}(\omega)\}$$

$$= P^o_{nn}\, \mathrm{sinc}(W\tau)$$

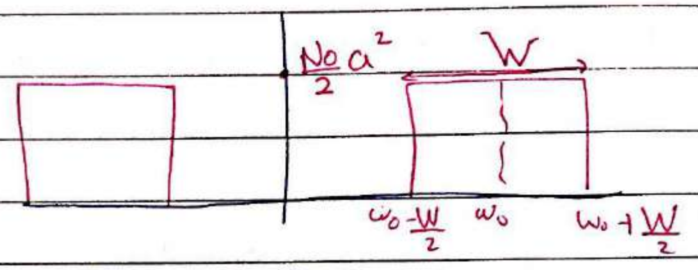


* Bandpass Noise





$$S_{nn}^o(\omega) = S_{nn}^i(\omega)\,|H(\omega)|^2$$



$$P_{nn}^o = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{nn}^o(\omega)\,d\omega = \frac{1}{2\pi}\left[W\,a^2\frac{N_0}{2}\right]\times 2$$

$$= \boxed{\frac{W a^2 N_0}{2\pi}}$$

$$R_{nn}^o(\tau) = FT^{-1}\{S_{nn}^o(\omega)\}$$

$$= P_{nn}^o \,\sin\left(\frac{W\tau}{2}\right)\cos(\omega_0\tau)$$

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