Fall2016

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Partial

Dr. Ahmad Abdullah

No.

9.3 Vector product (cross product) :-

$\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ w_2 & w_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ w_1 & w_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ w_1 & w_2 \end{vmatrix} \hat{k}$$

$$= (u_1 w_3 - u_3 w_2)\hat{i} - (u_1 w_3 - u_3 w_1)\hat{j} + (u_1 w_2 - u_2 w_1)\hat{k}$$

Results:-

① $\vec{u} \times \vec{w} = -\vec{w} \times \vec{u}$

② $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

③ In general

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ why ??

$A = |\vec{a} \times \vec{b}|$ $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$

9.5 Curves :-

Curve $C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ Vector form

$\langle -3 + 2t, -4t, 5 - t \rangle$

$L: x = -3 + 2t$, $y = 0 - 4t$, $z = 5 - t$ parametric form

Ex: $x^2 + 4y^2 = 4 \longrightarrow \frac{x^2}{2^2} + y^2 = 1$ (Ellipse)

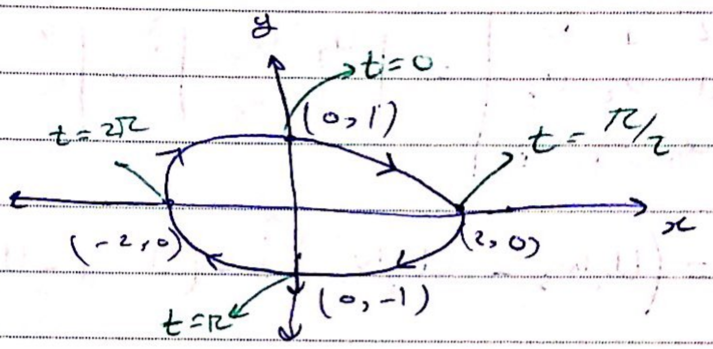
$\sin^2 x + \cos^2 x = 1$ $\frac{(2 \sin t)^2}{2^2} + \cos^2(t) = 1$

$x = 2 \sin t$, $y = \cos(t)$

No.

Continue to previous example :-

$\vec{r}(t) = \langle x(t), y(t) \rangle = \langle 2\sin(t), \cos(t) \rangle$, $0 \le t \le 2\pi$



* negative sence (clock wise)
* positive sence (Counter clock wise)

* if $x = 2\sin 4t$ $\rightarrow 0 \le t \le \pi/2$

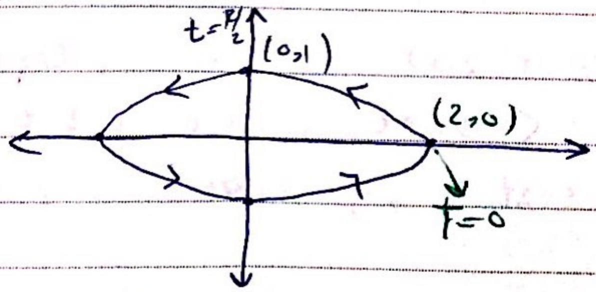
* if we choose $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

$x = 2\cos t$, $y = \sin(t)$ $\quad \frac{(2\cos(t))^2}{2^2} + \frac{(\sin(t))^2}{1^2} = 1$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$= \langle 2\cos(t), \sin(t) \rangle$

$0 < t < 2\pi$



Counter Clockwise (positive sense)

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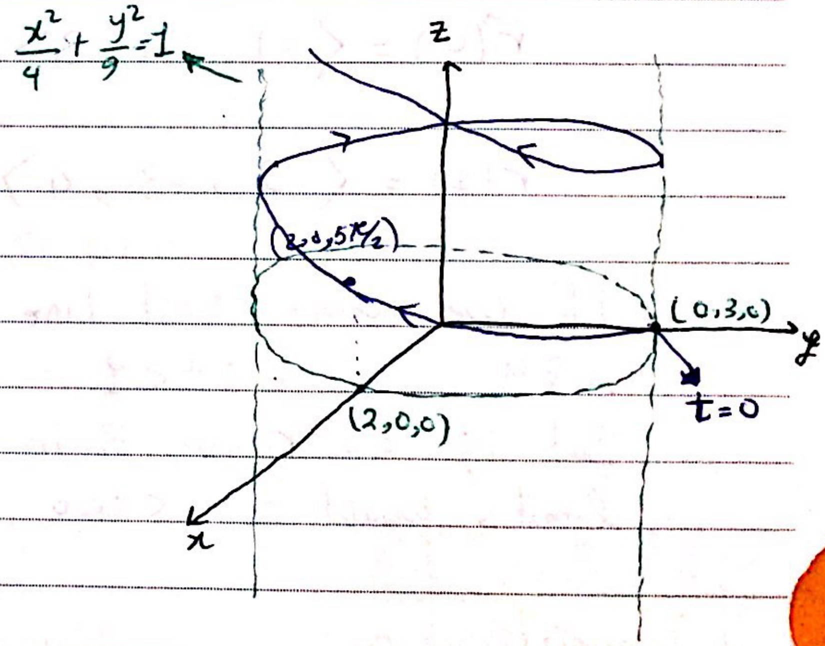
EX: $\vec{r}(t) = \langle 2\sin(t) , 3\cos(t) , 5t \rangle$

since we have just one variable , so this is curve , if we have $\vec{r}(t,s)$ then this is surface

Sol: $\vec{r}(t) = \langle x(t) , y(t) , z(t) \rangle$

In two dimentions ⇒ (without the z component) $\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \frac{(2\sin(t))^2}{4} + \frac{(3\cos(t))^2}{9} = 1$

Hellix ⟹



EX: Line : The equation of the line passing through point $A(a_1, a_2, a_3)$ and parallel to $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\vec{r}(t) = \langle a_1 + tb_1 , a_2 + tb_2 , a_3 + tb_3 \rangle$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$\vec{r}(t) = \langle x(t) , y(t) , z(t) \rangle$$

$$\vec{r}(t) = \vec{a} + t\vec{b}$$

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→ followed

No.

Ex: Find the equation of the line passing through $A(-1, 0, 3)$ and $B(2, -3, 4)$
$a_1 \; a_2 \; a_3 \qquad a_1 \; a_2 \; a_3$

Sol: $\vec{AB} = \langle 3, -3, 1 \rangle$ // line that we want
$b_1 \; b_2 \; b_3$

Line: $\vec{r}(t) = \langle -1+3t, 0-3t, 3+t \rangle$

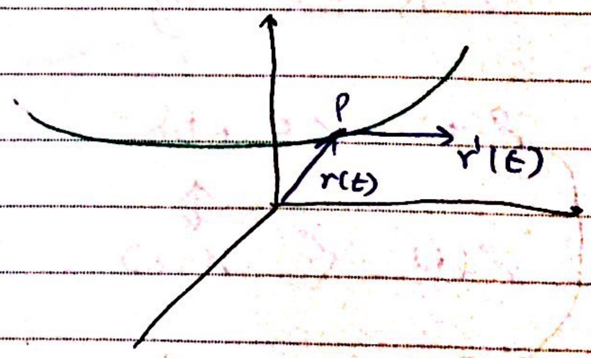
$\vec{r}(0) = \langle -1, 0, 3 \rangle$ point (A)

$\vec{r}(1) = \langle 2, -3, 4 \rangle$ point (B)

* If we want just line segement between A&B $\rightarrow$ $0<t<1$
But if we choose B in our equation, then the limits would be $-1<t<0$

* Tangent vector

The tangent vector to $\vec{r}(t)$ is given by $\vec{r}'(t)$



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* The tangent Line to curve $C : \vec{r}(t)$ at point (p) is given by :-

$\vec{r}(t) = \vec{a} + t\vec{b}$ represents the position vector to the point.

$q(w) = \vec{r}(t_0) + w\,\vec{r}\,'(t_0)$

fixed time → the line which is parallel to the line

EX: Find the tangent to the Ellipse :-

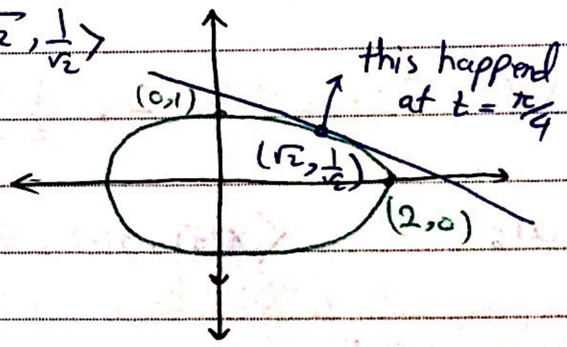
$$\frac{1}{4}x^2 + y^2 = 1 \quad @ \; p\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$$

Sol: $\vec{r}(t) = \langle 2\cos(t), \sin(t) \rangle, \quad 0 \leq t \leq 2\pi$

$\vec{r}(\pi/4) = \left\langle \frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle \sqrt{2}, \frac{1}{\sqrt{2}} \right\rangle$

$\vec{r}\,'(t) = \langle -2\sin t, \cos(t) \rangle$

$\vec{r}\,'(\pi/4) = \left\langle -\sqrt{2}, \frac{1}{\sqrt{2}} \right\rangle$



The tangent Line

$q(w) = \vec{r}(\pi/4) + w\,\vec{r}\,'(\pi/4)$

$q(w) = \left\langle \sqrt{2}, \frac{1}{\sqrt{2}} \right\rangle + w \left\langle -\sqrt{2}, \frac{1}{\sqrt{2}} \right\rangle$

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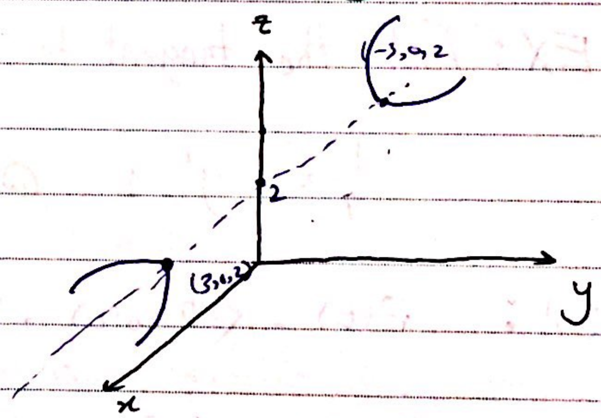
EX: what curves are represented by the following
$\vec{r}(t) = \langle 3\cosh t, 5\sinh t, 2 \rangle$

