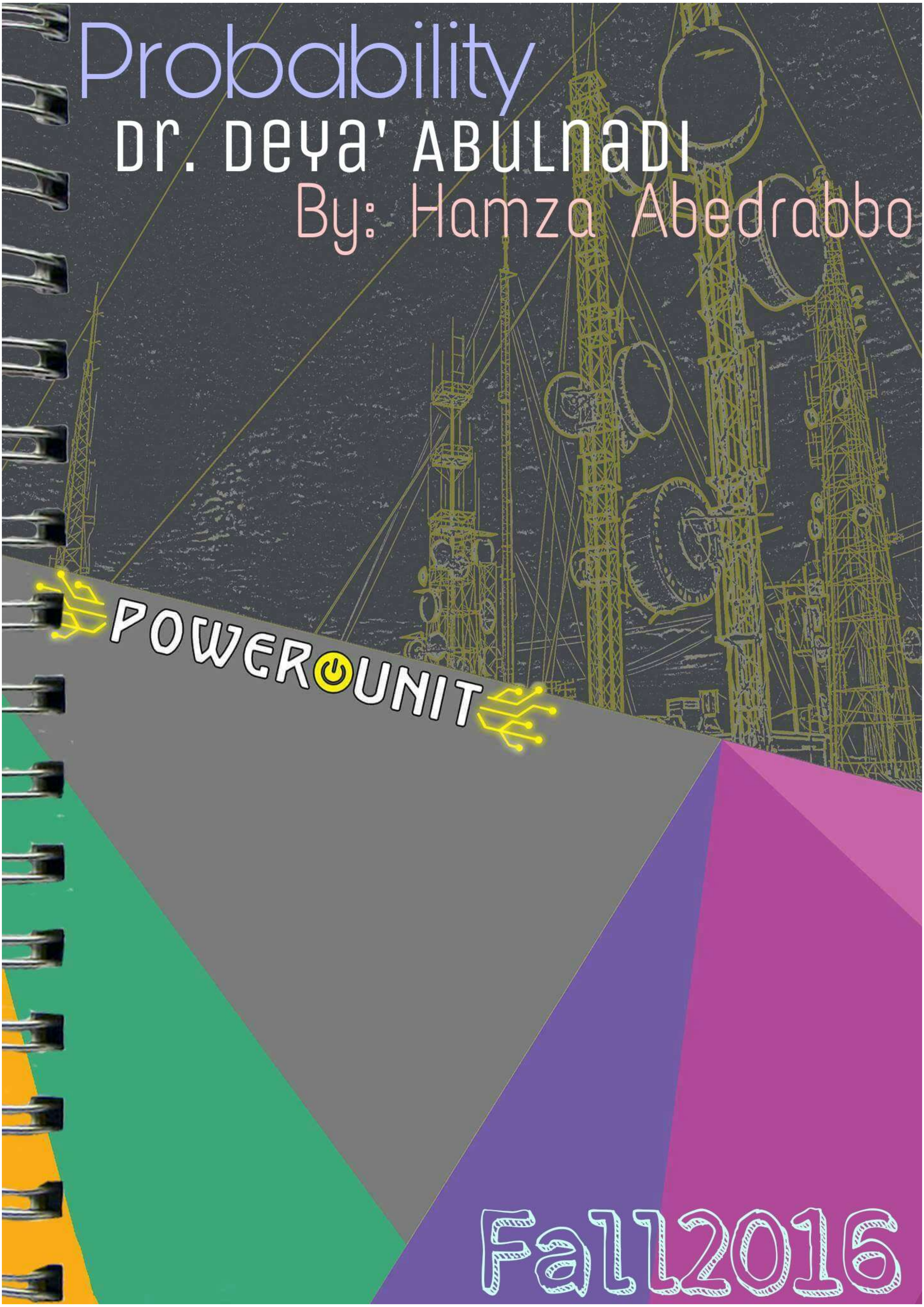

Probability
Dr. Deya' Abulnadi
By: Hamza Abedrabbo
POWERUNIT
Fall2016

8/9

No. Probability

chapter 1

Set theory

A: a set

a is an element in a set A

$a \in A$

belong to

$A = \{1, 2, 3\}$

A: is an integer $1 \leq n \leq 3$

$1 \in A$, $2 \in A$

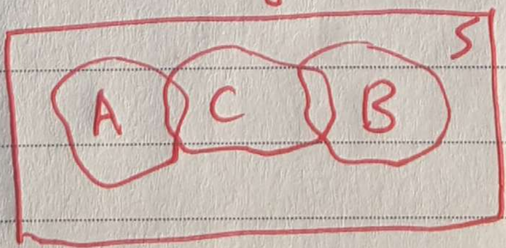
A set, B is a ~~[illegible]~~ subnet of A

$B \subset A$

contained

Set operations:

Venn diagram:



S: the Universal ~~[illegible]~~ set

Φ: the empty set

No.

Union and Intersection

Union of two sets: all the elements in the sets.

$$C = A \cup B$$

Intersection of sets: The common elements between the sets A & B.

~~D = A∩B~~ $D = \cancel{A\cap} A \cap B$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

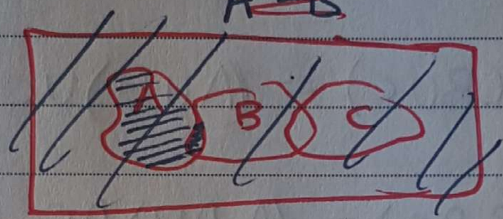
$$\bigcap_{i=1}^{N} A_i = A_1 \cap A_2 \cap \cdots \cap A_N$$

No.

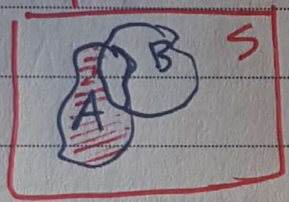
① The Difference of two sets

(A − B) : all the elements in A & not in B.

$\cancel{A-B}$



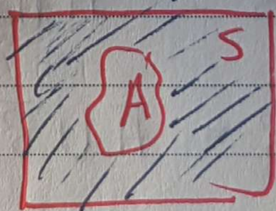
A − B



② complement

$\bar{A} = S - A$

$\bar{A}$



$\bar{A} \cup A = S$

Algebra of sets : sets, Union & intersections

I. commutative law :-

$A \cap B = B \cap A$

$A \cup B = B \cup A$

II. Distributive law :-

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

brackets ⇒ Distribute.

No.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

III. The Associative law

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$$

IV. Demorgan's law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Duality Principle:- every Identity has a dual.

- Replace union by Intersection & vice versa

- Replace S by Φ & Vice versa.

No.

ex

1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\Updownarrow$ Duality

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2) $\bar{A} \cup A = S$

$\Updownarrow$ Duality

$\bar{A} \cap A = \phi$

~~Probabilit~~

No.

Probability:

1. The Relative frequency definition of probability.

2. Axiometic definition of probability.

The experiment.

- The outcome of an experiment $\Rightarrow A$
- All possible outcomes of an experiment. $\Rightarrow S$ ~~: Sample Space~~
- The probability of the experiment.
$\Rightarrow P(A)$

*ex

$\rightarrow A = \{1\}$

$S = \{1, 2, 3, 4, 5, 6\}$

$P(A) = \frac{1}{6}$

$\rightarrow B = \{2 < b < 5\} = \{3, 4\}$

$P(B) = \frac{2}{6} = \frac{1}{3}$

A: The event

S: Sample Space

P(A): Probability of A

Scanned by CamScanner

No.

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n}$$

n_A: The number of occurance of A.

n: The number of all occurances.

2219

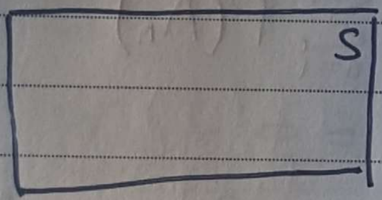
No.

~~A~~ Set $\underline{\underline{A}}$

Event A

$A = \{H\}$

S: universal Set



S: Sample Space

$S = \{H, T\}$

$S = \{1,2,3,4,5,6\}$

B: an integer between 3 & 6

$B = \{3,4,5,6\}$

Probability :-

1. Relative frequency definition

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

# of occaurances

$$P(H) = \lim_{n \to \infty} \frac{n_H}{n}$$

No.

2. The Axiomatic Definition :-

a. $P(A) \geq 0$ (Axiom 1)

b. $P(S) = 1$ (Axiom 2)

The certain event

c. $P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n)$ (Axiom 3)

if $A_m \cap A_n = \phi$

for all $m \neq n = 1, 2, \ldots, N$

$P(A \cup B) = P(A) + P(B)$

given that $A \cap B = \phi$ $A \neq B$

The impossible event

~~prove that~~

(*) $\boxed{P(\phi) = 0}$, $S \cup \phi = S$

$P(S \cup \phi)$

sol

by Axiom 3

$\cancel{P(S)} + P(\phi) = \cancel{P(S)}$

$P(\phi) = 0$

No.

$A \cup \bar{A} = S \rightarrow$ sample set

$A \cap \bar{A} = \phi \rightarrow$ empty set

$P(A \cup \bar{A}) = P(S) = 1$

$\cancel{P(A)} = P(A) + P(\bar{A}) = 1$

$$\boxed{P(\bar{A}) = 1 - P(A)}$$

Joint & Conditional Probability :-

intersection
A.B

Union
A+B

$P(A \cap B)$: The probability of joint event of A & B

$\cancel{P(A \cap B}$ $\boxed{P(A \cap B) = P(A) + P(B) - P(A \cup B)}$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$A = \{1, 2, 3\}$
$B = \{3, 4, 5\}$

$A \cap B = \{3\}$

$\cancel{A \cup B = \{1, 2, 4, 5\}}$

$A \cup B = \{1, 2, 3, 4, 5\}$

No.

~~Ex~~

The conditional probability :-

$P(B) > 0$

Then we define the conditional probability as $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

This means B already happened

→ Certain event

Then what's the probability of A happening.

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

FIVE APPLE

Scanned by CamScanner

25/9

No.

Ex: 100 resistors are found in a Box with the following values

Resistors (Ω)	Tolerance 5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Event A: draw a 47 Ω Resistor

Event B: draw a resistor with 5% tolerance

Event C: draw a 100 Ω resistor

$$P(A) = \cancel{P(47)}\ P(47\Omega) = \frac{44}{100}$$

$$P(B) = P(5\%) = \frac{62}{100}$$

$$P(C) = P(100\Omega) = \frac{32}{100}$$

FIVE APPLE

No.

The joint Probabilities:

* $P(A \cap B) = P(47\Omega \cap 5\%\ \text{tolerance})$

$= \frac{28}{100}$

* $P(A \cap C) = P(47\Omega \cap 100\Omega)$

$=$ zero

* $P(B \cap C) = P(5\%\ \text{tol} \cap 100\Omega)$

$= \frac{24}{100}$

The conditional Probabilities:

* $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{\frac{28}{100}}{\frac{62}{100}} = \frac{28}{62}$

* $P(A/C) = \frac{P(A \cap C)}{P(C)} =$ zero

* $\cancel{P(B/A) = \frac{24}{62}\ \frac{P(B \cap A)}{P(A)} = \frac{28}{47}}$

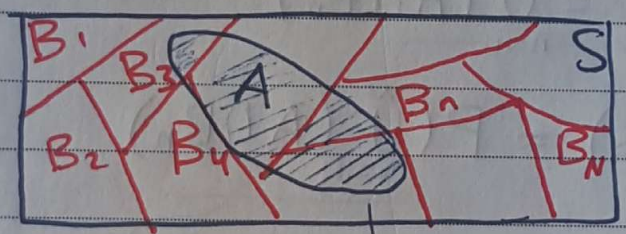
*

No.

* $P(B/C) = \dfrac{P(B \cap C)}{P(C)} = \dfrac{24}{32}$

*

Total Probability:



disjoint

~~B_n~~

$B_n : n = 1, 2, \ldots N$

such that:

$$\bigcup_{n=1}^{N} B_n = S \quad \text{and}$$

$$B_m \cap B_n = \Phi \quad \text{for } m \neq n = 1, 2, \ldots N$$

Prove that $P(A) = \sum_{n=1}^{N} P(A/B_n)\, P(B_n)$

$$P(A) = P(A/B_1)P(B_1) + \cancel{P(A/B_2 P} \; P(A/B_2)P(B_2) + P(A/B_3)P(B_3) + \ldots + P(A/B_N)P(B_N)$$

No.

Proof:

~~$A \cap S = A \cap$~~

$$A \cap S = A \cap \left[\bigcup_{n=1}^{N} B_n \right]$$

$$= \bigcup_{n=1}^{N} (A \cap B_n)$$

$$P(A \cap S) = P\left(\underline{\bigcup_{n=1}^{N} (A \cap B_n)} \right)$$

$A \cap B_n$ are disjoint

$$P(A) = P(A \cap S) = \sum_{n=1}^{N} P(A \cap B_n)$$

$$P(A/B_n) = \frac{P(A \cap B_n)}{P(B_n)}$$

$$\Rightarrow P(A) = \sum_{n=1}^{N} P(A \cap B_n) = \sum_{n=1}^{N} P(A/B_n)\,P(B_n)$$

Proven ✓

FIVE APPLE

Scanned by CamScanner

No.

Baye's theorem:

- $P(B_n/A) = \dfrac{P(B_n \cap A)}{P(A)}$

same

- $P(A/B_n) = \dfrac{P(A \cap B_n)}{P(B_n)}$

* $P(B_n/A) = \dfrac{P(A/B_n)\, P(B_n)}{P(A)}$

* $P(A) = P(A/B_1)\, P(B_1) + P(A/B_2)\, P(B_2) + \cdots P(A/B_N)\, P(B_N)$

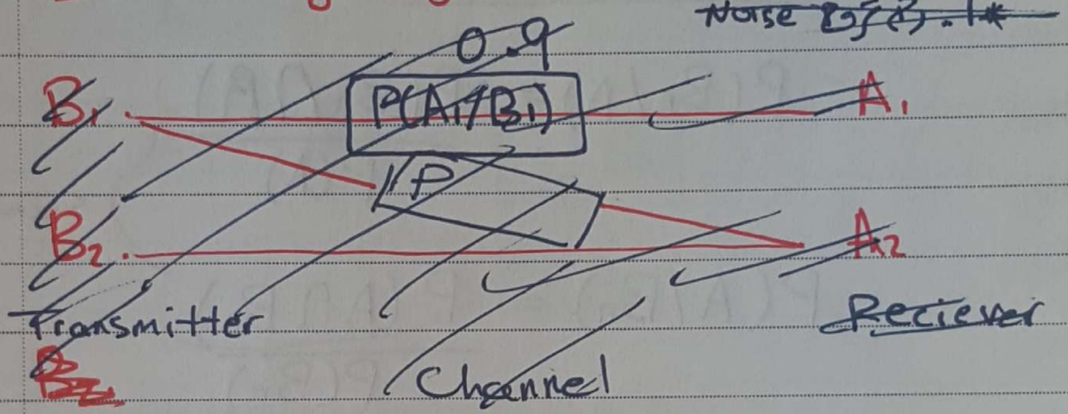
$$\boxed{P(B_n/A) = \frac{P(A/B_n)\, P(B_n)}{\sum_{n=1}^{N} P(A/B_n)\, P(B_n)}}$$

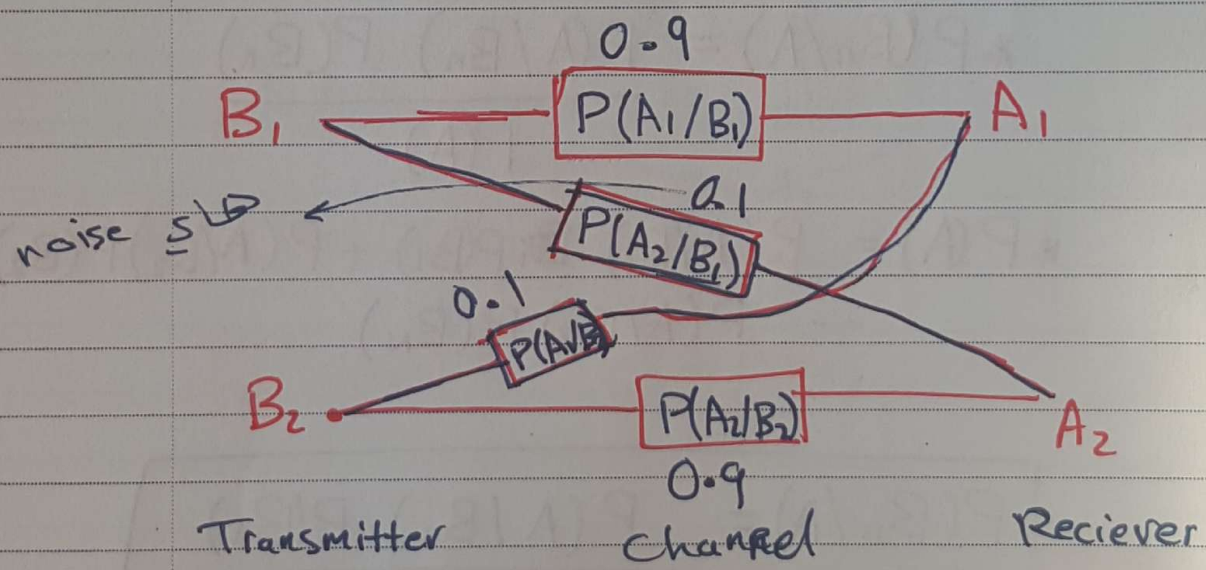
The Baye's theorem

ما تختصر لأنه في $\sum_{n=1}^{N}$ تحت وما في فوق

No.

Ex1: Binary Symmetric channel





$P(B_1) = .6$ → Given
$P(B_2) = .4$ →

another Ex

$P(B_1)=.6$ $B_1 — \boxed{P(A_1|B_1)} — A_1$ $P(A_1) = .6$

$P(B_2)=.4$ $B_2 — \boxed{P(A_2|B_2)} — A_2$ $P(A_2) = .4$

No.

Sol Ex1:

$* \ P(A_1) = P(A_1/B_1)\,P(B_1) \cancel{+ P(A_2/B_2)\,P(B_2)}$

$+ P(A_1/B_2)\,P(B_2)$

$= (.9)(.6) + (.1)(.4)$

$= .54 + .04$

$\cancel{P(A_1) = .4}$

$\boxed{P(A_1) = .58}$

$* \ P(A_2) = P(A_2/B_2)\,P(B_2)$

$+ P(A_2/B_1)\,P(B_1)$

$= (.9)(.4) + (.1)(.6)$

$\cancel{P(A_2)} = .36 + .06$

$\boxed{P(A_2) = .42}$

27/9

No.

Independent Events :

* Two events (Event A & Event B)

$$P(A/B) = P(A)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\boxed{P(A \cap B) = P(A)\ P(B)}$$

* Three events (A_1, A_2, A_3)

$$P(A_1 \cap A_2) = P(A_1)\ P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1)\ P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2)\ P(A_3)$$

$$\boxed{P(A_1 \cap A_2 \cap A_3) = P(A_1)\ P(A_2)\ P(A_3)}$$

No.

<u>Ex</u> A card is selected from a 52-card deck

define: Event A : select a King

Event B : Select a Jack or queen

Event C : select a heart

<u>Sol</u>

- $P(A) = \frac{4}{52}$

- $P(B) = \frac{8}{52}$

- $P(C) = \frac{13}{52}$

- $P(A \cap B) = \boxed{P(\phi) = 0}$

- $P(A \cap C) = \frac{1}{52}$

- $P(B \cap C) = \frac{2}{52}$

$P(A \cap B) = 0 \overset{?}{=} P(A)\,P(B)$

$= \frac{4}{52} \cdot \frac{8}{52}$

$= \frac{32}{(52)^2}$

A & B $\Rightarrow$ dependent

$P(A \cap C) = \frac{1}{52} \overset{?}{=} P(A)\,P(C)$

$= \frac{4}{52} \cdot \frac{13}{52}$

$= \frac{1}{52}$

A & C $\Rightarrow$ independent

* means that if i draw ~~choose~~ a king that will not affect it being a heart too.

$P(B \cap C) = \frac{2}{52} \overset{?}{=} P(B)\,P(C)$

$= \frac{8}{52} \cdot \frac{13}{52}$

$= \frac{2}{52}$

B & C $\Rightarrow$ independent

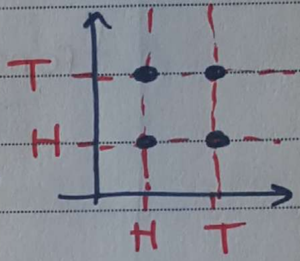
No.

Combined Experiment:

$S_1 = \{H, T\}$

$S_2 = \{H, T\}$

$S = \{HH, HT, TH, TT\}$



: Cartesian Product.

$S_1 = \{H, T\}$

$S_2 = \{1, 2, 3, 4, 5, 6\}$

$S = S_1 \times S_2$

$S = \{H1, H2, H3, H4, H5, H6\}$ ~~T1, T2, T3, T4, T~~

$\{, T1, T2, T3, T4, T5, T6\}$

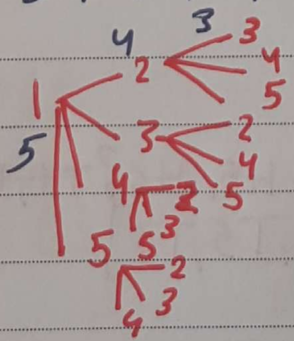
No.

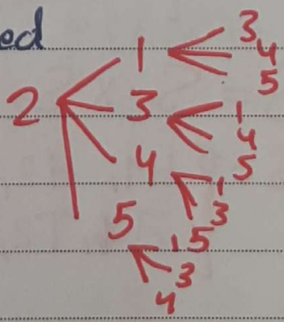
Permutations & Combinations :

الترتيب

$$\boxed{\text{Ordering of } r \text{ elements taken from } n = n(n-1)(n-2)\cdots\underbrace{(n-r+1)}_{n-(r-1)} = \boxed{\frac{n!}{(n-r)!} = P_r^n \quad r = 1, 2, \cdots, n}}$$

→ A plate number from only 3 digits [1, 2, 3, 4, 5], do not repeat a number are used





$5 \times 4 \times 3$

There are 60 combinations

No.

Combination:

* الترتيب غير مهم

$$\boxed{\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{P^n_r}{P^r_r}}$$

$\boxed{\cancel{0!=1}}$

* $0! = 1$

$\binom{n}{r}$ is called the binomial coefficient

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

* $$(x+y)^3 = \sum_{r=0}^{3} \binom{3}{r} x^r y^{3-r}$$

$$= \binom{3}{0} \underbrace{\cancel{x^0}}_{1} y^3 + \binom{3}{1} x\, y^2 + \cancel{\binom{3}{2}}$$

$$+ \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 \underbrace{\cancel{y^0}}_{1}$$

$$= y^3 + 3xy^2 + 3x^2y + x^3$$

$$\frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

Scanned by CamScanner

29/9

No.

Bernoulli Trials:

* Combined experiment :

Only two possible outcomes. A & $\bar{A}$

~~That~~

* The probability that A happened K times and of N trials.

$$\boxed{P(\underbrace{A\,A\,A \cdots A}_{K\text{ times}},\ \underbrace{\bar{A}\bar{A}\bar{A}\cdots\bar{A}}_{(N-K)\text{ times}}) = P^K(1-P)^{N-K}}$$

if $P(A)=p$ then $P(\bar{A})=1-p$

$$\boxed{P(\bar{A}) = 1 - P(A)}$$

$$\cancel{P(\underbrace{A\ A}_{k\text{ times}})} \Rightarrow P(\underbrace{A}_{k\text{ times}}\ \underbrace{\bar{A}}_{(N-k)\text{ times}}) = P(A)^K * (1-P(A))^{N-K}$$

~~P(A occurs K times out of N trials) = $\binom{N}{K} P^K (1-P)^{N-}$~~

$$\boxed{P(A\text{ occurs }K\text{ times out of }N\text{ times}) = \binom{N}{K} P^K (1-P)^{N-K}}$$

This is The Bernoulli Trials

FIVE APPLE

No.

Ex: A Submarine attempts to sink an aircraft carrier, It will succeed of two or more torpedes hit the carrier. ~~The~~ ~~Sub~~ The Submarine fires 3 torpedes and the probability of a hit is 0.4 for each torpedes

What is the probability that the Carrier will be sunk?

Sol:

$A = \{\text{torpede hits}\}$

$P(A) = .4$

$P(\bar{A}) = .6$

$\rightarrow P\{\text{exactly No hits}\} = \binom{3}{0}(.4)^0 * (.6)^3$

$= \frac{3!}{(3-0)!\,0!} * 1 * (.6)^3 = (.6)^3$

$\rightarrow P(\text{exactly 1 hit}\} = \binom{3}{1}(.4)^1(.6)^2$

$= \frac{3!}{(3-1)!\,1!} * (.4)^1(.6)^2 = 3 * .4 * (.6)^2$

No.

$$P(\text{Exactly 2 hits}) = \binom{3}{2}(.4)^2(.6)^1$$

$$= \frac{3!}{(3-2)!\,2!}(.4)^2(.6)^1$$

$$= \frac{3}{1}(.4)^2(.6)^1 = .288$$

$$P\{\text{Exactly 3 hits}\} = \binom{3}{3}(.4)^3(.6)^0$$

$$= \frac{3!}{0!\,3!}(.4)^3$$

$$= (.4)^3 = .064$$

$$P(\text{carrier will be sunk}) = P(2\text{hits}) + P(3\text{hits})$$

$$= .064 + .288 = .352$$

$$\boxed{\sum_{r=0}^{N}\binom{N}{r}p^r(1-p)^{N-r} = 1}$$

* In the previous example $N=3$

$$\boxed{(X+Y)^N = \sum_{r=0}^{N}\binom{N}{r}x^r y^{(N-r)}}$$

*See your textbook ~~ch 4 sec 2~~

Sterling Demovre laplace Not required

FIVE APPLES

Chapter 2

No.

The Random Variables

A function that maps the outcome of an experiment to the real numbers.

$S=\{H,T\}$ $P(H)=.4 \Rightarrow X(H)=5 \Rightarrow P(5)=0.4$

$P(T)=.6 \Rightarrow X(T)=-5 \Rightarrow P(-5)=0.6$

* We always represent Random Variable by upper case letters.

$X(S) \equiv$ a possible outcome of the experiment

* A discrete Random Variable, $\overset{R.V}{X}$ with two possible values $X=5$ & $X=-5$ with probabilities small

$P(X=5)=0.4$ $x \in X$

$P(X=-5)=0.6$ $y \in Y$

R.V X, Y, Z, W

elements $\in$ R.V x, y, z, w

4/10

No.

