* Cross product:

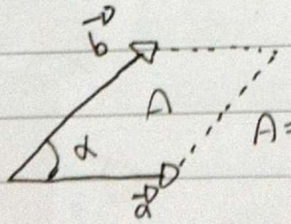
Def: Let $\vec{a} = [a_1, a_2, a_3]$ and
$\vec{b} = [b_1, b_2, b_3]$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

$$= [a_2 b_3 - a_3 b_2 \,,\, a_3 b_1 - a_1 b_3 \,,\, a_1 b_2 - a_2 b_1]$$

* Remarks:

1. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2. $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$
3. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \alpha$ represent the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$



$A = |\vec{a} \times \vec{b}|$

④ If $\vec{a} \times \vec{b} = 0$ then $\vec{a} \parallel \vec{b}$

$i \times j = k$ | $j \times i = -k$
$j \times k = i$ | $k \times j = i$
$k \times i = j$ | $i \times k = j$

ex: Let $\vec{a} = [1, 1, 0]$
$\vec{b} = [3, 0, 0]$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} \hat{k}$$

$$= 0\hat{i} - 0\hat{j} - 3\hat{k} \rightarrow [0, 0, -3]$$

* 9.4: vector and scalar function and their feild

Def: ① a vector function gives avector value for apoint p in space

$$\vec{v}(P) = [v_1(P), v_2(P), v_3(P)]$$

or

$$\vec{v}(x,y,z) = [v_1(x,y,z),\ v_2(x,y,z),\ v_3(x,y,z)]$$

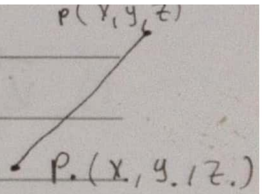
② A scalar function gives scalar value for apoint p:

$$f(P) = \alpha$$

③ A vector function defines avector field and a scalar functions defines ascalar field.

$$\vec{f}(x,y) = [\sin x, 3e^{y}]$$

~~$\vec{f}(x,y) = [\sin x, 3e^{y}]$~~

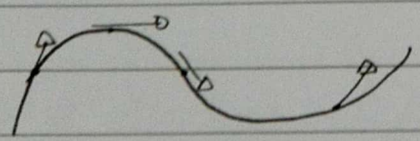


ex: (scalar function)

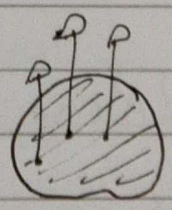
The distance from a fixed point P_0 to any point P is a scalar function

$$f(P) = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

ex: (vector field)



field of tangent vectors of a curve



field of normal vectors of a surface

* note that vector functions may also depend on time t:

$$\vec{v}(t) = [v_1(t), v_2(t), v_3(t)]$$

or

$$\vec{v}(t) = v_1(t)\hat{i} + v_2(t)\hat{j} + v_3(t)\hat{k}$$

$$\vec{v}(t)' = [v_1(t)' + v_2(t)' + v_3(t)']$$

* Differentiation rules:

1) $(\alpha \vec{v})' = \alpha \vec{v}'$
2) $(\vec{u} + \vec{v})' = \vec{u}' + \vec{v}'$
3) $(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$
4) $(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

ex: partial derivatives:

1) $\vec{v}(x,y) = [3\cos x, 3\sin x, y]$

$$\frac{\partial \vec{v}}{\partial x} = [-3\sin x, 3\cos x, 0]$$

$$\frac{\partial \vec{v}}{\partial y} = [0, 0, 1]$$

② $\vec{v}(x,y) = [e^x \cos y, e^x \sin y]$

~~0.8264 3.1699 0.6830~~

~~= 2000(P/F n=2) + 300(P/A n=4)(P/F n=4)~~

~~− 500(P/A n=4) − 1000(P/F n=5) 0.6209~~

~~− x(A/A) x(P/A n=3)(P/F n=7)~~

~~2.4869 0.5132~~

~~= 1652.8 + 649.5128 − 341.5~~

~~− 620.9 = 1.2763 x~~

③ $\vec{V} = [\cos x \cosh y, -\sin x \sinh y]$

- 9.5 Curves. Arc length

⊙ A curve C can be represented by a vector function with a parameter t

$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

parametric representation of a curve

⊙ the direction of the curve is determind by increasing values of t ex: find a parametric representation of the following curve:

$$x^2 - y = 0 \ / \ z = 3x - 1$$

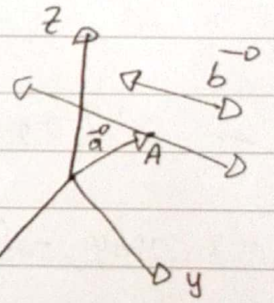
let $x = t \rightarrow y = t^2$ and $z = 3t - 1$

$$\vec{r}(t) = [t, t^2, 3t-1]$$

* parametric equation:

$\boxed{1}$ straight line: The parametric equation of a straight line in the direction of a vector $\vec{b} = [b_1, b_2, b_3]$ and passes through the point $A(a_1, a_2, a_3)$ is given by:

$$\vec{r(t)} = \vec{a} + \vec{b}t$$
$$= [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3]$$



ex: find the parametric equ of a straight line that passes through $p(2,-1,3)$ in the direction of $\vec{v} = 2\hat{i} - \hat{k}$

$\vec{a} = [2,-1,3]$, $\vec{b} = [2,0,-1]$

$\vec{r}(t) = [2+2t, -1, 3-t]$

* the parametric equ of a straight line is not a unique

غير محدود

ex: find the parametric equ of a straight line that passes through the point $P_1(3,4,-1)$ and $P_2(7,2,0)$

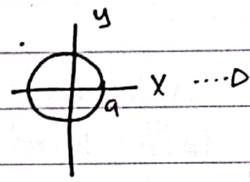
$\vec{a} = [3,4,-1]$, $\vec{b} = [7-3, 2-4, 0--1] = [4,-2,1]$

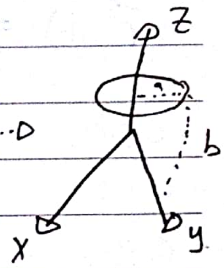
$\vec{r(t)} = [3+4t, 4-2t, -1+t]$

* line يكون محدود $1 \geq t \geq 0$

$\boxed{2}$ circle: the parametric equ of the circle $x^2+y^2=a^2$, $z=b$ is given by:

$\vec{r}(t) = [a\cos t, a\sin t, b]$, $0 \le t \le 2\pi$





ex: find the parametric equ of:

1) $x=3$, $y^2+z^2=4$

$\vec{r}(t) = [3, 2\cos t, 2\sin t]$

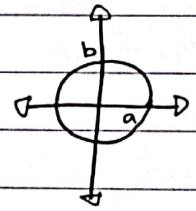
2) $(x-1)^2+y^2=9$, $z=0$ $\rightarrow$ $x-1=3\cos t$ $\rightarrow$ $x=1+3\cos t$

$y = 3\sin t$

$\vec{r}(t) = [1+3\cos t, 3\sin t, 0]$

3) $y^2+z^2+4z=5$, $x=1$ $\qquad (y^2+(z+2)^2=9)$, $x=1$

$\boxed{3}$ ellipse: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $z=c$



$\vec{r}(t) = [a\cos t, b\sin t, c]$ $\quad 0 \le t \le 2\pi$

ex: find the parametric eq of:

1) $\dfrac{y^2}{3}+\dfrac{z^2}{4}=1$, $x=2$ $\rightarrow$ $\vec{r}(t) = [2, \sqrt{3}\cos t, 2\sin t]$, $0 \le t \le 2\pi$

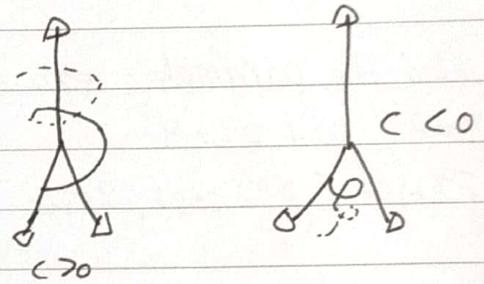
2) $(x-2)^2+16(y+3)^2=64$, $z=-1$

$\dfrac{(x-2)^2}{64}+\dfrac{(y+3)^2}{4}=1$ $\qquad z=-1$

$\boxed{4}$ circular helix:

$\vec{r}(t) = [a\cos t, b\sin t, ct]$ $\quad 0 \leq t \leq 2\pi$

- $c > 0$ right hand screw
- $c < 0$ left hand screw
- $c = 0$ ellipse

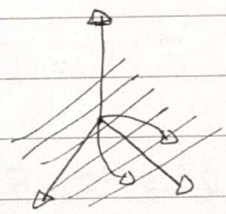


* curve:

(1) plane curve: is a curve that lies in a plane

ex: $y = x^2,\ z = 0$

plane كله تقع على كل ال

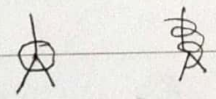


2) Twisted ~~fate~~ curve: is not a plane curve

3) simple curve: is a curve without multiple point (That is, without points at which the curve intersect or touches itself)

ما بقطع نفسه ولا بتقاطع بنقطة

ex:



4) Arc of a curve: is a portion between any two points of the curve.
for simplicity we say "curve" for curves as well as for arcs

* tangent to a curve:

- $\vec{r}'(t)$ is tangent vector
- equ of tangent line to the curve $\vec{r}(t)$ at t_0 is given by:
$\vec{q}(w) = \vec{r}_0 + w\,\vec{r}'(t)$

ex: find the tangent to the ellipse $\frac{1}{4}x^2 + y^2 = 1$ at $P(\sqrt{2}, \frac{1}{\sqrt{2}})$

$\vec{r}(t) = [2\cos t, \sin t, 0]$ at $P(\sqrt{2}, \frac{1}{\sqrt{2}}, 0)$

$2\cos t_0 = \sqrt{2} \rightarrow \frac{1}{\sqrt{2}} \rightarrow t_0 = \pi/4$

Now, $\vec{r}'(t) = [-2\sin t, \cos t, 0] \rightarrow$ Thus, $\vec{r}(t_0) = [\sqrt{2}, \frac{1}{\sqrt{2}}, 0]$
$\vec{r}'(t) = [-\sqrt{2}, \frac{1}{\sqrt{2}}, 0]$

the equ of tangent line $\rightarrow \vec{q}(w) = \vec{r}(t_0) + w\,\vec{r}'(t_0) = [\sqrt{2}, \frac{1}{\sqrt{2}}, 0] + w[-\sqrt{2}, \frac{1}{\sqrt{2}}, 0]$
$= [\sqrt{2}(1-w), \frac{1}{\sqrt{2}}(1+w), 0]$

9.7: Gradient of a scalar field Directional Derivative.

