Probability & Random Variables

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Notebook

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Second Exam

Q1| For $F_Y(y) = 0.25u(y-3) + b\,u(y-5) + 0.35\,u(y-8)$
given b is a constant, find the Total Average Power ?

Q2| For Two R.V's X&Y with mean a_x, a_y & variances σ_X^2, σ_Y^2. If X&Y are statistically independent
Find $E[(XY)^2]$?

Q3| Given $X \sim N(0,1)$ & $Y = e^X$. Then find $f_Y(y)$?

Q4| Given $X \sim N(2,16)$, $Y \sim N(1,9)$, $C_{XY} = -0.5$
$V = X + bY$
$W = X - Y$
, given that V&W are orthogonal.
what is the value of b ?

Q5| Given $f_{X,Y}(x,y) = \frac{1}{3\pi} e^{-\frac{1}{2}\left[\frac{x^2}{4} + \frac{y^2}{9} - \frac{xy\sqrt{15}}{12}\right]}$, find the value of M_{11} ?
X & Y with Zero mean

Q6| Given the Covariance Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, Given $Z = 3X_1 + 5X_2$
find the value of AC power of Z ?

Q7| for Q6 given that $\bar{X} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ with $Z = 2X_2 + X_3$.
Find $E[Z + 0.5]$?

Q8| Given $X \sim \exp(1,3)$. find $f_X(x \mid X < 7)$?

Q9| Given $M_X(v) = e^{5v^2}$. find the Variance ?

Q10| Given $f_X(x) = 2\,x^{-4}\,u(x-1)$, find σ_X^2 ?

Q11| Given a R.V X with values -1 & 1 if $P\{x=-1\}=0.5$
find the characteristic function ?

Q12| Given $x_i \sim U(0,1)$ & $Y = \sum_{i=1}^{6} x_i$.
$f_Y(y) = \frac{1}{\sqrt{\pi\alpha^2}}\, e^{-\frac{(y-c)^2}{\alpha^2}}$. Find the values of c & α ?

Q13| Given $Y = x_K = K^2$; $K = 1, 2, 3, 4, 5$
with probabilities respectively: $0.1, 0.2, 0.15, 0.4, 0.15$
find the expectation of Y ?

Q14| Given $f_{X,Y}(x,y) = \frac{1}{2}\, y\, e^{-xy}$. Find $E[2e^{y}]$?
$1 < X < \infty$

Q15| for a 30 resistors if the probability of drawing a metal resistor $= 0.03$, what is the Average of this case of drawing a metal-resistor ?

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Solutions:

Q1| first we find (b) by: $0.25 + b + 0.35 = 1 \Rightarrow \boxed{b = 0.4}$

Now: Total average power represented by m_2.

$m_2 = \sum x_i^2\, P\{X_i\} \Rightarrow$ where $x_i = \{3, 5, 8\}$

$= (9)(0.25) + (25)(0.4) + (64)(0.35) = \boxed{34.65}$

Q2

$X \rightarrow a_x$ & σ_x^2 since ind. $E[XY] = E[X] \cdot E[Y]$

$Y \rightarrow a_y$ & σ_y^2

$\Rightarrow E[(XY)^2] = E[(XY) \cdot (XY)] = E[XY] \cdot E[XY]$

$= E[X] \cdot E[Y] \cdot E[X] \cdot E[Y]$

$= \boxed{a_x^2 \cdot a_y^2}$

Q3

$f_Y(y)$ is given by: $f_Y(y) = f_X(T^{-1}) \frac{dT^{-1}}{dy}$

$T = Y = e^x \Rightarrow T^{-1} = X = \ln y \rightarrow \frac{dT^{-1}}{dy} = \frac{1}{y}$

f_X is given by: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Rightarrow f_X(T^{-1}) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(\ln y)^2}{2}}$

so $f_Y(y) = \frac{1}{\sqrt{2\pi}\, y} e^{\frac{-(\ln y)^2}{2}}$

Q4

$C_{xy} = -0.5 \Rightarrow R_{xy} = C_{xy} + \bar{X}\bar{Y} = -0.5 + (2)(1) = 1.5$

since V & W orth. $R_{VW} = 0 = E[VW]$

$E[X^2] = 16 + 4 = 20$

$E[Y^2] = 9 + 1 = 10$

$E[(X + bY)(X - Y)] = E[X^2 + -XY + bXY - bY^2]$

$= 20 - (1.5) + b(1.5) - b(10) = 0$

$-8.5b + 18.5 = 0 \Rightarrow \boxed{b = 2.176}$

Q5

$\mu_{11} = C_{xy} = \rho_{xy} \sigma_x \sigma_y$

from the law of $f_{X,Y}(x,y)$ for 2 R.V's.

$\Rightarrow \sigma_x = 2$ & $\frac{2\rho \cancel{xy}}{(1-\rho^2)\sigma_x\sigma_y} = \cancel{xy} \frac{\sqrt{15}}{12}$

$\sigma_y = 3$

$\Rightarrow 2\rho = \frac{6\sqrt{15}}{12}(1-\rho^2) = 1.94 - 1.94\rho^2$

$1.94\rho^2 + 2\rho - 1.94 = 0$ solving: $\rho_1 = 0.6$ & $\rho_2 = -1.64$ (can't be)

so $\rho = 0.6$

$\Rightarrow \mu_{11} = (0.6)(2)(3) \Rightarrow \mu_{11} = 3.6$

Q6

AC power $= \sigma_Z^2 = 9\,\sigma_{x_1}^2 + 25\,\sigma_{x_2}^2 + 2(3)(5)\,C_{x_1x_2}$