* $X(s)$ $\qquad$ $x = \{-5, 5\}$

$S = \{H, T\}$ $\qquad$ X, Y, Z, W, etc

$X(H) = 5$ $\qquad$ $x \in X$

$X(T) = -5$

S

*

(H,1)	(T,1)
(H,2)	(T,2)
(H,3)	(T,3)
(H,4)	(T,4)
(H,5)	(T,5)
(H,6)	(T,6)

$X(s)$

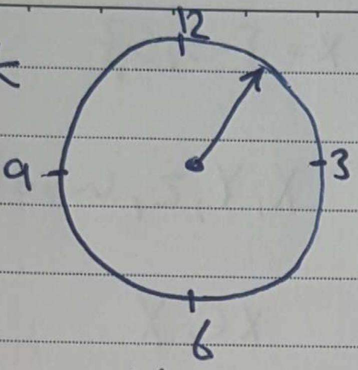
−12	−10	−8	−6	−4	−2	0	1	2	3	4	5	6
(H,6)	(H,5)	(H,4)	(H,3)	(H,2)	(H,1)		(T,1)	(T,2)	(T,3)	(T,4)	(T,5)	(T,6)

as if $H = -2$ & 1 are multiplied by H or T
$T = 1$ $\qquad$ 2, 3, 4, 5, 6

This is called Discrete Random Variables

$P(-12) = \frac{1}{12}$ $\qquad$ $P(6) = \frac{1}{12}$

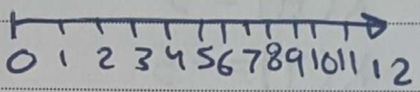
No.



$X(S)$

$S = \{0 < S \leq 12\}$

$\boxed{X = S}$



$X = \{0 < S \leq 12\}$

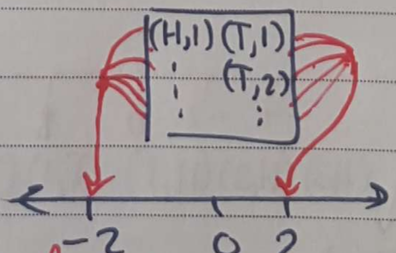
→ This is Continuos R.V

→ $P\{0 < X < 3\} = \frac{3}{12}$ $P\{X = 3\} = \frac{1}{\infty} = \text{zero}$

~~P{2 < X < 5} = 3~~

* Conditions for a function to have R.V

1. A function not a multivalued relation.



: many values of x to 1 value of y ✓

: " " " y to 1 value of x ✗

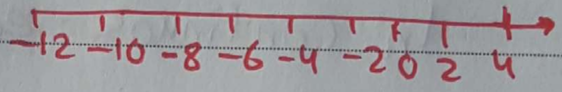
~~2. The set {X ≤ x} is an event.~~

FIVE APPLE

ch1
ex 1.3.1

No.

2. The set $\{X \leq x\}$ is an event.



$\{X \leq -3\} = \{-12, -10, -8, -6, -4\}$

$\{X \leq -20\} = \Phi$

$\{X \leq 4\} = S$

: Problem if not Real numbers

$S = \{R, G, B\}$

3. $P\{X = -\infty\} = 0$

$P\{X = \infty\} = 0$

* Take a look at ex 1.3.1 ch1

*The probability of a discrete event on a continuos R.V equals zero (clock example)
The equality in this does not matter.

FIVE APPLE

No.

In Discrete : equality matters

In continuos : equality does not matter

~~P{X~~

In cont. $\Rightarrow P\{0 < x \leq 3\} = P\{0 \leq x \leq 3\}$

In Discrete $\Rightarrow P\{x \leq 3\} \neq P\{x < 3\}$

Ex. $S = \{1, 2, 3, 4\}$

$X = S^3$

$P(1) = 3/24$ $\quad$ $P(3) = 7/24$

$P(2) = 4/24$ $\quad$ $P(4) = 10/24$

$X = \{1, 8, 27, 64\}$

$P\{x = 1\} = 3/24$

$P\{x = 8\} = 4/24$

$P\{x = 27\} = 7/24$

$P\{x = 64\} = 10/24$

FIVE APPLE

No.

The Distribution function:

The commulative probability distribution function

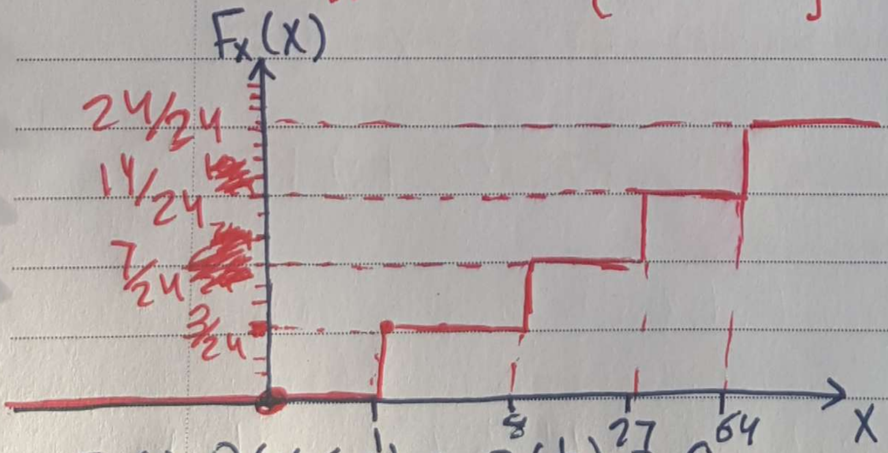
(CDF)

~~$F_X(x) = P\{X \leq x\}$~~

CDF $F_X(x) = P\{X \leq x\}$

$F_Y(y) = P\{Y \leq y\}$

$F_W(w) = P\{W \leq w\}$



$F_X(0) = P(X \leq 0) = P(\phi) \stackrel{?}{=} 0$

$F_X(.99) = P(X \leq .99) = P\phi = 0$

$F_X(1) = P(X \leq 1) = P((X<1) \cup (X=1))$

$= P(X<1) + P(X=1)$

$= 0 + 3/24 = 3/24$

No.

$F_X(5) = P\{X \leq 5\} = \cancel{P\{X \leq 5\} + P\{X \leq 5\}}$

$= P\{$

FIVE APPLE

6/10

No.

Prob

The Distribution function

Ex1 $F_X(x) = P\{X \leq x\}$

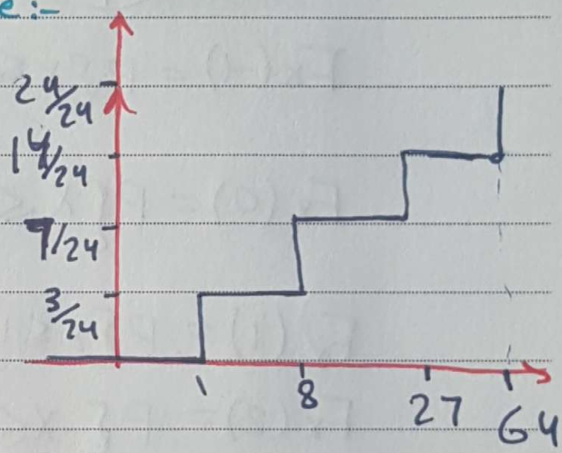
Discrete R.V Example:-

$P(X=1) = 3/24$

$P(X=8) = 4/24$

$P(X=27) = 7/24$

$P(X=64) = 10/24$



$F_X(-1) = P\{X \leq -1\} = 0$

$F_X(0) = P\{X \leq 0\} = 0$

$F_X(.99) = P\{X \leq .99\} = 0$

$F_X(1) = P\{X \leq 1\} = P\{(X<1) \cup (X=1)\} = 0 + \frac{3}{24} = \frac{3}{24}$

$F_X(7) = P\{X \leq 7\} = P\{(X<1) \cup (X=1) \cup (1<X \leq 7)\}$

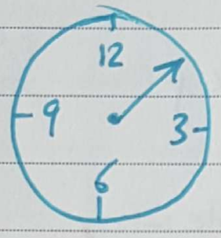
$= 0 + \frac{3}{24} + 0 = \frac{3}{24}$

FIVE APPLE

Scanned by CamScanner

No.

Ex2 Continuos R.V



$X = \{0 < x \leq 12\}$

$F_X(-1) = P\{x \leq -1\} = 0$

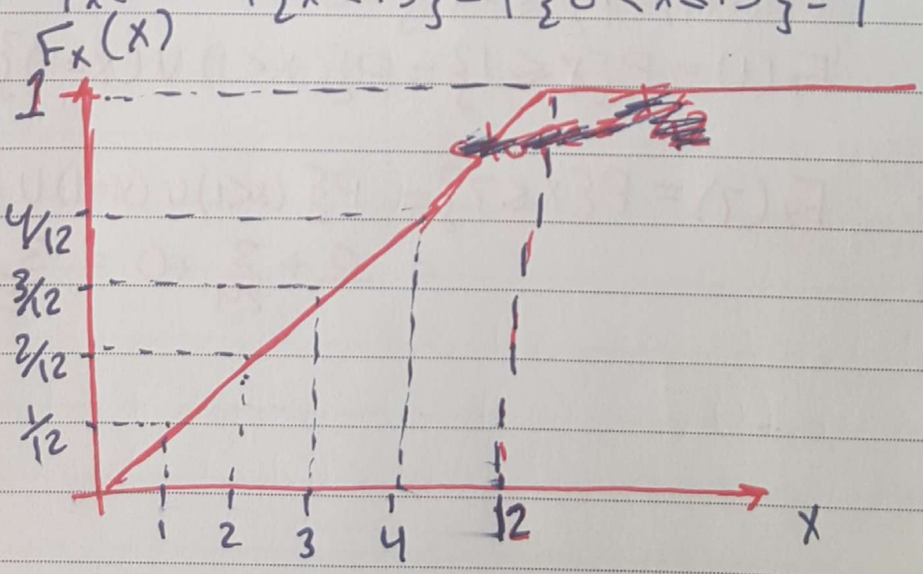
$F_X(0) = P\{X \leq 0\} = 0$

$F_X(1) = P\{x \leq 1\} = P\{0 < x \leq 1\} = \frac{1-0}{12-0} = \frac{1}{12}$

$F_X(2) = P\{x \leq 2\} = P\{0 < x \leq 2\} = \frac{2}{12}$

$F_X(12) = P\{x \leq 12\} = P\{0 < x \leq 12\} = 1 = 12/12$

$F_X(15)$ $P\{x \leq 15\} = P\{0 < x \leq 15\} = 1 = 12/12$



No.

Properties of Distribution function:

1) $F_X(-\infty) = 0$

$F_X(-\infty) = P\{X = -\infty\} = 0$

~~$F_X(\infty) = P\{...\}$~~

2) $F_X(\infty) = 1$

$F_X(\infty) = P\{X \leq \infty\} = 1$

~~3) $F_X(x) =$~~

3) $0 \leq F_X(x) \leq 1$

4) $F_X(x_1) \leq F_X(x_2)$, if $x_1 < x_2$

$F_X(x)$ is a non decreasing function

5) $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

for Ex1 Discrete RV

$P\{8 < X \leq 64\} = F_X(64) - F_X(8)$

$= 1 - 7/24 = 17/24$

$P\{8 < X < 64\} = 14/24 - 7/24 = 7/24$

*هاد بسبب المساواة عند الطرف الأكبر

FIVE APPLE

Prob

Scanned by CamScanner

No.

for Ex2 Continuos RV

$P\{2<X\leq 10\} = F_X(10) - F_X(2)$
$= 10/12 - 2/12 = 8/12$

$P\{2<x<10\} = F_X(10) - F_X(2)$
$= 10/12 - 2/12 = 8/12$

6) $F_X(x^+) = F_X(x)$

* Always continuos from the right
not necessarily from the left.

for Ex1

$F_X(1^-) \neq F_X(1)$

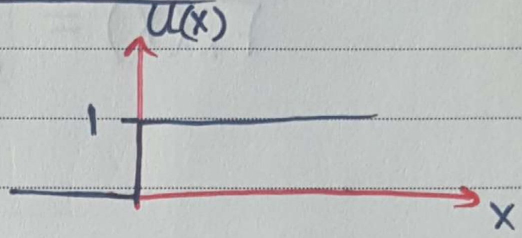
$F_X(1) = F_X(1^+)$

* Properties 1,2,4,6 are used to check if a Certain $G_X(x)$ is a Valid distribution function or not.

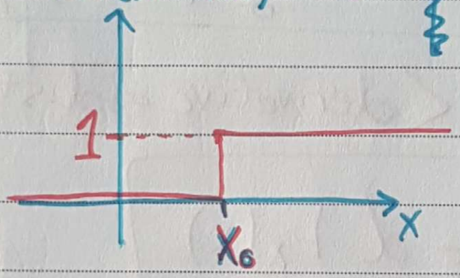
No.

The Unit step function:

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

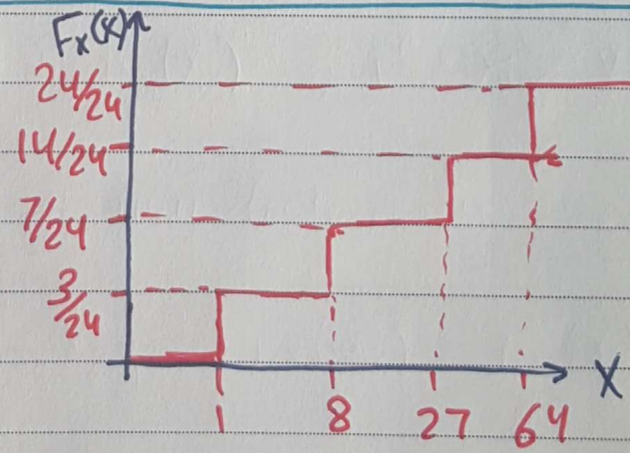


$u(x - x_0)$



$$u(x - x_0) = \begin{cases} 1, & x \geq x_0 \\ 0, & x < x_0 \end{cases}$$

For Ex1



$$F_x(x) = \frac{3}{24} u(x-1) + \frac{4}{24} u(x-8)$$

$$+ \frac{7}{24} u(x-27) + \frac{10}{24} u(x-64)$$

No.

$$F_X(x) = \sum_{i=1}^{N} P\{X = x_i\}\, u(x - x_i)$$

The Probability Desity function

(PDF) <derivative of distribution function>

$$\boxed{f_X(x) = \frac{d\,F_X(x)}{dx}}$$

FIVE APPLE

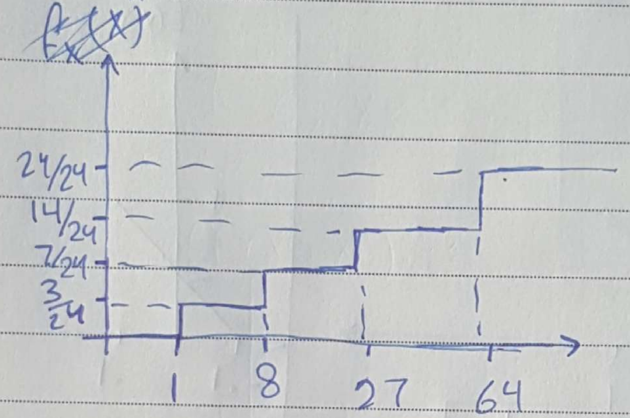
Scanned by CamScanner

9/10

No.

The probability density function PDF

$\rightarrow f_X(x)$



$$F_X(x) = \frac{3}{24} u(x-1) + \frac{4}{24} u(x-8) + \frac{7}{24} u(x-27) + \frac{10}{24} u(x-64)$$

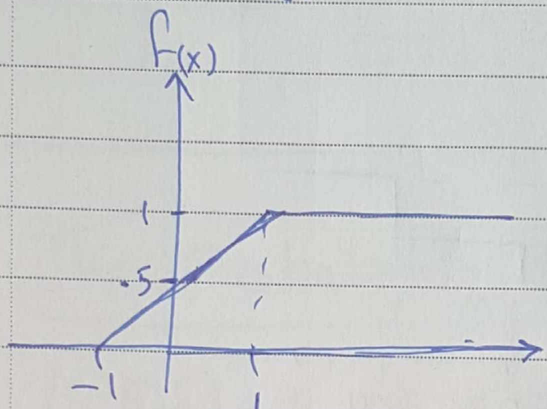
$$F_X(10) = \frac{3}{24} \overset{1}{u(9)} + \frac{4}{24} \overset{1}{u(2)} + \cancel{\frac{7}{24} \overset{0}{u(-17)}} + \cancel{\frac{10}{24} \overset{0}{u(-54)}}$$

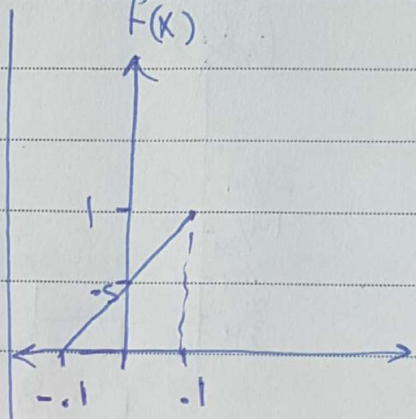
$$u(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

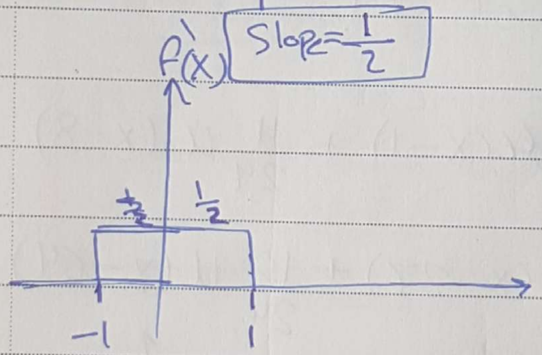
$$\boxed{F_X(10) = \frac{7}{24}}$$

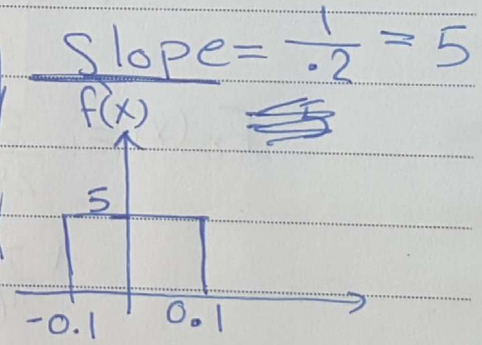
No.

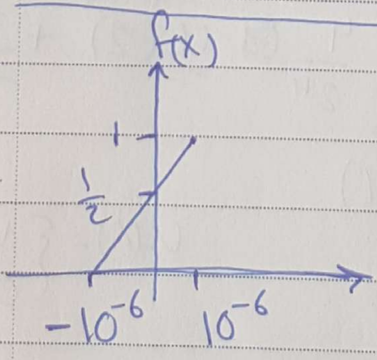
* Derivative of a function Graphically



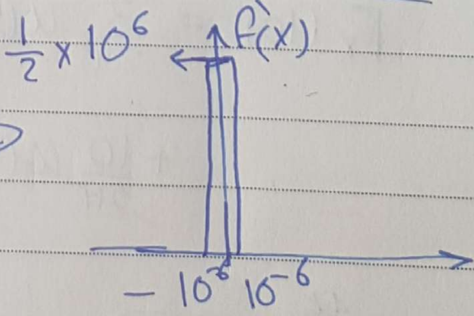


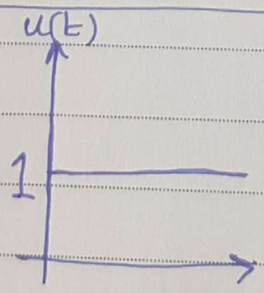




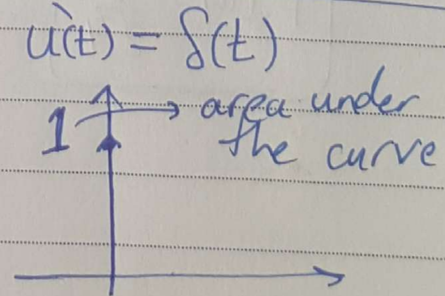


$\Rightarrow$





$\Rightarrow$

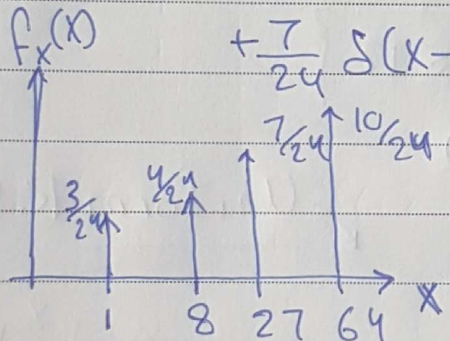


* Derivative of a step function $\Rightarrow$ Delta function (impulse)

No.

$$F_X(x) = \frac{3}{24}u(x-1) + \frac{4}{24}u(x-8) + \frac{7}{24}u(x-27) + \frac{10}{24}u(x-64)$$

$$f_X(x) = \frac{3}{24}\delta(x-1) + \frac{4}{24}\delta(x-8) + \frac{7}{24}\delta(x-27) + \frac{10}{24}\delta(x-64)$$

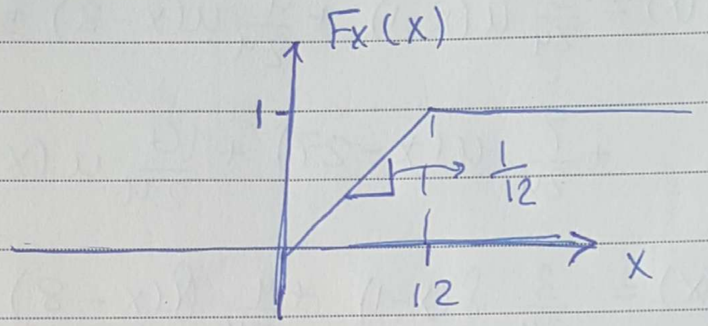


$$* \quad \boxed{f_X(x) = \sum_{i=1}^{N} P\{X = x_i\}\,\delta(x - x_i)}$$

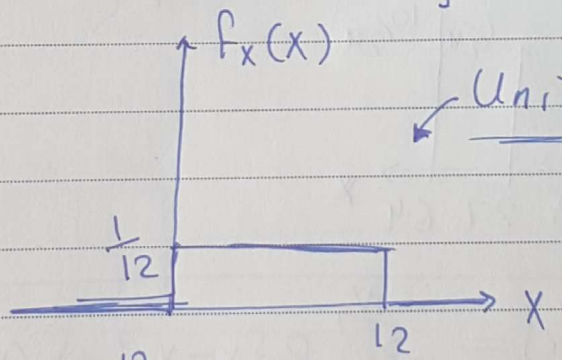
$$\int_0^\infty f_X(x)\,dx = \frac{3}{24} + \frac{4}{24} + \frac{7}{24} + \frac{10}{24} = 1$$

No.

Continuos Case:



$X = \{0 \leq x \leq 12\}$



Uniform R.V

لأن كل القيم X لها نفس

ال R.V

$f_X(x)$

$\int_0^{12} \frac{1}{12} dx = 1$

*Properties of PDF:

1. $f_X(x) \geq 0$, for all x

2. $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

3. $\int_{-\infty}^{x} f_X(\xi)\, d\xi = F_X(x)$

dummy Variable

FIVE APPLE

$$f_X(x) = \frac{d\,F_X(x)}{dx}$$

$$\int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} d\,F_X(x)$$

$$\int_{-\infty}^{x} f_X(\xi)\,d\xi = F_X(x) - \cancel{F_X(-\infty)}^{\,0}$$

4. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)\,dx = F_X(x_2) - F_X(x_1)$

~~properties 1&2 ar~~

*Properties 1&2 are used to check if a function $g_X(x)$ is a valid PDF or not.

Scanned by CamScanner

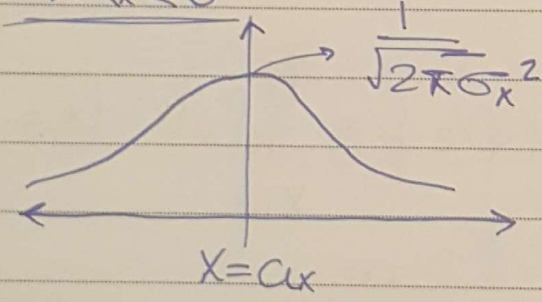
The Gaussian Random Variable

↕

The Normal R.V

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \, e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}$$

$-\infty < x < \infty$



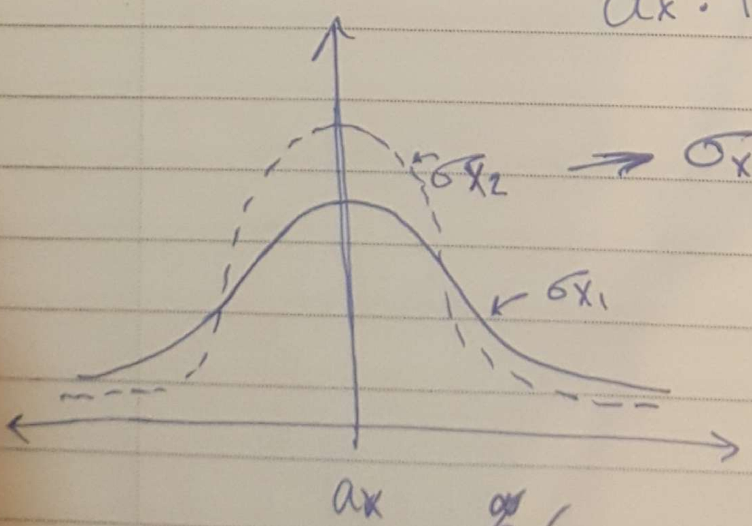
$\sigma_x \equiv$ constant

$a_x \equiv$ constant

σ_x : Standard deviation

σ_x^2 : The variance

a_x : The mean



$\rightarrow \sigma_{x_2} < \sigma_{x_1}$

$\cancel{F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}}} \quad \cancel{e^{-(x-\sigma_x)}}$

No.

$$F_x(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(\xi - a_x)^2}{2\sigma_x^2}} \cdot d\xi$$

~~[illegible]~~ 11/10

*Standard Gaussian ($a_x = 0$ and $\sigma_x^2 = 1$)
unit variance

Prototype Gaussian:

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-\xi^2}{2}} d\xi$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}, \quad -\infty < x < \infty$$

$$F_x(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-(\xi - a_x)^2/2\sigma_x^2} \cdot d\xi$$

let $u = \frac{\xi - a_x}{\sigma_x} \Rightarrow du = \frac{1}{\sigma_x} d\xi$

$$\boxed{d\xi = \sigma_x du}$$

FIVE APPLE

Scanned by CamScanner

No.

$$F_X(x) = \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = F\left(\frac{x-a_x}{\sigma_x}\right)$$

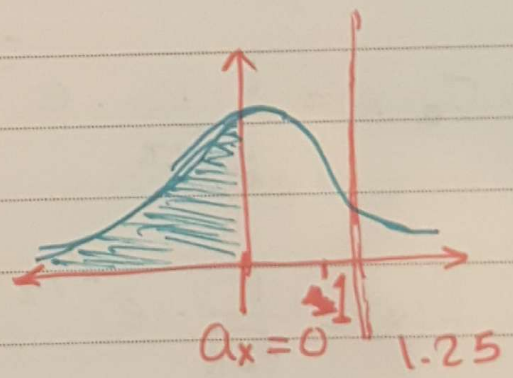
Table B.1 (418)

Values of F(x) for $0 \leq x \leq 3.89$ in steps of 0.01

$$F(0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$\boxed{\sigma_x = 1}$

$\boxed{a_x = 0}$



x	.00	.01	.02	.03	.04	.05	...
0.0							
0.1							
0.2							
⋮							
0.9							
1.0							
1.1							
1.2							
1.25							
⋮							

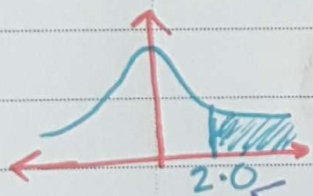
$0.8944 = F(1.25)$

Ex : The height of clouds above ground at some location is a Gaussian R.V (X) with $a_x = 1830\,m$ and $\sigma_x = 460\,m$

① Find the probability that clouds will be higher than 2750m

Sol :

$$P\{X > 2750\} = 1 - P\{X \le 2750\}$$



$$= 1 - F_X(2750)$$

$$F_X(2750) = F\left(\frac{2750-1832}{460}\right) = F(2.0)$$

$$F_X(2750) = 0.9773$$

$$P\{X > 2750\} = 1 - .9773 = 0.0227$$

② Find the probability that clouds will be lower than 900 m

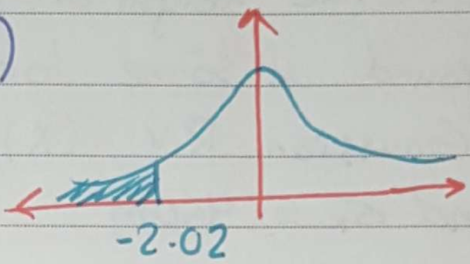
Sol

$$P\{X \le 900\} = F_X(900) = F\left(\frac{900-1830}{460}\right)$$

$$= F(-2.02)$$

No.