↳ surface عمودي vector

Def: The gradient of a scalar function $f(x,y,z)$ is defined as:
$$\text{grad } f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

ندخل scalar
و يطلع vector

- del operator is defined as:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\text{grad } f = \nabla f$$

ex: $f(x,y,z) = \sin x . e^{yz}$

grad $f = [\cos x\, e^{yz}, z \sin x\, e^{yz}, y \sin x\, e^{yz}]$

* Directional derivative: directional derivative of f at P in the direction of $\hat{b}$ is given by:

$$D_{\hat{b}} f(P) = \text{grad } f(P) . \hat{b}$$

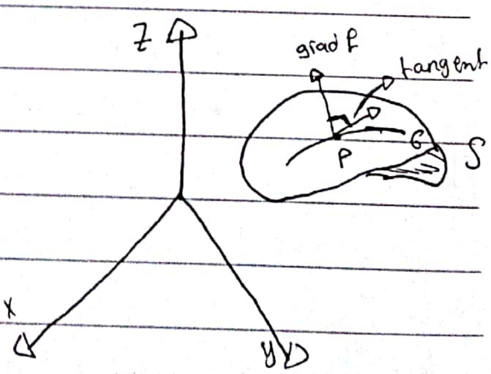
ex: find the directional of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $P(2,1,3)$ in the direction of $\vec{a} = \hat{i} - 2\hat{k}$

sol: grad $f = [4x, 6y, 2z] \rightarrow \nabla f(P) = [8,6,6]$

$$\hat{b} = \frac{\vec{a}}{|\vec{a}|} = \left[\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right]$$

$$\rightarrow D_{\hat{b}} f(P) = [8,6,6] . \left[\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right] \rightarrow \frac{8}{\sqrt{5}} + 0 - \frac{12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

* Gradient as surface normal vector:



A surface S: $f(x,y,z) = c$

A curve C: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Tangent vector of the curve C: $\vec{r}\,'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$

If G is on S, the surface eq becomes:

$$f[x(t), y(t), z(t)] = c$$

Now, diff w.r.t t:

$$\frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z' = 0$$

tangent vector

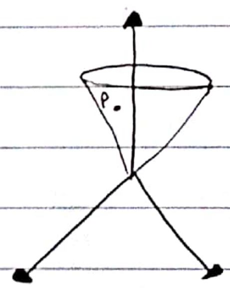
$$\text{grad } f \cdot \vec{r}\,' = 0$$

$\rightarrow$ gradient of f at the point p is a normal vector to the surface at the point p

ex: A cone is given by $z^2 = 4(x^2+y^2)$, find a normal vector at the point $P(1,0,2)$

عوض بـ ∇

$4(x^2+y^2) - z^2 = 0 \quad \rightarrow f(x,y,z) = 0$



$\text{grad } f = 8x\hat{i} + 8y\hat{j} - 2z\hat{k}$

$\vec{n} = \text{grad } f(P) = 8\hat{i} - 4\hat{k}$

Def: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ is called the laplacen of f

من Scalar لل Scalar

- $\nabla^2 = \nabla \cdot \nabla$

ex: $f(x,y,z) = 3x^2 y + e^z \quad \rightarrow \nabla^2 f = 6y + 0 + e^z = 6y + e^z$

properties:

1) $\nabla(f^n) = n f^{n-1} \cdot \nabla f$

2) $\nabla(fg) = f \nabla g + \nabla f\, g$

3) $\nabla(f/g) = \dfrac{g \nabla f - f \nabla g}{g^2}$

4) $\nabla^2(fg) = g \nabla^2 f + 2 \nabla f \nabla g + f \nabla^2 g$

9.8: Divergence of a vector field.

Def: The divergence of the vector function $\vec{v}(x,y,z) = v_1(x,y,z)\,\hat{i} + v_2(x,y,z)\,\hat{j} + v_3(x,y,z)\,\hat{k}$ is defind as:

$$\operatorname{div} \vec{v} = \frac{dv_1}{dx} + \frac{dv_2}{dy} + \frac{dv_3}{dz} \quad \text{(scalar function)}$$

من vector ← scalar

Using del operator:

$$\operatorname{div} \vec{v} = \left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right) \cdot \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right) = \nabla \cdot \vec{v}$$

ex: $\vec{v} = xe^{y}\,\hat{i} + \sin y\,\hat{j} + 3x^2\cosh(y+z)\,\hat{k}$

$$\operatorname{div} \vec{v} = e^{y} + \cos y + 3x^2 \sinh(y+z)$$

- $\operatorname{div}(\operatorname{grad} f) = \nabla \cdot \nabla f = \nabla^2 f$ (Laplacian of f)
- $\operatorname{div}(f\vec{v}) = f \operatorname{div} \vec{v} + \vec{\nabla} \cdot \nabla f$
- $\operatorname{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$

9.9: curl of a vector field

Def: The curl of a vector function $\vec{v}(x,y,z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is ~~def~~ defined as

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

من vector الى vector

dx = اشتقه الـ z بالنسبه لـ y والـ y لـ z واطرحهم

dy ~ ~ ~ ~ ~ ~ ~

dz ~ ~ ~ ~ ~ ~ ~

ex: $\vec{v}(x,y,z) = yz\,\hat{i} + 3zx\,\hat{j} + z\,\hat{k}$

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ yz & 3zx & z \end{vmatrix}$$

$$= (0-3x)\hat{i} - (0-y)\hat{j} + (3z-z)\hat{k}$$

$$= -3x\hat{i} + y\hat{j} + 2z\hat{k}$$

Theorem:

- $\text{curl}(\text{grad } f) = \nabla \times (\nabla f) = 0$
- $\text{div}(\text{curl } \vec{v}) = \nabla \cdot (\nabla \times \vec{v}) = 0$

properties:

- $\text{curl}(\vec{u} + \vec{v}) = \text{curl } \vec{u} + \text{curl } \vec{v}$
- $\text{curl}(f\vec{v}) = \nabla f \times \vec{v} + f \text{ curl } \vec{v}$
- $\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl } \vec{u} + \vec{u} \text{ curl } \vec{v}$

chap 10: vector Integral Calculus

10:1 line integrals

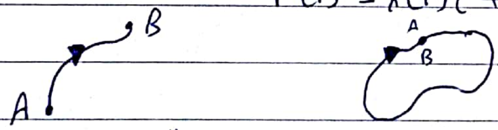
- A definite integral $\int_a^b f(x)\,dx$; $\longleftarrow [\ a \ \ldots \ b\ ] \longrightarrow \mathbb{R}$

Integrate the Integrand $f(x)$ from $x=a$ to $x=b$

- A line integral (or curve integral):

integration along a curve C in parametric representation:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$



"oriented curves"

- the direction from A to B in which t increases is called the positive direction

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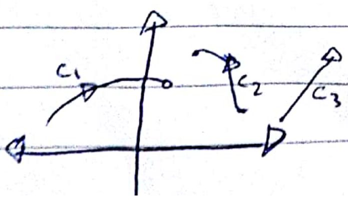
Def: A curve C : $\vec{r}(t)$ is said to be smooth if $\vec{r}'(t)$ is continuous

Def: A piecewise smooth curve has finitely many smooth curves



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* Definition and Evalution of Line integrals:

A line integral of a vector function $\vec{F}(x,y,z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, over a curve $G: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

is given by $\int_G F(\vec{r})\cdot dr = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

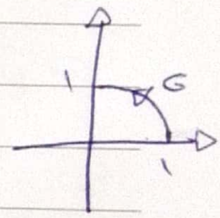
- since $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\rightarrow \int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_C (f_1\,dx + f_2\,dy + f_3\,dz)$$

$$= \int_a^b \left(f_1\frac{dx}{dt} + f_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right)dt$$

ex: (Line integral in the plane)

Find the line integral of $\vec{F}(\vec{r}) = -y\hat{i} - xy\hat{j}$ over the circular



$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le \pi/2$

$\vec{F}(\vec{r}) = -\sin t\,\hat{i} - \cos t \sin t\,\hat{j}$

$\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

$$\int_G \vec{F}(\vec{r})\cdot d\vec{r} = \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\,dt \rightarrow \pi/4 - 1/3$$

ex: (line integral in space) → Find the line integral $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ along a helix G:

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$, $0 \le t \le 2\pi$

$\vec{F}(\vec{r}) = 3t\,\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k}$

$\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 3\hat{k}$

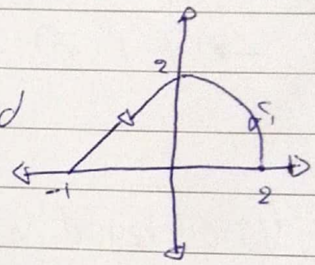
$$\int_G \vec{F}(\vec{r})\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r})\cdot\vec{r}\,'\,dt$$

$$= \int_0^{2\pi} -3t\sin t + \cos^2 t + 3\sin t\ (dt)$$

$$= 7\pi$$

- $\int_C \alpha \vec{F}\cdot d\vec{r} = \alpha \int_C \vec{F}\cdot d\vec{r}$
- $\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$
- $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$
- $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$

ex: evalute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and

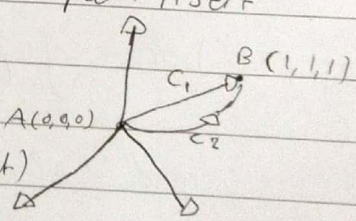


$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

* Path dependence:

Thm: The Line integral $\int_C \vec{F}\cdot d\vec{r}$ generally depends not only on $\vec{F}$ and endpoints of the path, but also on the path itself

ex: $\vec{F} = 5z\hat{i} + xy\hat{j} + x^2 z\hat{k}$



C_1: $\vec{r}_1(t) = t\hat{i} + t\hat{j} + t\hat{k}$, $0 \le t \le 1$ (straight Line segment)

C_2: $\vec{r}_2(t) = t\hat{i} + t\hat{j} + t^2\hat{k}$, $0 \le t \le 1$ (parabolic arc)

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 (5t\hat{i} + t^2\hat{j} + t^3\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})\,dt = \int_0^1 (5t+t^2+t^3)\,dt = \frac{1}{4}$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 (5t^2\hat{i} + t^2\hat{j} + \cancel{5}t^4\hat{k})\cdot(\hat{i}+\hat{j}+2t\hat{k})\,dt$$
$$= \int_0^1 6t^2 + 2t^5\,dt = 2/3$$

In general, aline integral depends on $\vec{F}$, A, B and path C

10.2 : path independence of the integral

اذا ما بدك parametrision نحل ←

- A Line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent if it has the same value for all curves C with the same endpoints, That is, its value depends only on the endpoints of C, not on C itself

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r}$$

Thereom : A line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent in a Domain D if $\vec{F} = \nabla f$ for some scalar function f defined in D

- if $\vec{F} = \nabla f$ then f is called a potential of $\vec{F}$, and in this case

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

ex: Show that $\int_C \vec{F} \cdot d\vec{r} = \int_C (2x\,dx + 2y\,dy + 4z\,dz)$ is path independent and find its value for endpoints $A(0,0,0)$ and $B(2,2,2)$

$\vec{F} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$ $\qquad$ $\vec{F} = \nabla f = x^2 + y^2 + 2z^2$

$\int_C \vec{F} \cdot d\vec{r}$ is path independent

$\rightarrow \int_C \vec{F} \cdot d\vec{r} = f(2,2,2) - f(0,0,0) = 16$

ex: find $\int_C \vec{f} \cdot d\vec{r} = \int_C (3x^2 dx + 2yz\, dy + y^2 dz)$ from $A(0,1,2)$ to $B(1,-1,7)$ by showing $\vec{f}$ has apotential

$$\vec{f} = [3x^2, 2yz, y^2]$$

$$\vec{f} = \nabla f \rightarrow \frac{df}{dx} = 3x^2 \rightarrow f = x^3 + g(y,z)$$

$$\frac{df}{dy} = 2yz \rightarrow f = x^3 + y^2 z + h(z)$$

$$\frac{df}{dz} = y^2 \rightarrow f = x^3 + y^2 z + c$$

$\int_C \vec{f} \cdot d\vec{r}$ is path independent

$$\int_C \vec{f} \cdot d\vec{r} = f(1,-1,7) - f(0,1,2) = 6$$

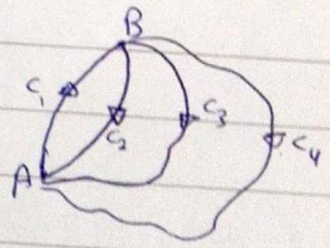
* integration around closed curves:

Thm: A Line integral of $\vec{F}$ is path independent in a domain D if $\int_C \vec{f} \cdot d\vec{r} = 0$ whenever C is a closed path in D

proof: $\int_{C_1} \vec{f} \cdot d\vec{r} + \int_{C_2} \vec{f} \cdot d\vec{r} = 0$

$\int_{C_1} \vec{f} \cdot d\vec{r} + \int_{C_3} \vec{f} \cdot d\vec{r} = 0$

$\int_{C_1} \vec{f} \cdot d\vec{r} + \int_{C_4} \vec{f} \cdot d\vec{r} = 0$

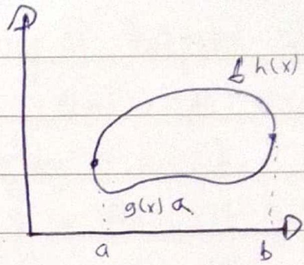


$\int_{C_2} \vec{f} \cdot d\vec{r} = \int_{C_3} \vec{f} \cdot d\vec{r} = \int_{C_4} \vec{f} \cdot d\vec{r}$

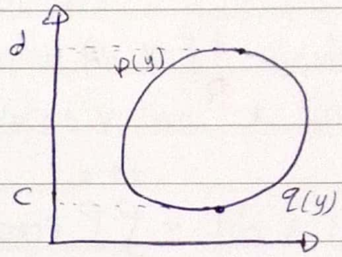
$\vec{f}$ is path independent

in this case $\vec{f}$ is called conservative

10.3 Double Integrals



$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$



$$\iint f(x,y)\,dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dx\,dy$$

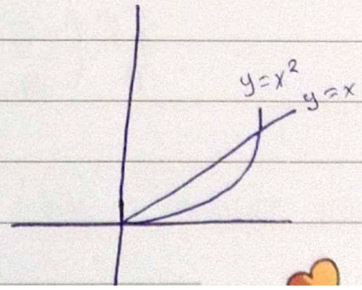
ex: $$\int_0^2 \int_x^{2x} (x+y)^2\,dy\,dx = \int_0^2 \left.\frac{(x+y)^3}{3}\right|_x^{2x} dx = \int_0^2 9x^3 - \frac{8}{3}x^3\,dx$$

$$= \left.\frac{19}{3}\cdot\frac{x^4}{4}\right|_0^2 = \frac{76}{3}$$

ex: $$\int_0^3 \int_{-y}^{y} (x^2+y^2)\,dx\,dy = \int_0^3 \left.\frac{x^3}{3} + y^2x\right|_{-y}^{y} dy$$

$$= \int_0^3 \left(\frac{y^3}{3} + y^3\right) - \left(\frac{-y^3}{3} - y^3\right) dy = 54$$

ex: evaluate $\iint (x+2y)\,dA$



$$\iint_D (x+2y)\,dA = \int_0^1 \int_{x^2}^{x} (x+2y)\,dy\,dx = \dots$$

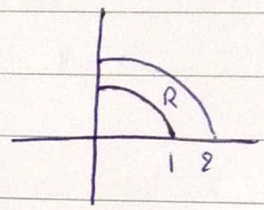
* Double integral in polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

If $R_3 = \{(r,\theta) : \alpha \le \theta \le \beta,\ g(\theta) \le r \le h(\theta)\}$

$$\iint f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

ex: evaluate $\iint_R x\,dA$ where R is the region between the two circles $x^2+y^2=1$ and $x^2+y^2=4$ in the first quadrant



$$\iint x\,dA = \int_0^{\pi/2}\int_1^2 r\cos\theta\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} (\cos\theta\,d\theta)\left(\int_1^2 r^2\,dr\right)$$

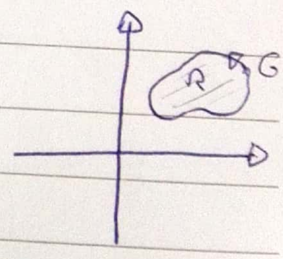
$$= 1 \times (8/3 - 1/3) = 7/3$$

10.4: Green Theorem in the plane

* Greens Theorem:

If R is a closed region in xy-plane with boundary C (with positive orientation)

If $\vec{F} = [f_1, f_2] = f_1\hat{i} + f_2\hat{j}$ is a vector function, then



$$\iint_R \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right) dx\,dy = \oint_C f_1\,dx + f_2\,dy$$

- Greens Thereom in vector form can be written as:

$$\iint_R \underset{(\nabla\times\vec{F})}{\text{curl}\,\vec{F}}\cdot k\,dx\,dy = \oint_C \vec{F}\cdot d\vec{r}$$

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ex (verification of Greens Thm):

let $\vec{f} = (\underbrace{y^2 - 7y}_{f_1})\hat{i} + (\underbrace{2xy + 2x}_{f_2})\hat{j}$ and $G: x^2 + y^2 = 1$ Then

(i) $\iint_R \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right) = \iint_R (2y+2) - (2y-7)\,dx\,dy = 9\iint_R dx\,dy = 9 \times$ area of R $= 9\pi$

Recall: $\iint_R dx\,dy =$ Area of R

(ii) $\vec{f}(\vec{r}) = (\sin^2 t - 7\sin t)\hat{i} + (2\cos t\sin t + 2\cos t)\hat{j}$

$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

$\oint_C \vec{f}\cdot d\vec{r} = \int_0^{2\pi} (-\sin^3 t + 7\sin^2 t + 2\cos^2 t\sin t + 2\cos^2 t)\,dt = 9\pi$

* Some Applications of Green's Thm:

(i) if $f_2 = x$ and $f_1 = 0 \rightarrow \iint_R \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right)dx\,dy = \iint_R dx\,dy = \oint_C x\,dy$

(ii) If $f_2 = 0$ and $f_1 = -y \rightarrow \iint_R \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right)dx\,dy = \iint_R dx\,dy = -\oint_C y\,dx$

$\therefore$ Area of a region R is: $A = \frac{1}{2}\oint x\,dy - y\,dx$

ex: Find the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

$\vec{C} : \vec{r}(t) = [3\cos t, 4\sin t],\ 0 \le t \le 2\pi$ $\qquad \oint_C y\,dx - x\,dy$

$A = \frac{1}{2} \oint_C x\,dy - y\,dx$

$= \frac{1}{2} \int_0^{2\pi} [-4\sin t, 3\cos t] \cdot [-3\sin t, 4\cos t]\, dt$

$= \frac{1}{2} \int_0^{2\pi} 12\sin^2 t + 12\cos^2 t \, dt$

$= 6 \int_0^{2\pi} dt = 12\pi$

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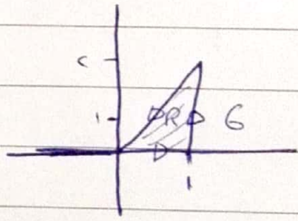
ex: evaluate $\oint_C xy\,dx + x^2y^3\,dy$, where C is the triangle with vertices (0,0), (1,0) and (1,2) with positive oriantation.