Sol: $\langle x(t), y(t), z(t) \rangle$

$$\frac{x^2}{9} - \frac{y^2}{25} = 1 \quad , \quad z = 2$$

$$\frac{(3\cosh t)^2}{9} - \frac{(5\sinh t)^2}{25} = 1 \quad , \quad \cosh^2 t - \sinh^2 t = 1$$

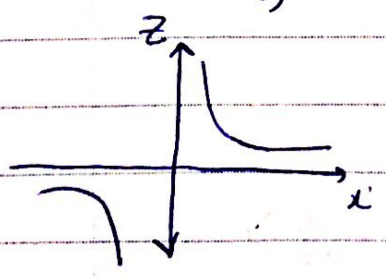


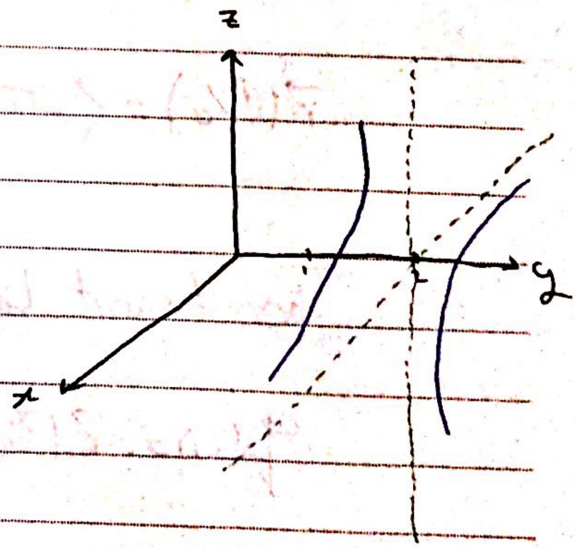
EX: $\vec{r}(t) = \langle t, 2, \frac{1}{t} \rangle$

Sol: $\langle x(t), y(t), z(t) \rangle$

$x(t) = t$, $y = 2$, $z(t) = \frac{1}{t}$

$z(t) = \frac{1}{x(t)}$, $y = 2$





→ followed

$f(x,y,z)=K \rightarrow x^2+y^2+z^2=9$ (sphere) (surface)

$z=f(x,y,z)=x^2+y^2$ (surface)

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4.7 Gradiant of a scaler function

Given a scaler function $f(x,y,z)$ which is defined and differentiable in a domain im 3-space with Cartesian Coordinates $x, y,$ and z.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

① The directional derivative of f at a point P in the direction of a unit vector $\vec{a}$ is given by

$$D_{\vec{a}} f(P) = \nabla f \cdot \vec{a}$$

② Given a surface $f(x,y,z)=K$ then the normal at P is given by $\vec{n} = \nabla f(P)$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$ → Gradiant operator

$\frac{15}{402}$ $f = 4x^2 + 9y^2 + z^2$, $P:(5,-1,-11)$

$\nabla f(5,-1,-11) = ?$

Sol: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 8x, 18y, 2z \rangle$

$\nabla f(5,-1,-11) = \langle 8(5), 18(-1), 2(-11) \rangle$

No.

$\frac{7-10}{402}$ prove: $\nabla(f^n) = n f^{n-1} \nabla f$

Sol: $\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle$

$\nabla(f^n) = \left\langle \frac{\partial f^n}{\partial x}, \frac{\partial f^n}{\partial y}, \frac{\partial f^n}{\partial z} \right\rangle$

$= \left\langle n f^{n-1} \frac{\partial f}{\partial x}, n f^{n-1} \frac{\partial f}{\partial y}, n f^{n-1} \frac{\partial f}{\partial z} \right\rangle \Rightarrow \frac{d}{dx}[F(x)]^n = n[F(x)]^{n-1} f'(x)$

$= n f^{n-1} \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$\nabla(f^n) = n f^{n-1} \nabla f$

$\frac{8}{402}$ $\nabla(fg) = f \nabla g + g \nabla f$

Sol: $\nabla(fg) = \left\langle \frac{\partial (fg)}{\partial x}, \frac{\partial (fg)}{\partial y}, \frac{\partial (fg)}{\partial z} \right\rangle$

$= \left\langle \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} g + f \frac{\partial f}{\partial z} \right\rangle$

$= \left\langle \frac{\partial f}{\partial x} g, \frac{\partial f}{\partial y} g, \frac{\partial f}{\partial z} g \right\rangle + \left\langle \frac{\partial g}{\partial x} f, \frac{\partial g}{\partial y} f, \frac{\partial g}{\partial z} f \right\rangle$

$= g \nabla f + f \nabla g$

followed

No.

4.8 Divergence of a Vector Field.

Let $\vec{v}(x,y,z) = \langle v_1(x,y,z), v_2(x,y,z), v_3(x,y,z) \rangle$ be a differentiable vector function then:-

$$\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\Rightarrow \operatorname{div}(\operatorname{grad} f) = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} + \frac{\partial^2 f}{\partial z}$$

prove: Let f be is a scaler function

Gradiant $\Rightarrow$ $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

Divergence $\Rightarrow$ $\nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} + \frac{\partial^2 f}{\partial z} = f_{xx} + f_{yy} + f_{zz}$

Laplace operation

* we call the divergence of a gradiant as (Laplacian) *

5/405 $\vec{v} = x^2y^2z^2 \langle x, y, z \rangle$ find the divergence.

Sol: $\vec{v} = \langle x^3y^2z^2, x^2y^3z^2, x^2y^2z^3 \rangle$

$$\nabla\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2$$

$$= 4x^2y^2z^2$$

followed

No. scaler vector

$\frac{4}{406}$ prove : (b) $\text{div}(f\vec{v}) = f\,\text{div}\,\vec{v} + \vec{v}\cdot\nabla f$

Sol: $\vec{v} = \langle v_1 , v_2 , v_3 \rangle \Rightarrow f\vec{v} = \langle fv_1 , fv_2 , fv_3 \rangle$

$$\text{div}(f\vec{v}) = \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)$$

$$= \frac{\partial f}{\partial x}v_1 + f\frac{\partial v_1}{\partial x} + \frac{\partial f}{\partial y}v_2 + f\frac{\partial v_2}{\partial y} + \frac{\partial f}{\partial z}v_3 + f\frac{\partial v_3}{\partial z}$$

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + \frac{\partial f}{\partial x}v_1 + \frac{\partial f}{\partial y}v_2 + \frac{\partial f}{\partial z}v_3$$

$$= f\,\nabla\vec{v} + \vec{v}\cdot\nabla f$$

(c) $\text{div}(f\nabla g) = f\nabla^2 g + \nabla g\cdot\nabla f$ → solve it !

$\frac{16}{406}$ $f = e^{xyz}$, find $\nabla^2 f$

Sol: $f_x = yz\,e^{xyz} \longrightarrow f_{xx} = (yz)^2 e^{xyz}$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= (yz)^2 e^{xyz} + (xy)^2 e^{xyz} + (xz)^2 e^{xyz}$$

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9.9 Curl of a vector Field:

Let $\vec{v}(x,y,z) = \langle v_1, v_2, v_3 \rangle$ be a differentiable vector function of the cartesian coordinates x, y and z.

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) K$$

EX: $\vec{v} = \langle x^2 y, e^{3z}, 2x + z^3 \rangle$, find $\nabla \times \vec{v}$

Sol: $\nabla \times \vec{v} = -3e^{3z} i - 2j - x^2 K$

$\frac{14}{409}$ Show that
(b) $\text{div}(\text{curl } \vec{v}) = 0$

Sol: $\vec{v} = \langle v_1, v_2, v_3 \rangle \Rightarrow \text{curl } \vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) K$

$$\text{div}(\text{curl } \vec{v}) = \frac{\partial}{\partial x}\left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right]$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial v_1}{\partial z \partial y}$$

Note that (not always) $\frac{\partial^2 v_3}{\partial x \partial y} = \frac{\partial^2 v_3}{\partial y \partial x}$ & $\frac{\partial^2 v_2}{\partial x \partial z} = \frac{\partial^2 v_2}{\partial z \partial x}$ & $\frac{\partial^2 v_1}{\partial y \partial z} = \frac{\partial^2 v_1}{\partial z \partial y}$

$= \text{Zero}.$

No.

(d) curl (grad f) = $\vec{0}$ do it @ home !

Chap 10 : Vector Integral Calculus

10.1 : Line Integral :-

1] Smooth curve : $C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
continously differentiable (The first derivative curve is continous)

2] piecewise smooth path : the curve is constructed of many sub carves & all of them is smoth.

* In this book every path of integration of a line integral is assumed to be piecewise smoth.

Definition of line integral :

A line integral of a vector function $\vec{F}(\vec{r})$ over a curve $C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $a < t < b$ is defined by

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\overset{\vec{r}(t)}{\vec{r}_0}) \cdot \vec{r}'(t)\, dt \quad \rightarrow \text{with } d\vec{r} = \langle dx, dy, dz \rangle$$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt$$

→ followed

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Notes:-

① $f'(x) = \frac{df}{dx} \longrightarrow df = f'(x)\,dx$

$\vec{r}'(t) = \frac{d\vec{r}}{dt} \longrightarrow d\vec{r} = \vec{r}'(t)\,dt$

② $\vec{r} = \langle x, y, z \rangle \rightarrow \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$d\vec{r} = \langle dx, dy, dz \rangle$

③ $\vec{F} = \langle F_1, F_2, F_3 \rangle \longrightarrow d\vec{r} = \langle dx, dy, dz \rangle$

$\vec{F}(\vec{r}) \cdot d\vec{r} = F_1\,dx + F_2\,dy + F_3\,dz$

④ $dx = x'\,dt$

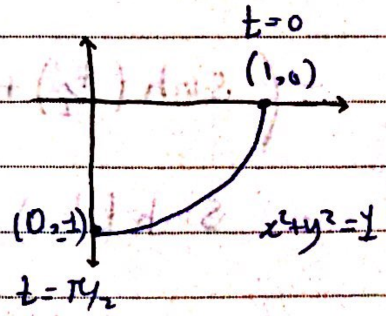
9th edition $\frac{1-13}{425}$ work done by a force, calculate $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$

$\frac{6}{425}$ $\vec{F} = \langle e^{x}, e^{y} \rangle$, clockwise along the circle with center (0,0) from (1,0) to (0,-1)