$F(-2.02)$



$$F(-2.02) = 1 - F(2.02)$$
$$= 1 - .9783$$
$$= 0.0217$$

③ $P\{800 < x \leq 2000\}$

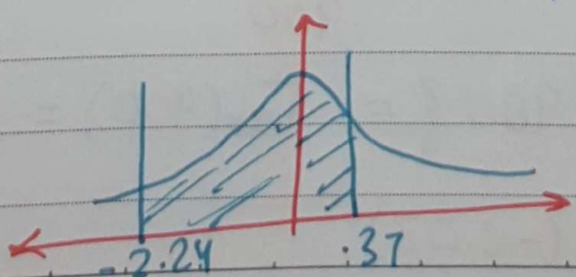
$$F_x(2000) - F_x(800)$$

$$= F\left(\frac{2000-1830}{460}\right) - F\left(\frac{800-1830}{460}\right)$$

$$= F(0.37) - F(-2.24)$$

$$= F(0.37) - (1 - F(2.24))$$

$$= F(0.37) + F(2.24) - 1$$



FIVE APPLE

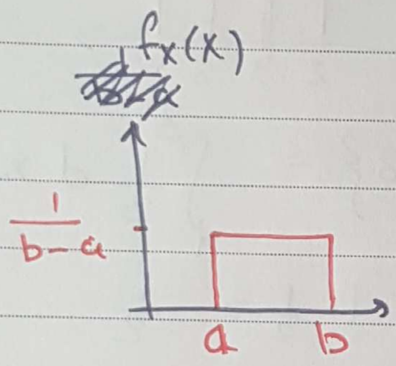
Scanned by CamScanner

13/10

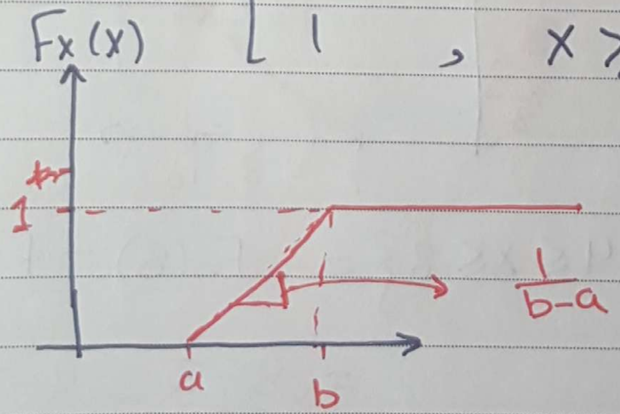
No.

Uniform R.V

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

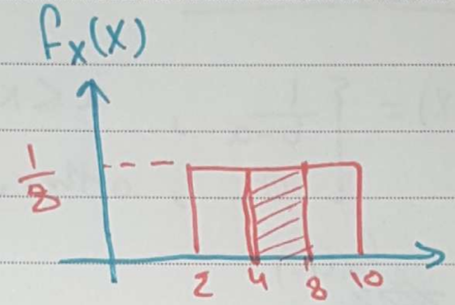


$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b \end{cases}$$



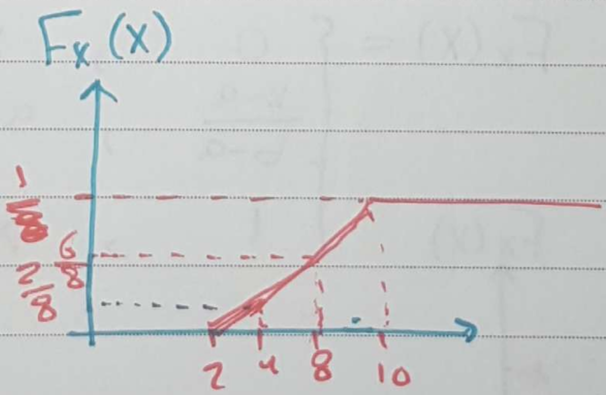
No.

Ex



$$P\{4 \le x \le 8\} = \int_4^8 \frac{1}{8}\, dx$$

$$= \frac{x}{8}\Big|_4^8 = 1 - \frac{1}{2} = \frac{1}{2}$$



$$P\{4 \le x \le 8\} = F_x(8) - F_x(4)$$

$$= \frac{8-2}{8} - \frac{4-2}{8} = \frac{1}{2}$$

No.

Gaussian (Normal) R.V

$$X \sim N(a_x , \sigma_x^2)$$

Ex $X \sim N(5,16)$ $\cancel{f_X(x) = \frac{1}{\sqrt{32\pi}}}$

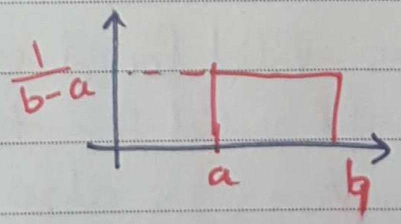
$$u = \frac{X-5}{4}$$

$$u \sim N(0,1)$$

$$f_X(x) = \frac{1}{\sqrt{32\pi}} e^{-\frac{(x-5)^2}{32}} , \quad -\infty < x < \infty$$

The uniform R.V

$$X \sim U(a,b)$$



No.

* Conditional distribution & density functions:

I) $P(A/B) = \dfrac{P(A\cap B)}{P(B)}$ $\qquad F_X(x) = P\{X \leq x\}$

$A = \{X \leq x\}$

$B = \{X \leq b\}$

$$\boxed{P\{X \leq x / B\} = F_X(x/B)}$$ conditional distribution

$$\boxed{\frac{d F_X(x/B)}{dx} = f_X(x/B)}$$ conditional density

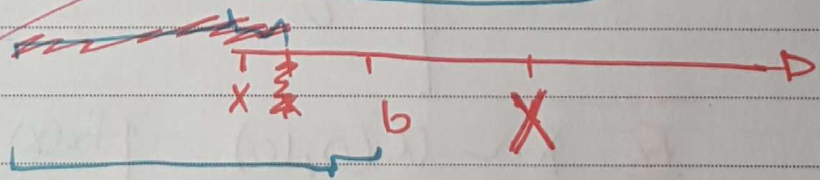
* All properties valid for distribution function are still valid for the conditional Distribution function, is also true for the conditional Density function.

No.

$$B = \{ X \leq b \}$$

$$\cancel{F_X(x/B) = P(A/B) = P\{(X \leq x)/(X \leq b)\}}$$

$$\boxed{\begin{aligned} F_X(x/B) &= P(A/B) = P\{(X \leq x)/(X \leq b)\} \Rightarrow \cancel{P\{}X \leq x \\ &= \frac{P\{(X \leq x) \cap (X \leq b)\}}{P\{X \leq b\}} \end{aligned}}$$



$$= \begin{cases} \dfrac{P\{X \leq x\}}{P\{X \leq b\}}, & x < b \\ \dfrac{P\{X \leq b\}}{P\{X \leq b\}} = 1, & x \geq b \end{cases}$$

No.

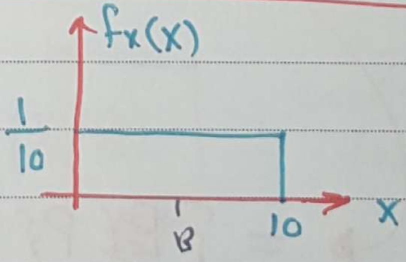
Ex $F_X(X/B) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & , x \leq b \\ 1 & , x \geq b \end{cases}$

$\cancel{f_X(X/B) = \begin{cases} \dfrac{f_X(x)}{F_X(b)} & , x < b \end{cases}}$

$$f_X(X/B) = \begin{cases} \dfrac{f_X(x)}{F_X(b)} = \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx} & , x < b \\ 0 & , x \geq b \end{cases}$$

Ex $X \sim U(0,10)$

$B = \{ X \leq 5 \}$

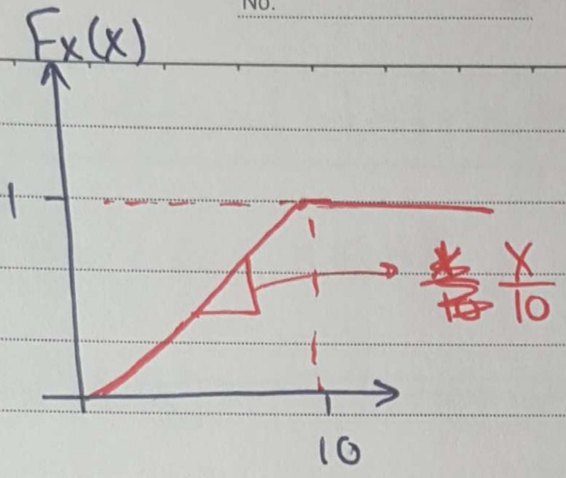


Find: 1) $f_X(X/B)$, 2) $F_X(X/B)$

2] $F_X(X/B)$

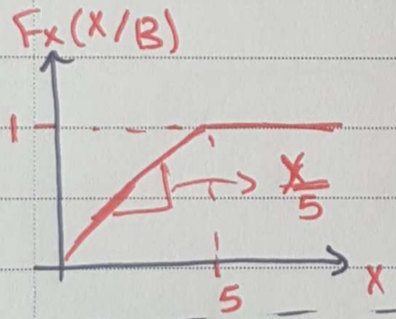
$$F_X(X/B) = \begin{cases} \dfrac{F_X(x)}{F_X(B)} & , x < b \\ 1 & , x \geq b \end{cases}$$

No.



$$F_X(x) = \begin{cases} 0 & , x < 0 \\ \frac{x}{10} & , 0 \le x < 10 \\ 1 & , x > 10 \end{cases}$$

2) $$F_X(x/B) = \begin{cases} 0 & , x \le 0 \\ \frac{\frac{x}{10}}{5/10} = \frac{x}{5} & , 0 \le x < 5 \\ 1 & , x \ge 5 \end{cases}$$



1) $$f_X(x/B) = \begin{cases} 0 & , x < 0 \\ \frac{\frac{1}{10}}{\frac{5}{10}} = \frac{1}{5} & , 0 < x < 5 \\ 0 & , x \ge 5 \end{cases}$$

16|10

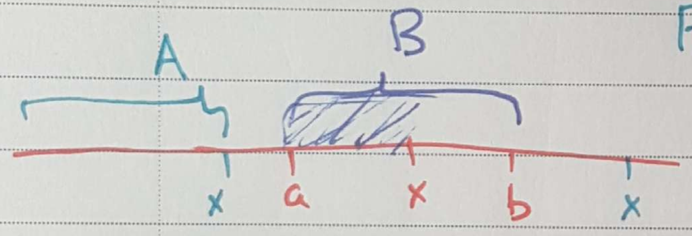
No.

*Conditional distribution and density function

II) $B = \{a < x \leq b\}$, $A = \{X \leq x\}$

$\cancel{P(A/B) = F_x(X/B) = P\{X \leq x\} \cap \{a}$

$$P(A|B) = F_x(X/B) = \frac{P\{X \leq x\} \cap \{a \leq x \leq b\}}{P\{a \leq x \leq b\}}$$



$$F_x(X/B) = \begin{cases} 0 & , x < a \\ \dfrac{F_x(X) - F_x(a)}{F_x(b) - F_x(a)} & , a \leq x < b \\ 1 & , b < x \end{cases}$$

$$\frac{P\{a \leq x \leq X\}}{P\{a \leq x \leq b\}}$$

$$f_x(X/B) = \begin{cases} 0 & , X < a \\ \dfrac{f_x(X)}{\int_a^b f_x(x)\,dx} & , a \leq X < b \\ 0 & , X > b \end{cases}$$

Scanned by CamScanner

No.

Ch3: Operations on one R.V:-

→ Expectation

$E[\cdot] \equiv$ Expectation operation

$E[X] \equiv$ The expected value of x

$\bar{X} \equiv$ The average value, The statistical mean

The mean

$$\boxed{\bar{X} = E[X]}$$

Ex: 20 Students

Grades	frequency (# of students who achieved this grade)
90	3
80	5
70	10
60	2

$$\text{grade Average} = \frac{90*3 + 80*5 + 70*10 + 60*2}{20}$$

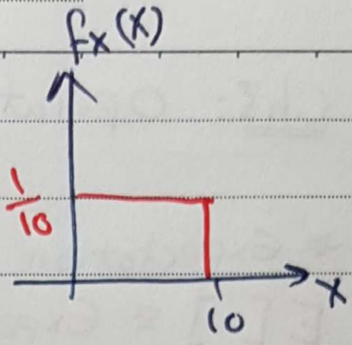
$$= 90*\frac{3}{20} + 80*\frac{5}{20} + 70*\frac{10}{20} + 60*\frac{2}{20}$$

$E[X] = \sum_{i=1}^{N} x_i \, P\{x_i\}$ for discrete R.V ~~for continuous R.V~~

$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$ for continuous R.V

No.

Ex : $X \sim U(0,10)$



$$E[X] = \int_0^{10} x \cdot \frac{1}{10}\, dx$$

$$= \frac{x^2}{20}\Big|_0^{10} = \frac{100}{20} = 5$$

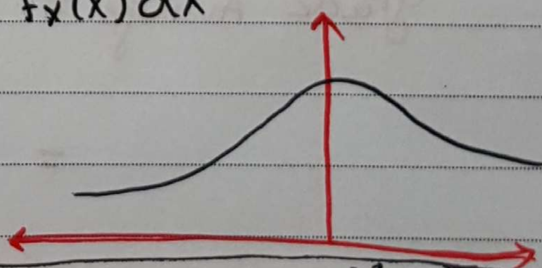
Ex : $X \sim N(a_x, \sigma_x^2)$

$$E[X] = a_x$$

$$\cancel{\bar{X} = 1 \cdot \frac{3}{24} + (8)\frac{4}{24} + 27\left(\frac{7}{24}\right) + 64\left(\frac{10}{24}\right)}$$

$$\cancel{= \frac{3}{24} + \frac{32}{24} + \frac{181}{24} + \frac{640}{24}}$$

$$E[X] = \int_{-\infty}^{\infty} x \; f_X(x)\, dx$$



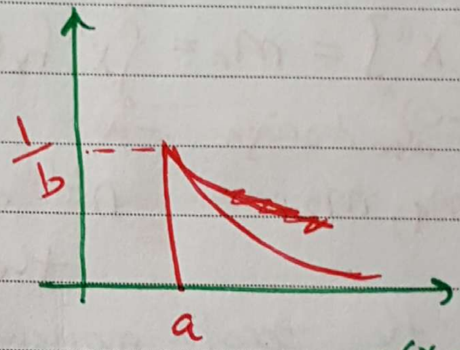
$$\boxed{E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-(x-a_x)^2/2\sigma_x^2} \cdot dx}$$

No.

Ex: Exponential R.V

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & , \ x > a \\ 0 & , \ x < a \end{cases}$$

Sol:



$$E[X] = \bar{X} = \int_0^\infty x \cdot \frac{1}{b} e^{-(x-a)/b} \, dx$$

$$= \frac{e^{a/b}}{b} \int_a^\infty x e^{-x/b} \, dx$$

let $u = x$
$du = dx$
$dv = e^{-\frac{x}{b}} dx$
$v = -b e^{-\frac{x}{b}}$

$$= \frac{e^{a/b}}{b} \left(-xbe^{-x/b} + b\int e^{-x/b} dx\right.$$

$$= \frac{e^{a/b}}{b} \left(-xbe^{-x/b} - b^2 e^{-x/b}\right) \Big|_a^\infty$$

$$= \frac{e^{a/b}}{b} \left(abe^{-a/b} + b^2 e^{-a/b}\right)$$

$$= a + b$$

Scanned by CamScanner

No.

*The Expected Value of a function of R.V

$g(x)$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_x(x)\,dx \Rightarrow \text{for continous R.V}$$

~~Ex: The Volt~~

$$E[g(x)] = \sum_{i=1}^{N} g(x_i)\, P\{x_i\}$$

Ex: The voltage across a resistor $R = 1\Omega$ is V which is uniform R.V between 0 & 5V find the average power delivedred to the Resistor??

Sol: $P = \frac{V^2}{R} = V^2 = g[V]$

$$f_V(V) = \begin{cases} \frac{1}{5} & , 0 < V < 5 \\ 0 & , \text{other wise} \end{cases}$$

$$E[g(V)] = \bar{P} = \int_0^5 \frac{1}{5} V^2\, dV$$

$$= \frac{1}{5} \frac{V^3}{3} \Big|_0^5$$

$$= \frac{1}{5}\left(\frac{125}{3}\right) - 0$$

$$= 25/3 \text{ Watt}$$

No.

Moments

I) Moments about the origin

$g(x) = x^n \quad , n = 0, 1, 2, 3 \ldots$

$$E[X^n] = m_n = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

$(X-0)^n$ about origin

$m_0, m_1, m_2 \ldots \qquad n^{th}$ moment about the origin

~~0th~~

$m_0 \equiv$ the zeroth moment about the origin

$m_1 \equiv$ the first moment about the origin

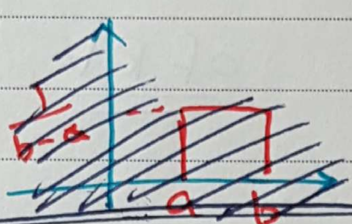
* $m_0 = E[X^0] = \int_{-\infty}^{\infty} x^0 f_X(x)\, dx = 1$

* $m_1 = E[X^1] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \bar{X} \equiv$ average

No.

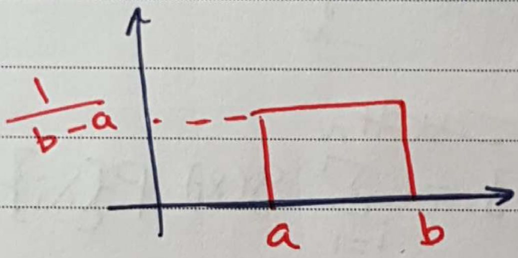
~~Ex find m_2 for $X \sim U(a,b)$~~

~~sol:~~



Ex: Find m_2 for $X \sim U(a,b)$

sol:



$$m_2 = E[X^2] = \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \cancel{(b^3 - a^3)} \left. \frac{x^3}{3} \right|_a^b$$

$$= \frac{1}{3} \frac{b^3 - a^3}{b-a} = \cancel{\frac{1}{3}} \cancel{(b^2 - a^2)}$$

$$= \frac{\cancel{(b-a)}(b^2 + ab + a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2 + ab + a^2}{3}$$

* Special case $u(0,1)$

$$m_2 = \frac{1}{3}$$

No.

II) Moments about the mean (central moments)

$$g(x) = (X - \bar{X})^n \quad , n = 0, 1, 2, 3 \ldots$$

$\mu_n = E[(X - \bar{X})^n]$ μ_n: the n^{th} order central moment

* $\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{x})^n f_x(x)\, dx$

~~$\mu_0 =$~~

* $\mu_0 = E[(X - \bar{X})^0] = \int_{-\infty}^{\infty} (x - \bar{x})^0 f_x(x)\, dx = 1$

* $\mu_1 = E[(X - \bar{X})^1] = \int_{-\infty}^{\infty} (x - \bar{x})\, f_x(x)\, dx$

$$= \int_{-\infty}^{\infty} x\, f_x(x)\, dx - \bar{X} \overbrace{\int_{-\infty}^{\infty} f_x(x)\, dx}^{1}$$

$$= \bar{X} - \bar{X} = \text{zero}$$

* Second Central Moment $\equiv$ Variance σ_x^2
* μ_2 (Variance)

$$\mu_2 = E[(X - \bar{x})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 f_x(x)\, dx$$

$$= E[x^2 - 2x\bar{x} + \bar{x}^2]$$

~~$= E[x^2]$~~

Scanned by CamScanner

No.

$$= E[x^2] - 2\bar{x}\,E[x] + E[\bar{x}^2]$$

$$= E[x^2] - 2\bar{x}\,\bar{x} + \bar{x}^2$$

$$= E[x^2] - 2\bar{x}^2 + \bar{x}^2$$

$$= E[x^2] - \bar{x}^2$$

$$\boxed{\cancel{\mu_2 = m_2 - m_1^2 = \sigma_x^2}}$$

$$\boxed{\underbrace{\mu_2}_{\text{Ac average power}} = \underbrace{m_2}_{\text{Total Average Power}} - \underbrace{m_1^2}_{\text{DC average Power}} = \sigma_x^2} \rightarrow \text{the Variance (standard deviation)}$$

* Skewness ~~$\mu_3 = E[$~~

$$\mu_3 = E[(x-\bar{x})^3]$$

Ex ~~u(a,b)~~ u(0,1)

find σ_x^2??

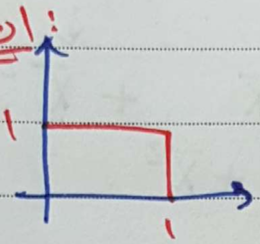
$$\mu_2 = m_2 - m_1^2$$

$$m_1 = \bar{x} = \int_0^1 x\,dx = \frac{1}{2}x^2 = \frac{1}{2}$$

No.

Ex $U(0,1)$ find variance σ_x^2 ??

sol:



$$\mu_2 = m_2 - m_1^2$$

$$m_1 = \bar{x} = \int_0^1 x \cdot 1 \, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}$$

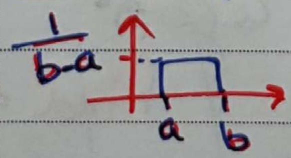
~~m_2~~

$$\mu_2 = E\left[\left(x - \frac{1}{2}\right)^2\right]$$

$$= \int_0^1 \left(x - \frac{1}{2}\right)^2 \cdot 1 \, dx = \int_0^1 x^2 - x + \frac{1}{4} \, dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x \Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

Ex $U(a,b)$ find variance σ_x^2 ??



$$\sigma_x^2 = m_2 - m_1^2$$

$$\sigma_x^2 = \underbrace{\frac{b^2 + ab + a^2}{3}}_{m_2} - \underbrace{\left(\frac{a+b}{2}\right)^2}_{m_1^2}$$

from prev. Example

$$= \frac{4(b^2 + ab + a^2) - 3(a^2 + b^2 + 2ab)}{12}$$

$$\boxed{\sigma_x^2 = (b-a)^2 / 12}$$

No.

*Function that give moments

The characteristic functions

$$\Phi_x(\omega) = E\left[e^{j\omega x}\right]$$

$$* \Phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx$$

$$* f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x} \Phi_x(\omega)\, d\omega$$

$$\frac{d\Phi(\omega)}{d\omega} = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx\right]$$

$$= \int_{-\infty}^{\infty} \frac{d}{d\omega}\left[e^{j\omega x} f_x(x)\right] dx$$

$$-j\,\frac{d\Phi_x(\omega)}{d\omega}\bigg|_{\omega=0} = \cancel{-j}\int_{-\infty}^{\infty} \cancel{j} x\, e^{j\omega x} f_x(x)\, dx\bigg|_{\omega=0}$$

$$* -j\,\frac{d\Phi_x(\omega)}{d\omega}\bigg|_{\omega=0} = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = m_1 = \bar{x}$$

$$* (-j)^2\,\frac{d^2\Phi_x(\omega)}{d\omega^2}\bigg|_{\omega=0} = \int_{-\infty}^{\infty} \cancel{j^2} x^2 e^{j\omega x} f_x(x)\, dx\bigg|_{\omega=0} * \cancel{(-j)^2}$$

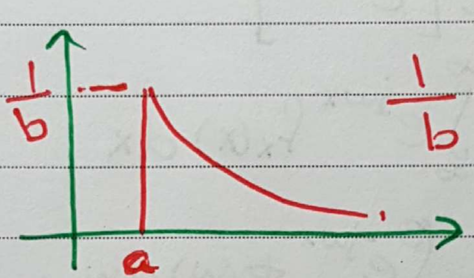
$$* (-j)^2\,\frac{d^2\Phi_x(\omega)}{d\omega^2}\bigg|_{\omega=0} = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx = m_2$$

Scanned by CamScanner

No.

$$\boxed{m_n = (-j)^n \left.\frac{d^n \Phi(w)}{dw^n}\right|_{w=0}}$$

Ex



$$\frac{1}{b} e^{-(x-a)/b}$$

$$\Phi_x(w) = \int_a^\infty e^{jwx} \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}} \, dx$$

$$= \frac{e^{a/b}}{b} \int_a^\infty e^{-x\left(\frac{1}{b} - jw\right)} \, dx$$

$$= \frac{e^{a/b}}{b} \left. \frac{e^{-x\left(\frac{1}{b} - jw\right)}}{-\left(\frac{1}{b} - jw\right)} \right|_a^\infty$$

$$\boxed{\Phi_x(w) = \frac{e^{jwa}}{1 - jwb}}$$

Scanned by CamScanner

No.

$$(-j)\frac{d\Phi_x(\omega)}{d\omega}\Big|_{\omega=0} = m_1 = E[x] = \bar{x}$$

$$= (-j)\,\frac{ja\cancel{e^{j\omega a}}(1-\cancel{j\omega b}) - \cancel{e^{j\omega a}}(-jb)}{(1-\cancel{j\omega b})^2}\Bigg|_{\omega=0}$$

$\cancel{m_1 = -j}$

$$m_1 = E[x] = \bar{x} = \frac{-j[(ja) - (-jb)]}{1}$$

$$= (ja+jb)(-j)$$

$\boxed{m_1 = a+b}$ → like before

$$\sigma_x^2 = m_2 - m_1^2$$

$$E[x^2] = (-j)^2\,\frac{d\Phi_x^2(\omega)}{d\omega}\Big|_{\omega=0}$$

20/10

No.