$\oint_C xy\,dx + x^2y^3\,dy = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx = \frac{2}{3}$

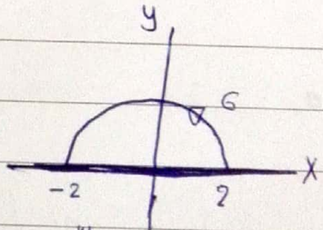


ex: evaluate $\oint_C (e^x + 4y)\,dx + (\sin^2 y + 5x)\,dy$ where C is the upper half of the circle $x^2+y^2=4$

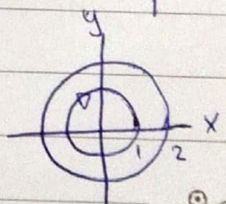
$\oint_C (e^x+4y)\,dx + (\sin^2 y + 5x)\,dy = \iint 5-4\,dx\,dy$

$=$ Area of R

$= \frac{1}{2} \times 4\pi = 2\pi$



ex: evaluate $\oint_C y^3\,dx - x^3\,dy$ where



$\oint_C y^3\,dx - x^3\,dy = \iint_R -3x^2 - 3y^2\,dx\,dy = -3\iint_R (x^2+y^2)\,dx\,dy$

$= -3 \int_0^{2\pi}\int_1^2 r^2 \cdot r\,dr\,d\theta = -3\left(\int_0^{2\pi} d\theta\right)\left(\int_1^2 r^3\,dr\right)$

$= -3 \times 2\pi \times \frac{15}{4} = \frac{-45\pi}{2}$

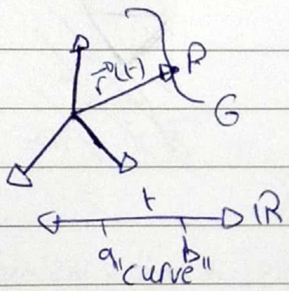
10.5 : surfaces for surface integrals

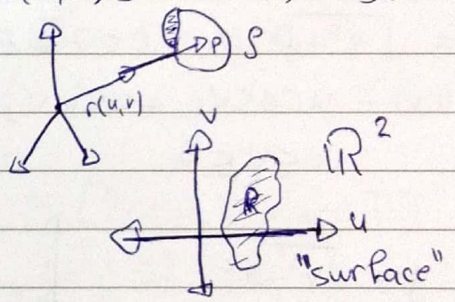
* Representations of surfaces in xyz-space:

$$z = f(x,y) \quad \text{or} \quad g(x,y,z)=0$$

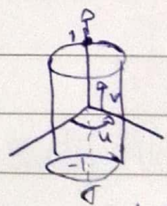
* parametric representation:

$$\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$





ex: parametric representation of a cylinder



$$x^2+y^2=4 \quad , \quad -1 \le z \le 1$$

$$\vec{r}(u,v) = 2\cos u\,\hat{i} + 2\sin u\,\hat{j} + v\hat{k}$$

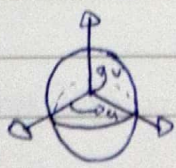
$$= [2\cos u, 2\sin u, v] , \quad 0 \le u \le 2\pi \quad , \quad -1 \le v \le 1$$

ex: ~ ~ ~ sphere

$$x^2+y^2+z^2=9$$

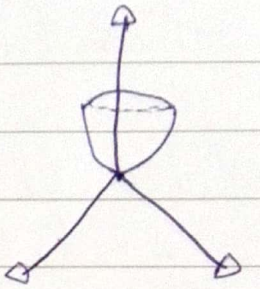
$$\vec{r}(u,v) = 3\cos v\cos u\,\hat{i} + 3\cos v\sin u\,\hat{j} + 3\sin v\,\hat{k}$$

$$0 \le u \le 2\pi \qquad -\pi/2 \le v \le \pi/2$$



ex: ~ ~ ~ elliptic paraboliod.

$z = x^2 + y^2 \quad , \quad 0 \le z \le 4$



$\vec{r}(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\,\hat{k}$

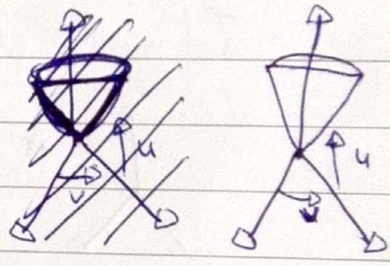
$0 \le u \le 2 \quad , \quad 0 \le v \le 2\pi$

ex: ~ ~ ~ a cone.

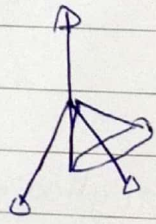
$z = \sqrt{x^2+y^2} \qquad 0 \le z \le 5$

$\vec{r}(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\,\hat{k}$

$0 \le u \le 5 \qquad 0 \le v \le 2\pi$



$y = \sqrt{x^2+z^2}$



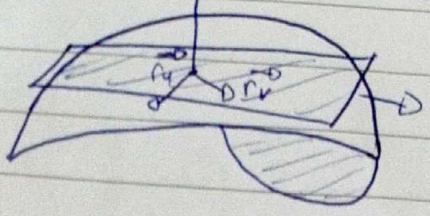
$= u\cos v\,\hat{i} + u\,\hat{j} + u\sin v\,\hat{k}$

* Tangent plane and surface normals

Def: tangent plane of a surface S at the point p is a plane containing tangent ~~surface~~ vectors of S at p.

Def: Normal vector of a surface S at the point p is a vector perpendicular to the tangent plane.

$\vec{n} \to$ unit normal vector



$\to$ tangent plane

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- A normal vector of the surface S at the point P is:

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$\hat{n} = \frac{\vec{N}}{|\vec{N}|} \quad \text{"unit normal vector"}$$

ex: $x^2+y^2=4$, $0 \le z \le 3$ "cylinder"

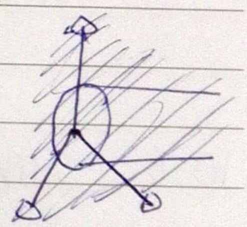
$\vec{r}(u,v) = [2\cos u, 2\sin u, v] \Rightarrow \vec{r}_u = [-2\sin u, 2\cos u, 0]$

$\vec{r}_v = [0,0,1]$

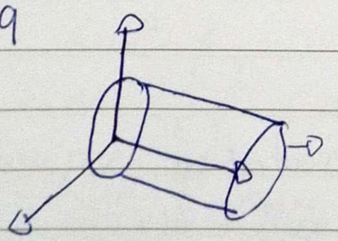
$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = [2\cos u, 2\sin u, 0]$$

$|\vec{N}| = 2$

$\hat{n} = \cos u\,\hat{i} + \sin u\,\hat{j}$



ex: $\frac{x^2}{4} + \frac{z^2}{9} = 1$ $0 \le y \le 4$



$\vec{r}(u,v) = [2\cos u, v, 3\sin u]$

Thm: if S is given by $g(x,y,z)=0$ then the surface normal vector is $\vec{N} = \nabla g$

ex: unit normal vector of a sphere $x^2+y^2+z^2=4$

let $g(x,y,z) = x^2+y^2+z^2-4$

$\vec{N} = \nabla g = 2x+2y+2z$ and $|\vec{N}| = 4$

$\therefore \hat{n} = [\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z]$

ex: unit normal of a cone

$z = \sqrt{x^2+y^2}$

let $g(x,y,z) = \sqrt{x^2+y^2} - z$

$\vec{N} = \nabla g = [\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1] = |\vec{N}| = \sqrt{2}$

$\hat{n} = \frac{1}{\sqrt{2}} [\quad]$

10.6 Surface Integrals

- A surface S in parametric representation is given by:

$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$

- the surface normal vector is:

$\vec{N} = \vec{r}_u \times \vec{r}_v$

- Unit normal vector:

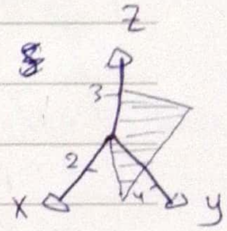
$\hat{n} = \frac{\vec{N}}{|\vec{N}|}$

Def: A surface integral of a vector function $\vec{f}(\vec{r})$ over the surface S is defined as:

$$\iint_S \vec{f}\cdot\vec{n}\, dA = \iint_R \vec{f}\cdot\vec{N}\, du\, dv$$

where R is the projection of S into the uv-plane

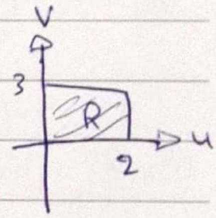
ex: evaluate $\iint_S \vec{f}\cdot\hat{n}\, dA$ where $\vec{f} = [3z^2, 6, 6xz]$ and S: $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$



$S: \vec{r} = [x, x^2, z]$

let $u = x$ $v = z$ $\rightarrow$ $\vec{r}(u,v) = [u, u^2, v]$, $0 \le u \le 2$, $0 \le v \le 3$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = [2u, -1, 0]$$

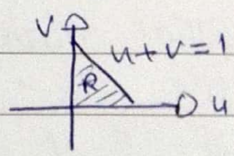


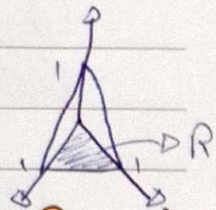
$\vec{f}(\vec{r}(u,v)) = [3v^2, 6, 6uv]$

$\vec{f}\cdot\vec{N} = 6uv^2 - 6$

$\therefore \iint_S \vec{f}\cdot\hat{n}\, dA = \int_0^3\int_0^2 6uv^2 - 6\; du\, dv = \int_0^3 3u^2v^2 - 6u \Big|_0^2 dv = 72$

ex: evaluate $\iint_S \vec{f}\cdot\vec{n}\, dA$ where $\vec{f} = [x^2, 0, 3y^2]$ and S is the portion of the plane $x + y + z = 1$ in the first octan





let $x = u$, $y = v \rightarrow z = 1 - x - y$
$= 1 - u - v$

$\vec{r}(u,v) = [u, v, 1-u-v]$ $\quad$ $0 \le u \le 1 - v$
$0 \le v \le 1$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = [1, 1, 1]$$

$$\vec{f}(\vec{r}(u,v)) = [u^2, 0, 3v^2]$$

$$\vec{f} \cdot \vec{N} = u^2 + 3v^2 \rightarrow \therefore \iint_S \vec{F} \cdot \hat{n}\, dA = \int_0^1 \int_0^{1-v} (u^2 + 3v^2)\, du\, dv$$

$$= \int_0^1 \left(\frac{u^3}{3} + 3v^2 u \Big|_0^{1-v} \right) dv = \int_0^1 \frac{(1-v)^3}{3} + 3v^2(1-v)\, dv$$

$$= -\frac{(1-v)^4}{12} + v^3 - \frac{3}{4} v^4 \Big|_0^1 = \frac{1}{3}$$

10.7 Divergence Thereom of Gouss. (cloused surface)

Triple integral $\leftrightarrow$ surface integral

Thm: let T be aclosed bounded region in space whose boundary is apiecewise smooth oriented surface S with positive orientation (outward)