Sol: $C: \vec{r}(t) = \langle \cos t, -\sin t \rangle \quad 0 \le t \le \pi/2$

$\vec{r}(0) = \langle 1, 0 \rangle$

$\vec{r}(\pi/2) = \langle 0, -1 \rangle$



$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$

$\vec{r}'(t) = \langle -\sin t, -\cos t \rangle$

$\vec{F}(\vec{r}(t)) = \langle e^{\cos t}, e^{-\sin t} \rangle$

$= \int_0^{\pi/2} (-\sin t\, e^{\cos t} - \cos t\, e^{-\sin t})\,dt$

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8/425 $\vec{F} = \langle \cosh x , \sinh y , e^{z} \rangle$, $C: \vec{r} = \langle t, t^2, t^3 \rangle$
from $(0,0,0)$ to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$
$t=0$ $\quad$ $t=\frac{1}{2}$

Sol: $\vec{r}(0) = \langle 0, 0^2, 0^3 \rangle$, $\vec{r}(\frac{1}{2}) = \langle \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3 \rangle$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{r}(t) = \langle t , t^2 , t^3 \rangle \Rightarrow \vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{F} = \langle \cosh x(t), \sinh y , e^{z} \rangle \Rightarrow \vec{F}(\vec{r}(t)) = \langle \cosh(t) , \sinh(t^2) , e^{t^3} \rangle$$

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_0^{1/2} \langle \cosh(t), \sinh(t^2), e^{t^3} \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt$$

$$= \int_0^{1/2} \left(\cosh(t) + 2t\sinh(t^2) + 3t^2 e^{t^3} \right) dt$$

$$= \left(\sinh(t) + \cosh(t^2) + e^{t^3} \right) \Big|_0^{1/2} = \text{......}$$

$$= \left(\sinh(\tfrac{1}{2}) + \cosh(\tfrac{1}{4}) + e^{\frac{1}{8}} \right) - \left(\sinh(0) + \cosh(0) + e^{0} \right)$$

$$= \sinh(\tfrac{1}{2}) + \cosh(\tfrac{1}{4}) + e^{1/8} - 0 - 1 - 1$$

$$= \sinh(\tfrac{1}{2}) + \cosh(\tfrac{1}{4}) + e^{1/8} - 2$$

9th edition

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9
425

$\vec{F}$ as in prob 8

C: the straight segment from $\overbrace{(\underset{a_1}{0}, \underset{a_2}{0}, \underset{a_3}{0})}^{A}$ to $\overbrace{(1/2, 1/4, 1/8)}^{B}$

Sol: we have to find first the equation of the line

$$\overrightarrow{AB} = \langle 1/2, 1/4, 1/8 \rangle - \langle 0, 0, 0 \rangle = \langle \underset{}{\overset{b_1}{1/2}}, \overset{b_2}{1/4}, \overset{b_3}{1/8} \rangle$$

$0 \le t \le 1$

$$L: \vec{r}(t) = \langle a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t \rangle = \langle \underbrace{\tfrac{1}{2}t}_{x(t)}, \underbrace{1/4\, t}_{y(t)}, \underbrace{1/8\, t}_{z(t)} \rangle$$

$\vec{r}(0) = \langle 0, 0, 0 \rangle$, $\vec{r}(1) = \langle 1/2, 1/4, 1/8 \rangle$, so our limits is correct

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$\vec{r}\,'(t) = \langle \frac{1}{2}, \frac{1}{4}, 1/8 \rangle$, $\vec{F}(\vec{r}(t)) = \langle \cosh(x(t)), \sinh(y(t)), e^{z(t)} \rangle$

$$\vec{F}(\vec{r}(t)) = \langle \cosh(1/2\, t), \sinh(1/4\, t), e^{1/8 t} \rangle$$

$$\int_0^1 \langle \cosh(1/2\, t), \sinh(1/4\, t), e^{1/8 t} \rangle \cdot \langle 1/2, 1/4, 1/8 \rangle dt$$

$$= \int_0^1 \left(\frac{1}{2}\cosh(1/2\, t) + \sinh(1/4\, t)\, 1/4 + \frac{1}{8} e^{1/8 t} \right) dt$$

$$= \sinh(1/2\, t) + \cosh(1/4\, t) + e^{1/8 t} \Big|_0^1$$

$$= \left[\sinh(1/2) + \cosh(1/4) + e^{1/8}\right] - \left[\sinh(0) + \cosh(0) + 1\right]$$

∴ هذه القيمة تساوي القيمة في السؤال السابق ، هذا لا يعني بتاتاً أنها لا تعتمد على المسار followed.

No.

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$\vec{F} = \langle y^2, x^2, \cos^2 z \rangle$, $C: \vec{r} = \langle \cos t, \sin(t), t \rangle$

from $\underbrace{(1,0,0)}_{t=0}$ to $\underbrace{(1,0,4\pi)}_{t=4\pi}$

Sol: $\vec{r}(0) = \langle 1,0,0 \rangle$, $\vec{r}(4\pi) = \langle 1,0,4\pi \rangle$

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$\langle F_1, F_2, F_3 \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle$$

$$= \int_a (F_1 x'(t) + F_2 y'(t) + F_3 z'(t))\, dt$$

$$\vec{r}(t) = \langle \overset{x(t)}{\cos t}, \overset{y(t)}{\sin(t)}, \overset{z(t)}{t} \rangle$$

$$\vec{F} = \langle (y(t))^2, (x(t))^2, \cos^2(z(t)) \rangle$$

$$\vec{F} = \langle \sin^2(t), \cos^2(t), \cos^2(t) \rangle$$

$x(t) = \cos(t) \longrightarrow x'(t) = -\sin(t)$

$y(t) = \sin(t) \longrightarrow \frac{dy(t)}{dt} = \cos(t)$

$z(t) = t \longrightarrow z'(t) = 1$

$$= \int_0^{4\pi} \left(-\sin^3(t) + \cos^3(t) + \cos^2(t)\right) dt$$

$$* \int_0^{4\pi} \cos^3(t)\, dt = \int_0^{4\pi} \cos^2(t)\cos(t)\, dt = \int_0^{4\pi} (1-\sin^2 t)\cos t\, dt$$

Let $u = \sin t$

$$* \int_0^{4\pi} \cos^2(t)\, dt = \int_0^{4\pi} \frac{1}{2}(1+\cos 2t)\, dt = \ldots\ldots$$

$\rightarrow$ followed

No.

NOTE: The forms of line Integral:-

1] $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

2] $\int_C \vec{F}(\vec{r})\,dt = \int_a^b \vec{F}(\vec{r}(t))\,dt$

scaler function

3] $\int_C f(\vec{r})\,dt = \int_a^b f(\vec{r}(t))\,dt$

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$\vec{F} = \langle (xy)^{1/3}, \left(\frac{y}{x}\right)^{1/3}, 0 \rangle$

$C: \vec{r} = \langle \cos^3 t, \sin^3 t, 0 \rangle, \quad 0 \leq t \leq \pi/4$

Find $\int_C \vec{F}(\vec{r})\,dt$

sol: $\vec{F}(\vec{r}(t)) = \left\langle (\cos^3(t)\sin^3(t))^{1/3}, \left(\frac{\sin^3(t)}{\cos^3(t)}\right)^{1/3}, 0 \right\rangle$

$= \left\langle \cos(t)\sin(t), \frac{\sin(t)}{\cos(t)}, 0 \right\rangle$

$\int_C \vec{F}(\vec{r})\,dt = \int_0^{\pi/4} \left\langle \frac{1}{2}\sin(2t), \frac{\sin(t)}{\cos(t)}, 0 \right\rangle dt$

$= \left\langle \frac{-1}{4}\cos(2t), -\ln|\cos(t)|, 0 \right\rangle \Big]_0^{\pi/4}$

$= \left\langle \frac{1}{4}, -\ln\left|\frac{1}{\sqrt{2}}\right|, 0 \right\rangle$ → note that in this type of line integral the answer is vector not scaler!

Thursday | Dr. Ahmad Abdullah
29/9/2016

No.

10.2 path independence of Line Integrals

Theorem 1: A Line integral $\int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ (1)
in a domain D is path independent
if and only if

$\vec{F} = \nabla f$ for some function f in D.

Definition: The differential form $\vec{F}(\vec{r})\cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$ (4)
is exact in D if and only if

$\vec{F} = \nabla f$ in D for some f.

Theorem 3*: The integral (1) is path independent in a domain D if and only if the differential form (4) is exact, and has continous coefficients F_1, F_2, and F_3.

$\frac{1-8}{432}$ Show that the form under integral sign is exact and evaluate the integral.

$\frac{6}{432}$ $$\int_{(0,0,0)}^{(1,1,0)} e^{x^2+y^2-2z}(x\,dx + y\,dy - dz)$$

sol: $$\int_{(0,0,0)}^{(1,1,0)} \Big(\underbrace{xe^{x^2+y^2-2z}}_{F_1} dx + \underbrace{ye^{x^2+y^2-2z}}_{F_2} dy - \underbrace{e^{x^2+y^2-2z}}_{F_3} dz \Big)$$

F_1 & F_2 & F_3 are continous on all $\mathbb{R}^3$!

→ followed

No.

$\Rightarrow$ We can conclude that: $f = \frac{1}{2} e^{x^2+y^2-2z}$

$\Rightarrow$ to check: $f_x = x e^{x^2+y^2-2z} = F_1$

$f_y = y e^{x^2+y^2-2z} = F_2$

$f_z = -e^{x^2+y^2-2z} = F_3$

By Inspection

* We want to find f such that

$$\vec{F} = \nabla f \Rightarrow \langle F_1, F_2, F_3 \rangle = \langle f_x, f_y, f_z \rangle$$

$$F_1 = f_x \rightarrow x e^{x^2+y^2-2z} = f_x$$

$$F_2 = f_y \rightarrow y e^{x^2+y^2-2z} = f_y$$

$$F_3 = f_z \rightarrow -e^{x^2+y^2-2z} = f_z$$

We have to solve These equations

$\Rightarrow$ (*) $f = \int x e^{x^2+y^2-2z} dx = \frac{1}{2} e^{x^2+y^2-2z} + g(y,z)$

$$f_y = y e^{x^2+y^2-2z} + \frac{\partial g(y,z)}{\partial y} = y e^{x^2+y^2-2z}$$

$$\frac{\partial g(y,z)}{\partial y} = \text{Zero} \longrightarrow g(y,z) = \int 0 \, dy = h(z)$$

(*) $f = \frac{1}{2} e^{x^2+y^2-2z} + h(z)$ (**)

$$f_z = -e^{x^2+y^2-2z} + \frac{dh(z)}{dz} = -e^{x^2+y^2-2z} \longrightarrow \frac{dh(z)}{dz} = \text{Zero}$$

$$\longrightarrow h(z) = \int 0 \, dz = C$$

$\rightarrow$ followed

No.

