Department of Electrical Engineering
Faculty of Engineering & Technology
University of Jordan

Course Title: Probability & Random Variables	**Exam:** 2^{nd} Exam	**Date:** Dec/20/2015
Course No.: 0903321	**Semester:** 1^{st} Term 2015-2016	**Time Period:** 1:30 Hr.
Instructor: Dr. Ahmad Atieh & Prof Mohammed Khasawneh		

Q.1	Q.2	Q.3	Q.4	Total /30
8	6	3	3.5	20.5

Student Name: [redacted]

Student Number:

Section: [redacted]

POWERUNIT

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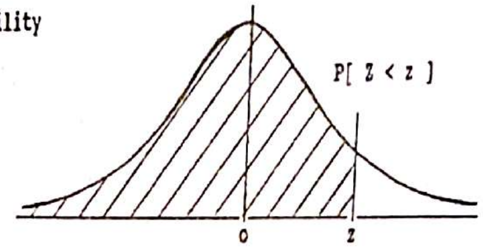
STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[\,Z < z\,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

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Q1) (7 marks)

A discrete random variable X has the probability density function

$$p_x[x_i] = \begin{cases} \frac{1}{2} & x_1 = -1 \\ \frac{1}{4} & x_2 = -\frac{1}{2} \\ \frac{1}{8} & x_3 = 0 \\ \frac{1}{16} & x_4 = \frac{1}{2} \\ \frac{1}{16} & x_5 = 1 \end{cases}$$

a) If $Y = \cos(\pi X)$, find the density function for Y?
b) Find the probability density function for X and Y and sketch them?
c) Calculate the coefficient of skewness for X?

$$F_X(x) = \frac{1}{2} u(x+1) + \frac{1}{4} u\left(x+\frac{1}{2}\right) + \frac{1}{8} u(x)$$
$$+ \frac{1}{16} u\left(x-\frac{1}{2}\right) + \frac{1}{16} u(x-1)$$

Y ~~x_1~~ $X_1 = -1 \Rightarrow Y_1 = \cos(-\pi) = -1$

$X_2 = -\frac{1}{2} \Rightarrow Y_2 = 0$

$X_3 = 0 \Rightarrow Y_3 = 1$

$X_4 = \frac{1}{2} \Rightarrow Y_4 = 0$

$X_5 = 1 \Rightarrow Y_5 = -1$

1

$$F_Y(y) = \left(\frac{1}{2} + \frac{1}{16}\right) \delta(y+1)$$
$$+ \left(\frac{1}{4} + \frac{1}{16}\right) \delta(y)$$
$$+ \frac{1}{8} \delta(y-1)$$

a) $F_Y(y) = \frac{9}{16} \delta(y+1) + \frac{5}{16} \delta(y)_3 + \frac{2}{16} \delta(y-1)$

Look @ page 5

part (c) ## ## Conf. of skewness = $\frac{M_3}{\sigma_x^3}$

$= .5074457$

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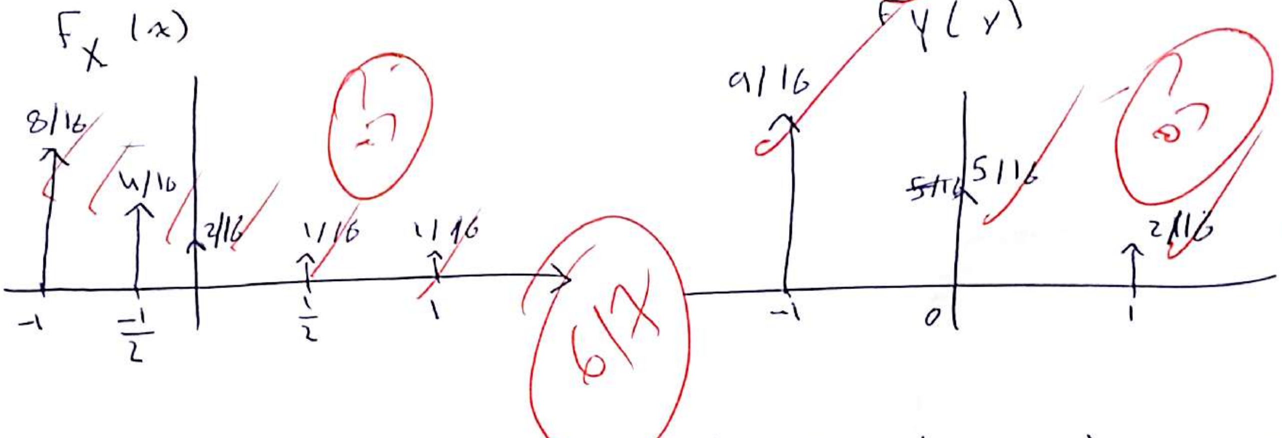
Q1

b) $F_X(x) = \frac{1}{2}\delta(x+1) + \frac{1}{4}\delta(x+\frac{1}{2}) + \frac{1}{8}\delta(x) + \frac{1}{16}\delta(x-\frac{1}{2}) + \frac{1}{16}\delta(x-1)$