- let $\vec{f}(x,y,z)$ be acontineous vector function and has cont first ~~[illegible]~~ parial derivate in T. Then: $\iiint_T \operatorname{div}(\vec{F})\, dv = \iint_S \vec{F} \cdot \hat{n}\, dA$

$$\iiint_T \left[\frac{df_1}{dx} + \frac{df_2}{dy} + \frac{df_3}{dz} \right] dx\, dy\, dz$$

$$= \iint_S f_1\, dy\, dz + f_2\, dz\, dx + f_3\, dx\, dy$$

where $\vec{f} = [f_1, f_2, f_3]$

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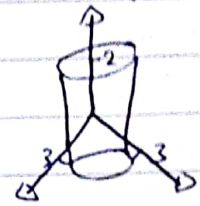
ex: evalute $\iint_S \vec{F}\cdot\hat{n}\,dA$ where $\vec{F}=[x^3,y^3,z^3]$ and $S = x^2+y^2=9$

"top and bottom are included" $0\le z\le 2$ "cylinder"



$\iint \vec{F}\cdot\hat{n}\,dA = \iiint_T div(\vec{F})\,dv$

$= \iiint_T [3x^2+3y^2+3z^2]\,dv$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2+y^2=r^2$

$= \int_0^2\int_0^{2\pi}\int_0^3 [3r^2+3z^2]\,r\,dr\,d\theta\,dz = 315\pi$ "cylindrical coordinate"

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ex: $S : z=\sqrt{4-x^2-y^2}$ (upper hemisphere) and $x^2+y^2\le 4$

"sphereical coordinate"

$\iint_S \vec{F}\cdot\hat{n}\,dA = \int_0^2\int_0^{\pi/2}\int_0^{2\pi} (3\rho)^2\rho^2\sin\theta\,d\theta\,d\varphi\,d\rho = (192/5)\pi$

ex: evaluate $\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where $S: x^2+y^2=16,\ 0\le z\le 3$ and "sides and bottom are included"

$\iint_S \vec{F}\cdot\hat{n}\,dA = \iiint_T [3x^2+x^2+x^2]\,dv = \iiint_T 5x^2\,dv$ "cylindrical coordinate"

$= \int_0^3\int_0^{2\pi}\int_0^4 (5r^2\cos^2\theta)\,r\,dr\,d\theta\,dz$

$= \left(\int_0^3 dz\right)\left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_0^4 5r^3\,dr\right) = 960\pi$

H.w : evaluate $\iint_S \vec{F}\cdot\hat{n}\,dA$ where $\vec{F}=[xy,\ y^2+\sin(xz),\ 3e^x\cos y]$

and $S: z=1-x^2,\ -1\le x\le 1,\ 0\le y\le 2$

$\iint_S \vec{F}\cdot\hat{n}\,dA = \iiint_T 3y\,dv$

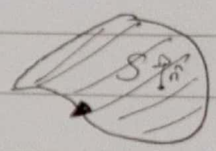
$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^2 3y\,dy\,dz\,dx$

- stoke's theorem (closed curve)

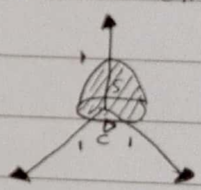
thm: let S be a piesewise smooth oriented surfase and let its boundary be a piecewise smoth simple closed curve G.

* Let $\vec{f}(x,y,z)$ be a cont vector function with cont partial first derivatives. Then $\iint_S \operatorname{curl}(\vec{f})\cdot\hat{n}\,dA = \oint_C \vec{f}\cdot d\vec{r}$

$\operatorname{curl}(\vec{f}) \to \nabla\times\vec{f}$



ex: (verifaction of stoke's theorem) - Let $\vec{f} = [y, z, x]$ and $S: z = 1-(x^2+y^2)$, $z \geq 0$ "paraboloid"



(i) $\operatorname{curl}\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y & z & x \end{vmatrix} = -\hat{i}-\hat{j}-\hat{k} = [-1,-1,-1]$

$\vec{N} = \nabla(z+x^2+y^2-1) = [2x, 2y, 1]$

$\iint_S \operatorname{curl}\vec{f}\cdot\hat{n}\,dA = \iint_S \operatorname{curl}\vec{f}\cdot\vec{N}\,dx\,dy = \iint_S -2x-2y-1\,dx\,dy$

$= \int_0^{2\pi}\int_0^1 (-2\cos\theta -2\sin\theta -1)\,r\,dr\,d\theta$

(ii) $z=0 \to x^2+y^2=1$ $\quad G: \vec{r}(t) = [\cos t, \sin t, 0]$, $0 \leq t \leq 2\pi$

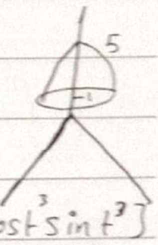
$\vec{f}(\vec{r}(t)) = [\sin t, 0, \cos t] \to \vec{r}'(t) = [-\sin t, \cos t, 0]$

$\vec{f}\cdot\vec{r}' = -\sin^2 t$

$\therefore \oint_C \vec{f}\cdot d\vec{r} = \int_0^{2\pi} \vec{f}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_0^{2\pi} -\sin^2 t\,dt = \frac{-1}{2}\int_0^{2\pi}(1-\cos 2t)\,dt$

$= \frac{-1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = -\pi$

ex: Use Stoke's Thm to evaluate $\iint_S \text{curl}\ \vec{F}\cdot\hat{n}\,dA$ where
$\vec{F} = [z^2, -3xy, x^3y^3]$ and S: $z = 5-x^2-y^2$, $z \geq 1$



$z = 1 \to x^2+y^2 = 4$

C: $\vec{r}(t) = [2\cos t, 2\sin t, 1]$ $\qquad \vec{F}(\vec{r}(t)) = [1, -12\cos t\sin t, 64\cos^3 t\sin^3 t]$

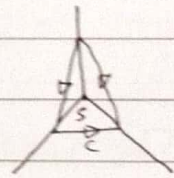
$\vec{r}\,'(t) = [-2\sin t, 2\cos t, 0]$

$\vec{F}(\vec{r}(t)\cdot\vec{r}(t)') = -2\sin t - 24\cos^2 t\sin t$

$\therefore \iint_S \text{curl}\ \vec{F}\cdot\hat{n}\,dA = \int_0^{2\pi} -2\sin t - 24\cos^2 t\sin t\ dt = 0$

H.w: use stokes' Thm to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = [z^2, y^2, x]$
and C is the triangle with vertices (1,0,0), (0,1,0) and (0,0,1)
with counter-clockwise rotation.

$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\ \vec{F}\cdot\hat{n}\,dA = \int_0^1\int_0^{1-x} 1-2x-2y\ dy\,dx$



$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} \to [0, 2z, 0]$

$x+y+z=1$

$[1, 1, 1]$

~~zz~~ $z = 1-x-y$

$\hat{n} = [1, 1, 1]$

$\to \iint 2z\ dA$

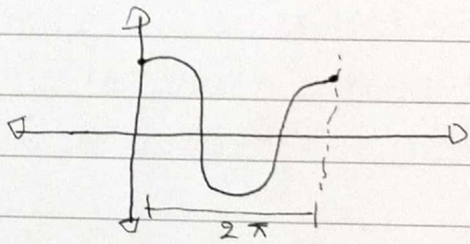
$\int_0^1\int_0^{1-x} 2-2x-2y\,dx\,dy$

Chapter 11: Fourier Analysis

11.1: Fourier series.

Def: Function f is said to be periodic with $p>0$ if $f(x) = f(x+p)$

ex: $f(x) = \cos x$



* Remark: If a periodic function f is periodic with period p, then its also periodic with period $2p, 3p, \ldots$

- The smallest period of $f(x)$ is called the fundamental period.

* Recall:

1) If $f(-x) = f(x)$, then f is called even function.
2) If $f(-x) = -f(x)$, then f is called odd function.
3) $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$ (f even function)
4) $\int_{-L}^{L} f(x) = 0$ (f odd function)

Defn: Two function $f(x)$ and $g(x)$ are called orthogonal on $[a,b]$ if $\int_a^b f(x)g(x)\,dx = 0$

- A set of function is said to be multually orthogonal if each pair of function in the set is ~~[illegible]~~ orthogonal.