(**) $f = \frac{1}{2} e^{x^2+y^2-2z} + C$, Assume $c =$ Zero

$f = \frac{1}{2} e^{x^2+y^2-2z}$

$F_1 = f_x$ & $F_2 = f_y$ & $F_3 = f_z$

$\vec{F} = \nabla f \Rightarrow \vec{F}(\vec{r}) \cdot d\vec{r}$ is exact

$\Rightarrow \int_{(0,0,0)}^{(1,1,0)} \cdots$ is path indep.

$$\int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = \int_A^B \underbrace{\nabla f \cdot d\vec{r}}_{\text{path indep.}} = \int_A^B \langle f_x, f_y, f_z \rangle \cdot \langle dx, dy, dz \rangle$$

$$= \int_A^B (f_x dx + f_y dy + f_z dz)$$

$df \equiv$ total differential

$$= \int_A^B df = f(B) - f(A)$$

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4/10/2016 Tuesday | Dr. Ahmad Abdullah

No.

$\frac{8}{432}$ $\displaystyle\int_{(2,0,1)}^{(4,4,0)} \left[2x(y^3-z^3)\,dx + 3x^2y^2\,dy - 3x^2z^2\,dz\right]$

sol: we want to find f such that $\nabla f = \vec{F}$

$\boxed{\begin{array}{l} F_1 = \frac{\partial f}{\partial x} \\ F_2 = \frac{\partial f}{\partial y} \\ F_3 = \frac{\partial f}{\partial z} \end{array}} \Longrightarrow \begin{array}{l} f_x = 2x(y^3-z^3) \\ f_y = 3x^2y^2 \quad (1) \\ f_z = -3x^2z^2 \quad (2) \end{array}$

(*) $f = \int 2x(y^3-z^3)\,dx = x^2(y^3-z^3) + g(y,z)$

$f_y = \overset{(1)}{3x^2y^2} = 3x^2y^2 + \frac{\partial g(y,z)}{\partial y} \longrightarrow \frac{\partial g(y,z)}{\partial y} = 0$

$\longrightarrow g(y,z) = \int 0\,dy = h(z)$

(*) $f = x^2(y^3-z^3) + h(z)$

$f_z = \overset{(2)}{-3x^2z^2} = -3x^2z^2 + \frac{dh(z)}{dz} \longrightarrow \frac{dh(z)}{dz} = 0$

$\longrightarrow h(z) = \int 0\,dz = c$ (Take the constant equal zero)

(*) $f = x^2(y^3-z^3)$ so $\vec{F} = \nabla f$ (check!)

* so the integral is independant of path

$\displaystyle\int_{(2,0,1)}^{(4,4,0)} [\cdots\cdots] = f(4,4,0) - f(2,0,1) = 4^2(4^3-0^3) - 2^2(0^3-1^3)$

(4) $\vec{F}\cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$

(6') $\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0$, $\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} = 0$, $\frac{\partial F_2}{\partial x} - \frac{\partial F}{\partial y} = 0$

No.

Theorem 2: The integral $\int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ (1) is path independant in a domain D, so its value around every closed curve is Zero.

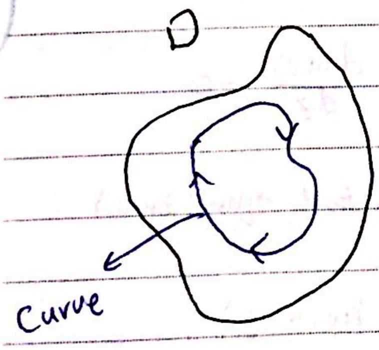
Theorem 3: Let F_1, F_2, F_3 in the line integral (1) be continous and have continous first partial derivatives in a domain D in space; Then:

(a) if the differential form (4) is exact in D and thus (1) is path indep.; then in D

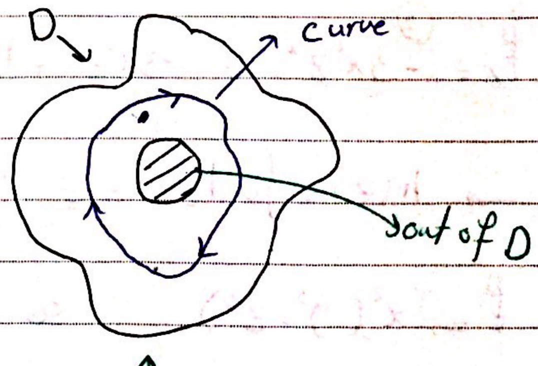
$$\text{curl}\,\vec{F} = \vec{0} \quad (6)$$

(b) if (6) holds in D and D is $\boxed{\text{simply connected}}$ then (4) is exact in D and thus (1) is path indep.

* A domain D is called simply connected if: every closed curve in D can be continously shrunk to any point in D without leaving D.



simply connected



not simply connected

*sphere inside sphere in 3D would be (simply connected)

→ Followed

No.

$\frac{11-19}{432}$ Check for path independence and if indep. integrate from $(0,0,0)$ to (a,b,c)

$\frac{12}{432}$ $(3x^2e^{2y} + x)\,dx + 2x^3e^{2y}\,dy + 0\,dz$

$\quad F_1 \qquad F_2 \qquad F_3$

Sol: $\nabla \times \vec{F} = \overrightarrow{Zero}$ (check!)

* Since the Domain in our case $|R^3|$, so it is simple connected; so

curl $\vec{F} = 0 \longrightarrow$ (1) is path indep.

curl $\vec{F} = 0 \Rightarrow \int_{(0,0,0)}^{(a,b,c)} [(3x^2e^{2y} + x)dx + (2x^3e^{2y})dy + 0\,dz]$

is path indep.

$\Rightarrow$ Find f such that $\vec{F} = \nabla f$ ..... we find $f = x^3e^{2y} + \frac{x^2}{2}$ (check!)

$\Rightarrow \int_{(0,0,0)}^{(a,b,c)} \cdots = f(a,b,c) - f(0,0,0)$

$= a^3e^{2b} + \frac{a^2}{2} - 0 + 0 = a^3e^{2b} + \frac{a^2}{2}$

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No.

14/432 $2x\sin y\,dx + x^2\cos y\,dy + y^2dz$
F_1 F_2 F_3

Sol: $\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = (2y-0)\,i - (\;)\,j + (\;)\,k \neq \vec{0}$

$\nabla \times \vec{F} \neq \vec{0} \Rightarrow$ so the integral is path dependent.

10.3 Double integrals:-

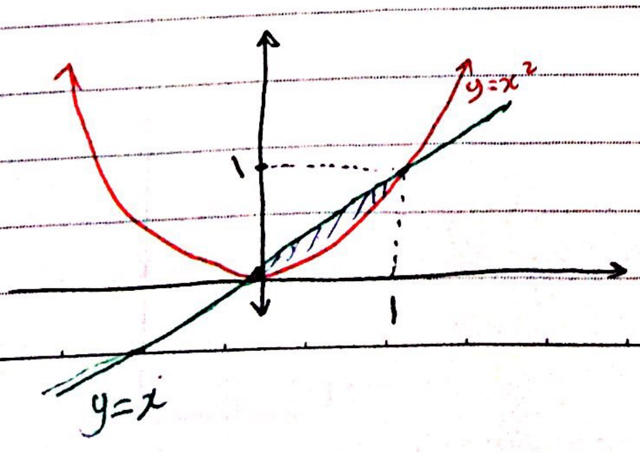
3/438 $\int_0^1 \int_{x^2}^{x} (1-2xy)\,dy\,dx$

Sol: $\int_0^1 \left[y - \frac{2xy^2}{2} \Big]_{x^2}^{x} \right] dx = \int_0^1 \left(x - x^3 - (x^2 - x^5) \right) dx$

$= \int_0^1 (x - x^3 - x^2 + x^5)\,dx =$ ..... Continue

4/438 As prob.3 order reversed

$\int_0^1 \int_{x^2}^{x} (1-2xy)\,dy\,dx = \int_0^1 \int_{x=y}^{x=\sqrt{y}} (1-2xy)\,dx\,dy$



$x=y$ & $x=\sqrt{y}$ (نأخذ المرجعية الأفقية)

$\Rightarrow$ then continue as the previous problem.

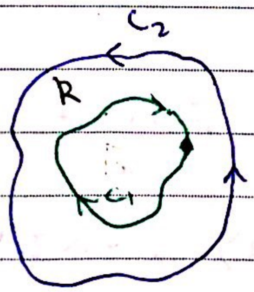
No.

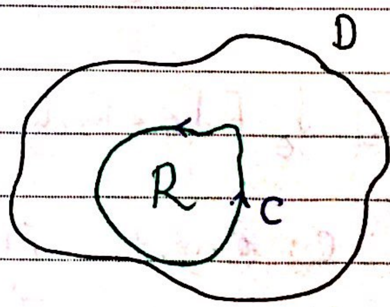
10.4 Green's Theorem in The plane

Theorem 1: Let R be a closed bounded region in the x-y plane whose boundary C consist of finitely many smoth curves.
Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continous and have continous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ every where in some domain containing R, Then:

$$\boxed{\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \oint_C (F_1 dx + F_2 dy)}$$

Here we integrate along the entire boundry c of R in such a sense that R is on the left as we advance in the direction of integration.



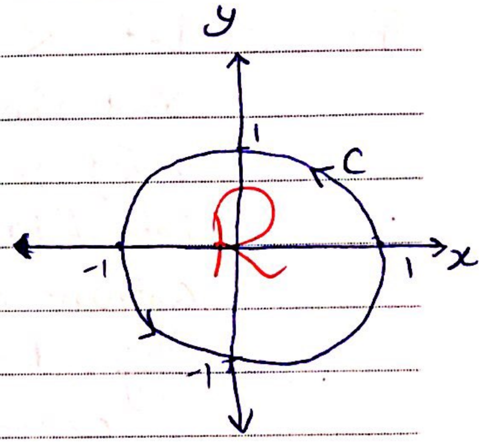


No.

EX1: Verify Green's Theorem for

$F_1 = y^2 - 7y \quad , \quad F_2 = 2xy + 2x$

C the circle $x^2+y^2=1$



Sol: $\boxed{1}$ $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$

$= \iint_R \left(2y+2 - (2y-7)\right) dx\,dy$

$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 9 \, dx\,dy$ (9 multiplied by the area of the circle)

$= \int_{-1}^{1} (18\sqrt{1-y^2})\, dy$ ...... solve it

$= 9\pi$ units^2

$\boxed{2}$ $\oint_C F_1 dx + F_2 dy = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$C: x^2+y^2=1 \quad \vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad , \quad 0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle \quad , \quad 0 \leq t \leq 2\pi$

$\vec{F} = \langle y^2 - 7y , 2xy + 2x \rangle = \langle \sin^2(t) - 2\sin(t), 2\cos(t)\sin(t) + 2\cos(t) \rangle$

$= \int_0^{2\pi} \langle \sin^2(t) - 7\sin(t) , 2\cos(t)\sin(t) + 2\cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$

$= \ldots = 9\pi$

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9/10/2016

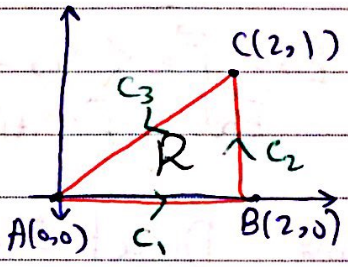
No.