Moment Generated function :-

$$M_x(s) = E[e^{sx}] \qquad \boxed{s = \sigma + jw}$$

$$M_x(s) = \int_{-\infty}^{\infty} e^{sx} f_x(x)\, dx$$

$$\left.\frac{d M_x(s)}{ds}\right|_{s=0} = \left.\int_{-\infty}^{\infty} x e^{sx} f_x(x)\, dx\right|_{s=0} = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

$$\left.\frac{d^2 M_x(s)}{ds^2}\right|_{s=0} = \left.\int_{-\infty}^{\infty} x^2 e^{sx} f_x(x)\, dx\right|_{s=0} = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx = m_2$$

$$\boxed{m_n = \left.\frac{d^n M_x(s)}{ds^n}\right|_{s=0}}$$

Scanned by CamScanner

No.

~~Ex:~~ $\cancel{f_x(x) = \{ 1/b \; e^{-\left(\frac{x-a}{b}\right)}}$

Ex $f_x(x) = \begin{cases} \frac{1}{b} e^{-\left(\frac{x-a}{b}\right)} & , x > a \\ 0 & , x < a \end{cases}$

~~Sol:~~

$$M_x(s) = E[e^{sx}] = \int_a^{\infty} \frac{1}{b} e^{sx} e^{-\left(\frac{x-a}{b}\right)} dx$$

$$M_x(s) = \frac{e^{\frac{a}{b}}}{b} \int_a^{\infty} e^{-\left(\frac{1}{b}-s\right)x} \cdot dx$$

$$\cancel{= \frac{e^{a/b}}{b} \cdot \frac{e^{-\left(\frac{1}{b}-s\right)x}}{-\left(\frac{1}{b}-s\right)}}$$

$$= \frac{e^{a/b}}{b} \left. \frac{e^{-\left(\frac{1}{b}-s\right)x}}{-\left(\frac{1}{b}-s\right)} \right|_a^{\infty}$$

~~$M_x(s) =$~~

$$M_x(s) = \frac{e^{a/b}}{b} \frac{e^{-(1/b-s)a}}{\left(\frac{1}{b}-s\right)} = \frac{e^{sa}}{1-bs}$$

No.

$$E[X] = m_1 = \frac{d M_x(s)}{ds}\bigg|_{s=0}$$

$$m_1 = \frac{a e^{sa}(1-bs) - e^{sa}(-b)}{(1-bs)^2}\bigg|_{s=0} = a+b$$

---

* Transformation of a R.V :-

R.V $\qquad X \longrightarrow \boxed{T(x)} \xrightarrow{\;Y\;}$ R.V

$f_x(x) \qquad\qquad\qquad\qquad f_y(y)$

$F_x(x) \qquad\qquad\qquad\qquad F_y(y)$

$$\boxed{Y = T(x)}$$

Ex : $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $T(x) = 2x+1$, $Y = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

* *Appendix G (P:444)

Leibniz Rule:

let $G(u)$ represent the integral

$$G(u) = \int_{\alpha(u)}^{B(u)} H[x, u]\, dx$$

No.

$$\boxed{G(u) = \int_{\alpha(u)}^{B(u)} H[x,u]\,dx}$$

$$\cancel{\frac{d\,B(u)}{d(u)} = H[B(u),u]\,\frac{d\,B(u)}{du} - H[\alpha(u),u]\,\frac{d\,\alpha(u)}{du}}$$

$$\boxed{\frac{d\,G(u)}{d(u)} = H[B(u),u]\,\frac{d\,B(u)}{du} - H[\alpha(u),u]\,\frac{d\,\alpha(u)}{du} + \int_{\alpha(u)}^{B(u)} \frac{d\,H[x,u]}{du}\,.dx}$$

*Special case: $B(u)$, $\alpha(u) \rightarrow$ constants

$$G(u) = \int_a^b H[x,u]\,dx$$

$$\frac{d\,G(u)}{du} = \int_a^b \frac{d\,H[x,u]}{du}\,dx$$

23/10

No.

* Transformation of a R.V

$Y = T(x)$

$$X \longrightarrow \boxed{T(X)} \longrightarrow Y$$

$f_X(X)$ $\qquad$ $f_Y(y)$

$F_X(X)$ $\qquad$ $F_Y(y)$

* Leibniz rule

$$G[u] = \int_{\alpha(u)}^{\beta(u)} H[x,u]\, dx$$

$$\frac{dG[u]}{du} = H[\beta(u), u]\frac{d\beta(u)}{du} - H[\alpha(u), u]\frac{d\alpha(u)}{du} + \int_{\alpha(u)}^{\beta(u)} \frac{dH[x,u]}{du} \cdot dx$$

No.

* Motonically increasing function

$T(x) = y$

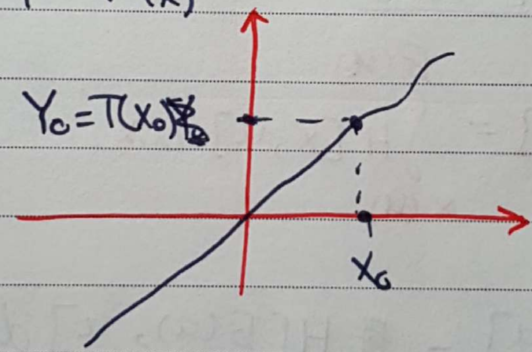
① One to One correspondance

for each x there is one and only one $T(x) = Y$

② for $x_1 < x_2 \Rightarrow T(x_1) < T(x_2)$

Increasing function

$Y = T(x)$



Find $F_y(y_0) = P\{Y \leq y_0\} = P\{X \leq X_0\}$

$$\boxed{F_y(y_0) = F_x(X_0)}$$

$T(x) = 2X$ مثلا تكون لما بيزبط بس هاد*

$X - 3$

$5X + 5$

$\frac{1}{2}X$

$\underbrace{\qquad}$ increasing & one to one correspondance

$$\boxed{\cancel{F_y(y_0)} \quad \cancel{X_0 = T^{-1}(y_0)} \quad \cancel{\int_{-\infty}^{y_0} f_y(y)\,dy = \int_{-\infty}^{x_0} f_x(x)\,dx}}$$

No.

$$\frac{d}{dy_0}\left[\int_{-\infty}^{y_0} f_y(y_0)\, dy = \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_x(x)\, dx \right]$$

$\boxed{\begin{matrix} u = y_0 \\ x = y \end{matrix}}$ $\boxed{\begin{matrix} u = y_0 \\ x = x \end{matrix}}$

apply leibniz rule

~~$f_y(y_0) \cdot (1) - (0) + 0 =$~~

~~$f_y(y_0) \cdot (1) - 0 + 0 = f_x(T^{-1}(y_0)) \cdot dT^{-1}(y)$~~

$$f_y(y_0) \cdot 1 - 0 + 0 = f_x(T^{-1}(y_0)) \cdot \frac{dT^{-1}(y_0)}{dy_0} - 0 + 0$$

$$\boxed{f_y(y_0) = f_x(T^{-1}(y_0)) \cdot \frac{dT^{-1}(y_0)}{dy_0}}$$

$$f_y(y) = f_x(T^{-1}(y)) \cdot \frac{dT^{-1}(y)}{dy}$$

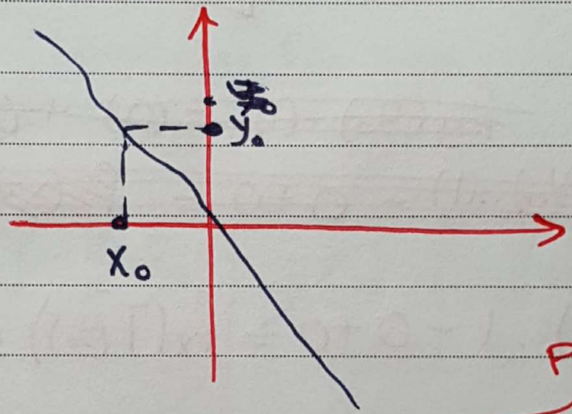
General rule for motonically increasing function

No.

* monotonically decreasing function

① One to one Correspondance.

② $x_1 < x_2 \Rightarrow T(x_1) > T(x_2)$



$$F_y(y_0) = P\{y \le y_0\} = 1 - \underbrace{P\{x \le x_0\}}_{P\{x \geq x_0\}}$$

$$\boxed{F_y(y_0) = 1 - F_x(x_0)}$$

$$\frac{d}{dy_0}\left[F_y(y_0) = 1 - F_x(x_0)\right]$$

$$\frac{d}{dy_0}\left[\int_{-\infty}^{y_0} f_y(y)\,dy = 1 - \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_x(x)\,dx\right]$$

No.

$$\frac{d}{dy_0}\left[\int_{-\infty}^{y_0} f_y(y)\,dy = 1 - \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_x(x)\,dx\right]$$

$u = y_0$ Apply leibniz rule $u = y_0$

$x = y$ $x = x_0$

$$\boxed{f_y(y_0) = -f_x(T^{-1}(y_0)) \cdot \frac{d\,T^{-1}(y_0)}{dy}}$$

$$\boxed{f_y(y) = -f_x(T^{-1}(y)) \cdot \frac{d\,T^{-1}(y)}{dy}}$$

General rule for motonically decreasing function

---

* General rule for motonic functions [increasing or decreasing]

$$\boxed{f_y(y) = f_x(T^{-1}(y)) * \left|\frac{d\,T^{-1}(y)}{dy}\right|}$$

No.

Ex: $T(x) = ax+b$

$y = ax+b$

Sol:

~~T(y)~~

$$T^{-1}(y) = x = \frac{y-b}{a}$$

$$\frac{dT^{-1}(y)}{dy} = \frac{1}{a}$$

$$f_y(y) = f_x\left(\frac{y-b}{a}\right) \cdot \left|\frac{1}{a}\right| \quad , y_1 < y < y_2$$

Not entirely correct. ← ? ?

$$\boxed{\begin{aligned} y_1 &= ax_1+b \\ y_2 &= ax_2+b \end{aligned}}$$

27/10

No.

Transformation of Monotonic functions:-

$$\boxed{f_y(y) = f_x(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|}$$

Ex: $T(x) = Y = ax+b$

$$T^{-1}(y) = x = \frac{Y-b}{a}$$

$$\frac{dT^{-1}(y)}{dy} = \frac{dx}{dy} = \frac{1}{a}$$

$$f_y(y) = f_x\left(\frac{Y-b}{a}\right) \cdot \left|\frac{1}{a}\right|$$

$< y <$

Range of y

Ex: $f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-\mu_x)^2/2\sigma_x^2}$, $-\infty < x < \infty$

$T(x) = ax+b$ find $f_y(y)$??

Sol:

$$\boxed{f_y(y) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{y-b}{a} - \mu_x\right)^2/2\sigma_x^2} * \left|\frac{1}{a}\right| \quad , -\infty < y < \infty}$$

No.

$$f_y(y) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{y-b}{a} - \frac{a a_x}{a}\right)^2 / 2\sigma_x^2} * \left|\frac{1}{a}\right| , \quad -\infty < y < \infty$$

$|a| = \sqrt{a^2}$

$$= \frac{1}{\sqrt{2\pi \sigma_x^2 * a^2}} e^{-(y - \boxed{a a_x + b})^2 / 2 a^2 \sigma_x^2} , \quad -\infty < y < \infty$$

a_y , σ_y^2

$$\boxed{f_y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \cdot e^{-(y-a_y)^2/2\sigma_y^2} , \quad -\infty < y < \infty}$$

Such that:

$\boxed{\sigma_y = a\sigma_x}$, $\boxed{a_y = a a_x + b}$

*

$$a_y = E[Y] = E[ax+b]$$
$$= aE[x+b]$$
$$= a a_x + b$$

$$\sigma_y^2 = E[Y] = E[(Y - a_y)^2]$$
$$= E[(ax+b) - (a a_x + b)]$$
$$= E[(a(x - a_x))^2]$$
$$= E[a^2 (x - a_x)^2]$$
$$= a^2 E[(x - a_x)^2]$$
$$= a^2 \sigma_x^2$$

FIVE APPLE

$$\underset{\substack{\uparrow \\ a_x\ \ \sigma_x^2}}{\text{Ex: } N(0,1)} \longrightarrow \underset{\substack{\uparrow \\ a_y\ \ \sigma_y^2}}{N(5,4)}$$

find $T(x)$??

$5 = a(0) + b \rightarrow a_y = a\, a_x + b$

$\boxed{b = 5}$

$4 = a^2(1) \longrightarrow \sigma_y^2 = a^2 \sigma_x^2$

$\boxed{a = \pm 2}$

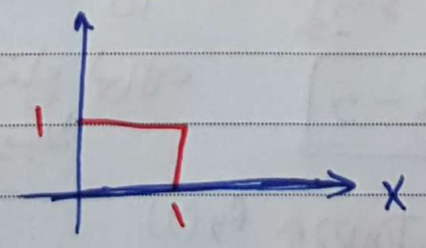
$* Y = T(x) = ax + b$ المطلوب :

$$\boxed{T(x) = \pm 2X + 5}$$

Ex: $X \sim U(0,1)$

$T(x) = ax + b \Rightarrow$

$f_X(x)$



$$f_y(y) = f_x(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right| , \text{ Range of } y$$

$\cancel{f_y(y) = 1 * \frac{1}{|a|}}$, $\cancel{b < y < a+b}$

$f_y(y) = 1 * \frac{1}{|a|}$, $a > 0 \rightarrow b < y < a+b$

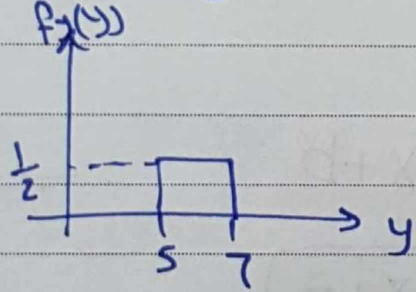
, $a < 0 \rightarrow b - a < y < b$

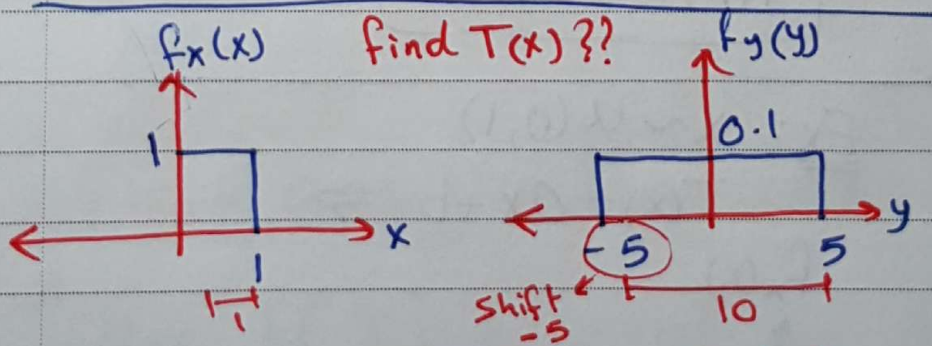
No.

Ex: $y = 2x + 5$

$$f_y(y) = 1 \times \frac{1}{2}$$

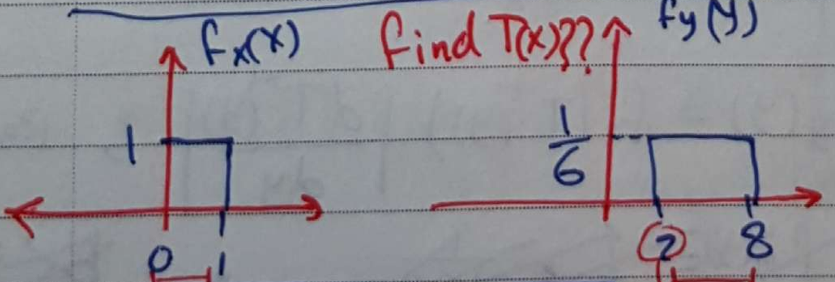
$$f_y(y) = \frac{1}{2} \quad , \; 5 < y < 7$$





$$\boxed{Y = T(x) = 10X - 5}$$

الارتفاع لحاله
بيزبط



$$\boxed{Y = T(x) = 6x + 2}$$

shift +2

$$f_y(y) = \begin{cases} \frac{1}{6} & , 2 < y < 8 \\ 0 & , \text{otherwise} \end{cases}$$

FIVE APPLE

Scanned by CamScanner

No.

Try at home:

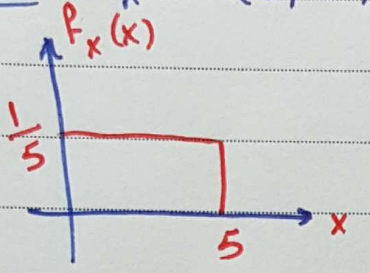
$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & , x > a \\ 0 & , x < a \end{cases}$$

$$Y = T(x) = ax + b$$

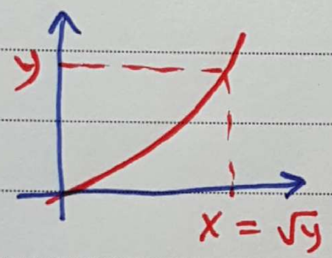
11/1

No.

Ex: $X \sim \mathcal{U}(0, 5)$



$T(x) = x^2$



$$f_Y(y) = f_X(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|$$

$$T^{-1}(y) = x = \sqrt{y}$$

$$\frac{dT^{-1}(y)}{dy} = \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = 0.2 * \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{10\sqrt{y}}, \quad 0 < y < 25$$

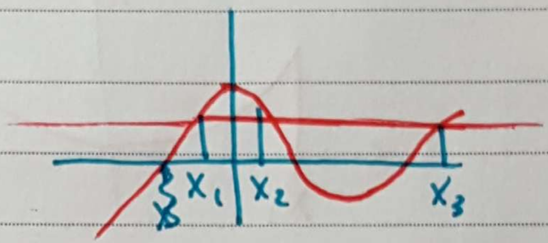
check:

$$\int_0^{25} \frac{1}{10\sqrt{y}}\, dy$$

$$= \frac{2y^{\frac{1}{2}}}{10} \Big|_0^{25} = \frac{10-0}{10} = 1$$

No.

Nonmonotonic transformation of Continuos R.V



$$f_y(y) = \sum_n \frac{f_x(x_n)}{\left| \frac{dT(x)}{dx} \right|}\Bigg|_{x=x_n}$$

* in this the sum is taken so ~~any~~ as to include all the roots y_n, $n = 1,2,3 --$ which are the solution of the equation $\boxed{y = T(x)}$

Ex:

$y = ax^2 + bx + c$

$ax^2 + bx + c - y = 0$

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4a(c-y)}}{2a}$$

Ex $y = x^2$

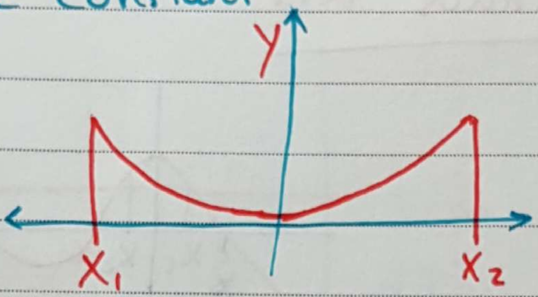
$x_1, x_2 = \pm\sqrt{y}$

$x_1 = -\sqrt{y}$

$x_2 = +\sqrt{y}$

No.

Ex $y = T(x) = Cx^2$ where C is a +ve constant



$$Y = Cx^2$$

$$X^2 = Y/c \quad , \quad X = \pm\sqrt{y/c}$$

$$X_1 = \sqrt{y/c} \quad , \quad X_2 = -\sqrt{y/c}$$

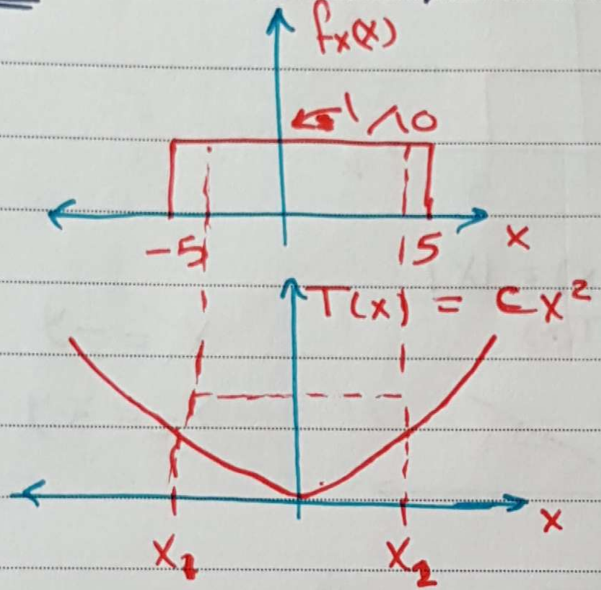
$$\frac{dT(x)}{dx} = 2CX$$

$$f_y(y) = \frac{f_x(-\sqrt{y/c})}{|2C(-\sqrt{y/c})|} + \frac{f_x(+\sqrt{y/c})}{|2C(+\sqrt{y/c})|}$$

$$f_y(y) = \frac{f_x(-\sqrt{y/c}) + f_x(+\sqrt{y/c})}{2\sqrt{cy}} \quad , \quad y \geq 0$$

No.

Ex $X \sim U(-5, 5)$



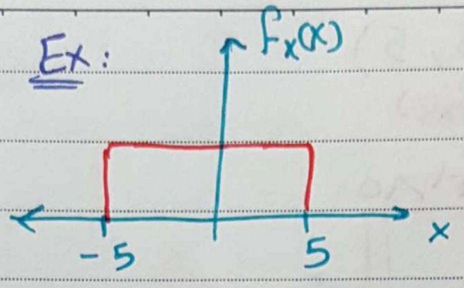
$$f_Y(y) = \frac{0.1 + 0.1}{2\sqrt{cy}} \quad , \quad 0 \leq y \leq 25c$$

$$f_Y(y) = \frac{0.1}{\sqrt{cy}} \quad , \quad 0 \leq y \leq 25c$$

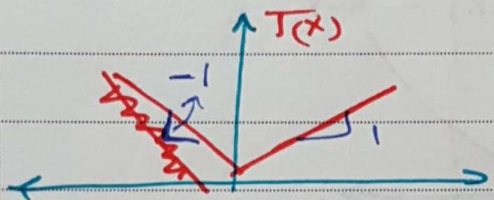
$$f_Y(y) = \frac{1}{10\sqrt{cy}} \quad , \quad 0 \leq y \leq 25c$$

No.

Ex:



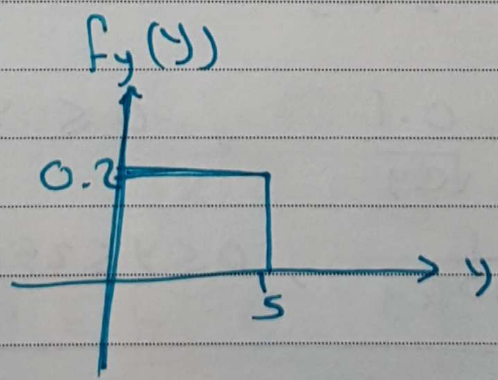
$Y = T(x) = |X|$



$X_1 = -y$

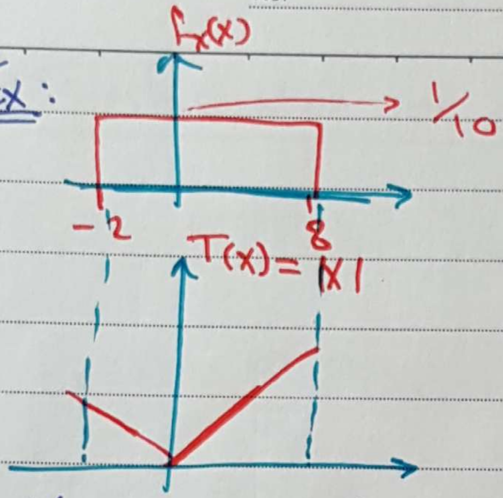
$X_2 = +y$

$$f_Y(y) = \begin{cases} 0.2, & 0 < y \leq 5 \\ 0, & \text{otherwise} \end{cases}$$



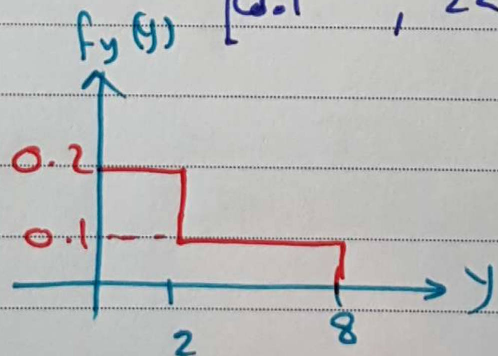
No.

Ex:



$$Y = T(x) = |x| \,, \quad -2 < x < 2$$
$$Y = T(x) = x \,, \quad 2 < x < 8$$

$$f_Y(y) = \begin{cases} 0.2 \,, & 0 < y < 2 \\ 0.1 \,, & 2 < y < 8 \end{cases}$$



* Solve at home

$T(x) = x^2$

~~f(x)~~ $x \sim U(-2, 8)$

3/11

No.