$F_Y(y) = \frac{9}{16}\delta(y+1) + \frac{5}{16}\delta(y) + \frac{2}{16}\delta(y-1)$



c) $m_1 = E[X] = \bar{X} = \frac{1}{2}\cdot -1 + \frac{1}{4}\cdot\frac{-1}{2} + \frac{1}{8}\cdot 0 + \frac{1}{16}\cdot\frac{1}{2} + \frac{1}{16}\cdot 1$

$= -.53125$ 1.0

$m_2 = E[X^2] = (\frac{1}{2})\cdot(-1)^2 + (\frac{1}{4})\cdot(\frac{-1}{2})^2 + 0 + \frac{1}{16}\cdot(\frac{1}{2})^2 + \frac{1}{16}\cdot(1)^2$

$= .640625 = \sigma_x^2 \Rightarrow \sigma_x = \sqrt{.640625}$ 1

$m_3 = E[X^3] = \frac{1}{2}\cdot(-1)^3 + \frac{1}{4}\cdot(\frac{-1}{2})^3 + 0 + \frac{1}{16}\cdot(\frac{1}{2})^3 + \frac{1}{16}\cdot(1)^3$

$= -.4609375$ 0.7

$M_3 = E((X-\bar{X})^3) = E[(X^2 - 2X\bar{X} + \bar{X}^2)(X - \bar{X})]$ 0.

$E[X^3 - 2X^2\bar{X} + \bar{X}^2X - \bar{X}X^2 + 2X\bar{X}^2 - \bar{X}^3] \Rightarrow E[X^3 - 3\bar{X}X^2 + 3X\bar{X}^2 - \bar{X}^3]$

$\Rightarrow m_3 - 3m_1m_2 + 2m_1^3$ $= .2660193 = M_3$ 0.5

Conf. of skewness $= M_3/\sigma_x^3 = .5074$

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Q2) (8 marks)

An analog-to-digital converter (ADC) with L levels is used in a digital communication system. **Binary digits** are also used in the mapping process of each sample to "a" proper ADC level. The sample is mapped to an upper level if the sample value is greater than or equal to half the step size (Δv); **see the figure below**. Assume each level is represented by **n** bits and the dynamic range of the A/D converter is M. Quantization error usually occurs if any sample lies between two successive quantization levels. Assume that the quantization error is a random variable with uniform probability density function, and "on the average" equal numbers of 0 and 1 bits are transmitted over the channel.

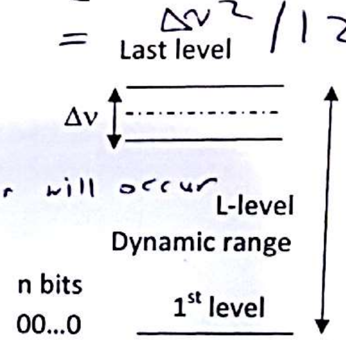
Derive the quantization noise variance?

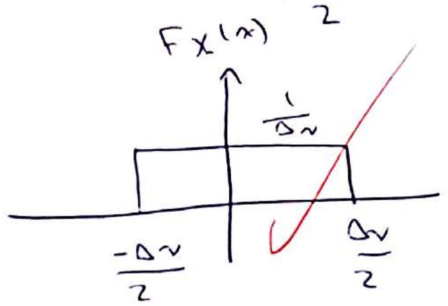
So η_2 = Variance $= \Delta v^2 / 12$

The value is $x \Rightarrow$ if $\not< x$

$-\frac{\Delta v}{2} < x$

$-\frac{\Delta v}{2} < x < \frac{\Delta v}{2}$ error will occur





$$f_X(x) = \begin{cases} \frac{1}{\Delta v} & -\frac{\Delta v}{2} < x < \frac{\Delta v}{2} \\ 0 & o.w \end{cases}$$