* orthogonality of trigonometric functions:

1) $$\int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi y/L)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 2L & \text{if } m = n = 0 \end{cases}$$

2) $$\int_{-L}^{L} \cos(m\pi x/L)\sin(n\pi x/L)\,dx = 0$$

3) $$\int_{-L}^{L} \sin(m\pi y/L)\sin(n\pi x/L)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \end{cases}$$

* fourier series: If f has period $2L$ defined on $[-L,L]$, Then:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(n\pi x/L) + b_n\sin(n\pi x/L)\right)$$

where, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx \; ; n = 0,1,2,\dots$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx \; ; n = 1,2,3\dots$

* Remark: If $L=\pi$, then $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$

where,
$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n=0,1,\dots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n=1,\dots \end{aligned}}$$

ex: compute the fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \text{ (even)} \\ x, & 0 < x < \pi \text{ (odd)} \end{cases}$

$f(x) = 0 \rightarrow$ even
$f(x) = x \rightarrow$ odd

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\cancel{\int_{-\pi}^{0} 0\,dx} + \int_{0}^{\pi} x\,dx\right) \Rightarrow a_0 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\cos(nx)\,dx} + \int_{0}^{\pi} x\cos(nx)\,dx\right]$$

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$$a_n = \frac{\cos(n\pi) - 1}{\pi n^2} = \frac{(-1)^n - 1}{\pi n^2}, \quad n = 1,2,\dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0 \times \sin(nx)\,dx} + \int_{0}^{\pi} x\sin(nx)\,dx\right]$$

$$= \frac{-\cos(n\pi)}{n}$$

* $b_n = \frac{(-1)^{n+1}}{n}, \quad n = 1,2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx\right]$$

ex: find the fourier series for $f(x) = \begin{cases} -1 , & -\pi < x < 0 \\ 1 , & 0 < x < \pi \end{cases}$

$f(-x) = f(x)$ even

$f(-x) = -f(x)$ odd

odd function

$\rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0$ " odd function "

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx = \frac{-2}{\pi} \frac{\cos(nx)}{n} \Big|_0^{\pi}$

$\rightarrow \frac{-2}{\pi} \frac{\cos(n\pi) + 1}{n} \Rightarrow \frac{2}{\pi} \left(\frac{1-(-1)^n}{n}\right), n = 1, 2, 3$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{1-(-1)^n}{n}\right) \sin(nx)$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)} \sin((2n-1)x)$$

Fourier convergence theorem

Assume that f is periodic with a period $2L$ and piecewise continous on $[-L, L]$

Then:

the corresponding fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2$$

converges to the average

$$\frac{f(x^+) + f(x^-)}{2}$$

where $f(x_0^-) = \lim\limits_{\substack{h\to 0\\ h>0}} f(x_0 - h)$ and $f(x_0^+) = \lim\limits_{\substack{h\to 0\\ h>0}} f(x_0 + h)$

lim from left

limit from right

11.3 function of any period $(p = 2L)$

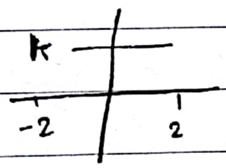
fourier series:

$$f(x) = \frac{a_0}{L} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, \dots$$

ex: find the fourier series of $f(x) = \begin{cases} 0 & , -2 \le x \le -1 \\ k & , -1 \le x \le 1 \\ 0 & , 1 \le x \le 2 \end{cases}$



$p = 4 = 2L \rightarrow L = 2$

* period $(-\pi, \pi) = 2\pi$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} k\,dx = k$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cdot\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{k}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-1}^{1} = \frac{2k}{2\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-1}^{1} k\sin\left(\frac{n\pi x}{2}\right)dx = 0$$

odd function

$$f(x) = \frac{k}{L} + \sum_{n=1}^{\infty} \frac{2k}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{2} x\right)$$

$$= \frac{k}{L} + \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)\pi} \cos\left(\frac{(2n-1)\pi}{2} x\right)$$

ex: $f(x) = \begin{cases} -k , & -2 \leq x \leq 0 \\ k , & 0 \leq x \leq 2 \end{cases}$ $\qquad P = 4 \rightarrow L = 2$

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx \qquad a_n = 0 \text{ for } n = 0, 1, 2, \dots$$

f "odd function"

$$b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin\left(\frac{n\pi}{2} x\right) dx$$

$$= \frac{1}{2} \left[\int_{-2}^{0} k \sin\left(\frac{n\pi}{2} x\right) dx + \int_{0}^{2} k \sin\left(\frac{n\pi}{2} x\right) dx \right]$$

$$= \frac{1}{2} \left[\frac{4k - 4k\cos(n\pi)}{n\pi} \right] = \frac{2k - 2k\cos(n\pi)}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2k - 2k(-1)^n}{n\pi} \sin\left(\frac{n\pi}{2} x\right)$$

11.4 even and odd function. Half-range expansions

* If $f(x)$ is an even periodic function with period $2L$, then the furier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $n = 0, 1, \ldots$

$0.0 = E$
$0.E = 0$
$E.E = E$

* If $f(x) =$ is an odd periodic function with period $2L$, then the fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$. $n = 1, 2, \ldots$

ex: $f(x) = |x|$, $-1 \le x \le 1$

$$f(x) = \begin{cases} -x, & -1 \le x \le 0 \\ x, & 0 \le x \le 1 \end{cases}$$ "even function"

$p = 2 \rightarrow L = 1$

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 x\,dx = 1$$

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)dx = 2\int_0^1 \cos(2\pi x)$$

$$= \frac{2\cos(n\pi) - 1}{n^2\pi^2} = \frac{2(-1)^n - 1}{n^2\pi^2}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-n}{(2k-1)^2\pi^2}\cos((2n-1)\pi)$$

ex:

$$f(x) = \begin{cases} \frac{2k}{L} x, & 0 \le x < \frac{1}{2} L \\ \frac{2k}{L}(L-x), & \frac{1}{2} L \le x \le L \end{cases}$$

even extension

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx = k$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx = \frac{4k}{n^2 \pi^2} \left[2\cos\left(\frac{n\pi}{2}\right) - \underbrace{\cos(n\pi)}_{(-1)^n} - 1 \right]$$

$$f_e(x) = \frac{k}{2} - \frac{16}{k\pi^2} \left[\frac{1}{(2)^2} \cos\left(\frac{n\pi x}{L}\right) + \frac{1}{(6)^2} \cos\left(\frac{6\pi}{L} x\right) + \dots \right]$$

odd extension:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx = \frac{8k}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{8k}{n^2 \pi^2} (-1)^{n+1}, & n \text{ odd} \end{cases}$$

$$f(x) = \frac{8k}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi}{L} x\right)$$

11.7 Fourier Integrals

let

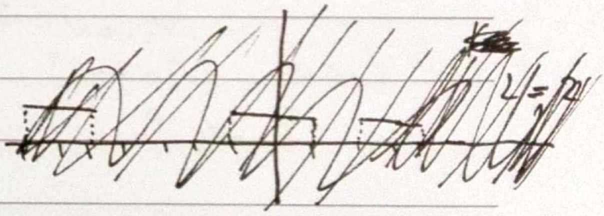
$f_L(x)$ be aperiodic function of period $2L$, then $f(x)$ can be represent by a furier series.

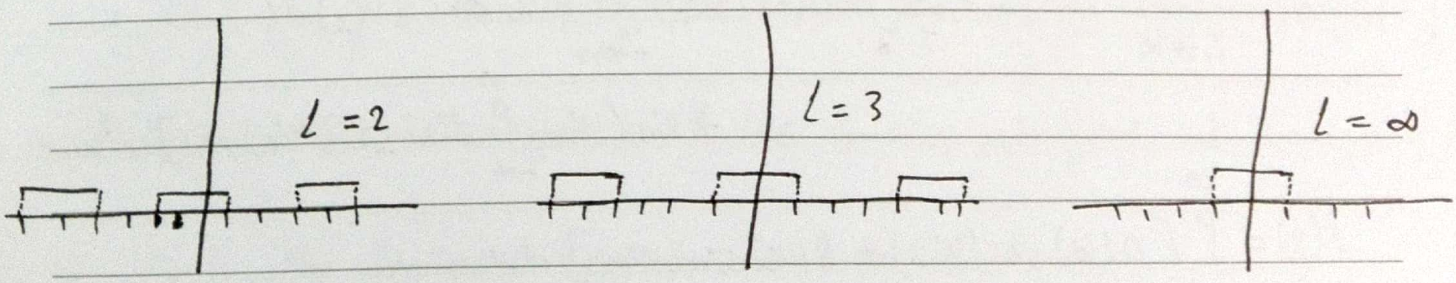
$$f_L(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(w_n x) + b_n \sin(w_n x) \right] \quad \dots \quad (*)$$

where $w_n = \frac{n\pi}{L}$

ex: $$f(x) = \begin{cases} 0 , & -L < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < L \end{cases}$$

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 , & -1 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$





If we insert a_n and b_n in $(*)$ then

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(x)\,dx + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos(w_n x)\int_{-L}^{L} f_L(x)\cos(w_n x)\,dx$$

$$+ \sin(w_n x)\int_{-L}^{L} f_L(x)\sin(w_n x)\,dx\Big]$$

Now:- $\Delta w = w_{(n+1)} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

Thus: $f_L(x) = \frac{1}{2L}\int_{-2}^{2} f_L(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\cos(w_n x)\Delta w\int_{-L}^{L} f_L(x)\,dx$

$$+ \sin(w_n x)\,\Delta w\int_{-L}^{L} f_L(x)\sin(w_n x)\,dx\Big]$$

let $L \to \infty$ $\quad (\Delta w \to 0)$ $\quad w_1 \to w$

$\sum \to \int$!

$$f(x) = \lim_{L\to\infty} f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\cos(w_x)\int_{-\infty}^{\infty} f(x)\cos(w_x)\,dx$$

$$+ \sin(w_x)\int_{-\infty}^{\infty} f(x)\sin(w_x)\,dx\Big]\,dw$$

$$f(x) = \int_0^{\infty}\big[A(w)\cos(wx) + B(w)\sin(wx)\big]\,dw$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)\,dx$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)\,dx$

fourier integral representation of $f(x)$

Theorem: If f and f' are piecewise contineous, then the fourier integral converges to

$$\frac{f(x^+) + f(x^-)}{2}$$ at points discontinous

ex: $f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

- find the fourier integral representation of $f\omega$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx = \frac{1}{\pi}\int_0^1 x\cos(\omega x)\,dx = \frac{1}{\pi}\left[\frac{\omega\sin(\omega) + \cos(\omega) - 1}{\omega^2}\right]$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx = \frac{1}{\pi}\int_0^1 x\sin(\omega x)\,dx = \frac{1}{\pi}\left[\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right]$$

fourier integral represntation of $f(\omega)$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\omega\sin\omega + \cos\omega - 1}{\omega^2}\right]\cos(\omega x) + \left(\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right)\sin(\omega x)\Big]\,d\omega$$