Transformation of a discrete R.V

$$f_X(x) = \sum_n P(x_n)\, \delta(x - x_n)$$

$$F_X(x) = \sum_n P(x_n)\, u(x - x_n)$$

$$Y = F_X(x)$$

$$f_Y(y) = \sum_n P(y_n)\, \delta(y - y_n)$$

$$F_Y(Y) = \sum_n P(y_n)\, u(y - y_n)$$

where $y_n = T(x_n)$

$P(y_n) = P(x_n)$

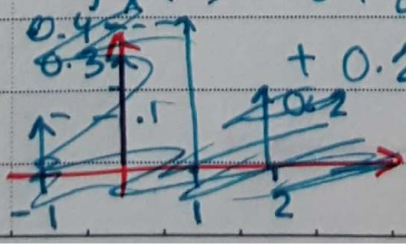
Ex: $X = -1, 0, 1, 2$

$P(-1) = 0.1$, $P(1) = 0.4$

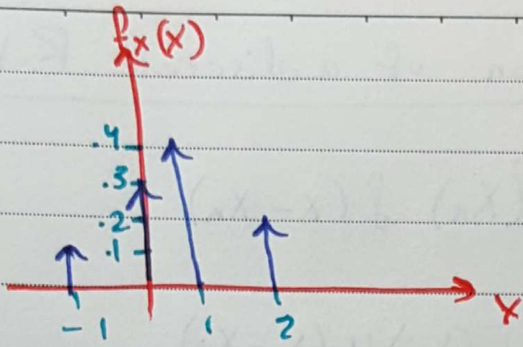
$P(0) = 0.3$, $P(2) = 0.2$

$$Y = 2 - X^2 + \frac{X^3}{3}$$

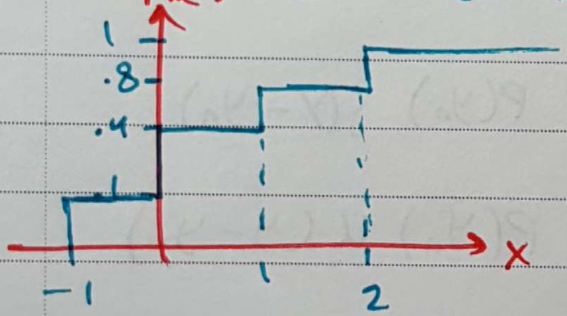
$$f_X(x) = 0.1\, \delta(x+1) + 0.3\, \delta(x) + 0.4\, \delta(x-1) + 0.2\, \delta(x-2)$$



No.



$$F_x(x) = 0.1\, u(x+1) + 0.3\, u(x) + 0.4\, u(x-1) + 0.2\, u(x-2)$$



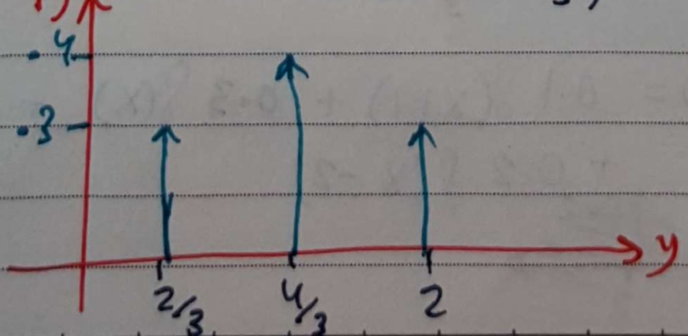
$$T(-1) = 2 - (-1)^2 + \frac{(-1)^3}{3} = 2 - 1 - \frac{1}{3} = \frac{2}{3}$$

$$T(0) = 2$$

$$T(1) = \frac{4}{3}$$

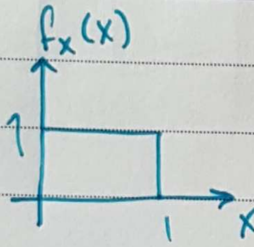
$$T(2) = \frac{2}{3}$$

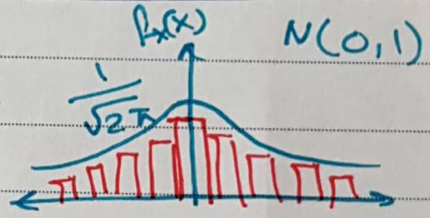
$$f_y(y) = 0.1\,\delta(y - 2/3) + 0.3\,\delta(y-2) + 0.4\,\delta(y - 4/3) + 0.2\,\delta(y - 2/3)$$



~~Fy~~

$$F_Y(y) = 0.3\,u(y-\tfrac{2}{3}) + 0.4\,u(y-\tfrac{4}{3}) + 0.3\,u(y-2)$$

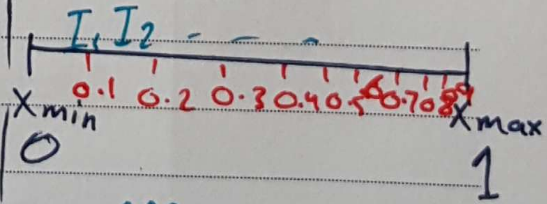




<u>data vector X</u>

X = rand(10,000, 1) | <u>hist(X)</u> hist(X,M)

10,000 [⋮]



$I_1 = 998$

$I_2 = 1002$

Y = randn(10,000, 1)

Y = [⋮]

5/11

No.

Chapter 4 : Multiple R.Vs

* Joint Distribution function

$F_X(x) = P\{X \leq x\}$, $\{X \leq x\} = A$

~~$F_Y(y)$~~

$F_Y(y) = P\{Y \leq y\}$, $\{Y \leq y\} = B$

$F_{XY}(x, y) = P\{X \leq x , Y \leq y\}$

$F_{XYZ}(x, y, z) = P\{X \leq x, Y \leq y, Z \leq z\}$

~~$F_{X_1, X_2 \cdots X_N}(x_1, x_2, \cdots x_N)$~~

~~$F_{X_1, X_2 \cdots X_N}$~~

$F_{X_1, X_2 \cdots X_N}(x_1, x_2, \cdots x_N) = P\{X_1 \leq X , X_2 \leq X , \cdots\}$

Scanned by CamScanner

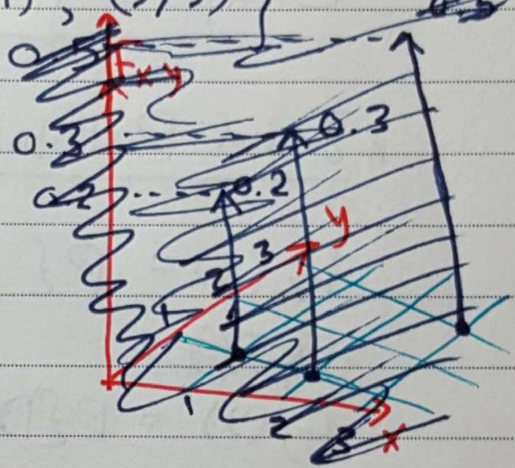
No.

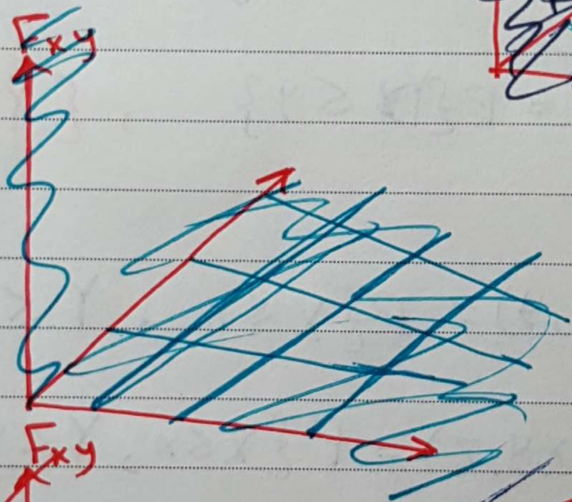
Ex $\{(1,1), (2,1), (3,3)\}$

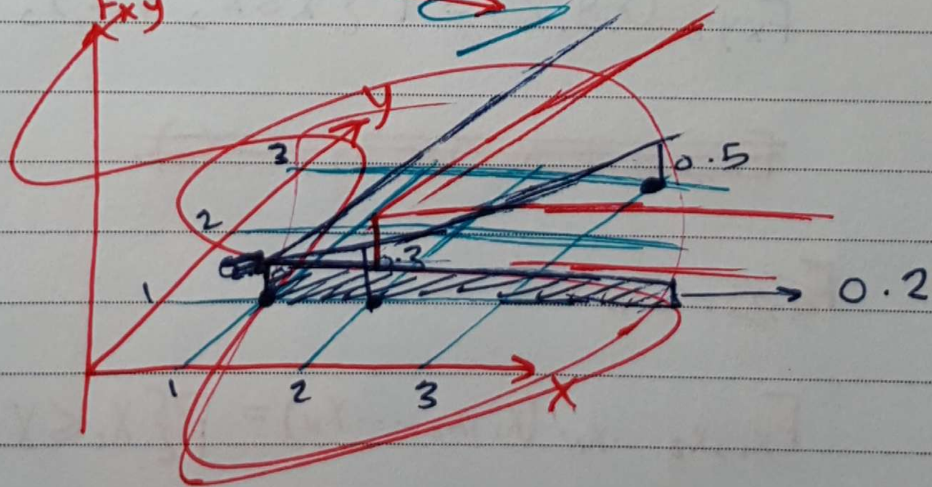
$P(1,1) = 0.2$

$P(2,1) = 0.3$

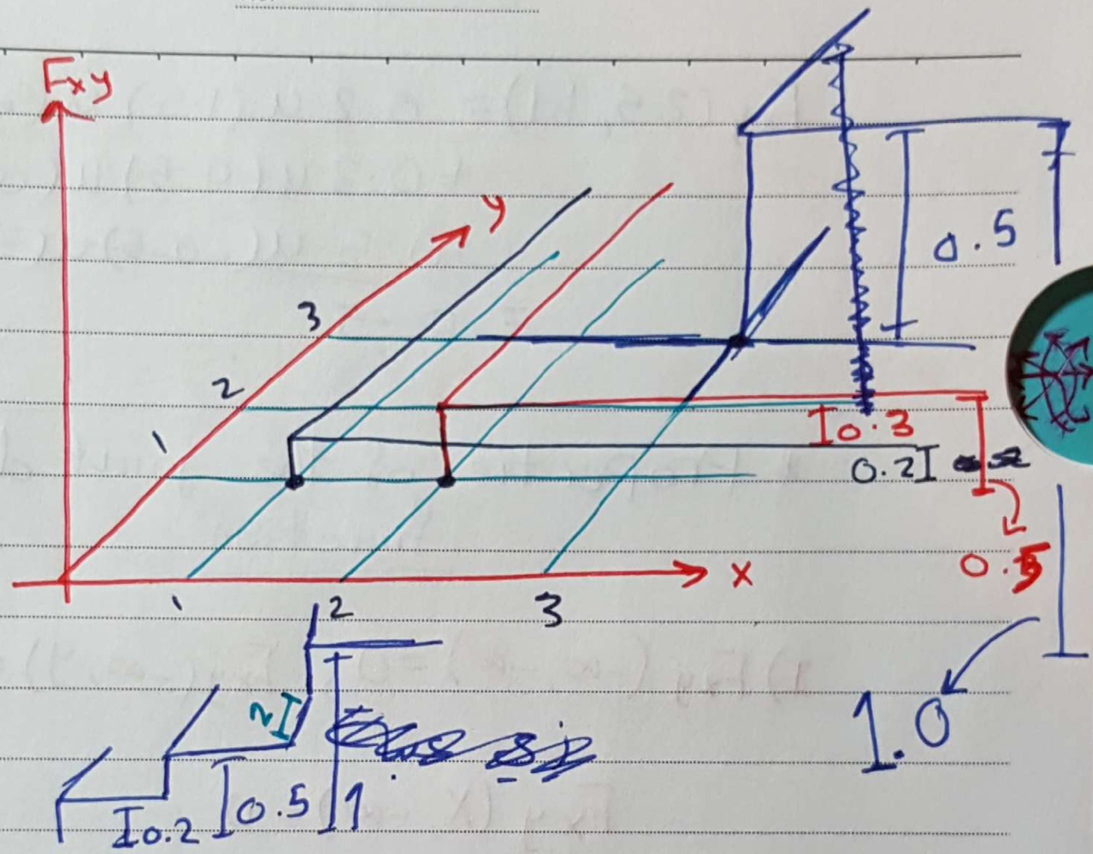
$P(3,3) = 0.5$







No.



زي عتبة بيت

$F_{xy}(1.1, 1.5) = P\{X \leq 1.1, Y \leq 1.5\} = 0.2$

$P\{X \leq 2.5, Y \leq 3.0\} = F_{xy}(2.5, 3.0) = 0.5$

$$F_{xy}(x,y) = 0.2\, u(x-1)\,u(y-1) + 0.3\, u(x-2)\,u(y-1) + 0.5\, u(x-3)\,u(y-3)$$

In Discrete Equality matters

$P\{X < 2, Y < \infty\} = 0.2$

$P\{X \leq 2, Y < \infty\} = 0.5$

No.

$F_{xy}(2.5, 1.1) = 0.2\, u(1.5)\, u(0.1)$
$\quad + 0.3\, u(0.5)\, u(0.1)$
$\quad + 0.5\, \cancel{u(-0.5)\, u(-31.9)}$ zero
$\quad = 0.5$

---

* Properties of the joint distribution function

I) $F_{xy}(-\infty, -\infty) = 0, \quad F_{xy}(-\infty, y) = 0$

$F_{xy}(x, -\infty) = 0$

~~3) $F_{xy}\{ P\{X \le x, Y \le -\infty\}\}$~~

II) $F_{xy}(\infty, \infty) = 1$

$F_{xy}(x, \infty) = F_x(x)$

$F_{xy}(\infty, y) = F_y(y)$

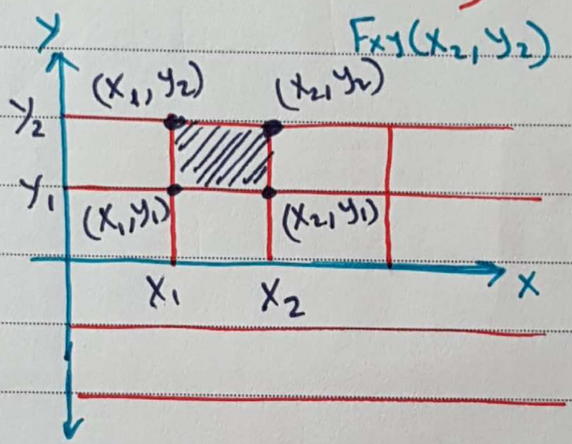
III) $0 \le F_{xy}(x, y) \le 1$

~~IV) $F_{xy}(x_2, y_2) + F_{xy}(x_1, y_1)$~~

No.

IV) $F_{xy}(x_2, y_2) + F_{xy}(x_1, y_1) - F_{xy}(x_1, y_2) - F_{xy}(x_2, y_1)$

$= P\{x_1 \leq X \leq x_2 \; , \; y_1 \leq Y \leq y_2\}$

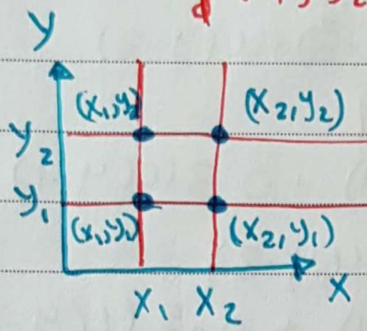


8/11

No.

IV) $P\{x_1 \le X < x_2 \,,\; y_1 \le Y \le y_2\}$

$\cancel{= F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1) + F_{XY}(x_2, y_1) + F_{XY}(x_2, y_2)}$



$$= F_{XY}(x_2, y_2) + F_{XY}(x_1, y_1) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1)$$

V) $F_{XY}(x,y)$ is nondecreasing function in both x & Y

VI) $\underbrace{F_{XY}(x,\infty) = F_X(x) \,,\; F_{XY}(\infty,y) = F_Y(y)}_{\text{The marginal distribution functions}}$

* From a previous Example

$$F_{XY}(x,y) = 0.2\,u(x-1)\,u(y-1) + 0.3\,u(x-2)\,u(y-1) + 0.5\,u(x-3)\,u(y-3)$$

$$F_{XY}(x,\infty) = 0.2\,u(x-1)\,\cancel{u(\infty-1)}^{1} + 0.3\,u(x-2)\,\cancel{u(\infty-1)}^{1} + 0.5\,u(x-3)\,\cancel{u(\infty-3)}^{1}$$

$$= F_X(x) = 0.2\,u(x-1) + 0.3\,u(x-2) + 0.5\,u(x-3)$$

No.

$$F_Y(y) = F_{XY}(\infty, y)$$

$u(y-1)$

$$= 0.2\, u(\overset{1}{\cancel{\infty-1}})\, u(y-1) + 0.3\, u(\overset{1}{\cancel{\infty-2}})$$

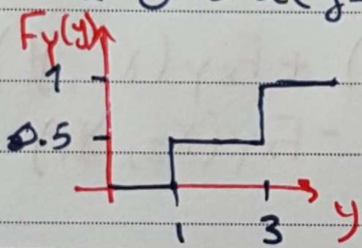
$$+ 0.5\, u(\cancel{\infty-3})\, u(y-3)$$

1

$$= 0.2\, u(y-1) + 0.3\, u(y-1)$$

$$+ 0.5\, u(y-3)$$

$$= 0.5\, u(y-1) + 0.5\, u(y-3)$$



The joint density function:-

$$\cancel{f(x,y) = \frac{d^2 F(x,y)}{dx\,dy}}$$

$$\boxed{f_{XY}(x,y) = \frac{d^2 F_{XY}(x,y)}{dx\,dy} = \frac{d^2 F_{XY}(x,y)}{dy\,dx}}$$

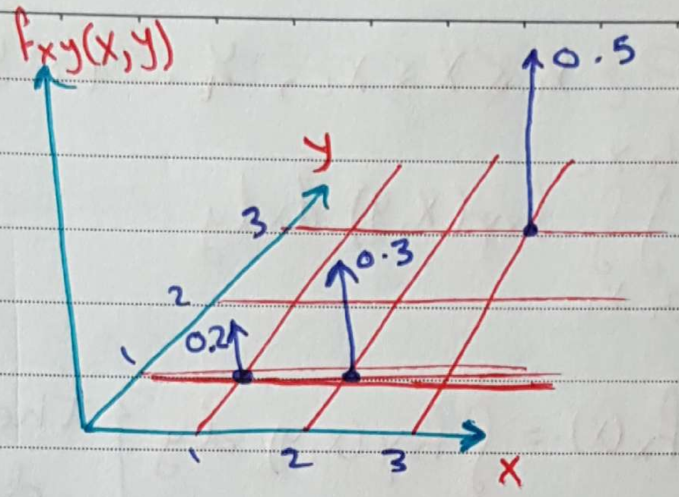
for previous example

$$f_{XY}(x,y) = 0.2\, \delta(x-1)\, \delta(y-1)$$

$$+ 0.3\, \delta(x-2)\, \delta(y-1)$$

$$+ 0.5\, \delta(x-3)\, \delta(y-3)$$

No.



* Properties of the joint density function

I) $f_{xy}(x,y) \geq 0$

II) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\, dx\, dy = 1$

III) $F_{xy}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{xy}(\Sigma_1, \Sigma_2)\, d\Sigma_1\, d\Sigma_2$

* we use dummy variables because we used x, y as upper limits

~~IV) $F_x(x) = \iint f_{xy}(x,y)$~~

IV) $F_x(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{xy}(\Sigma_1, \Sigma_2)\, d\Sigma_2\, d\Sigma_1$

$F_y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{xy}(\Sigma_1, \Sigma_2)\, d\Sigma_1\, d\Sigma_2$

Scanned by CamScanner

No.

V) $P\{x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2\}$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{xy}(x,y)\,dx\,dy$$

VI) $$\left.\begin{aligned} f_x(x) &= \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy \\ f_y(y) &= \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx \end{aligned}\right\}$$ The marginal density functions.

$$f_x(x) = \frac{d\,F_x(x)}{dx}, \quad f_y(y) = \frac{d\,F_y(y)}{dy}$$

Ex $f_{xy}(x,y) = u(x)\,u(y)\,x e^{-x(y+1)}$

Find the marginal densities ??

~~$f_x(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{xy}$~~

$$f_x(x) = \int_{0}^{\infty} u(x)\,x e^{-x(y+1)}\,dy$$

$$= \int_{0}^{\infty} u(x)\,x e^{-xy} \cdot e^{-x}\,dy$$

$$= \frac{u(x)\,x e^{-xy}}{-x} e^{-x} \Big|_0^{\infty}$$

FIVE APPLE

No.

$$\cancel{= -u(x) e^{-x} e^{-xy} \Big|_0^{\infty}}$$

$$\cancel{= -u(x) e^{-x}}$$

$$= u(x) \cancel{x} \cdot e^{-x} \left(\frac{e^{-xy}}{-\cancel{x}} \right) \Big|_0^{\infty}$$

$$= u(x) e^{-x}$$

$$f_y(y) = \int_0^{\infty} u(y) x e^{-x(y+1)} dx$$

$$= u(y) \int_0^{\infty} \underbrace{x}_{u} \underbrace{e^{-x(y+1)}}_{dv} dx$$

by parts

$U.V - \int v.du$

$$= \left(u(y) \left(-x \frac{e^{-x(y+1)}}{y+1} \right) + \int \frac{e^{-x(y+1)}}{y+1} dy \right) \Big|_0^{\infty}$$

10/11

No.

Statistical Independence,

Event A & Event B

~~P(A B) = joint~~

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$

$$P(A/B) = P(A)$$

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$A = \{X \leq x\}$, $B = \{Y \leq y\}$

$$P\{X \leq x \underbrace{,}_{\text{intersection}} Y \leq y\} = P\{X \leq x\} \cdot P\{Y \leq y\}$$

$$= \underbrace{F_X(x) \cdot F_Y(y)}_{\text{product of marginal functions}}$$

$$f_{XY}(x,y) = \frac{d^2 F_{XY}(x,y)}{dx\,dy}$$

$$= \frac{d^2 F_X(x) \cdot F_Y(y)}{dx \quad dy}$$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

Scanned by CamScanner

No.

Ex: $f_{XY}(x,y) = u(x).u(y) \cancel{e}.x\, e^{-x(y+1)}$

from previous example we found $\begin{cases} f_X(x) = u(x)e^{-x} \\ f_Y(y) = \dfrac{u(y)}{(y+1)^2} \end{cases}$

$$f_X(x).f_Y(y) = \frac{u(x)\,u(y)\,e^{-x}}{(y+1)^2} \neq f_{XY}(x,y)$$

which means X & Y are not independent

Ex: $f_{XY}(x,y) = \frac{1}{12}u(x)\,u(y)\,e^{-\frac{x}{4}-\frac{y}{3}}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^{\infty} \frac{1}{12}u(x)\,e^{-x/4-y/3}\,dy$$

$$= \frac{1}{12}u(x)e^{-x/4}(-3e^{-y/3})\Big|_0^{\infty}$$

$$= \cancel{\tfrac{1}{12}} -\frac{1}{4}u(x)e^{-x/4}(0-1)$$

$$\boxed{f_X(x) = \frac{1}{4}u(x)e^{-x/4}}$$

$$\boxed{f_Y(y) = \frac{1}{3}u(y)e^{-y/3}}$$

we found it by doing the same integral

No.

cont. ex

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

X & Y are independent

The Distribution & Density of a Sum of R.Vs

① The sum of 2 R.Vs.

$$W = X+Y$$

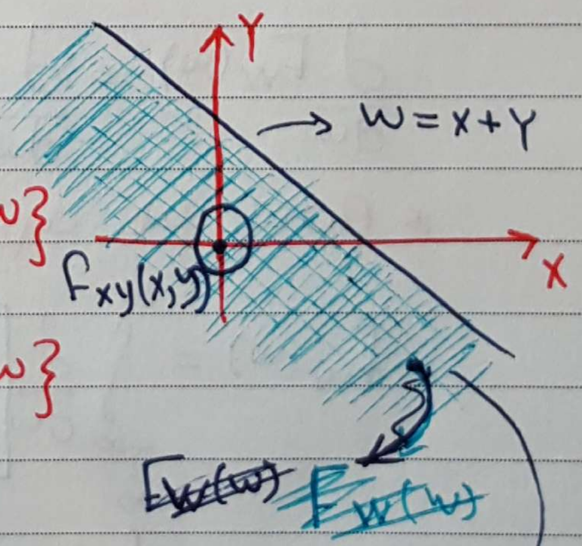
~~we have the~~

We have $f_{XY}(x,y)$ and $F_{XY}(x,y)$

$f_W(w)$??

$$F_W(w) = P\{W \leq w\}$$

$$= P\{X+Y \leq w\}$$



$F_W(w)$

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_{XY}(x,y)\, dx\, dy$$

No.

$$\boxed{f_W(w) = \frac{dF_W(w)}{dw} \cancel{= \frac{d}{dw}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_{XY}(x,y)\,dx\,dy\right]} = \frac{d}{dw}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_{XY}(x,y)\,dx\,dy\right]}$$

→ Assume X & Y are Independent

then

$$\cancel{\frac{d}{dw}}\, F_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_X(x) . f_Y(y)\,dx\,dy$$

$$\frac{d}{dw} F_W(w) = \frac{d}{dw}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_X(x) . f_Y(y)\,dx\,dy\right]$$

* now we apply leibnitz rule

$$f_W(w) = \int_{-\infty}^{\infty} \frac{d}{dw}\left[\int_{-\infty}^{w-y} f_X(x)\, \underbrace{f_Y(y)}_{\text{* constant for inner integration}}\, dx\right] dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, \frac{d}{dw}\left[\int_{-\infty}^{w-y} f_X(x)\,dx\right] dy$$

$$\frac{d}{dw}\left[\int_{-\infty}^{w-y} f_X(x)\,dx\right] = f_X(w-y) . 1$$

No.

$$\boxed{f_W(w) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(w-y)\, dy}$$ → This is a convolution Integral

~~equal~~ equal

$$\boxed{f_W(w) = f_X(x) * f_Y(y)}$$ → convolution

$$\boxed{f_W(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx}$$

~~convolution~~ ~~is~~

13/11

No.

The sum of Two R.Vs

$$W = X + Y$$

$$\cancel{F_W(w) = \int_{-\infty}^{\infty} F_Y(y)}$$

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy$$

$$= \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

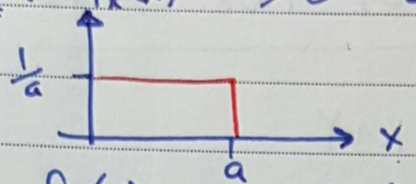
$$= \cancel{F_X} \; f_Y(y) * f_X(x)$$

$$= f_X(x) * f_Y(y)$$

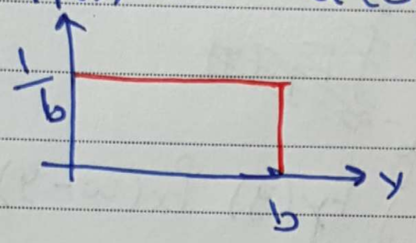
convolution integral

No.