8/8

$$\eta_2 = E\left[x^2 - 2x\bar{x} + \bar{x}^2\right] = E[x^2] - \bar{x}^2 = \frac{\Delta v^2}{12} - 0 = \frac{\Delta v^2}{12}$$

$$E[x] = \int_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}} x \cdot \frac{1}{\Delta v}\, dx \Rightarrow \left.\frac{x^2}{2\Delta v}\right|_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}}$$

$$\boxed{m_2 = \frac{\Delta v^2}{12}}$$

$$= \frac{1}{2\Delta v}\left(\frac{\Delta v^2}{4} - \frac{\Delta v^2}{4}\right) = 0$$

$$E[x^2] = \frac{1}{\Delta v}\int_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}} x^2\, dx \Rightarrow \left.\frac{x^3}{3\Delta v}\right|_{-\frac{\Delta v}{2}}^{\frac{\Delta v}{2}} \Rightarrow \frac{1}{3\Delta v}\left(\frac{\Delta v^3}{8} + \frac{\Delta v^3}{8}\right)$$

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Q3) (7 marks)

In a digital communication channel, information is transmitted in the form of "1"s and "0"s. In this context 5-volts represent logic '1' as 0-volts represent logic '0'. Further, the signal is contaminated by a source of Gaussian noise with mean equal zero and variance equal to 4. Suppose there are twice as many "ones" as there are "zeros". In a fairly simplistic detection model, the decision threshold is commonly located half way between the transmitted logic levels. Find:

$a_x = 0$ $\sigma_x^2 = 4$

a. The entropy of the source generating this type of signals
b. The error that comes out as a result of the noise source

$$F_X(x) = \frac{1}{\sqrt{2\pi \cdot 4}} e^{\frac{-(x)^2}{2 \cdot 4}} = \frac{1}{\sqrt{8\pi}} e^{\frac{-x^2}{8}}$$

$$\text{Entropy} = E\left[\frac{1}{\log_2(x)}\right] = E\left[-\log_2(x)\right]$$

$$p(\text{one}) = \frac{2}{3}$$

$$p(\text{zero}) = \frac{1}{3}$$

$$= \int \sum_{i=1}^{2} p(x_i)\, -\log_2(x_i)$$

-4

~~information sent by~~

(a)

$$\text{Entropy} = \sum_{x=1} -\log_2(x_j)\ ;$$ ~~x that could be~~

$i = 1 \quad x_i = 1 \Rightarrow$ one sent

$i = 2 \quad x_i = 0 \Rightarrow$ zero sent

$$\text{Expected value} = \sum_{x_i}^{2^{x_j}} -\log_2(x_i)\ p(x_i)$$

$E[\text{Entropy}]$

$x_i \Rightarrow$ one sent

$x_j \Rightarrow$ zero sent

$$-\log_2(1)\frac{2}{3} + -\log_2(2)\frac{1}{3} = -\frac{2}{3}\log(1) - \frac{1}{3}$$

$$-\frac{2}{3}\log(x_i) - \frac{1}{3}\log(x_j)$$

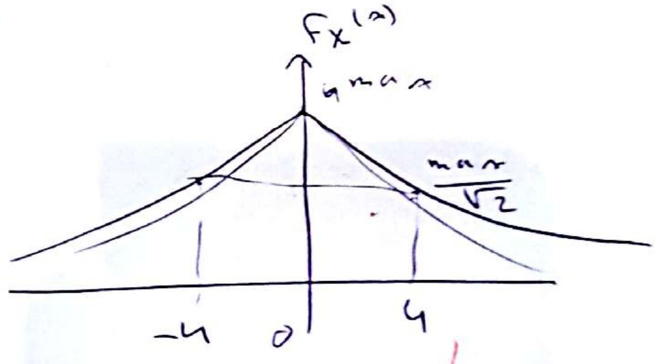
(b)

$+ P\{2.5 < x \leq 5\}$

$P\{0 < x \leq \overset{2.5}{5}\} = P\{\cancel{error}\}$ — 5 — 2.5 — 0

$= \cancel{F_X(5) - F_X(0)} \equiv 2(p(2.5) - p(0))$

Since $F_X(x)$ is a gaussian R.V



$2(.9938 - .5) = .9876 = P\{No\ error\}$

$F_X(x)\ \varepsilon(error) = $ Gaussian R.V

error will occur if x is between $(0 \to 2.5)$ or $(2.5 \to 5)$

$P\{error\} = 1 - .9876 = .0124$

it's value between $\boxed{\cancel{-2.5} \to 2.5}$ $\boxed{0 \to 2.5}$

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Q4 (8 marks)

If the joint density function of the two random variables ***X and Y*** is given by:

$$f_{XY}(x,y) = \begin{cases} 4y(x-y)e^{-(x+y)}, & 0 < x < \infty, 0 \le y \le x \\ 0, & elsewhere \end{cases}$$

Find E[***X***/***Y*** = *y*].

$$f_X(x \mid Y = y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{\infty}^{\infty} 4y(x-y)\, e^{-(x+y)}\, dx \qquad e^{-x} e^{-y}$$