- Determine the convergence of the fourier integral at $x=-1$, $x=0$, $x=1$

at $x=-1$ the fourier integral converges to $f(-1) = -1$

$x = 0$ ~ ~ ~ ~ ~ $f(0) = 0$

$x = 1$ ~ ~ ~ ~ ~

$f(1) = f(1^+) + f(1^-) = 1$

ex: find the fourier integral representation of

$$f(x) = \begin{cases} 0 , & x < -1 \\ 1 , & -1 < x < 1 \\ 0 , & x > 1 \end{cases} = f(x) \begin{cases} 1 , & |x| < 1 \\ 0 , & |x| > 1 \end{cases}$$

$$A(w) = \frac{1}{\pi} \int_{-1}^{1} \cos(wx)\, dx = \frac{2}{\pi} \int_{0}^{1} \cos(wx)\, dx = \frac{2 \sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi} \int_{-1}^{1} \sin(wx)\, dx = 0$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin w}{\pi w} \cos(wx)\, dw$$

$x = -1 \quad f(-1) = \dfrac{f(-1)^{+} + f(-1)^{-}}{2} = \dfrac{1}{2}$

$x = 1 \quad f(1) = \dfrac{f(1)^{+} + f(1)^{-}}{2} = \dfrac{1}{2}$

the fourier integral converges to

* fourier cosine integral

If $f(x)$ is an even function, then:

$$f(x) = \int_0^\infty A(w) \cos(w_x)\, dw$$

where

$$A(w) = \frac{2}{\pi} \int_0^\infty f(x) \cos(w_x)\, dw$$

* fourier sine Integral

If $f(x)$ is an odd function, then

$$f(x) = \int_0^\infty B(w) \sin(w_x)\, dw$$

where

$$B(w) = \frac{2}{\pi} \int_0^\infty f(x) \sin(wx)\, dw$$

11.8: fourier cosine and sine transforms:

$$\mathcal{F}_C\{f(x)\} = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(w_x)\, dx$$

is called fourier cosine transform of $f(x)$ and

$$\mathcal{F}_c^{-1}\{\hat{f}_c(w)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos(w_x)\, dw$$

is called inverse fourier cosine transform

$$\mathcal{F}_s\{f(x)\} = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(w_x)\, dx$$

is called fourier sine transform

$$\mathcal{F}_s^{-1}\{\hat{f}_s(w)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin(w_x)\, dw$$

is called inverse fourier sine transforms ex: $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & x > a \end{cases}$

1) $F_c\{f(x)\} = f_c^{\wedge}(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos(wx)\, dx$

$= \sqrt{\frac{2}{\pi}} \, \frac{k \sin aw}{w}$

2) $F_s\{f(x)\} = f_s^{\wedge}(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin(wx)\, dx$

$= \sqrt{\frac{2}{\pi}} \, k \left(\frac{1 - \cos(aw)}{w}\right)$

* $F_c\{\alpha f(x) + \beta g(x)\} = \alpha F_c\{f(x)\} + \beta F_c\{g(x)\}$, $\alpha, \beta \in R$

$F_s\{\alpha f(x) + \beta g(x)\} = \alpha F_s\{f(x)\} + \beta F_s\{g(x)\}$

* $F_c\{f'(x)\} = w F_s\{f(x)\} - \sqrt{\frac{2}{\pi}} \cdot f(0)$

$F_s\{f'(x)\} = -w F_c\{f(x)\}$

* $F_c\{f''(x)\} = -w^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

$F_s\{f''(x)\} = -w^2 F_s\{f(x)\} + \sqrt{\frac{2}{\pi}} w f(0)$

* اذا اعطاني F_s نحل على المشتقة الأولى

* اذا ما اعطاني F_s نحل على المشتقة الثانية

11.10 fourier Transform

* fourier transform is useful in solving PDEs
* we define the fourier transform for a piecewise continous absolutely integrable

$\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, function $f(x)$ by

$$\mathcal{F}\{f(x)\} = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

and the inverse fourier transform by

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

ex: compute the fourier transform of

$$f(x) = \begin{cases} e^{-2x} & x \geq 0 \\ e^{2x} & x < 0 \end{cases}$$

$$\mathcal{F}\{f(x)\} = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(2-iw)x}\, dx + \int_{0}^{\infty} e^{-(2+iw)x}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\frac{1}{2-iw} + \frac{1}{2+iw}}_{i=j \text{ conject}} \right] = \frac{1}{\sqrt{2\pi}} \left(\frac{4}{4+w^2} \right)$$

* fact: $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

ex: compute the fourier transform of $f(x) = e^{-2x^2}$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x^2 + \frac{i\omega x}{2}\right)}\, dx$$ اكمال مربع

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left[x^2 + \left(\frac{i\omega x}{2}\right) + \left(\frac{i\omega}{4}\right)^2 - \left(\frac{i\omega}{4}\right)^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left[\left(x + \frac{i\omega}{4}\right)^2 + \frac{\omega^2}{16}\right]}\, dx$$

$$= \frac{e^{-\omega^2/8}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x + \frac{i\omega}{4}\right)^2}\, dx$$

let $z = \sqrt{2}\left(x + \frac{i\omega}{4}\right)$ $\quad dz = \sqrt{2}\, dx$

$$F\{f(x)\} = \frac{e^{-\omega^2/8}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-z^2}\, dz \rightarrow \frac{e^{-\omega^2/8}}{2}$$

ex: find the furier transform of:

$$f(x) = \begin{cases} 1 & |x| < 1 \rightarrow -1 < x < 1 \\ 0 & |x| > 1 \end{cases}$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-i\omega x}}{i\omega x} \Big]_{-1}^{1}$$

$$F\{f(x)\} = \frac{2}{\omega\sqrt{2\pi}} \left(\frac{e^{i\omega} - e^{-i\omega}}{2i} \right) \quad + (\cos)$$

$$= \frac{2}{\omega\sqrt{2\pi}} \sin(\omega)$$

- Thereom (Linearity of fourier trans form)

$$F\{\alpha f(x) + \beta g(x)\} = \alpha F\{f(x)\} + \beta F\{g(x)\} \qquad \alpha, \beta \in R$$

Thereom

1) $F\{f'(x)\} = i\omega F\{f(x)\}$

Proof

Fourier transform table

$f(x)$	$\hat{f}(w)$
1) $\begin{cases} 1, & -b<x<b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin(bw)}{w}$
2) $\begin{cases} 1, & b<x<c \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{-ibw} \quad e^{icw}}{iw\sqrt{2\pi}}$
3) $\begin{cases} e^{-ax}, & x>0 \\ 0, & \text{otherwise} \end{cases}$ $a>0$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
4) $\frac{1}{x^2+a^2}$ $a>0$	$\sqrt{\frac{\pi}{2}}\,\frac{e^{-a\vert w\vert}}{a}$
5) $\begin{cases} e^{ax} & b<x<c \\ 0 & \text{other} \end{cases}$	$\frac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}\,(a-iw)}$
6) $\begin{cases} e^{iaw} & -b<x<b \\ 0 & \text{other} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin b(w-a)}{w-a}$
7) e^{-ax^2} $a>0$	$\frac{1}{\sqrt{2a}}\,e^{-w^2/4a}$
8) $\frac{\sin ax}{x}$ $a>0$	$\begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } \vert w\vert<a \\ 0 & \vert w\vert>a \end{cases}$

chapter 12 : Partial Differential equation "PDEs"

12.1 : Basic concepts of PDEs

- Let us agree to take for the time being two independent variable
  $x \sim$ space variable
  $t \sim$ time variable

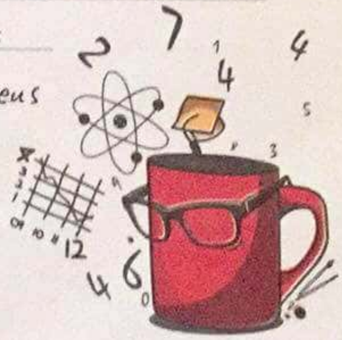
* if u depends on x and t, then:
$$u_x = \frac{du}{dx}, \quad u_{xt} = \frac{d^2u}{dt\,dx}$$

* we will assume that all derivatives are continuous on a specific domain under conserfation thus we can interchange the order of ~~differ~~ differntion
$$u_{xtx} = u_{txx} = u_{xxt}$$

Def: A partial differential equation is on equation contains finite number of partial Derivatives but at least one

Def: The order of the PDE is the order of the highest derivative

Def: If each term contains u or one of its derivative, then the PDE is called homogeneus



Some Important second-order PDEs

1) $u_{tt} = c^2 u_{xx}$ c: constant
"one dimension wave equation"

2) $u_t = c^2 u_{xx}$ "one dimension heat equation"

3) $u_{xx} + u_{yy} = 0$ "two dimension laplace equ"

4) $u_{xx} + u_{yy} = f(x,y)$ " two ~ poisson equ"

5) $u_t = c^2(u_{xx} + u_{yy})$ " ~ ~ wave ~ "

6) $u_{xx} + u_{yy} + u_{zz}$ " three ~ laplace ~ "

Remark: the set of solution can be very large and on needs some constrains (boundary conditions of initial conditions) to restric the solution to have physical meaning

ex: $u_{xx} + u_{yy} = 0$

is satisfied by

$u(x,y) = x^2 - y^2$

$u(x,y) = e^x \cos y$

$u(x,y) = \sinh x \cosh y$

$u(x,y) = \ln(x^2+y^2)$

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superposition principle:

If u_1 and u_2 are solution of the homogeneous PDE, then
$u = c_1 u_1 + c_2 u_2$ is also solution

ex: find solution depending on x and y of

1) $u_{xx} - u = 0$ since y doesn't appear, then we may assume

$$u'' - u = 0$$

change $y^2 - 1 = 0 \rightarrow y = \pm 1$

$u(x,y) = c_1(y)e^{-x} + c_2(y)e^{x} \rightarrow$ general solution

2) $u_{yy} + 4u_y + 4u = 0$ x doesnt appear

$$u'' + 4u' + 4u = 0$$

$y^2 + 4y + 4 = 0 \rightarrow y = -2$ $x^2 + 4x + 4 = 0$

general solution $u(x,y) = c_1(x)e^{-2y} + c_2(x)\,y\,e^{-2y}$

3) $u_{xx} + 2u_x + 25u = 0$ since y doesnt appear, then we may assume

$$u'' + 2u' + 25u = 0$$

change $y^2 + 2y + 25 = 0$

$b^2 - 4ac \rightarrow$ $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$y = -1 \pm 24i$$