Ex: $f_X(x) \sim U(0,a)$

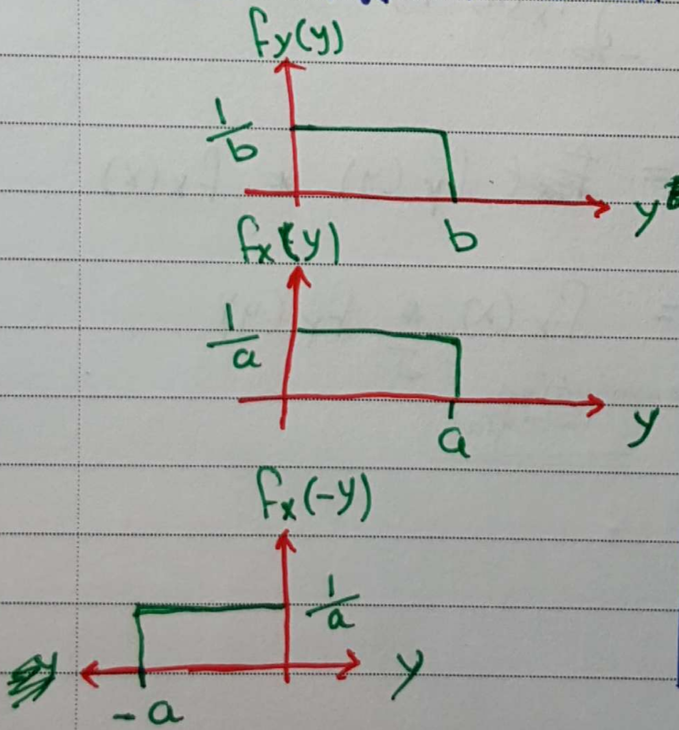


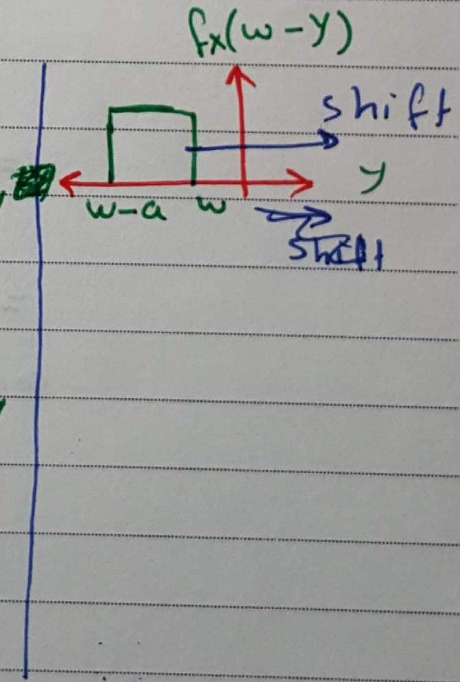
$f_Y(y) \sim U(0,b)$ ~~$U(0,b)$~~



$b > a$

find $f_W(w)$ such that $w = x + y$

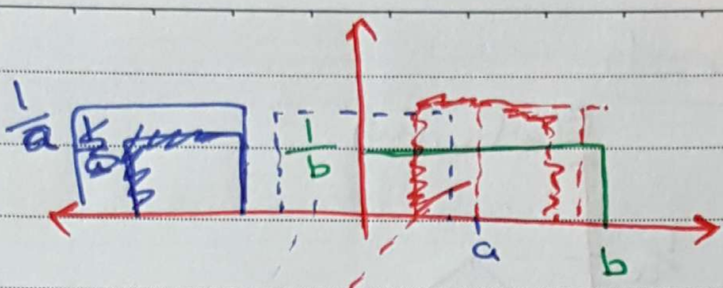




$$f_W(w) = \begin{cases} 0 & , \ w < 0 \\ \dfrac{w}{ab} & , \ 0 \leq w < a \\ \dfrac{1}{b} & , \ a < w < b \end{cases}$$

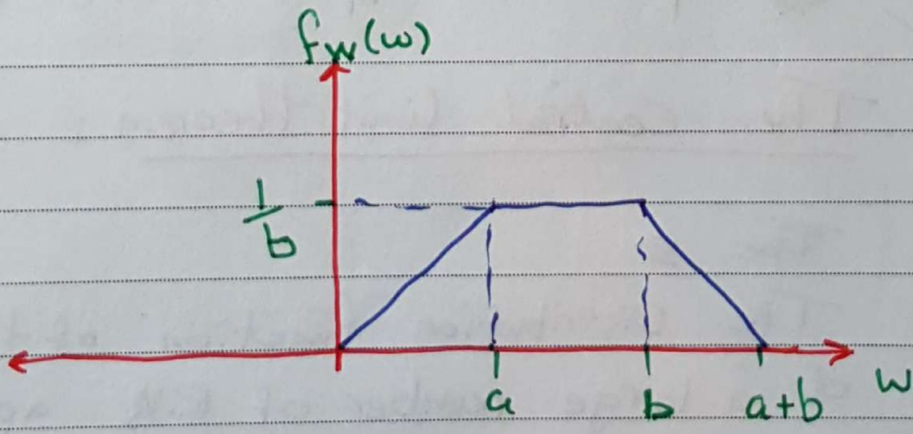
ارتفاع الأول x ارتفاع الثاني x العرض ←

No.



$$
f_W(w) = \begin{cases} 0 & , \quad w < 0 \\ \dfrac{w}{ab} & , \quad 0 < w < a \\ \dfrac{1}{b} & , \quad a \leq w < b \\ \dfrac{a+b-w}{ab} & , \quad b \leq w < a+b \\ 0 & , \quad w \geq a+b \end{cases}
$$

ارتفاع الأول x ارتفاع الثاني
x العرض



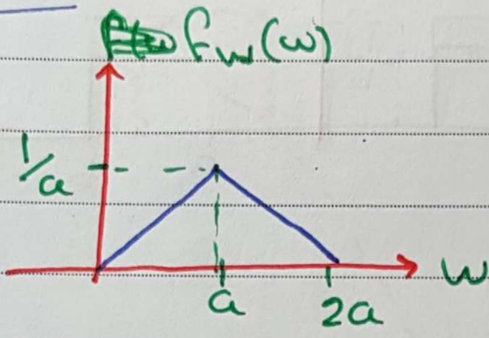
Area under the curve $= \frac{1}{2} a \cdot \frac{1}{b} + (b-a) \cdot \frac{1}{b} + \frac{1}{2} a \cdot \frac{1}{b}$

$$= \frac{a}{b} - \frac{a}{b} + 1$$

$$= 1$$

No.

If $a = b$



The sum of several R.Vs :

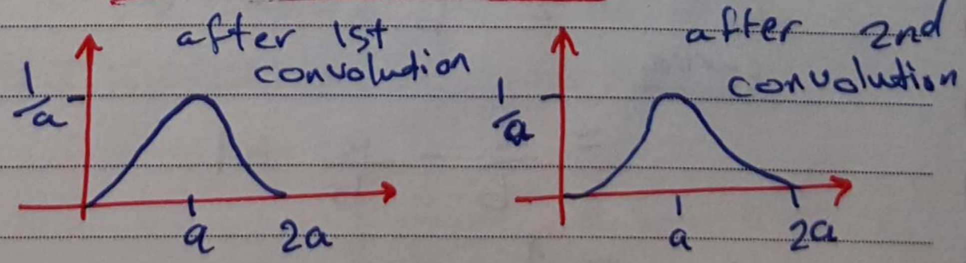
$$Y = \underbrace{(X_1 + X_2)}_{w_1} + X_3$$

$$Y = w_1 + X_3$$

$$\Rightarrow f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3)$$

The Central limit theorem :

The Distribution function of the sum of a large number of R.Vs approaches a Gaussian Distribution.



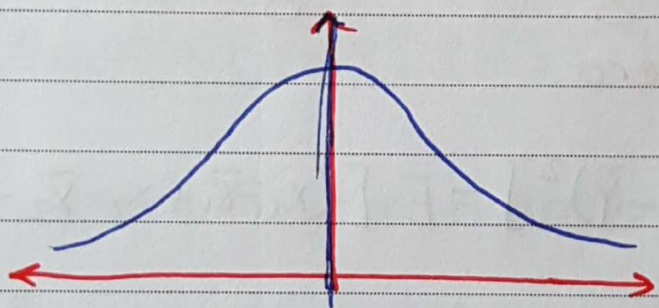
No.

~~12 Uniform R.Vs U(0,1)~~

* 12 Uniform R.Vs $U(0,1)$

$$Y = X_1 + X_2 + \dots + X_{12}$$

Gives Gaussian using matlab



$a_x = 6$

$$E[Y] = E[X_1 + X_2 + \dots X_{12}]$$

$$= \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} \dots + \tfrac{1}{2} \quad \} \text{ 12 times}$$

$$a_x = 6$$

$$\sigma_{x_1}^2 ?? = \frac{1}{12}$$

$$\sigma_{x_1}^2 = m_2 - m_1 = \frac{1}{12}$$

$$= \frac{1}{3} - \frac{1}{4}$$

No.

σ_y^2 ??

$$E[(Y-\bar{Y})^2] = E[(X_1 - \bar{X}_1 + X_2 - \bar{X}_2 - - - X_{12} - \bar{X}_{12})^2]$$

$$E[(X_1 - \bar{X}_1)(X_2 - \bar{X}_2)]$$

$$= \underbrace{E[(X_1 - \bar{X}_1)]}_{0} \cdot \underbrace{E[(X_2 - \bar{X}_2)]}_{0}$$

= zero

$$E[(Y-\bar{Y})^2] = E[(X_1 - \bar{X}_1 + X_2 - \bar{X}_2 - - - X_{12} - \bar{X}_{12})^2]$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 - - - \sigma_{x_{12}}^2$$

$$= \left(\frac{1}{12}\right) 12$$

$$= 1$$

17/11

No.

~~Ch 5~~

Chapter 5 :

Operations on Multiple R.Vs.

Expected value of a function of multiple R.Vs

$g(x,y)$

$g(x_1, x_2, --, x_N)$

~~$\bar{g} = E[g(x,y)]$~~

$$\bar{g} = E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{xy}(x,y)\, dx\, dy$$

$$\bar{g} = E[g(x_1, x_2 \dots, x_N)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2 \dots x_N)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2 \dots x_N)\, f_{x_1 x_2 \dots x_N}(x_1, x_2, \dots x_N)\, dx_1 dx_2 \dots dx_N$$

~~$g(x_1, x_2 \dots x_n) = g(x_1)$~~

~~$g(x)$~~

No.

Joint moments about the origin

$$m_{nk} = E[X^n Y^k] \quad , \quad g(x,y) = X^n Y^k$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{xy}(x,y)\,dx\,dy$$

$(n+k)$ is the order of the moment.

$$\boxed{\begin{aligned} m_{10} &= E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{xy}(x,y)\,dx\,dy = \bar{X} \\ m_{01} &= E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{xy}(x,y)\,dx\,dy = \bar{Y} \end{aligned}}$$

Both m_{10} and m_{01} are the 1st moments about the origin.

$$\boxed{\begin{aligned} m_{20} &= E[X^2 Y^0] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{xy}(x,y)\,dx\,dy = E[X^2] \\ m_{02} &= E[X^0 Y^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{xy}(x,y)\,dx\,dy = E[Y^2] \end{aligned}}$$

The 2nd moments about the origin.

No.

$$\boxed{m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} XY\, f_{xy}(x,y)\,dx\,dy}$$

↳ The correlation R_{xy}

If X & Y are Statistically independent

$$R_{xy} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_x(x)\, f_y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x\, f_x(x)\,dx \cdot \int_{-\infty}^{\infty} y\, f_y(y)\,dy$$

$$R_{xy} = E[XY] = \bar{X}\,\bar{Y} = E[X]\,E[Y]$$

X & Y are uncorrelated

* Statistically independent ⟶ Uncorrelated

* Uncorrelated ⟶ Statistically independent only if joint Gaussian

17/11 continue

No.

* $R_{xy} = E[XY] = 0$

X & Y are orthogonal.

* Joint central Moments

$$\mu_{nk} = E[(x-\bar{x})^n (y-\bar{y})^k]$$

(n+k) is the order of the moment

$$\mu_{20} = E[(x-\bar{x})^2] = \sigma_x^2$$

$$\mu_{02} = E[(y-\bar{y})^2] = \sigma_y^2$$

$$\mu_{10} = E[(x-\bar{x})] = 0$$

$$\mu_{01} = E[(y-\bar{y})] = 0$$

$$\mu_{11} = E[(x-\bar{x})(y-\bar{y})] = C_{xy} : \text{The covariance.}$$

$$= E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}]$$

$$C_{xy} = E[XY] - \bar{X}\bar{Y}$$

Scanned by CamScanner

No.

$$\boxed{C_{xy} = R_{xy} - \bar{x}\,\bar{y}}$$

Relation between covariance & correlation

* If X & Y are uncorrelated

$$\boxed{C_{xy} = 0} \quad , \quad \boxed{R_{xy} = \bar{x}\,\bar{y}}$$

* The correlation coefficient ρ

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{02}\,\mu_{20}}} = \frac{C_{xy}}{\sqrt{\sigma_x^2\,\sigma_y^2}}$$

$$\boxed{\rho = \frac{C_{xy}}{\sigma_x\,\sigma_y}}$$

$$-1 \leq \rho \leq 1$$

always

100% correlation

inversly proportional

directly proportional

20/11

No.

Ex: X is a R.V

$Y = aX$ (Y is also a R.V)

$$\rho = \frac{C_{XY}}{\sqrt{\sigma_X^2 \sigma_Y^2}} = \frac{C_{XY}}{\sigma_X \sigma_Y}$$

* The variance of $X = \sigma_X^2$

~~* The variance of Y =~~

* The variance of $Y = E[(Y-\bar{Y})^2]$

$$= E[(aX-\overline{aX})^2]$$

$$= a^2 E[(X-\bar{X})^2]$$

$$= a^2 \sigma_X^2$$

$$C_{XY} = E[(X-\bar{X})(Y-\bar{Y})]$$

$$= E[(X-\bar{X})(aX-\overline{aX})]$$

$$= a E[(X-\bar{X})(X-\bar{X})] = a\sigma_X^2$$

$$\cancel{\rho = \frac{a\sigma_X^2}{\sqrt{\sigma_X^2 a^2 \sigma_X^2}}}$$

FIVE APPLE

No.

$$\rho = \frac{C_{XY}}{\sqrt{\sigma_x^2 \sigma_y^2}}$$

$$\rho = \frac{a\sigma_x^2}{\sqrt{\sigma_x^2 \, a^2 \sigma_x^2}} = \frac{a}{|a|} = \pm 1$$

* $\rho = 1$ if $a > 0$, $\rho = -1$ if $a < 0$

---

* Jointly Gaussian R.V :-

N _ R.V

$X_1, X_2, --- X_N$ are called jointly gaussian if their joint density funct. can be written as:-

$$\cancel{f_{X_1, X_2, \ldots, X_N}(x_1, x_2 \ldots x_N) =} \cancel{\left|(C_x)^{-1}\right|^{\frac{1}{2}}}$$

$$\cancel{\frac{\left|(C_x)^{-1}\right|^{\frac{1}{2}}}{(2\pi)^{N/2}} \cdot \exp\left[-(X-\bar{X})^T (C_x)^{-1}\right.}$$

No.

$$f_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots x_N) =$$

one element

(1*N) (N*N) (N*1)

$$\frac{\left|(C_x)^{-1}\right|^{\frac{1}{2}}}{(2\pi)^{N/2}} \cdot \exp\left[\frac{-(x-\bar{x})^T (C_x)^{-1} (x-\bar{x})}{2}\right]$$

where we define matrices :-

$$[X-\bar{X}] = \begin{bmatrix} X_1-\bar{X}_1 \\ X_2-\bar{X}_2 \\ \vdots \\ X_N-\bar{X}_N \end{bmatrix}$$

$$[X-\bar{X}]^T = \left[(X_1-\bar{X}_1), (X_2-\bar{X}_2), \ldots (X_N-\bar{X}_N)\right]$$

$$[C_x] = E\left[[X-\bar{X}]\,[X-\bar{X}]^T\right]$$

$$= E\left[\begin{bmatrix} X_1-\bar{X}_1 \\ \vdots \\ X_N-\bar{X}_N \end{bmatrix} \cdot \left[(X_1-\bar{X}_1) \ldots (X_N-\bar{X}_N)\right]\right]$$

No.

$$= E\begin{bmatrix} (x_1-\bar{x}_1)(x_1-\bar{x}_1), \dots, (x_1-\bar{x}_1)(x_N-\bar{x}_N) \\ (x_2-\bar{x}_2)(x_1-\bar{x}_1), \dots, (x_2-\bar{x}_2)(x_N-\bar{x}_N) \\ \vdots \\ (x_N-\bar{x}_N)(x_1-\bar{x}_1), \dots, (x_N-\bar{x}_N)(x_N-\bar{x}_N) \end{bmatrix}$$

$$= E\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \dots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \dots & C_{NN} \end{bmatrix}$$

Where,

$$C_{ij} = E\left[(x_i-\bar{x}_i)(x_j-\bar{x}_j)\right]$$

$$= \begin{cases} \sigma_{x_i}^2 & , i=j \\ \underbrace{C_{x_i x_j}}_{\rho\,\sigma_{x_i}\sigma_{x_j}} & , i\neq j \end{cases}$$

FIVE APPLE

$$[C_x] = \begin{bmatrix} \sigma_{x_1}^2 & \rho\,\sigma_{x_1}\sigma_{x_2} \\ \rho\,\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 \end{bmatrix}$$

determinant $\leftarrow \Delta = \sigma_{x_1}^2\sigma_{x_2}^2 - \rho^2\sigma_{x_1}^2\sigma_{x_2}^2 = \sigma_{x_1}^2\sigma_{x_2}^2(1-\rho^2)$

$$\cancel{[C_x]^{-1} = \frac{1}{\Delta}[C_x]}$$

$$[C_x]^{-1} = \frac{1}{\Delta}\begin{bmatrix} \sigma_{x_2}^2 & -\rho\,\sigma_{x_1}\sigma_{x_2} \\ -\rho\,\sigma_{x_1}\sigma_{x_2} & \sigma_{x_1}^2 \end{bmatrix}$$

$$= \frac{1}{(1-\rho^2)}\begin{bmatrix} 1/\sigma_{x_1}^2 & \dfrac{-\rho}{\sigma_{x_1}\sigma_{x_2}} \\ \dfrac{-\rho}{\sigma_{x_1}\sigma_{x_2}} & 1/\sigma_{x_2}^2 \end{bmatrix}$$

$$\left|[C_x]^{-1}\right| = \frac{1}{(1-\rho^2)^2}\left[\frac{1}{\sigma_{x_1}^2\sigma_{x_2}^2} - \frac{\rho^2}{\sigma_{x_1}^2\sigma_{x_2}^2}\right]$$

$$= \frac{1}{(1-\rho^2)\,\sigma_{x_1}^2\sigma_{x_2}^2}$$

No.

$$\frac{|(C_x)^{-1}|^{1/2}}{(2\pi)^{N/2}} \cdot \exp\left[-\frac{(x-\bar{x})^T (C_x)^{-1}(x-\bar{x})}{2}\right]$$

$$\boxed{|[C_x]^{-1}|^{1/2} = \frac{1}{\sqrt{1-\rho^2}\;\sigma_{x_1}\,\sigma_{x_2}}}$$

$$\cancel{\left(\frac{1}{1-\rho^2}\right)[(X_1-\bar{X}_1)(X_2-\bar{X}_2)]}$$

$$\left(\frac{1}{1-\rho^2}\right)[(X_1-\bar{X}_1)(X_2-\bar{X}_2)]\begin{bmatrix} \frac{1}{\sigma_{x_1}^2} & \frac{-\rho}{\sigma_{x_1}\sigma_{x_2}} \\ \frac{-\rho}{\sigma_{x_1}\sigma_{x_2}} & \frac{1}{\sigma_{x_2}^2} \end{bmatrix}\begin{bmatrix} (X_1-\bar{X}_1) \\ (X_2-\bar{X}_2) \end{bmatrix}$$

$$= \frac{1}{(1-\rho^2)}\left[\frac{(X_1-\bar{X}_1)}{\sigma_{x_1}^2} - \frac{\rho(X_2-\bar{X}_2)}{\sigma_{x_1}\sigma_{x_2}}\right]\left[\frac{-\rho(X_1-\bar{X}_1)}{\sigma_{x_1}\sigma_{x_2}^2}\right.$$

$$\left. + \frac{(X_2-\bar{X}_2)}{\sigma_{x_2}^2}\right]\begin{bmatrix} (X_1-\bar{X}_1) \\ (X_2-\bar{X}_2) \end{bmatrix}$$

No.

$$= \frac{1}{(1-\rho^2)} \left[\frac{(x_1 - \bar{x}_2)^2}{\sigma_{x_1}^2} - \frac{\rho (x_1 - \bar{x}_2)(x_2 - \bar{x}_2)}{\sigma_{x_1} \sigma_{x_2}} - \frac{\rho (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)}{\sigma_{x_1} \sigma_{x_2}} + \frac{(x_2 - \bar{x}_2)^2}{\sigma_{x_2}^2} \right]$$

$$\boxed{f_{x_1, x_2}(x_1, x_2) = \frac{1}{2\pi \sigma_{x_1} \sigma_{x_2} \sqrt{1-\rho^2}} * \exp\left[\frac{-1}{2(1-\rho^2)} \left(\frac{(x_1 - \bar{x}_1)^2}{\sigma_{x_1}^2} - \frac{2\rho (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)}{\sigma_{x_1} \sigma_{x_2}} + \frac{(x_2 - \bar{x}_2)^2}{\sigma_{x_2}^2} \right) \right]}$$

22/11

No.

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} *$$

$$\exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\bar{x})^2}{\sigma_x^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})^2}{\sigma_y^2}\right)\right]$$

If X & Y are uncorrelated $\longrightarrow \rho = 0$

$$\left(\frac{C_{xy}}{\sqrt{\sigma_x^2\sigma_y^2}} = \frac{C_{xy}}{\sigma_x\sigma_y}\right)$$

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} * \exp\left[\frac{-1}{2}\left(\frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sigma_y\sqrt{2\pi}} * \exp\left(\frac{-1}{2}\frac{(x-\bar{x})^2}{\sigma_x^2}\right)$$

$$* \exp\left(\frac{-1}{2}\frac{(y-\bar{y})^2}{\sigma_y^2}\right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left(\frac{-1}{2}\frac{(x-\bar{x})^2}{\sigma_x^2}\right)}_{f_x(x)} * \underbrace{\frac{1}{\sqrt{2\pi\sigma_y^2}}\exp\left(\frac{-1}{2}\frac{(y-\bar{y})^2}{\sigma_y^2}\right)}_{f_y(y)}$$

* One dimensional gaussian
* X & Y are statistically Independent

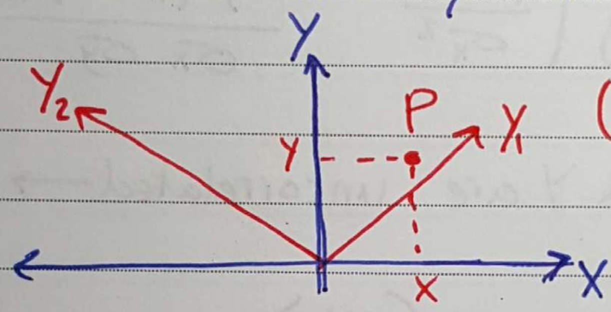
FIVE APPLE

No.

Ex $Y_1 = X\cos\theta + Y\sin\theta$

$Y_2 = -X\sin\theta + Y\cos\theta$

X & Y are known, $\bar{X}$ & $\bar{Y}$ are also known



$(P = x\bar{a}_x + y\bar{a}_y)$

X & Y are jointly gaussian:

$$f_{xy}(x,y) \sim N(\bar{X}, \bar{Y}, \sigma_x^2, \sigma_y^2, \rho)$$

$E[X] = \bar{X}$, $E[Y] = \bar{Y}$

$f_{Y_1 Y_2}(y_1, y_2) \longrightarrow Y_1$ & Y_2 are independent

$(C_{Y_1 Y_2} = 0)$

$\bar{Y}_1 = \bar{X}\cos\theta + \bar{Y}\sin\theta$

$\bar{Y}_2 = -\bar{X}\sin\theta + \bar{Y}\cos\theta$

$$C_{Y_1 Y_2} = E\left[(Y_1 - \bar{Y}_1)(Y_2 - \bar{Y}_2)\right] = 0$$

FIVE APPLE

Scanned by CamScanner

$$= E\Big[\big((x\cos\theta + y\sin\theta) - (\bar{x}\cos\theta + \bar{y}\sin\theta)\big) * \big((-x\sin\theta + y\sin\theta) - (\bar{x}\sin\theta + \bar{y}\cos\theta)\big)\Big]$$

$$= E\Big[\big((x-\bar{x})\cos\theta + (y-\bar{y})\sin\theta\big) * \big(-(x-\bar{x})\sin\theta + (y-\bar{y})\cos\theta\big)\Big]$$

$$= E\Big[-(x-\bar{x})^2\sin\theta\cos\theta + (y-\bar{y})^2\sin\theta\cos\theta - (x-\bar{x})(y-\bar{y})\sin^2\theta + (x-\bar{x})(y-\bar{y})\cos^2\theta\Big] = 0$$

$$= (\sigma_y^2 - \sigma_x^2)\sin\theta\cos\theta + C_{xy}(\cos^2\theta - \sin^2\theta)$$

$*$ $\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$

$$= (\sigma_y^2 - \sigma_x^2)\left(\tfrac{1}{2}\sin(2\theta)\right) + C_{xy}\cos(2\theta) = 0$$

$$C_{xy}\cos(2\theta) = \frac{(\sigma_x^2 - \sigma_y^2)\sin(2\theta)}{2}$$

$$\theta = \tfrac{1}{2}\tan^{-1}\left(\frac{2C_{xy}}{\sigma_x^2 - \sigma_y^2}\right) = \tfrac{1}{2}\tan^{-1}\left(\frac{2\rho\,\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right)$$

$$E\big[(x-\bar{x})(y-\bar{y})\big] = C_{xy}$$

Section 5.4 X مو مطلوب

No.