-4.5

$$= 4y \left(\cancel{-4y} \underbrace{\int_0^{\infty} e^{-(x+y)}\, x\, dx}_{\text{Term \#1}} - \int_0^{\infty} e^{-(x+y)}\, y\, dx \right)$$

$\cancel{\text{Term \#1}}$

$$\int_0^{\infty} x\, e^{-(x+y)}\, dx$$

$$e^{-x} e^{-y}$$

Term # 1

$$f_Y(y) = 4y e^{-y} \left(\int_0^{\infty} x e^{-x}\, dx - \int_0^{\infty} y e^{-x}\, dx \right)$$

$\boxed{\text{Term \# 1}}$ Term # 2

$$\int_0^{\infty} x e^{-x}\, dx \Rightarrow$$

$$-x e^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x}\, dx$$

$$x e^{-x} \Big|_{\infty}^{0} - e^{-x} \Big|_0^{\infty}$$

$$x e^{-x} \Big|_{\infty}^{0} + e^{-x} \Big|_{\infty}^{0}$$

$= 1$

$\boxed{\text{Term \# 2}}$

$$y \int_0^{\infty} e^{-x}\, dx$$

$$\Rightarrow y (-) e^{-x} \Big|_0^{\infty}$$

$$y\, e^{-x} \Big|_{\infty}^{0} = y$$

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$$F_Y(y) = 4ye^{-y}(1-y) \qquad e^{-x} \quad e^{-y}$$

$$F_X(x \mid Y=y) = \frac{4y(x-y)e^{-(x+y)}}{4ye^{-y}(1-y)}$$

$$F_X(x \mid Y=y) = \frac{(x-y)e^{-x}}{1-y}$$

$$E[F_X(x \mid Y=y)] = \int_{-\infty}^{\infty} F_X(x \mid Y=y)\, F_X(x)\, dx$$

$$\& \Rightarrow F_X(x) = \int_{-\infty}^{\infty} 4y e^{-x} e^{-y}(x-y)\, dy$$

$$4e^{-x}\left(\int_0^x y e^{-y} x\, dy - \int_0^{x} y^2 e^{-y}\, dy\right)$$

$$-yxe^{-y}\Big|_0^x + x\int_0^x e^{-y}\, dy \Rightarrow xye^{-y}\Big|_x^0 + xe^{-y}\Big|_x^0$$

$$\text{Term } (1) = (-x^2e^{-x}) \mp xe^{-x}$$

$$\int_0^x y^2 e^{-y}\, dy \Rightarrow y^2e^{-y}\Big|_x^0 + 2\int_0^x ye^{-y}\, dy$$

$$y^2e^{-y}\Big|_x^0 + 2\left(ye^{-y}\Big|_x^0 + e^{-y}\Big|_x^0\right) \Rightarrow \underbrace{x^2e^{-x} + 2xe^{-x} + 2(1-e^{-x})}_{\text{Term } (2)}$$

$$4e^{-x} + \left(-(x^2 e^{-x} + x e^{-x}\right) - \left(x^2 e^{-x} + 2xe^{-x} + 2(1-e^{-x})\right)$$

$$F_X(x) = -4e^{-x}\left(x^2 e^{-x} + x e^{-x} + x^2 e^{-x} + 2x e^{-x} + 2 - 2e^{-x}\right)$$

$$F_X(x) = -4e^{-x}\left(2x^2 e^{-x} + 3x e^{-x} + 2 - 2e^{-x}\right)$$

$$E_{xpec.} = \int_0^\infty -4e^{-x}\left(2x^2 e^{-x} + 3x e^{-x} + 2 - 2e^{-x}\right) \frac{(x-y)}{1-y} dx$$

$$\int_0^\infty \frac{-4e^{-2x}}{1-y}\left(2x^2 e^{-x} + 3xe^{-x} + 2 - 2e^{-x}\right)(x-y)\, dx$$

$$\frac{-4}{1-y}\left(\int 2x^2 e^{-3x} dx + \int 3x e^{-3x} dx + \int 2e^{-2x} dx - \int 2e^{-2x} dx\,(x-y)\right)$$

$$\frac{-4}{1-y}\Big[\int_0^\infty 2x^3 e^{-3x} dx + \int_0^\infty 3x^2 e^{-3x} dx + \int_0^\infty 2xe^{-2x} dx - \int_0^\infty 2x e^{-2x} dx$$

$$-y\left(\int_0^\infty 2x^2 e^{-3x} dx + \int_0^\infty 3xe^{-3x} dx + \int_0^\infty 2e^{-2x} dx - \int_0^\infty 2xe^{-2x} dx\right)\Big]$$

Subtitute in the Rules

$$\int x e^{ax} dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$$

$$\int x^2 e^{ax} = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$