Q] Using Greens Theorem, Evaluate $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ counterclockwise around the boundary curve C of the region R

$\vec{F} = \langle e^{-y}, e^{x} \rangle$

R is the tringle with vertices $A(0,0)$, $B(2,0)$, $C(2,1)$

Sol: Green's Theorem



$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

$$= \int_0^2 \int_0^{\frac{1}{2}x} \left(\frac{\partial e^{x}}{\partial x} - \frac{\partial e^{-y}}{\partial y} \right) dy\,dx$$

$$= \int_0^2 \int_0^{\frac{1}{2}x} (e^{x} + e^{-y})\,dy\,dx = \int_0^2 \left(ye^{x} - e^{-y} \right) \Big|_0^{\frac{1}{2}x} dx$$

$$= \int_0^2 \left[\left(\frac{1}{2}xe^{x} - e^{-\frac{1}{2}x} \right) - (-1) \right] dx = \int_0^2 \left(\frac{1}{2}xe^{x} - e^{-\frac{1}{2}x} + 1 \right) dx$$

$$= \frac{1}{2}\int_0^2 xe^{x}\,dx + 2e^{-\frac{1}{2}x}\Big|_0^2 + 2 \quad \ldots\ldots \text{Continue}$$

Direct Method :-

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

let starts with C_2 : $\int_{C_2} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\,dt$, $C_2 : x = 2$

C_2 : $\vec{r}(t) = \langle \overset{x}{2}, \overset{y}{t} \rangle \quad 0 \leq t \leq 1$

$\vec{r}\,'(t) = \langle 0, 1 \rangle$

$\vec{F} = \langle e^{-y}, e^{x} \rangle$

=> Followed

No.

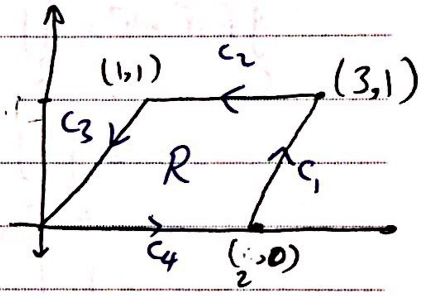
$$\vec{F}(\vec{r}(t)) = \langle e^{-t}, e^{2} \rangle$$

$$\text{The integral} = \int_0^1 \langle e^{-t}, e^{2} \rangle \cdot \langle 0, 1 \rangle \, dt$$

$$= \int_0^1 e^{2} \, dt = e^{2}$$

EX: $\vec{F} = \langle y, -x \rangle$

Find $\int_c \vec{F}(\vec{r}(t)) \cdot d\vec{r}$ for the give curve; $C_1 \cup C_2 \cup C_3$ just.



Sol: Direct Method:-

$$\int_c = \int_{c_1} + \int_{c_2} + \int_{c_3}$$

Green's Theorem:-

$$\int_c \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy - \int_{c_4} \vec{F}(\vec{r}) \cdot d\vec{r}$$

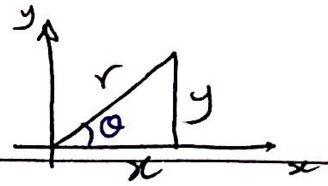
$$= \int_0^1 \int_y^{y+2} (-1-1) \, dx \, dy \; -$$

C_4: $y = 0$

$$C_4 : \vec{r}(t) = \langle \overset{x}{t}, \overset{y}{0} \rangle \quad 0 \leq t \leq 2$$ ... continue

→ followed

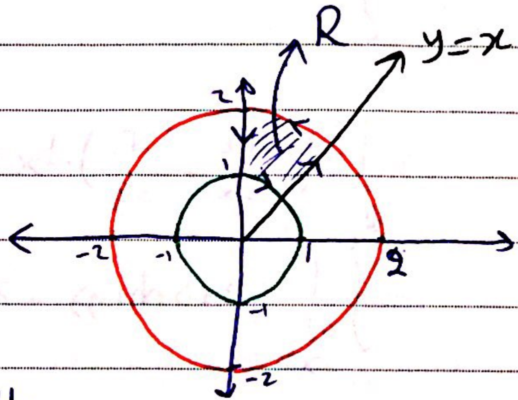
$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2+y^2}$



No.

12 / 444	$\vec{F} = \langle \underbrace{x^2y^2}_{F_1}, \underbrace{\frac{-x}{y^2}}_{F_2} \rangle$

$R: 1 \leq x^2+y^2 \leq 4 \quad , \quad x \geq 0 \; , \; y \geq x$



sol: $\oint \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$

$= \iint_R \left(\frac{-1}{y^2} - 2x^2y \right) dy\,dx$

$\Rightarrow$ Use polar coordinates to solve this Integral !

$= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=1}^{r=2} \left(\frac{-1}{(r\sin(\theta))^2} - 2(r\cos(\theta))^2(r\sin\theta) \right) \underline{r\,dr\,d\theta}$

don't forget it !

$= \int_{\pi/4}^{\pi/2} \int_1^2 \left(-\frac{1}{r}\csc^2(\theta) - 2r^4\cos^2(\theta)\sin(\theta) \right) dr\,d\theta$

$= \int_{\pi/4}^{\pi/2} \left(\left[-\csc^2(\theta)\ln(r) \right]_1^2 - \left[\frac{2r^5}{5}\cos^2(\theta)\sin(\theta) \right]_1^2 \right) d\theta$

$= \int_{\pi/4}^{\pi/2} \left(\left[-\csc^2(\theta)\ln(2) \right] - \left[\frac{2^6}{5}\cos^2(\theta)\sin(\theta) - \frac{2}{5}\cos^2(\theta)\sin(\theta) \right] \right) d\theta$

$= \int_{\pi/4}^{\pi/2} \left(\left[-\csc^2(\theta)\ln(2) + \frac{62}{5}\cos^2(\theta)\sin(\theta) \right] \right) d\theta$

$= \ln(2)\cot(\theta) \Big]_{\pi/4}^{\pi/2} + \frac{62}{5} \frac{\cos^3(\theta)}{3} \Big]_{\pi/4}^{\pi/2}$ ... continue

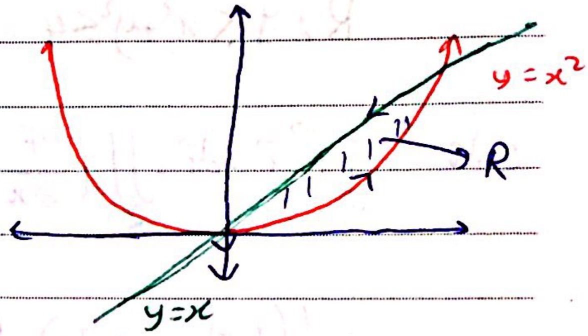
$\cot \pi/2 = 0$

$\Rightarrow$ followed

No.

$\frac{6}{444}$ | $\vec{F} = \langle x\cosh(y), x^2\sinh(y) \rangle$, $R: x^2 < y < x$

Sol: $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$



$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

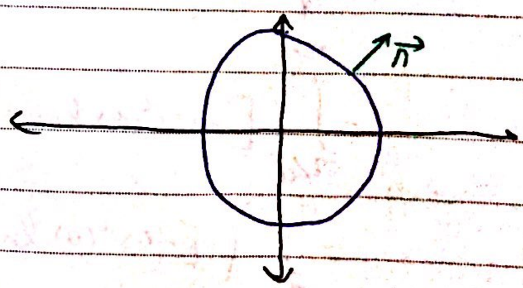
$$= \int_0^1 \int_{x^2}^{x} \left(2x\sinh(y) - x\sinh(y) \right) dy\,dx$$

$$= \int_0^1 \left(x\cosh(y) \Big|_{x^2}^{x} \right) dx = \int_0^1 \left(x\cosh(x) - x\cosh(x^2) \right) dx$$

$$= \underbrace{\int_0^1 x\cosh(x)}_{\text{by parts}} - \frac{1}{2}\sinh(x^2) \Big|_0^1 \quad \text{---- Continue}$$

Result From Green's Theorem :-

$$\iint_R \nabla^2 w \, dx\,dy = \oint_C \frac{\partial w}{\partial n} \, ds$$

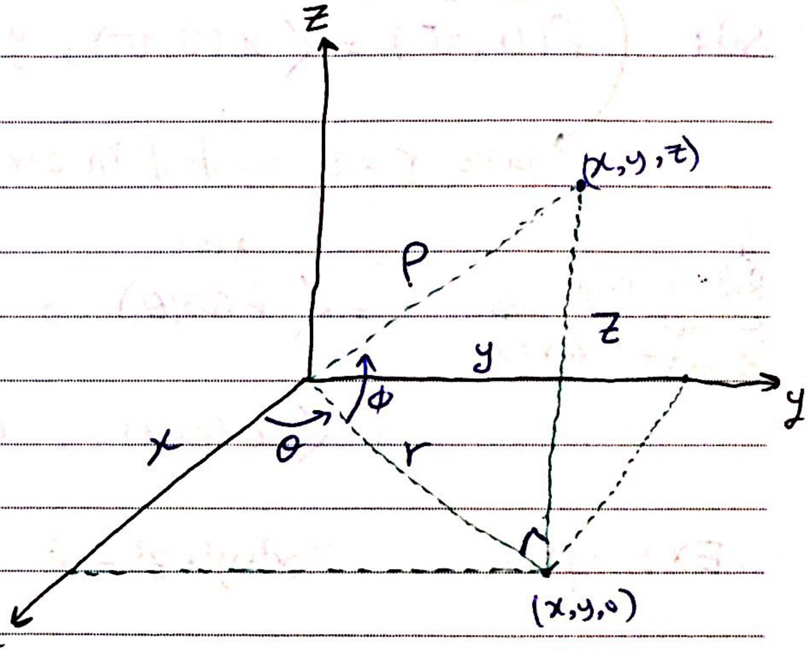


Tuesday 11/10/2016 | Dr. Ahmad Abdullah

No.