Chapter6 : Random process /Stochastic Process

$X(S,t)$

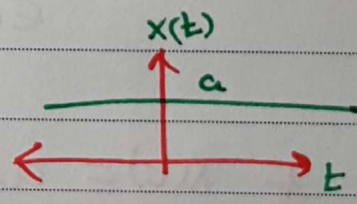
$X(S)$: Random Variable

~~X(t): Stochastic time process variable~~

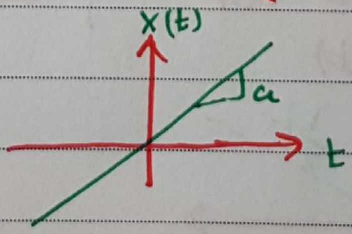
Random process is a Random variable including time

$$X(S,t) \Rightarrow X(t)$$

* Deterministic $X(t)=a$



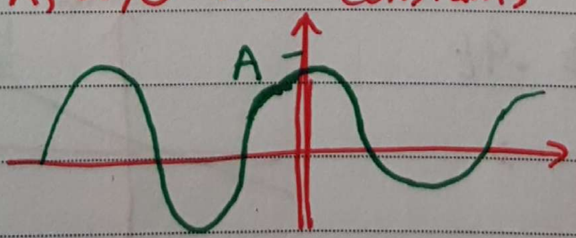
$X(t) = at$



$a>0$

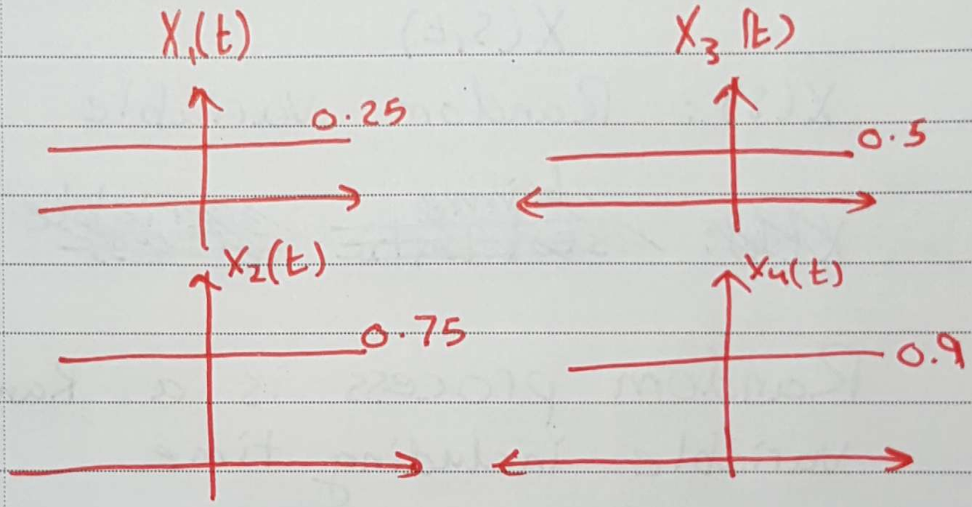
$X(t) = A\cos(\omega t + \theta)$

A, ω, θ are constants



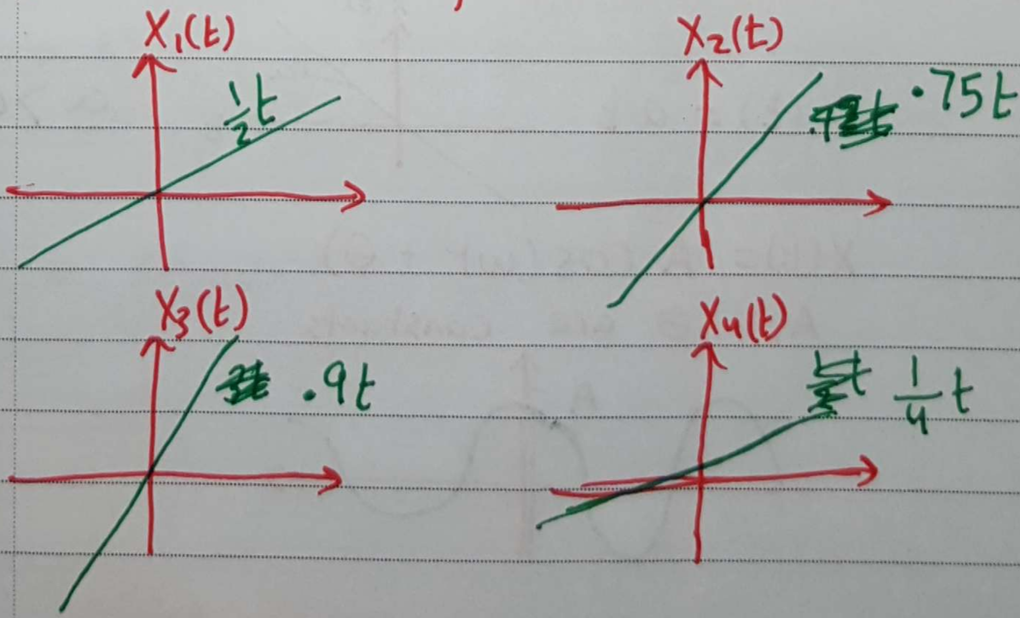
No.

* $X(t) = a \quad , \quad a \sim u(0,1)$



Ensemble of functions

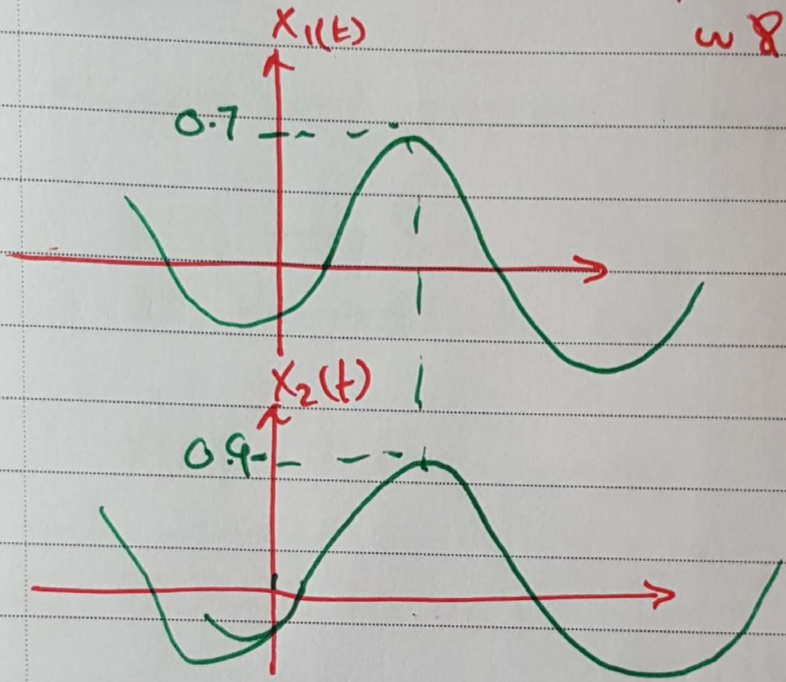
* $X(t) = at \quad , \quad a \sim u(0,1)$



No.

$X(t) = A\cos(\omega t + \theta)$, $A \sim U(0,1)$

ω & $\theta \equiv$ constants



Scanned by CamScanner

29/11

No.

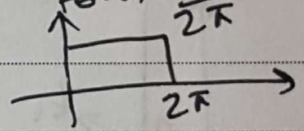
~~Chapter 6~~

*Random process temporal characteristics

$X(S) \longrightarrow X$ R.V

$X(S,t) \longrightarrow X(t)$ Random process

$X(t_1) = A\cos(\omega t_1 + \Theta)$ Random Variable

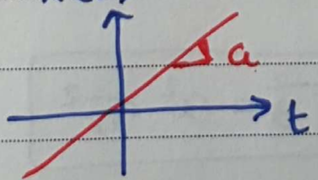


Ex: $\Theta \sim U(0, 2\pi)$

$X = T(\Theta) = \cos(\Theta)$ $f_\Theta(\theta)$

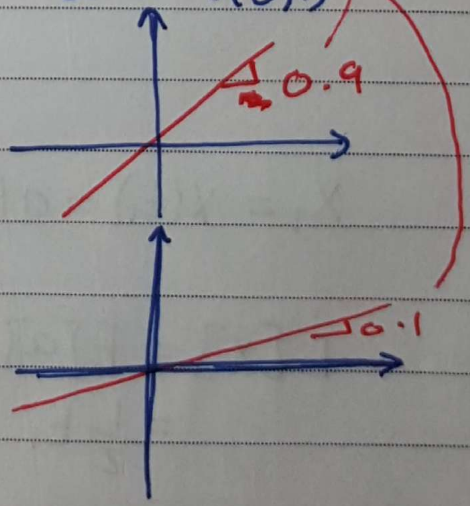
$f_X(x)$??

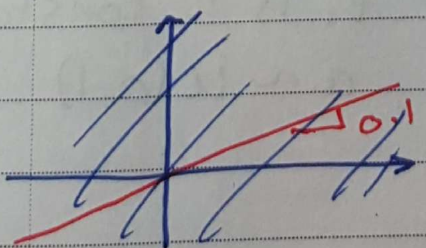
$X(t) = at$



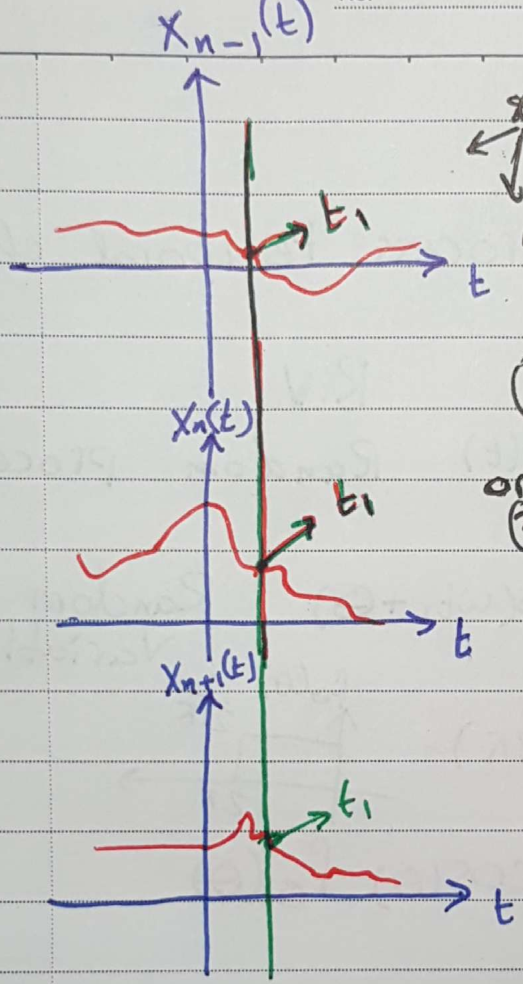
$X(t) = at$ such that

$a \sim U(0,1)$





No.



* The family of curves is called ensemble.

* each is either called ① member of the ensemble or ② Realization or ③ Sample function.

$X(t_1)$ = Random Variable

* $Y = bx$

$$\boxed{\sigma_y^2 = b^2 \sigma_x^2}$$

* $Y = ax + b$

$$\boxed{\sigma_y^2 = a^2 \sigma_x^2}$$

$X_1 = X(t_1) = at_1$ t_1 is a constant

$a \sim u(0,1)$

$E[x] = E[a]\, t_1$

$= \frac{1}{2} t_1$

$$\boxed{\sigma_{x_1}^2 = \sigma_a^2 t_1 = \frac{1}{12} t_1}$$

No.

*Classifications of Random processes

1) continuos Random Process.

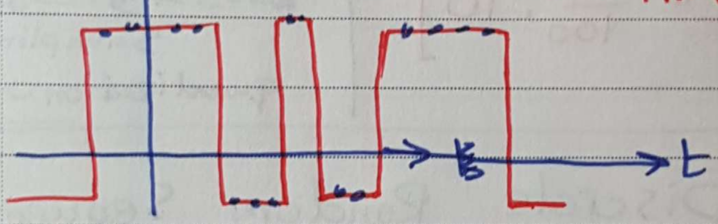
* continuos in Amplitude, * continuos in time

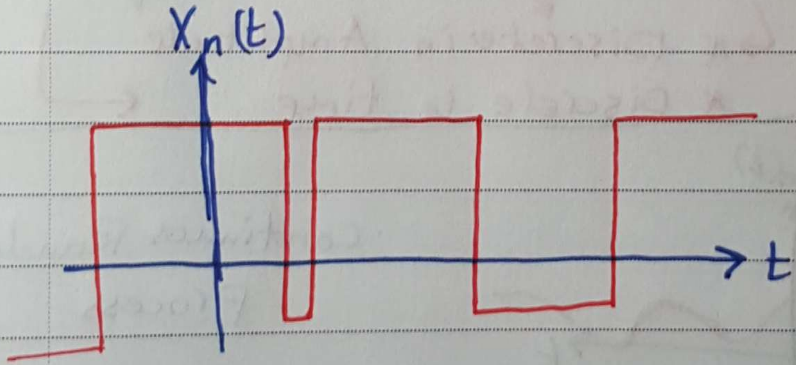
2) Discrete Random Process.

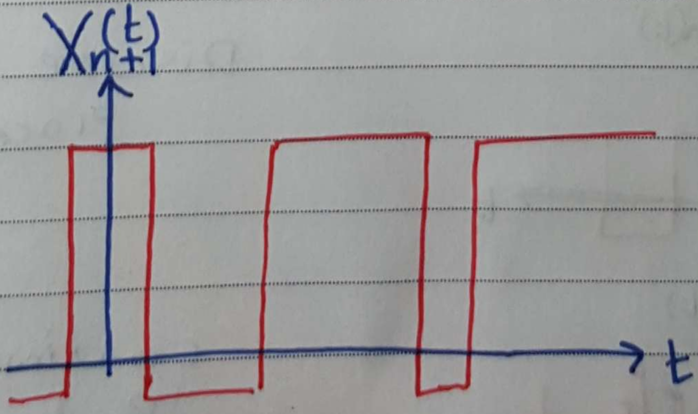
sample & hold

$X_{n-1}(t)$

* Discrete in Amplitude.
* continuos in time.







No. time

3) Discrete Random Process
(continuos Random Sequence)
↳ * continuos in Amplitude
* Discrete in time

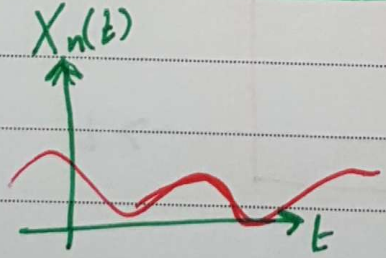
كيف نمثلها

>> $X(t) = A\cos(\omega t + \Theta)$, Sampling frequency = 100Hz

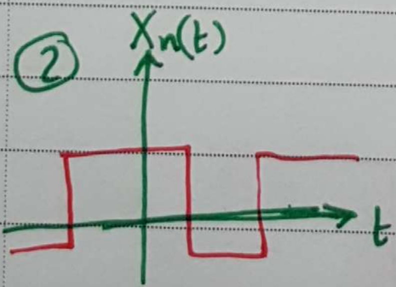
>> $t = [0 : \frac{1}{100} : 10]$

*Comm1
*الفكرة هون الواحد بيعمل Sampling بدون quantization

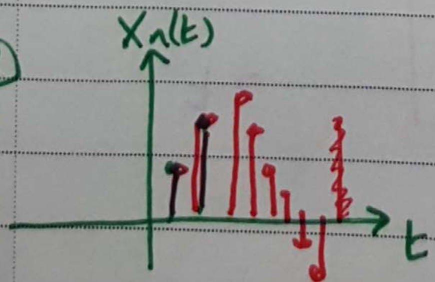
4) Discrete Random Sequence
↳ * Discrete in Amplitude
* Discrete in time

① 

Continuos Random Process

② 

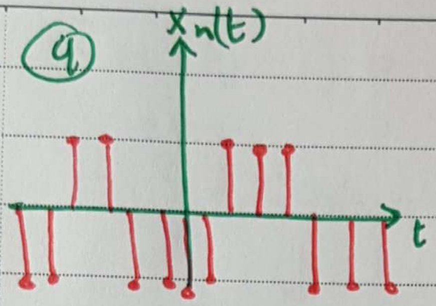
Discrete Random Process

③ 

continuos Random Sequence

No.

(9)



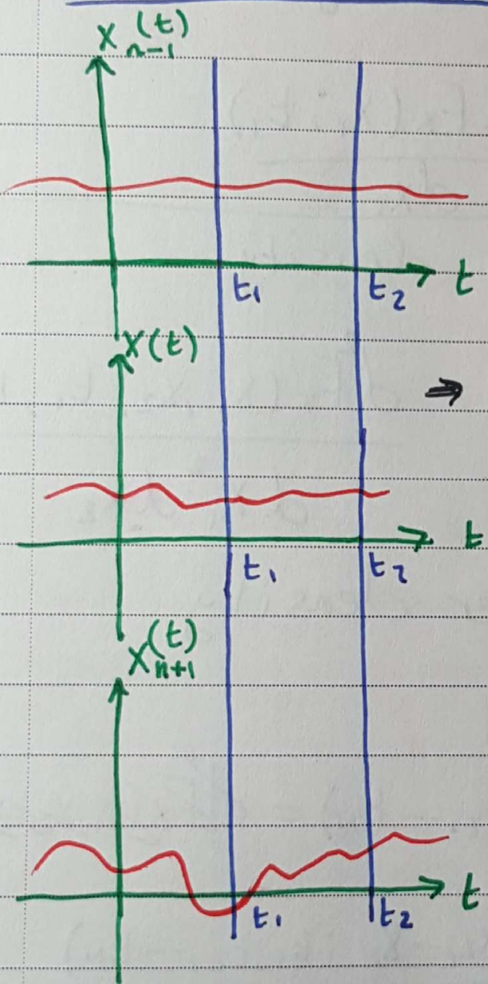
Discrete Random Sequence

Scanned by CamScanner

1/12

No.

*Distribution & Density function:



$X(t_1) = X_1 \equiv$ Random Variable

$X(t_2) = X_2 \equiv$ Random Variable

⇒ For X_1:

$F_x(x_1; t_1) = P\{X(t_1) \leq X_1\}$

~~1st~~ 1st order distribution

$F_x(x_1, x_2, t_1, t_2) =$

$P\{X(t_1) \leq X_1, X(t_2) \leq X_2\}$

2nd order distribution

$F_x(x_1, x_2, \dots X_N; t_1, t_2, \dots t_N) =$

$P\{X(t_1) \leq X_1, X(t_2) \leq X_2, \dots, X(t_N) \leq X_N\}$

Nth order distribution

→ The 1st order density

$$f_X(x_1;t_1) = \frac{d F_X(x_1;t_1)}{dx_1}$$

→ The 2nd order density

$$f_X(x_1, x_2; t_1, t_2) = \frac{d^2 F_X(x_1, x_2; t_1, t_2)}{dx_1\, dx_2}$$

→ The nth order density

$$f_X(x_1, x_2 \ldots x_N, t_1, t_2, \ldots t_N) = \cancel{d F_X(x_1, x_2 \ldots x_N}$$

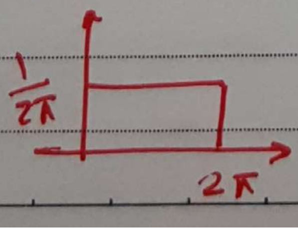
$$\frac{d^N F_X(x_1, x_2, \ldots x_N; t_1, t_2, \ldots t_N)}{dx_1\, dx_2 \ldots dx_N}$$

$$X(t_1) = X_1 = A \cos(\omega t_1 + \Theta)$$

$$\Theta \sim u(0, 2\pi)$$

$$E[X_1] = \int_0^{2\pi} A \cos(\omega t_1 + \Theta) \cdot \frac{1}{2\pi}\, d\Theta$$

$f_\Theta(\theta)$



No.

$$= \frac{A}{2\pi} \sin(\omega t_1 + \theta) \Big|_0^{2\pi}$$

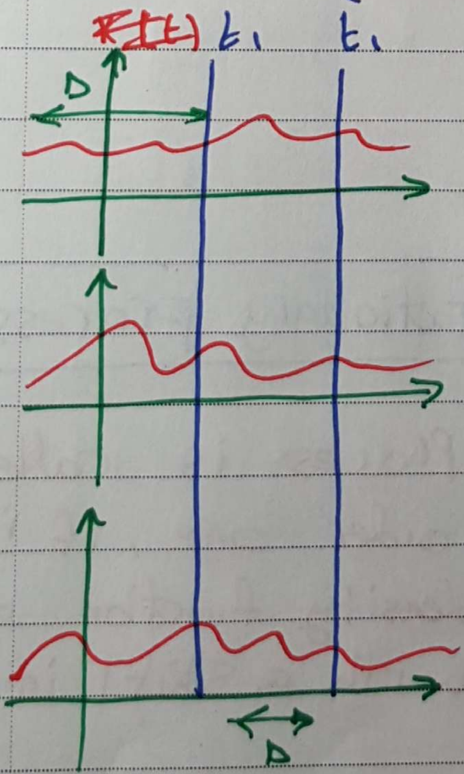
$$= 0$$

---

* Stationarity :

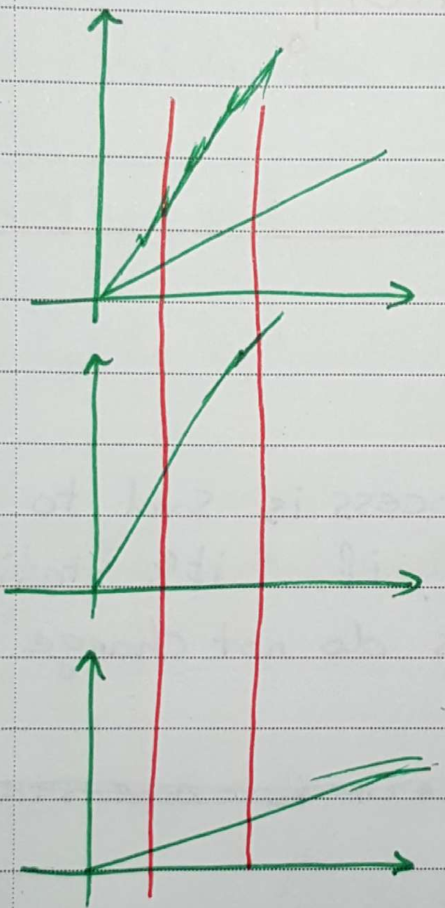
The Random Process is said to be stationary Process, if it's statistical ~~process~~ properties do not change with time.

~~Ex for non-stationary~~



No.

* Ex for non Stationary



$X(t) = at$

$a \sim u(0,1)$

$X(t_1) = at_1$

$E[X(t_1)] = E[X_1] =$

$= E[a]\,t_1$

$= \frac{1}{2} t_1$

This changes with time.

* 1st order Stationary process

A Random Process is called stationary to order one, if it's first order density function does not change with a shift in time origin.

No.

$$f_x(x_1;t_1) = f_x(x_1;t_1+\Delta)$$

→ for any t_1 and any real number Δ.

* <u>Consequence</u> :

$$E[X(t_1)] = E[X_1] = \bar{X} = \text{constant}$$

<u>proof</u> :

where $t_2 = t_1 + \Delta$

$$E[X(t_1)] = E[X(t_2)] = \bar{X} = \text{constant}$$

$$E[X(t_1)] = E[X_1] = \int_{-\infty}^{\infty} x_1 \, f_x(x_1;t_1) \, dx_1$$

$$E[X(t_2)] = E[X_2] = \int_{-\infty}^{\infty} x_2 \, f_x(x_2;t_2) \, dx_2$$

$$= \int_{-\infty}^{\infty} \Sigma \, f_x(\Sigma, t_2) \, d\Sigma$$

$$= \int_{-\infty}^{\infty} x_1 \, f_x(x_1, t_2) \, dx_1$$

No.

* 2nd Order Stationary process

~~* Consequence~~

$$\cancel{R_{xx}(t, t+\tau) = E[x(t), x(t+\tau)] = R_{xx}(\tau)}$$

Same thing goes here but for 2nd order. (see 1st order)

$$f_X(x_1, x_2 ; t_1, t_2) = f_X(x_1, x_2 ; t_1+\Delta, t_2+\Delta)$$

Correlation $R_{X_1X_2}(t_1, t_2)$

$$R_{X_1X_2}(t_1, t_2) = E[X_1(t_1)\, X_2(t_2)]$$

* Consequence Auto correlation

$$R_{xx}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$R_{xx}(\underset{t_1}{t}, \underset{t_2}{t+\tau}) = E[X(t)\, X(t+\tau)] = R_{xx}(\tau)$$

$\rightarrow t_2 - t_1 = \tau$

* Auto correlation is a function of τ only. (Time difference)

$$\boxed{\tau = t_2 - t_1}$$

FIVE APPLE

No.

* This gives first order Stationary process

* Nth order Stationary process is also kth order Stationary such that $K \leq N$

* **Wide Sense Stationary (WSS)**

2 conditions to be WSS

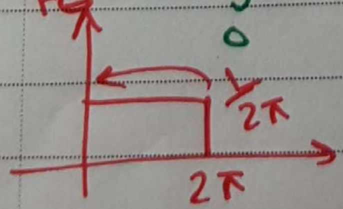
① $E[X(t)] = \bar{X} = \text{constant}$

② $R_{xx}(t, t+\tau) = E[X(t) \cdot X(t+\tau)] = R_{xx}(\tau)$

Ex: $X(t) = A\cos(\omega_0 t + \theta)$, ~~where A and θ~~ where A and ω_0 are constants, $\theta \sim U(0, 2\pi)$ Show that $X(t)$ is WSS process.

1. $E[X(t)] = E[A\cos(\omega_0 t + \theta)]$

$f_\Theta(\theta) = \int_0^{2\pi} A\cos(\omega_0 t + \theta) \cdot \frac{1}{2\pi} \, d\theta$



No.

$$E[X(t)] = \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega_0 t + \theta)\,d\theta$$

$$= \frac{A}{2\pi} \sin(\omega_0 t + \theta)\Big|_0^{2\pi}$$

$$= \frac{A}{2\pi}\Big(\underbrace{\sin(\omega_0 t + 2\pi) - \sin(\omega_0 t)}_{\text{zero}}\Big)$$

$$\rightarrow E[X(t)] = \text{zero} = \text{constant}$$

~~2. $R_{xx}(t, t+\tau)$~~

$$2.\ R_{xx}(t, t+\tau) = E[X(t) \cdot X(t+\tau)]$$

$$= \int_0^{2\pi} \frac{A\cos(\omega_0 t + \theta) \cdot A\cos(\omega_0(t+\tau) + \theta)}{2\pi}\,d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}\Big(\overbrace{\cos(2\omega_0 t + \omega_0\tau + 2\theta)}^{\text{zero}} + \underbrace{\cos(\omega_0\tau)}_{\text{constant}}\Big)\,d\theta$$

$$= \frac{A^2}{\frac{4\pi}{2}} \cos(\omega_0\tau) \cdot \cancel{2\pi}$$

$$= \frac{A^2}{2}\cos(\omega_0\tau)$$

~~Therefore~~

FIVE APPLE

No.

$$R_{xx}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$= \int_0^{2\pi} A\cos(\omega_0 t_1 + \theta)\, A\cos(\omega_0 t_2 + \theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{A^2}{2\pi} \int_0^{2\pi} \frac{1}{2} \left[\overbrace{\cancel{\cos(\omega_0(t_1+t_2)+2\theta)}}^{\text{zero}} + \underbrace{\cos(\omega_0(t_1-t_2))}_{\text{constant}} \right] d\theta$$

$$= \frac{A^2}{\cancel{2\pi}(2)} * \cos(\omega_0(t_1-t_2)) * \cancel{2\pi}$$

$$= \frac{A^2}{2} \cos(\omega_0(t_1-t_2)), \quad \tau = t_1 - t_2 \text{ or } t_2 - t_1$$

* this will not make a ~~differen~~ difference because $R_{xx}(\tau)$ is an even function

* Therefore the process is WSS

No.

* N^{th} order Stationary process

$$f_X(x_1, x_2 \cdots x_N;\ t_1, t_2 \cdots t_N)$$

$$= f_X(x_1, x_2 \cdots x_N;\ t_1+\Delta, t_2+\Delta \ldots t_N+\Delta)$$

Strict order Stationary

→ The process is Stationary to all orders.

* Jointly WSS

two conditions

1. $R_{xy}(t_1, t_2) = E[X(t_1)\,Y(t_2)]$

2. $R_{xy}(t, t+\tau) = E[X(t)\,Y(t+\tau)] = R_{xy}(\tau)$

FIVE APPLE

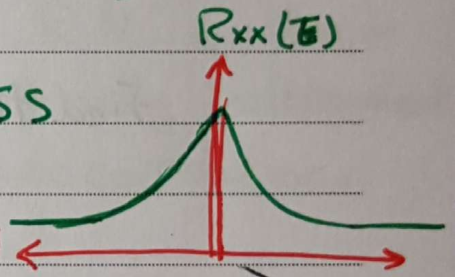
No.

* correlation functions

A) Auto correlation function & it's properties.