$u(x,y) = c_1(y)e^{-t}\cos[24x] + c_2(y)e^{-t}\sin[24x]$

4) $u_{xy} = -u_x$

let $v = u_x$

$v_y = u_{xy} \rightarrow v_y = -v$

$\frac{dv}{dy} = -v \qquad \int \frac{1}{v}\, dv = \int -dy$

$\ln v = -y + c$

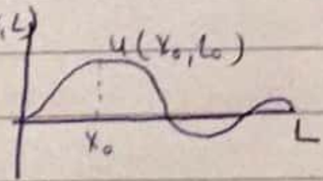
$v = e^{-y}\, c(x)$

$u_{xy} = \int c(x)\, e^{-y}\, dx + c_2(y)$

$u(x,y) = \bar{c}_1(x)\, e^{-y} + c_2(y)$

12.3 vibrating string wave equation

- consider a string of length L
- the model of the vibrating string consists of one-dimensional wave equation



$$u_{tt} = c^2 u_{xx}$$

and boundary conditions

$u(0,t) = 0$

$u(L,t) = 0$

and initial conditions:

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

The solution has three steps:

1) seprating of variables — فصل المتغيرات

2) satisfing the boundary conditions

3) satisfing the intial conditions

Remark: we are secking for a solution $u(x,t) \neq 0$

ex: Solve the following initial boundary value problem

PDE: $u_{tt} = c^2 u_{xx}$ $\quad 0 < x < L \quad L > 0 \quad$ 1)

boundary BC's: $u(0,t) = u(L,t) = 0 \quad L > 0 \quad$ 2)

initial IC's: $\left.\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right] \quad 0 \leq x < L \quad$ 3)

Solu: let us look for asolution of the form

$$u(x,t) = f(x) \cdot G(\cancel{x}\, t) \quad 4)$$

put (4) in (1) to get

$$f(x)\, g''(t) = c^2 f''(x)\, G(t)$$

using the boundary conditions:

$$u(0,t) = f(0)\, G(t) = 0 \rightarrow f(0) = 0 \quad 5)$$

$$u(L,t) = f(L)\, G(t) = 0 \rightarrow f(L) = 0 \quad 6)$$

$$G \neq 0 \rightarrow u \neq 0$$

$$\frac{f(x)''}{f(x)} = \frac{G''(t)}{c^2\, G(t)} = \alpha \quad 7)$$

$f'' - \alpha f = 0 \quad$ 8)

$G'' - c^2 \alpha G = 0 \quad$ 9)

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The constant y has the following cases

$$\alpha = k^2 \qquad 10)$$

or $$\alpha = 0 \qquad 11)$$

or $$\alpha = -k^2 \qquad 12)$$

where $k > 0$

If equation (k) holds, then from (8)

$$f(x) = c_1 e^{-kx} + c_2 e^{kx} \qquad 13)$$

put (5) and (6) in (13) $\rightarrow$ $c_1 = c_2 = 0$ $\rightarrow$ $f(x) = 0$

if equ (11) holds, then from (8)

$$f(x) = c_1 + c_2 x \qquad 14)$$

put (5) and (6) in (14) $\rightarrow$ $c_1 = c_2 = 0$ $\rightarrow$ $f(x) = 0$

If equation (12) holds, then from (8)

$$f(x) = c_1 \sin(kx) + c_2 \cos(kx) \quad \cdots \quad 15)$$

put 5) in 15) $\rightarrow$ $c_2 = 0$

$$f(x) = c_1 \sin(kx)$$

$\sin(kL) = 0$ $\qquad$ $kL = n\pi$ $\rightarrow$ $k = \dfrac{n\pi}{L}$ $\qquad n = 0, 1, \dots$ 16)

$$f_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad 17)$$

now, to find $G(t)$, put (16) in (9)

$$G_n''(t) + \left(\frac{cn\pi}{L}\right)^2 G_n = 0$$

$$\to G_n(t) = a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right) \qquad 18)$$

put (17) and (18) in (4)

$$u_n(x,t) = \sin\left(\frac{n\pi}{L} x\right)\left[a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)\right]$$

$$u(x,t) = \sum_{n=1}^{\infty}$$

*ايجاد a_n, b_n من initial

ex: $u_{tt} = c^2 u_{xx}$ $\quad 0 < x < L$

BCs $\quad u(0,x) = 0 \quad u(L,t) = 0$

IC's $\quad u(x,0) = f(x)$

$\quad u_t(x,0) = g(x)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right)\left[a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)\right] \quad \cdots 19$$

using the IC's (3) we have

$$u(x,0) = \sum_{n=1}^{\infty} a_n \left(\frac{n\pi x}{L}\right) = f(x)$$ "fourier sine series"

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad 20)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \left(\frac{cn\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

$$\left(\frac{cn\pi}{L}\right) b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

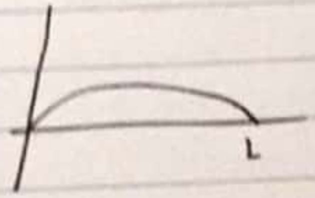
$$b_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad 21)$$

Sabsiting (20) and (21) in (19) gives the solution

Remark: In the previous ex

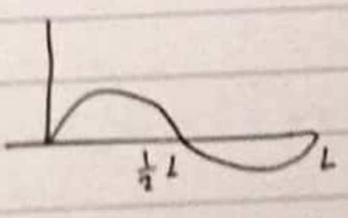
$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L}\right) t\right] \qquad n = 0, 1 \ldots$$

If $n=1$

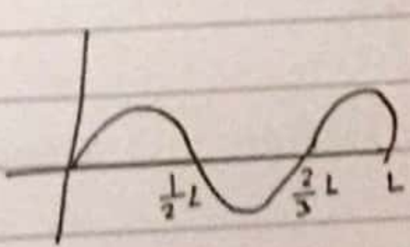


" standing wave "

If $n=2$



If $n=3$



ex:

PDE $u_{tt} = 5u_{xx}$ $\quad 0 < x < 7 \quad t > 0$

BC's $= u(0,t) = 0 \qquad u(7,t) = 0$

EC's $= u(x,0) = 2\sin\left(\frac{3\pi x}{7}\right) + \sin\left(\frac{17\pi}{7}x\right)$, $\quad u_t(x,0) = 0$

Sol: $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{7}\right)\left[a_n \cos\left(\frac{\sqrt{5}\,n\pi}{7}t\right) + b_n \sin\left(\frac{\sqrt{5}\,n\pi}{7}t\right)\right]$

using the EC's we have

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{7}\right) = 2\sin\left(\frac{3\pi x}{7}\right) + \sin\left(\frac{17\pi}{7}x\right)$

$a_3 = 2$ and $a_{17} = 1$ and $a_n = 0$ for $n \neq 3, 17$

$u_t(x,0) = \sum_{n=1}^{\infty} \left(\frac{\sqrt{5}\,\pi n}{7}\right) b_n \sin\left(\frac{n\pi}{7}x\right) = 0$

$b_n = 0$ for $n = 1, 2, \ldots$

Solu: $u(x,t) = 2\sin\left(\frac{3\pi}{7}x\right)\cos\left(\frac{3\sqrt{5}\,\pi}{7}t\right) + \sin\left(\frac{17\pi}{7}x\right)\cos\left(\frac{17\sqrt{5}\,\pi}{7}t\right)$

ex:

PDE $u_{tt} = c^2 u_{xx}$ $\quad 0<x<L$ $\quad t>0$ 1)

BC's $u_x(0,t) = 0,\ u_x = (L,t) = 0$ $\quad t>0$ 2)

Ec's $u(x,0) = f(x),\ u_t(x,0)\, g(x)$ $\quad 0 \le x \le L$ 3)

Assume $u_{xt} = f(x)\, g(t)$ 4)

4) in 1) gives

$$\frac{f''}{F} = \frac{G''}{c^2 G} = \alpha$$

$f'' - \alpha f = 0$ 5)

$G'' - c^2 \alpha G = 0$ 6)

Put (2) in (4) to get

$f'(0) = 0$ 7)

$f'(L) = 0$ 8)

Now, the constant α has the following cases

$\alpha = k^2$ 9)

or $\alpha = 0$ 10)

or $\alpha = k^2$ 11)

where $k > 0$

If (9) hold, then

$$f(x) = c_1 e^{-kx} + c_2 e^{kx}$$

using (7) and (8) $\rightarrow$ $c_1 = c_2 = 0$

$$f(x) = 0$$

If (10) holds, then

$$f(x) = c_1 + c_2 x$$

using (7) and (8) $\rightarrow$ $c_2 = 0$ and c_1 free

$$f(x) = 1$$

If (11) holds, then

$$f(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

using (7) $\quad c_2 = 0 \rightarrow f(x) = c_1 \cos(kx)$

$$f(x) = \cos(kx)$$

using (8) $\quad \sin(kL) = 0$

$$k = \frac{n\pi}{L} \qquad n = 1, 2 \qquad 12)$$

$$f_n(x) = \cos\left(\frac{n\pi}{L} x\right) \qquad 13)$$

To find $G_n(t)$

put (10) in (16) to obtain

$$G_0 = 0 \rightarrow G_0(t) = At + B$$

put (12) in (6)

$$G_n'' + \left(\frac{cn\pi}{L}\right)^2 G_n = 0$$

$$G_n(t) = a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)$$

general solu

$$u(x,t) = f_0(x)\, G_0(t) + \sum_{n=1}^{\infty} f_n(x)\, G_n(t)$$

$$= At + B + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L} x\right) \left[a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{nc\pi}{L} t\right)\right]$$

using the Ec's

$$A = \frac{1}{L} \int_0^L f(x)\, dx \qquad u(x,0) = f(x)$$

$$B = \frac{1}{L} \int_0^L g(x)\, dx \qquad u_t(x,0) = g(x)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx \qquad u(x,0) = f(x)$$

$$b_n = \frac{2}{cn\pi} \int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx \qquad u_t(x,0) = g(x)$$