10.5 Surfaces of surface integral:-

Cylindrical coordinates:-



$(x,y,z) \longleftrightarrow (r,\theta,z)$

$x = r\cos\theta$ $\qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\qquad \tan\theta = \frac{y}{x}$

$z = z$ $\qquad z = z$

$r \geq 0$

$0 \leq \theta \leq 2\pi$

spherical coordinates:-

$(x,y,z) \longleftrightarrow (\rho,\theta,\phi)$

$x = \rho\cos\phi \; \cos\theta$ $\qquad \rho = \sqrt{x^2+y^2+z^2}$

$y = \rho\cos\phi\,\sin\theta$ $\qquad \tan\theta = \frac{y}{x}$

$z = \rho\cos\phi$ $\qquad \cos\phi = \frac{r}{\rho} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$

$r = \rho\cos\phi$

$0 \leq \theta \leq 2\pi$

$-\pi/2 \leq \phi \leq \pi/2$

$\rho \geq 0$

→ followed

No.

EX: cylinder $x^2+y^2=a^2$, $-1 \le z \le 1$

write it in parametric form

Sol: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ (θ, z)

since $r = a$ constant in our case!

Replace each $\theta \to u$, $z \to v$ & $r=a$ $\Rightarrow$

$$= \langle \underset{x(t)}{r\cos(\theta)}, \underset{y(t)}{r\sin(\theta)}, \underset{z(t)}{z} \rangle$$

$$= \langle a\cos(u), a\sin(u), v \rangle \quad \begin{matrix} 0 \le \overset{\theta}{u} \le 2\pi \\ -1 \le \overset{z}{v} \le 1 \end{matrix}$$

EX: sphere $x^2+y^2+z^2=a^2$

Sol: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$\Rightarrow$ Note that $\rho = a$ is constant

the order not neccussary

$$\vec{r}(\overset{\theta}{u}, \overset{\phi}{v}) = \langle \rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi \rangle$$

$\rho \to a$
$\phi \to v$
$\theta \to u$

$$= \langle a\cos v\cos u, a\cos v\sin u, \rho\sin v \rangle$$

$0 \le u \le 2\pi$
$-\pi/2 \le v \le \pi/2$

* u & v are dummy variables, you can use whatever you want.

→ followed

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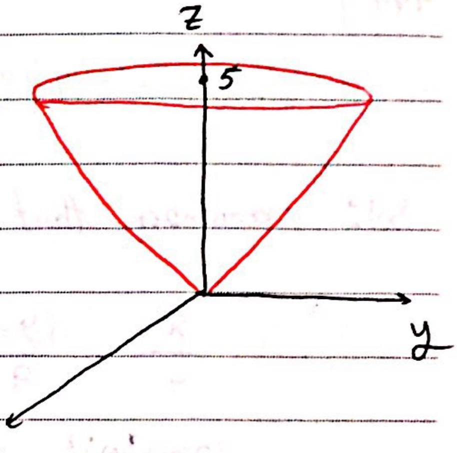
No.

Ex: Cone $\quad z=\sqrt{x^2+y^2} \quad\quad 0\leq z\leq 5$



Sol: $\quad \to \phi = \text{constant} = 45^\circ = \pi/4$

$$\vec{r}(\overset{\rho}{u},\overset{\theta}{v})=\langle x(u,v), y(u,v), z(u,v)\rangle$$

$$=\langle \rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi\rangle$$

$$=\langle u\cos(v)\frac{1}{\sqrt2}, u\sin(v)\frac{1}{\sqrt2}, u\frac{1}{\sqrt2}\rangle$$

To find the limits

$\Rightarrow \quad 0\leq z\leq 5 \quad , \sin\phi=\frac{z}{\rho}=\frac{1}{\sqrt2} \Rightarrow z=\frac{\rho}{\sqrt2}$

$\Rightarrow \quad 0\leq \frac{\rho}{\sqrt2}\leq 5 \Rightarrow 0\leq\rho\leq\sqrt2\,5 \Rightarrow u=\rho \quad 0\leq u\leq\sqrt2\,5$

$\Rightarrow \quad 0\leq\theta\leq 2\pi \Rightarrow 0\leq v\leq 2\pi$

Another method:-

$$\vec{r}(\overset{z}{u},\overset{\theta}{v})=\langle \underset{z\cos\theta}{x(u,v)}, \underset{z\sin\theta}{y(u,v)}, \underset{z}{z(u,v)}\rangle$$

$$=\langle u\cos v, u\sin v, u\rangle$$

Thursday
13/10/2016

Dr. Ahmad Abdullah

No.

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Hyperbolic cylinder

$$S:\ 9x^2 - 4(y+3)^2 = 36$$

Sol: remember that $\cosh^2(t) - \sinh^2(t) = 1$

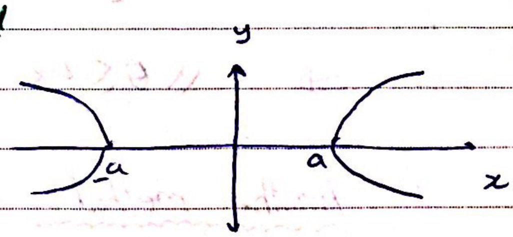
$$\frac{x^2}{4} - \frac{(y+3)^2}{9} = 1$$

$$\frac{(2\cosh(u))^2}{4} - \frac{(3\sinh(u) - 3 + 3)^2}{9} = 1$$

$$S: \vec{r}(u,v) = \langle 2\cosh(u),\ 3\sinh(u) - 3,\ v \rangle \qquad -\infty < v < \infty, \quad ? < u < ?$$

$\Rightarrow$ to find the limits of $\underline{u}$

$$\frac{x^2}{a^2} - \frac{y^2}{b} = 1 \Rightarrow$$



$a \leq x$ & $x \leq -a$

$2 \leq 2\cosh(u)$ & $2\cosh(u) \leq -2$

$\vdots$

then find $\underline{\underline{u}}$

$-\infty < y < \infty$

$-\infty < 3\sinh(u) - 3 < \infty$

→ followed

No.

10.6 Surface Integrals:-

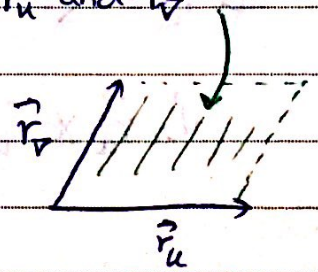
Given a surface S is parametric form $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
where (u,v) varies over a region R in the uv-plane.
Assume S to be piecewise smoth so that S has the Normal

$$\vec{N} = \vec{r}_u \times \vec{r}_v \qquad \text{\& Unite Normal vector: } \vec{n} = \frac{\vec{N}}{|\vec{N}|}$$

For a given vector function $\vec{F}$ we can define the surface integral over S by

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N}\, du\, dv$$

Here $\vec{N} = |\vec{N}|\,\vec{n}$ & $|\vec{N}| = |\vec{r}_u \times \vec{r}_v|$
& $|\vec{r}_u \times \vec{r}_v|$ represents the area of the parallelogram with sides $\vec{r}_u$ and $\vec{r}_v$



Hence

$$\vec{n}\, dA = \vec{n}|\vec{N}|\, du\, dv = \vec{N}\, du\, dv$$

and we see that

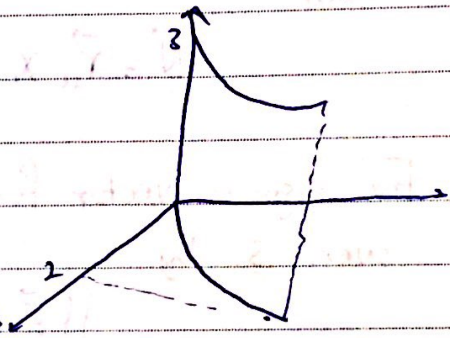
$dA = |\vec{N}|\, du\, dv$ is the element of area of S

No.

Example: compute the flux of water through the parapolic cylinder S: $y=x^2$, $0\le x\le 2$, $0\le z\le 3$ if the velocity is

$$\vec{v}=\vec{F}=\langle 3z^2, 6, 6xz\rangle$$

Sol: $\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r}(u,v))\cdot\vec{N}\,du\,dv$



S: $y=x^2$, $0\le x\le 2$, $0\le z\le 3$

S: $\vec{r}(u,v)=\langle x(u,v), y(u,v), z(u,v)\rangle$

we assumed $u=x$ $\Rightarrow$

$\vec{r}(u,v)=\langle u, u^2, v\rangle$ $\quad 0\le u\le 2$, $0\le v\le 3$

$\Rightarrow \vec{r}_u=\langle 1, 2u, 0\rangle$ & $\vec{r}_v=\langle 0,0,1\rangle$

$$\vec{N}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2u & 0\\ 0 & 0 & 1\end{vmatrix}=2u\hat{i}-\hat{j}=\langle 2u,-1,0\rangle$$

or $y-x^2=0 \Rightarrow \nabla f=\langle -2x, 1, 0\rangle$ Let $x=u$
$=\langle -2u, 1, 0\rangle=\vec{N}$

$\Rightarrow \vec{F}=\langle 3z^2, 6, 6xz\rangle$ $\quad\Rightarrow \vec{F}(\vec{r}(u,v))=\langle 3v^2, 6, 6uv\rangle$

$\Rightarrow \int_0^3\int_0^2 \langle 3v^2, 6, 6uv\rangle\cdot\langle 2u,-1,0\rangle\,du\,dv$

$\Rightarrow \int_0^3\int_0^2 [6uv^2-6]\,du\,dv$

Sunday | Dr. Ahmad Abdullah
16/10/2016

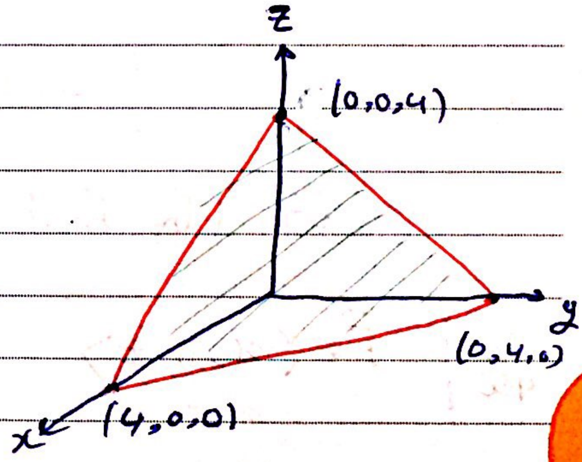
No.