$R_{xx}(t, t+\tau) = E[X(t)\,X(t+\tau)]$

if the process is WSS



$R_{xx}(t, t+\tau) = R_{xx}(\tau)$

properties of Auto correlation function.

1. $R_{xx}(\tau) \leq R_{xx}(0)$ max value at 0

2. $R_{xx}(\tau) = R_{xx}(-\tau)$ ⇒ even function

3. $R_{xx}(0) = E[(X(t))^2]$ ⇒ average Power of the process

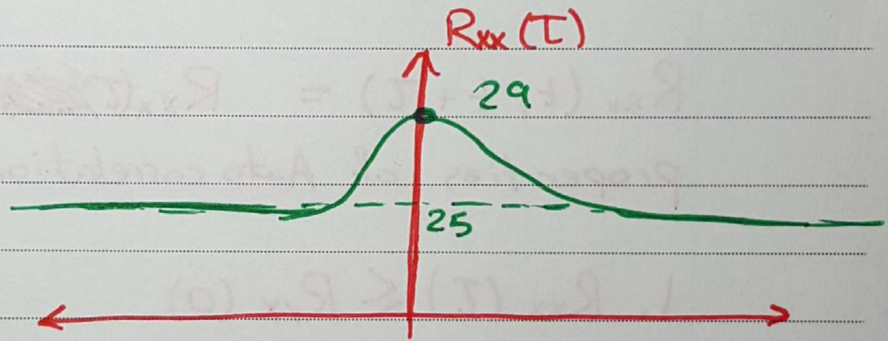
4. $\lim\limits_{|\tau| \to \infty} R_{xx}(\tau) = \bar{X}^2$

No.

Ex : Given the Auto correlation function of the following WSS process $X(t)$ as:

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

Find the mean & the variance of the process



$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{X}^2 = 25$$

$$\bar{X} = \pm\sqrt{25} = \pm 5$$

$$E[X^2(t)] = 29 = R_{xx}(0)$$

$$\sigma_x^2 = E[X^2(t)] - \bar{X}^2$$

$$\boxed{\sigma_x^2 = 4}$$

No.

Ex: $x(t)$ is WSS process with ~~$R_{xx}(t)$~~

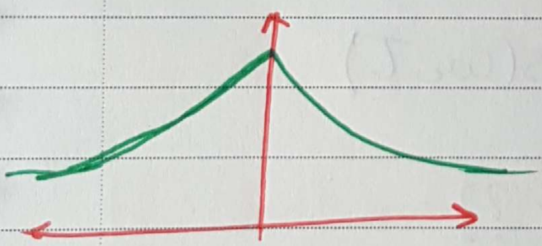
$$R_{xx}(\tau) = e^{-a|\tau|}$$

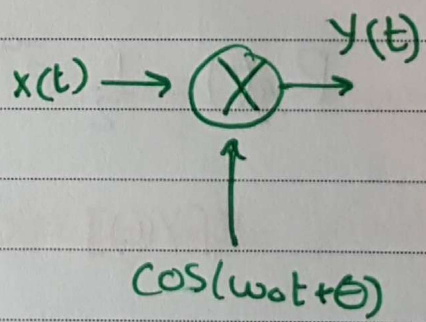
where $a > 0$ is a constant.

$X(t)$ amplitude modulates a carrier $\cos(\omega_0 t + \Theta)$, where ω_0 is constant and Θ is a R.V uniform on $(-\pi, \pi)$ that is statistically independent of $X(t)$, Determine the Autocorrelation function of $y(t)$ that

$$y(t) = X(t)\cos(\omega_0 t + \Theta).$$

Sol





$\rightarrow$ ~~$R_{yy}(t)$~~ $R_{yy}(\tau)$

~~$R_{yy}(t, t+\tau)$~~

$$R_{yy}(t, t+\tau) = E[y(t)\; y(t+\tau)]$$

$$\cancel{= E[X(t)\cos(\omega_0 t + \Theta) * X(t+\tau)\cos(\omega_0 t + \tau + \Theta)]}$$

FIVE APPLE

Scanned by CamScanner

No.

$$= E\left[X(t)\cos(\omega_0 t+\theta)\cdot X(t+\tau)\cos(\omega_0(t+\tau)+\theta)\right]$$

$$= \underbrace{E[X(t)\,X(t+\tau)]}_{R_{xx}(\tau)}\ \cancel{E[\cos(\omega_0 t)\cos}$$

$$* E\left[\cos(\omega_0 t+\theta)\cos(\omega_0(t+\tau)+\theta)\right]$$

$$= R_{xx}(\tau)\cdot E\left[\tfrac{1}{2}\left(\cos\omega_0\tau + \overset{0}{\cos(2\omega_0 t+\omega_0\tau+2\theta)}\right)\right]$$

$$= \frac{1}{2} R_{xx}(\tau)\cos(\omega_0\tau)$$

$$\cancel{\frac{1}{2} e^{-a|\tau|}\cos(\omega_0\tau)}$$

$$R_{yy}(\tau) = \frac{1}{2} e^{-a|\tau|}\cos(\omega_0\tau)$$

$E[Y(t)] = \text{constant}$??

$$E[Y(t)] = E[X(t)\cos(\omega_0 t+\theta)]$$

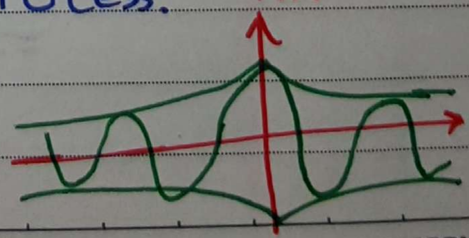
$$= E[\underbrace{\cancel{X(t)}}_{\text{zero}}]\ E[\cos(\omega_0 t+\theta)]$$

$$\bar{Y} = 0$$

$Y(t)$ is WSS process.

$R_{yy}(\tau)$

~~$R_{xy}(t)$~~



13/12

No.

B) The cross correlation function and it's properties

Two processes $X(t)$ and $Y(t)$

$\cancel{R_{xy}(t, t+\tau) = E[X(t)\, Y(t)]}$

$$R_{xy}(t, t+\tau) = E\left[X(t)\; Y(t+\tau)\right]$$

* If $X(t)$ and $Y(t)$ are atleast jointly wss, then $R_{xy}(\tau) = E[X(t)\; Y(t+\tau)]$

* If $R_{xy}(t, t+\tau) = 0$ (orthogonal processes)

* If $X(t)$ and $Y(t)$ are jointly wss and Independent
  $R_{xy}(t, t+\tau) = \bar{X}\; \bar{Y}$, which is a constant.

No.

Properties of the cross-correlation function :

1) $R_{xy}(-\tau) = \cancel{R_{xy}}\ R_{yx}(\tau)$

$R_{xy}(-\tau) = E[X(t)\ Y(t-\tau)]$

$t-\tau = t'$

$\cancel{R_{xy}t} \quad = E[X(t'+\tau)\ Y(t')]$

$= R_{yx}(\tau)$

2) $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)}$ * geometric mean

3) $|R_{xy}(\tau)| \leq \frac{1}{2}[R_{xx}(0)+R_{yy}(0)]$ * Arithmetic mean

* Arithmetic mean > geometric mean

~~always~~

$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)} \leq \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$

~~4) $E[(Y(t+\tau) + \alpha X(t))^2] \geq 0$~~

to prove 2

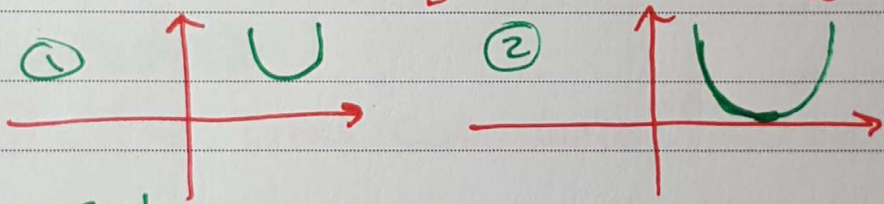
$$E\left[\cancel{\xi}\left(Y(t+\tau)+\alpha X(t)\right)^2\right] \cancel{\leq} \geq 0$$

$$E\left[\cancel{\xi}\, Y^2(t+\tau) + 2\alpha X(t)Y(t+\tau) + \alpha^2 X^2(t)\right] \geq 0$$

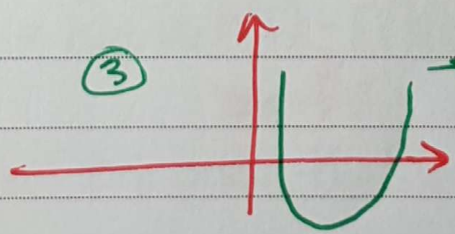
$$E[Y^2(t+\tau)] + 2\alpha E[X(t)Y(t+\tau)] + \alpha^2 E[X^2(t)] \geq 0$$

لأنها مع نفسها

$$\underbrace{R_{YY}(0)}_{c} + 2\alpha \underbrace{R_{XY}(\tau)}_{b} + \alpha^2 \underbrace{R_{XX}(0)}_{a} \geq 0 \qquad \geq 0$$



only ~~one of~~ these two solutions are possible.



This is impossible

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$b^2-4ac \leq 0$ for solutions ① & ② to be possible

No.

$$\cancel{4} R_{xy}^{2}(\tau) - \cancel{4} R_{xx}(0) R_{yy}(0) \leq 0$$

$$R_{xy}^{2}(\tau) \leq R_{xx}(0) R_{yy}(0)$$

$$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) R_{yy}(0)}$$

15/12

No.

* Covariance functions

I. Auto covariance function (C_{xx})

~~$C_{xx}(t, t+\tau) = E[(X(t) - E[X(t)])($~~

$$C_{xx}(t, t+\tau) = E\left[\{X(t) - E[X(t)]\} * \{X(t+\tau) - E[X(t+\tau)]\}\right]$$

$$= R_{xx}(t, t+\tau) - E[X(t)]\,E[X(t+\tau)]$$

~~$C_{xx}(t, t+\tau) =$~~

II. The cross covariance function

$$C_{xy}(t, t+\tau) = E\left[\{X(t) - E[X(t)]\} * \{Y(t+\tau) - E[Y(t+\tau)]\}\right]$$

$$= R_{xy}(t, t+\tau) - E[X(t)]\,E[Y(t+\tau)]$$

*For a process that is atleast WSS

$\Rightarrow C_{xx}(\tau) = R_{xx}(\tau) - \bar{X}^2$

$\Rightarrow C_{xy}(\tau) = R_{xy}(\tau) - \bar{X}\,\bar{Y}$

No.

توضيح للـ cross-covariance

$\cancel{E[\{X(t) - E[X(t)]\} \cdot \{E[Y(t+}$

$$E\left[\left(X(t) - E[X(t)]\right) \cdot \left(Y(t+\tau) - E[Y(t+\tau)]\right)\right]$$

$$= E\left[X(t)\,Y(t+\tau) - E[X(t)]\,Y(t+\tau) - E[Y(t+\tau)]\,X(t) + E[X(t)]\,E[Y(t+\tau)]\right]$$

$$= E[X(t)Y(t+\tau)] - E[X(t)]\,E[Y(t+\tau)] - E[Y(t+\tau)]\,E[X(t)] + E[X(t)]\,E[Y(t+\tau)]$$

$$= R_{xy}(\tau) - \bar{X}\,\bar{Y}$$

FIVE APPLE

No.

*Ergodicity

New operator: Time average:

$A[\cdot]$: Time average

$$A[\cdot] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\cdot]\, dt$$

$$\bar{X} = A[\underbrace{X(t)}_{\text{one sample function}}] = \lim \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

$\bar{X}$: Time average mean.

$$\mathcal{R}_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

↳ Time average autocorrelation function

*If the statistical averages ~~...~~

$\bar{X} = \bar{x}$ and $R_{xx}(\tau) = \mathcal{R}_{xx}(\tau)$

$\Rightarrow$ (Ergodic process)

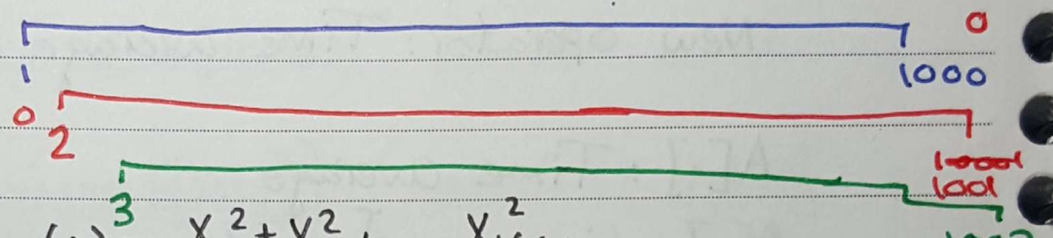
Time average mean

Time average autocorrelation function.

No.

Explanation

Data Sheet with a 1000 pt



$$R_{xx}(0) = \frac{X_1^2 + X_2^2 + \dots X_{1000}^2}{1000}$$

عددهم ← 1000

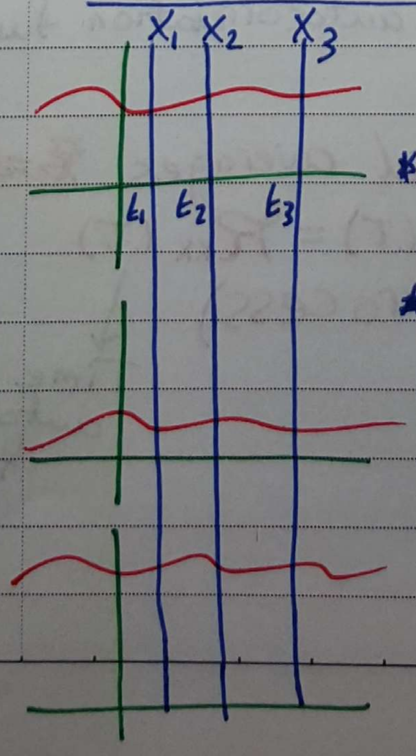
$$R_{xx}(1) = \frac{X_1 X_2 + X_2 X_3 \dots X_{999} X_{1000}}{999}$$

shift by 1

$$R_{xx}(2) = \frac{X_1 X_3 + X_2 X_4 \dots + X_{998} X_{999}}{998}$$

---

* Gaussian Random process



$$* \; f_{X_1}(x_1; t_1) = \frac{1}{\sqrt{2\pi\sigma_{x_1}^2}} e^{-(x_1 - a_{x_1})^2 / 2\sigma_{x_1}^2}$$

* * N^{th} order jointly Gaussian

C_N: covariance matrix

FIVE APPLE

Scanned by CamScanner

No.

$$f_X(x_1, x_2 \cdots x_N ; t_1, t_2 \cdots t_N) = \frac{\exp\left[2^{-1}[X-\overline{X}]^T [C_X]^{-1} [X-\overline{X}]\right]}{\sqrt{(2\pi)^N \left|[C_X]\right|}}$$

$$C_X = E\left[\begin{bmatrix} X_1-\overline{X}_1 \\ X_2-\overline{X}_2 \\ \vdots \\ X_N-\overline{X}_N \end{bmatrix} \left[X_1-\overline{X}_1, X_2-\overline{X}_2, \ldots X_N-\overline{X}_N\right]\right]$$

18/12

No.

Problem 6.3-37: Determine the largest K(constant) such that the function

$R_{xy}(\tau) = K e^{-\tau^2} \sin(\pi\tau)$

given that $E[X^2(t)] = 6$, $E[Y^2(t)] = 4$

Sol:

$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)\, R_{yy}(0)}$

$E[X^2(t)] = R_{xx}(0)$, $E[Y^2(t)] = R_{yy}(0)$

$$\frac{dR_{xy}(\tau)}{d\tau} = \cancel{K} e^{-\tau^2}(\pi)\cos(\pi\tau) + K\sin(\pi\tau) \cdot e^{-\tau^2} \cdot (-2\tau) = 0$$

in rad

$$\tan(\pi\tau) = \frac{\pi}{2\tau}$$

$$\tan(\pi\tau) - \frac{\pi}{2\tau} = 0 \quad \text{(by trial \& error)}$$

$\Rightarrow \tau = 0.41734$

$$R_{xy}(0.41734) = K e^{-(0.41734)^2} \sin(\pi(0.41734))$$

$R_{xy}(0.41734) = 0.812\,K$

$0.812\,K \leq \sqrt{(4)(6)}$

$\Rightarrow K \leq 6.0334$

FIVE APPLE

Scanned by CamScanner

No.

Problem ~~[illegible]~~ 6.5-1

A gaussian random process has an Auto correlation function

$$R_{xx}(\tau) = 6e^{-|\tau|/2}$$

Determine a Covariance Matrix for the Random Variables $X(t), X(t+1)$ $X(t+2)$ and $X(t+3)$.

$$\begin{bmatrix} X(t)-\bar{X}_0 \\ X(t+1)-\bar{X}_1 \\ X(t+2)-\bar{X}_2 \\ X(t+3)-\bar{X}_3 \end{bmatrix} \quad \text{where} \quad \begin{aligned} \bar{X}_0 &= E[X(t)] \\ \bar{X}_1 &= E[X(t+1)] \\ \bar{X}_2 &= E[X(t+2)] \\ \bar{X}_3 &= E[X(t+3)] \end{aligned}$$

$$\lim_{|\tau|\to\infty} R_{xx}(\tau) = \bar{X}^2 = 0$$

$$\Rightarrow \bar{X} = 0 \rightarrow \bar{X}_0 = \bar{X}_1 = \bar{X}_2 = \bar{X}_3 = 0$$

$$C = E\left[\begin{bmatrix} X(t) \\ X(t+1) \\ X(t+2) \\ X(t+3) \end{bmatrix} [X(t), X(t+1), X(t+2), X(t+3)]\right]$$

FIVE APPLE

No.

$$C = \begin{bmatrix} E[X^2(t)] & E[X(t)X(t+1)] & E[X(t)\,X(t+2)] & E[X(t)X(t+3)] \\ E[X(t+1)\,X(t)] & E[X^2(t+1)] & E[X(t+1)X(t+2)] & E[X(t+1)X(t+3)] \\ E[X(t+2)X(t)] & E[X(t+2)X(t+1)] & E[X^2(t+2)] & E[X(t+2)X(t+3)] \\ E[X(t+3)X(t)] & E[X(t+3)X(t+1)] & E[X(t+3)X(t+2)] & E[X^2(t+3)] \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 6e^{-\frac{1}{2}} & 6e^{-1} & 6e^{-3/2} \\ 6e^{-1/2} & 6 & 6e^{-\frac{1}{2}} & 6e^{-1} \\ 6e^{-1} & 6e^{-\frac{1}{2}} & 6 & 6e^{-\frac{1}{2}} \\ 6e^{-3/2} & 6e^{-1} & 6e^{-\frac{1}{2}} & 6 \end{bmatrix}$$

FIVE APPLE

20/12

No.

حل امتحان السكند:

Q1: Random variable X

$Y = g(x)$

a) Find $f_y(y)$ if $g(x) = 2F_x(x) + 4$

Sol

g(x) non decreasing &

$4 < g(x) < 6$

because continuos ⇒ monotonically increasing

$$f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} = \frac{\cancel{f_x(x_1)}}{2\cancel{f_x(x_1)}} = \frac{1}{2} \quad , 4<x<6$$

$$f_y(y) = \begin{cases} \frac{1}{2} & , 4<x<6 \\ 0 & , o.w \end{cases}$$

b) Find $y = g(x)$ such that y is uniform between (6, 10) , $6 < Y < 10$

$Y = g(x) = 4F_x(x) + 6$

FIVE APPLE

No.

Q2 Poisson density function

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

a) Find $\Phi_X(\omega)$

$$\Phi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} \left(e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k) \right) e^{j\omega x} dx$$

$$\cancel{= \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k) \int_{-\infty}^{\infty}}$$

$$= e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \int_{-\infty}^{\infty} e^{j\omega x} \delta(x-k) \, dx$$

$$= e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} e^{j\omega k}$$

$$= e^{-b} \sum_{k=0}^{\infty} \frac{(b e^{j\omega})^k}{k!}$$ → Taylor Series

~~Taylor~~

$$\Phi_X(\omega) = e^{-b(1-e^{j\omega})}$$

No.

b) Find mean & variance

$$m_1 = (-j)\frac{d\Phi_x(\omega)}{d\omega}\Big|_{\omega=0} = b$$

mean

$$m_2 = (-j)^2 \frac{d^2\Phi_x(\omega)}{d\omega^2}\Big|_{\omega=0} = b + b^2$$

$$\sigma_x^2 = m_2 - m_1^2 = b + b^2 - b^2 = b$$

variance

---

coordinate rotation

<u>Q3</u> $\theta = \pi/6$, X & Y R.Vs

$C_{xy} = 0.8$, $\sigma_x = 1.9$

$$Y_1 = X\cos\theta + Y\sin\theta$$

$$Y_2 = -X\sin\theta + Y\cos\theta$$

a) find Y_1 & Y_2 in terms of X & Y

$$Y_1 = \frac{\sqrt{3}}{2}X + \frac{1}{2}Y$$

$$Y_2 = -\frac{1}{2}X + \frac{\sqrt{3}}{2}Y$$

No.

b) Find σ_y^2

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2C_{xy}}{\sigma_x^2 - \sigma_y^2}\right)$$

$$\frac{\pi}{6} = \frac{1}{2}\tan^{-1}\left(\frac{2(0.8)}{(1.9)^2 - \sigma_y^2}\right)$$

$$\boxed{\sigma_y^2 = 2.686}$$

Q4 $\cancel{f_{xy}(x,y) = 0.4\,\delta(x+\alpha)}\,\delta(y-2)$

$$f_{xy}(x,y) = 0.4\,\delta(x+\alpha)\,\delta(y-2) + 0.3\,\delta(x-\alpha)\,\delta(y-2) + 0.1\,\delta(x-\alpha)\,\delta(y-\alpha) + 0.2\,\delta(x-1)\,\delta(y-1)$$

a) X & Y are uncorrelated

$C_{xy} = 0$, $C_{xy} = R_{xy} - \bar{X}\bar{Y}$, $R_{xy} = \bar{X}\bar{Y}$

$$E[XY] = E[X] \cdot E[Y]$$

$$E[X] = 0.4(-\alpha) + 0.3(\alpha) + 0.1(\alpha) + 0.2(1)$$
$$= 0.2$$

$$E[Y] = 0.4(2) + 0.3(2) + 0.1(\alpha) + 0.2(1)$$
$$= 0.1\alpha + 1.6$$

FIVE APPLE

No.

$E[XY] = 0.4(-\alpha)(2) + 0.3(\alpha)(2)$
$+ 0.1(\alpha)(\alpha) + 0.2(1)(1)$

$\cancel{= 0.8\alpha}$

$= -0.8\alpha + 0.6\alpha + 0.1\alpha^2 + 0.2$

$= 0.1\alpha^2 - 0.2\alpha + 0.2$

$E[XY] - \bar{X}\bar{Y} = 0.1\alpha^2 - 0.2\alpha + 0.2 - (0.2)(0.1\alpha + 1.6)$

$\boxed{\alpha_1 = 2.65}$ $\boxed{\alpha_2 = -0.45}$

b) X & Y are orthogonal

$R_{xy} = 0$, $E[XY] = 0$

$0.1\alpha^2 - 0.2\alpha + 0.2 = 0$

$\boxed{\alpha_1 = 1 + j}$

$\boxed{\alpha_2 = 1 - j}$

No.

Q5 X_1, X_2 and X_3 are Statistically independent

$\overline{X}_1 = -1$, $\sigma_{x_1}^2 = 2.0$

$\overline{X}_2 = 0.6$, $\sigma_{x_2}^2 = 1.5$

$\overline{X}_3 = 1.8$, $\sigma_{x_3}^2 = 0.8$

$Y = X_1 + X_2 + X_3$

$\overline{Y} = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 = 1.4$

$\sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2 = 4.3$

$$f_y(y) = \frac{1}{\sqrt{2\pi(4.3)}} e^{-(y-1.4)^2/2(4.3)} \qquad -\infty < y < \infty$$

$$Y \sim N(1.4, 4.3)$$

FIVE APPLE

22/12

No.

Chapter 7

Random processes: spectral characteristics

$$X(t) = X(S, t)$$

Deterministic Signal $x(t)$ Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) \xleftrightarrow{F} X(\omega)$$

No.

* The power density Spectrum

For a Random process $X(t)$,

$X_T(t)$ be defined as a portion of a sample function $x(t)$ that exists between $-T$ and T.

$$X_T(t) = \begin{cases} x(t) & , -T < t < T \\ 0 & , \text{otherwise} \end{cases}$$

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

---

* The energy contained in $x(t)$ in the interval $(-T, T)$:

$$E(T) = \int_{-T}^{T} x_T^2(t)\, dt = \int_{-T}^{T} x^2(t)\, dt$$

* The parseval's relation (Appendix D)

~~Appe~~

$$\int_{-\infty}^{\infty} x_1^*(\tau)\, x_2(\tau)\, d\tau = \cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega) X_2(\omega)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1^*(\omega)\, X_2(\omega)\, d\omega$$

No.

If $x_1(t) = x_2(t) = x(t)$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$E(T) = \int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

divide by 2T

$$P(T) = \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

The average power in the process

$$P_{xx} = \lim_{T\to\infty} \int_{-T}^{T} E[x^2(t)]\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[X_T^2(t)]}{2T} d\omega$$

$$\rho_{xx}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_{xx}(\omega)\, d\omega$$

frequency domain

No.

$$P_{xx} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[x^2(t)]\, dt$$

~~$P_{xx} = A[E[x^2(t)]]$~~

$P_{xx} = A[E[X^2(t)]]$, time domain

$P_{xx}(\omega)$: The power density Spectrum.
(The powers spectral density)

Ex: The Random process
$X(t) = A_0 \cos(\omega_0 t + \Theta)$
A_0 and ω_0 are constants.
Θ is uniform R.V , $\Theta \sim U(0, \frac{\pi}{2})$
Find P_{xx} ??

$$E[X^2(t)] = E[A_0^2 \cos^2(\omega_0 t + \Theta)]$$

$$= E\left[\frac{A_0^2}{2} + \frac{A_0^2}{2}\cos(2\omega_0 t + 2\Theta)\right]$$

$$= \frac{A_0^2}{2} + \frac{A_0^2}{2} \int_0^{\pi/2} \cos(2\omega_0 t + 2\Theta) \cdot \frac{1}{\pi}\, d\Theta$$

No.

$$= \frac{A_0^2}{2} - \frac{A_0^2}{\pi} \sin(2\omega_0 t)$$

$$\cancel{\frac{A_0^2}{2}\int_0^{\pi/2} \cos(2\omega_0 t + 2\theta)\,d\theta}$$

$$\cancel{= \frac{A_0^2}{2} \sin(2\omega_0 t + 2\theta)}$$

$$P_{xx} = A\left[E\left[X^2(t)\right]\right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{A_0^2}{2} - \frac{A_0^2}{\pi} \sin(2\omega_0 t) \right) dt$$

$$\boxed{P_{xx} = \frac{A_0^2}{2}}$$

FIVE APPLE