$C_{x_1x_2} = 1$, $\sigma_{x_1}^2 = 1$, $\sigma_{x_2}^2 = 4$ $\Rightarrow \sigma_Z^2 = (9)(1) + (25)(4) + 30(1) = \boxed{139}$

Q7

$E[Z+0.5] = E[Z] + 0.5$

$E[Z] = 2\,\bar{X}_2 + \bar{X}_3 = (2)(3) + 5 = 11$

$\Rightarrow \boxed{E[Z+0.5] = 11.5}$

Q8

$X \sim \exp(1,3) \Rightarrow f_X(x) = \begin{cases} \frac{1}{3} e^{-\frac{(x-1)}{3}} & , x > 1 \\ 0 & , x < 1 \end{cases}$

$F_X(x) = \begin{cases} 1 - e^{-\frac{(x-1)}{3}} & , x > 1 \\ 0 & , x < 1 \end{cases}$

$f_X(x \mid X<7) = \frac{dF}{dx}$.

$F(x \mid X<7) = \begin{cases} \frac{F_X(x)}{F_X(7)} & , 1<x<7 \\ 1 & , x > 7 \\ 0 & , x < 1 \end{cases}$; $F_X(7) = \underline{\underline{1 - e^{-2}}}$

$\frac{F_X(x)}{1-e^{-2}} = \frac{1 - e^{-\frac{(x-1)}{3}}}{1-e^{-2}}$

$f_X(x \mid X<7) = \begin{cases} \frac{e^{-\frac{(x-1)}{3}}}{3(1-e^{-2})} & , 1<x<7 \\ 0 & , \text{o.w} \end{cases}$

Q9

$M_X(v) = e^{5v^2} \Rightarrow m_1 = \frac{dM}{dv}\Big|_{v=0} = e^{5v^2} \cdot 10v \Big|_{v=0} = \text{Zero}.$

$m_2 = \frac{d^2M}{dv^2}\Big|_{v=0} = 10e^{5v^2} + 10v \cdot e^{5v^2} \cdot 10v \Big|_{v=0} = \underline{\underline{10}}$

so $\sigma_X^2 = m_2 - m_1^2 = 10$

Q10

$\sigma_X^2 = m_2 - m_1^2$

$m_1 = \int_{-\infty}^{\infty} x \cdot 2x^{-4} u(x-1)\,dx$

$= 2\int_1^{\infty} x^{-3}\,dx = 2\frac{x^{-2}}{-2}\Big|_1^{\infty} = \left(\frac{1}{1} - \frac{1}{\infty}\right) = \boxed{1}$

$m_2 = \int_{-\infty}^{\infty} x^2 \cdot 2x^{-4} u(x-1)\,dx = 2\int_1^{\infty} x^{-2}\,dx$

$= 2\,\frac{x^{-1}}{-1}\Big|_1^{\infty} = 2\left(\frac{1}{1} - \frac{1}{\infty}\right) = \boxed{2}$

$\sigma_X^2 = 2 - 1 = \boxed{1}$

Q11 $f_X(x) = 0.5\,\delta(x-1) + 0.5\,\delta(x+1)$

$$\phi_X(w) = \int_{-\infty}^{\infty} e^{jwx} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{2} e^{jwx}\,\delta(x-1)\,dx + \int_{-\infty}^{\infty} \frac{1}{2} e^{jwx}\,\delta(x+1)\,dx.$$

$$= \frac{1}{2}e^{jw} + \frac{1}{2}e^{-jw} = \frac{e^{jw}+e^{-jw}}{2} = \boxed{\cos(w)}$$

Q12 $Y = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$; $U(0,1) \Rightarrow \bar{X} = \frac{1}{2}$

$\bar{Y} = \bar{X}_1 + \dots + \bar{X}_6 = \frac{1}{2} + \dots + \frac{1}{2} = 6 * \frac{1}{2} = 3$; $\sigma_X^2 = \frac{(1-0)^2}{12} = \frac{1}{12}$

$\sigma_Y^2 = \sigma_{X_1}^2 + \dots + \sigma_{X_6}^2 = \frac{1}{12} * 6 = \frac{1}{2}$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\cdot\frac{1}{2}}}\, e^{\frac{-(y-3)^2}{2\cdot\frac{1}{2}}} \equiv \frac{1}{\sqrt{\pi\alpha^2}}\, e^{\frac{-(y-c)^2}{\alpha^2}}$$

$\boxed{c = 3}$

$\boxed{\alpha = 1}$

Q13 $Y = X_K = K^2 = \{1, 4, 9, 16, 25\}$

$E[Y] = (1)(0.1) + (4)(0.2) + (9)(0.15) + (16)(0.4) + (25)(0.15)$

$= \boxed{12.4}$.

Q14 $E[2e^y] \Rightarrow$ let $g(y) = 2e^y$, $f_{X,Y}(x,y) = \frac{1}{2} y\, e^{-xy}$

$$E[g(y)] = \int_{-\infty}^{\infty} g(y)\, f_Y(y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_1^{\infty} \frac{1}{2} y e^{-yx}\,dx = \frac{1}{2}\cancel{y}\, \frac{e^{-yx}}{\cancel{-y}}\Big|_{\infty}^{1} = \frac{1}{2}(e^{-y} - 0) = \frac{1}{2}e^{-y}$$

$$E[g(y)] = \int_{-\infty}^{\infty} \cancel{\tfrac{1}{2}} e^{-y} \cdot \cancel{2} e^{y} \cdot dy = \boxed{1}$$

Q15 Average for Binomial $= NP = (30)(0.03) = \boxed{0.9}$

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Good Luck.