$\frac{1-12}{456}$ | Evaluate $\int_S \vec{F} \cdot \vec{n}\, dA$

$\frac{2}{456}$ | $\vec{F} = \langle x^2, y^2, z^2 \rangle$, S: $x + y + z = 4$, $x \geq 0$
$y \geq 0$
$z \geq 0$

Sol: S: $\vec{r}(u,v) = \langle u, v, 4 - u - v \rangle \Rightarrow$ $x = u$
$y = v$
$z = 4 - x - y$
$= 4 - u - v$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N}\, du\, dv$$



$\vec{r}_u = \langle 1, 0, -1 \rangle$

$\vec{r}_v = \langle 0, 1, -1 \rangle$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \hat{i} + \hat{j} + \hat{k} = \langle 1, 1, 1 \rangle$$

or $x + y + z = 4 \longrightarrow \nabla F = \langle 1, 1, 1 \rangle$

$\vec{F} = \langle x^2, y^2, z^2 \rangle \longrightarrow \vec{F}(\vec{r}(u,v)) = \langle u^2, v^2, (4 - u - v)^2 \rangle$

$\longrightarrow$ followed

No.

$$\Rightarrow \int_0^4\int_0^{4-u} \langle u^2, v^2, (4-u-v)^2 \rangle \cdot \langle 1,1,1 \rangle \, dv\, du$$

$$= \int_0^4\int_0^{4-u} \left(u^2 + v^2 + (4-u)^2 + v^2 - 2v(4-u) \right) dv\, du$$

$$= \int_0^4 \int_0^{4-u} (2v^2 + 2u^2 + 2vu - 8v - 8u + 16)\, dv\, du$$

$$= \int_0^4 \Big(\frac{3}{2}(4-u)^3 + 2u^2 + u(4-u)^2 - 4(4-u)^2$$
$$- 8u(4-u) + 16(4-u) \Big)\, du$$

continue.

$= 64$ (The Final answer)

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$\vec{F} = \langle x, y, z \rangle \qquad S: \vec{r} = \langle u\cos(v), u\sin(v), u^2 \rangle$

$0 \leq u \leq 4, \quad -\pi \leq v \leq \pi$

Sol: $\vec{r}_u = \langle \cos(v), \sin(v), 2u \rangle$

$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle -2u^2\cos(v), -2u^2\sin(v), u \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle u\cos(v), u\sin(v), u^2 \rangle$

$\iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N}\, du\, dv =$

No.

$$= \int_0^4 \int_{-\pi}^{\pi} \langle 4\cos(v), 4\sin(v), u^2 \rangle \cdot \langle -2u^2\cos(v), -2u^2\sin(v), u \rangle \, dv\,du$$

$$= \int_0^4 \int_{-\pi}^{\pi} \left(-8u^2\cos^2(v) - u^2\sin^2(v) + u^3 \right) dv\,du$$

$$= \int_0^4 \int_{-\pi}^{\pi} \left(-8u^2 \underbrace{\left(\cos^2(v) + \sin^2(v)\right)}_{=1} + u^3 \right) dv\,du$$

$$= \int_0^4 \int_{-\pi}^{\pi} \left(-8u^2 + u^3 \right) dv\,du = \int_0^4 2\pi\left(u^3 - 8u^2\right) du$$

$$= 2\pi\left(\frac{u^4}{4} - \frac{8u^3}{3}\right)\Bigg|_0^4 = 2\pi\left[\left(\frac{4^4}{4} - \frac{8(4^3)}{3}\right) - (0)\right]$$ continue ...

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456

$\vec{F} = \langle \cosh(y), 0, \sinh(x) \rangle$ $\quad S: z = x + y^2$

$0 \leq y \leq x$

$0 \leq x \leq 1$

Sol: $S: \vec{r}(u,v) = \langle u, v, u + v^2 \rangle \Rightarrow$ $x = u, v = y$

$0 \leq u \leq 1$

$0 \leq v \leq u$

$$\vec{F}(\vec{r}(u,v)) = \langle \cosh(v), 0, \sinh(u) \rangle$$

$$\vec{r}_u = \langle 1, 0, 1 \rangle, \quad \vec{r}_v = \langle 0, 1, 2v \rangle$$

No.

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2v \end{vmatrix} = \hat{i}(-1) - \hat{j}(2v) + \hat{k}(1)$$

$$= \langle -1, -2v, 1 \rangle$$

$$\iint_S \vec{F}(\vec{r}(u,v)) \cdot \vec{N}\, du\, dv = \int_0^1 \int_0^u \langle \cosh(v), 0, \sinh(u) \rangle \cdot \langle -1, -2v, 1 \rangle\, du$$

$$= \int_0^1 \int_0^u \left(-\cosh(v) + \sinh(u) \right) dv\, du$$

$$= \int_0^1 \left(-\sinh(v) + v\sinh(u) \Big|_0^u \right) = \int_0^1 \left(-\sinh(u) + u\sinh(u) \right) du$$

$$= -\cosh(u) \Big|_0^1 + u\cosh(u) \Big|_0^1 - \int_0^1 \cosh(u)\, du$$

$$= \left(\cosh(0) - \cosh(1) \right) + \cosh(1) - \sinh(1)$$

$$= \cosh(0) - \sinh(1) = 1 - \sinh(1)$$

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18/10/2016 | Dr. Ahmad Abdullah
Tuesday

No.

10.7 Triple Integrals :

Divergence theorem of Guss :-

Theorem : Let T be closed bounded region in space (solid) whose boundary is piecewise smoth oriantable surface. Let $\vec{F}(x,y,z)$ be a vector function that is a continous and has continous first partial derivatives in some containing T then :

$$\iiint_T \operatorname{div}\vec{F}\, dv = \iint_S \vec{F}\cdot\vec{n}\, dA$$

If $\vec{F} = \langle F_1, F_2, F_3 \rangle$ and the outer normal vector in $\vec{n} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$ of S, The formula becomes :-

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iint_S \left(F_1\cos(\alpha) + F_2\cos(\beta) + F_3\cos(\gamma) \right) dA$$

$$= \iint_S \left(F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy \right)$$

No.

EX: Verification of the Divergence Theorem

Evaluate

$$\iint_S (7x\hat{i} - z\hat{k}) \cdot \vec{n}\, dA$$ over the surface of the sphere $S: x^2+y^2+z^2=4$

Sol: our volume is $T: x^2+y^2+z^2 \leq 4$

$\boxed{1}$ $\vec{F} = \langle \overset{F_1}{7x}, \overset{F_2}{0}, \overset{F_3}{-z} \rangle \Rightarrow \nabla \cdot \vec{F} = \text{div}\vec{F} = \underline{\underline{6}}$

using divergence theorem

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \text{div}\vec{F}\, dv$$

$= \iiint_T 6\, dv$ ⇒ since the integrand is constant (6) we can just find the volume of the sphere then multiply it by (6)

* remember: the volume of the sphere $\frac{4}{3}\pi(r)^2$

$$\iiint_T 6\, dv = (6)\left(\frac{4}{3}\pi(2)^2\right) = \frac{192}{3}\pi = \underline{\underline{64\pi}}$$

⇒ or you can find the volume of the sphere either using cartesian coordinates or using spherical coordinates (which is easiar)

→ followed

No.

$\boxed{2}$ without using divergence theorem (Direct method)

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r}(u,v))\cdot\vec{N}\,du\,dv$$

$$S: x^2+y^2+z^2=4 \Rightarrow S: \vec{r}(u,v)=\langle \overbrace{2\cos(u)\cos(v)}^{x}, \underbrace{2\cos(v)\sin(u)}_{y}, \underbrace{2\sin(v)}_{z}\rangle$$

$0 \leq u \leq 2\pi$

$-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$\vec{r}_u = \langle -2\sin(u)\cos(v), 2\cos(v)\sin(u), 0\rangle$$

$$\vec{r}_v = \langle -2\cos(u)\sin(v), -2\sin(v)\sin(u), 2\cos(v)\rangle$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \dots = \langle \overbrace{4\cos^2(v)\cos(u)}^{x}, \overbrace{4\cos^2(v)\sin(u)}^{y}, \underbrace{4\cos(v)\sin(u)}_{z}\rangle$$

$$\vec{F} = \langle 7x, 0, -z\rangle \Rightarrow \vec{F}(\vec{r}(u,v)) = \langle 14\cos(u)\cos(v), 0, -2\sin(v)\rangle$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \langle 14\cos(u)\cos(v), 0, -2\sin(v)\rangle \cdot \underbrace{\langle \quad \rangle}_{\vec{N}}\,du\,dv$$

⋮

continue.

→ followed

No.

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Evaluate surface integral $\iint_S \vec{F}\cdot\hat{n}\,dA$ by divergence theorem

$\vec{F} = \langle x^2, 0, z^2 \rangle$, S: the surface of the box

$|x| \leq 1$, $|y| \leq 3$, $0 \leq z \leq 2$

$-1 \leq x \leq 1$ $\quad -3 \leq y \leq 3$

Sol: $\iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv$

$\nabla\cdot\vec{F} = \operatorname{div}\vec{F} = 2x + 2z$

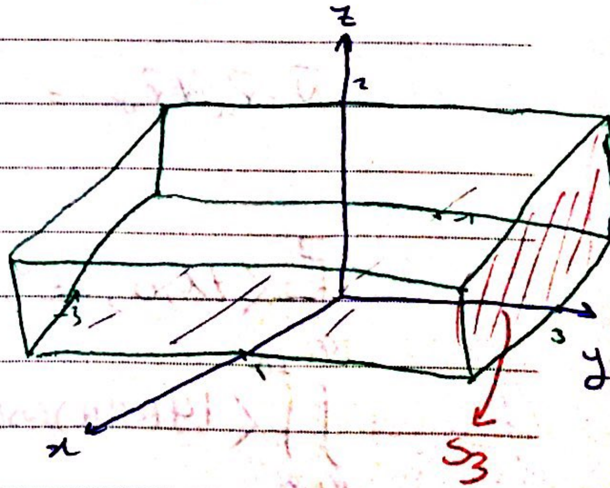
$$= \int_{-1}^{1}\int_{-3}^{3}\int_{0}^{2} (2x + 2z)\,dz\,dy\,dx = \int_{-1}^{1}\int_{-3}^{3}\left((2zx + z^2)\Big|_0^2\right)dy\,dx$$

...... then continue the integral.

if we want to solve this question without using divergence theorem

$S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

(ستة وجوه للمكعب)



for S_3: $y = 3$, $-1 \leq x \leq 1$, $0 \leq z \leq 2$

$\Rightarrow \vec{r}(u,v) = \langle u, 3, v \rangle$ (x, y, z)

$-1 \leq u \leq 1$, $0 \leq v \leq 2$

$\vec{r}_u = \langle 1, 0, 0 \rangle$, $\vec{r}_v = \langle 0, 0, 1 \rangle$, $\vec{N} = \langle 0, 1, 0 \rangle$ continue

$\Rightarrow$ then we have to do similar thing for the all six surfaces.

Thursday | Dr. Ahmad Abdullah
20/10/2016

$x^2+y^2=25$, $0\le z\le 2$ → بدون الغطاء العلوي أو السفلي (فقط الجانبي)

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$$\vec{F}=\langle \overset{F_1}{3xy^2},\ \overset{F_2}{yx^2-y^3},\ \overset{F_3}{3zx^2}\rangle$$

S is the surface of $x^2+y^2\le 25$, $0\le z\le 2$

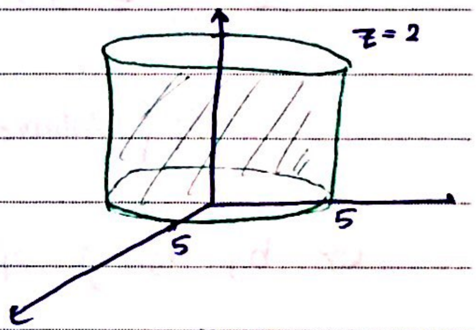
Sol: Using Divergence Theorem:-

$$\iint_S \vec{F}\cdot\vec{n}\,dA=\iiint_T \operatorname{div}\vec{F}\,dv$$

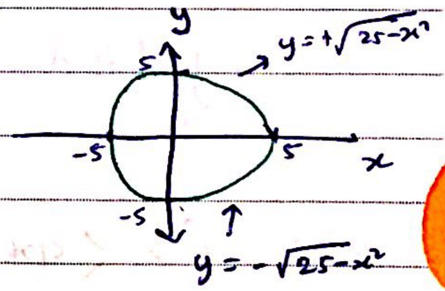
$$\operatorname{div}\vec{F}=3y^2+x^2-3y^2+3x^2=\underline{\underline{4x^2}}$$

$$\iiint_T 4x^2\,dx\,dy\,dz$$

$$=\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{0}^{2} 4x^2\,dz\,dy\,dx$$



But we will use cylindrical cordinates to find the integral



$$=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=5}\int_{z=0}^{z=2} 4(r\cos(\theta))^2\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{5}(2)4r^3\cos^2(\theta)\,dz\,dr\,d\theta=\int_0^{2\pi}\left(2r^4\cos(\theta)\Big|_0^5\right)d\theta$$

$$=\int_0^{2\pi}\left(2(5)^4\cos^2(\theta)\right)d\theta$$

$$=1250\int_0^{2\pi}\frac{1}{2}\left(1+\cos(2\theta)\right)d\theta=1250\pi$$

→ followed

if the surface (15): $x^2+y^2=z^2$, $0\leq z\leq 2$ ← Just the sides without the (Top) of the cone.

$x^2+y^2\leq 4$ & $z=2$ ← NO الغطاء العلوي عبارة

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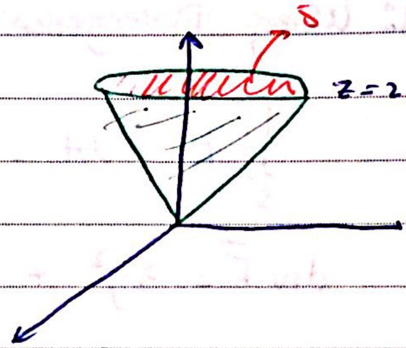
18 / 463	$\vec{F}=\langle \overset{F_1}{4x}, \overset{F_2}{3z}, \overset{F_3}{5y}\rangle$, S: is the surface $\boxed{\text{of}}$ the cone $x^2+y^2 \leq z^2$, $0\leq z\leq 2$

sol: $\iint_S \vec{F}\cdot\hat{n}\,dA = \iiint_V \operatorname{div}\vec{F}\,dv$



$\operatorname{div}\vec{F} = 4+0+0 = 4$

$= \iiint_T 4\,dv$

الارتفاع أقصى نصف قطر

$= 4(\text{Volume of cone}) = 4\left(\frac{1}{3}\pi(2)^2(2)\right)$

or by using spherical coordinates

$= \iiint_T 4\,dv = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^{2/\sin\phi} 4\rho\cos^2\phi\,d\rho\,d\theta\,d\phi$

EX: $\vec{F}=\langle 4x, 3z, 5y\rangle$ S^*: is the surface $x^2+y^2=z^2$
$0\leq z\leq 2$

sol: $\iint_{S^*}\vec{F}\cdot\hat{n}\,dA = \iint_S \vec{F}\cdot\hat{n}\,dA - \iint_{S^{**}}\vec{F}\cdot n\,dA$

from previus exampl $= \left(4\frac{1}{3}\pi(2^2)(2)\right) - \dots$

S^{**}: $x^2+y^2\leq 4$, $z=2$ $\quad S^{**}$: $\vec{r}(u,v)=\langle \overset{x}{u\cos(v)}, \overset{y}{u\sin(v)}, \overset{z}{2}\rangle$

$0\leq u\leq 2$

$0\leq v\leq 2\pi$

→ followed

No.

$\vec{r}_u = \langle \cos(v), \sin(v), 0 \rangle$

$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \text{——} = \langle 0, 0, u \rangle$

العامودي يجب أن يكون خارج من السطح!

$\vec{F}(\vec{r}(u,v)) = \langle 4u\cos(v), 3(2), 5u\sin(v) \rangle$

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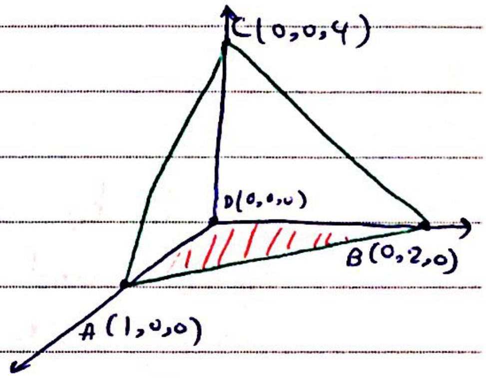
24/463 $\vec{F} = \langle 4x^2, y^2, -2\cos(\pi z) \rangle$

S: the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,4)$

Find $\iint_S \vec{F}\cdot\vec{n}\, dA$ using the Divergence theorem

Sol: $\iint_S \vec{F}\cdot\vec{n}\, dA = \iiint_T \operatorname{div}\vec{F}\, dv$

$$= \int_0^1 \int_0^{y=2-2x} \int_{z=0}^{z=4-4x-2y} (8x + 2y + 2\pi \sin(\pi z))\, dz\, dy\, dx$$



$\Rightarrow$ we have to find the equation of the plane, first we find a vector perpendicular to plane & a point at the plane

$\vec{AB} = \langle -1, 2, 0 \rangle$ & $\vec{AC} = \langle -1, 0, 4 \rangle$

$$\vec{AB} \times \vec{AC} = \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = 8\hat{i} + 4\hat{j} + 2\hat{k}$$

chose any point (A or B or C)

plane: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

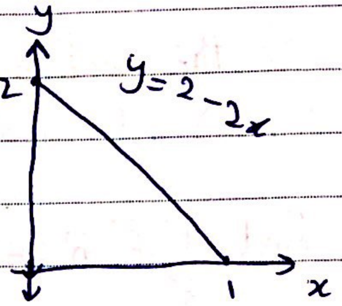
$$8(x-1) + 4(y-0) + 2(z-0) = 0$$

$$z = 4 - 4x - 2y$$

No.

To find the limits of integration for y & x

$\Rightarrow$ Then find the integral ---



10.9 Stokes's Theorem :-

Theorem : Let S be a piecewise smoth oriented surface in space. and Let the boundary of S be a piecewise smoth simple curve C.
Let $\vec{F}(x, y, z)$ be a continuous vector function that has continous first partial derivatives in a domain in space containing S, Then :-

$$\iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, dA = \oint_C \vec{F} \cdot \vec{r}'(s)\, ds$$

Here $\vec{n}$ is a unite normal vector of S and, depending on $\vec{n}$, the integration around C is taken in the sense shown in figure 251.
$\vec{r}' = \frac{d\vec{r}}{ds}$ is the unite tangent vector and s is the arc length of C.

$\rightarrow$ followed

No.

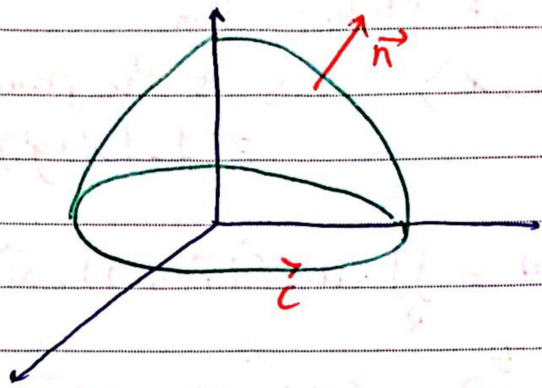


Figure 25).

Summary of the theorems :-

Green's Theorem :- $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dy\,dx = \oint_C \vec{F} \cdot \vec{r}\,dt$ (two dimintions)

Divergence Theorem :- $\iint_S \vec{F} \cdot \vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv$

Stokes's Theorem :- $\iint_S \operatorname{curl}\vec{F} \cdot \vec{n}\,dA = \oint_C \vec{F} \cdot \vec{r}'(s)\,ds$

* Green's theorem is a special case of stokes's Theorem but in two dimintions *

→ Followed

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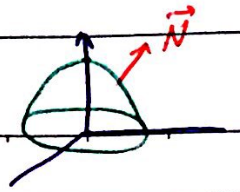

Ex: Verification of stoke's Thm

$\vec{F} = \langle y, z, x \rangle$, S is the paraboloid $z = 1 - (x^2+y^2)$, $z \geq 0$

$\boxed{1}$ $\iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, dA = \iint_R \text{curl}\,\vec{F}(\vec{r}(u,v)) \cdot \vec{N}\, du\, dv$

$S: \vec{r}(\overset{r}{u}, \overset{\theta}{v}) = \langle \overset{x}{u\cos(v)}, \overset{y}{u\sin(v)}, \overset{z}{1-u^2} \rangle$ $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$

$\vec{r}_u = \langle \cos(v), \sin(v), -2u \rangle$

$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \text{----} = \langle 2u^2\cos(v), 2u^2\sin(v), u \rangle$

$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -1\hat{i} - 1\hat{j} - 1\hat{k} = \langle -1, -1, -1 \rangle$

$= \int_0^{2\pi}\int_0^1 \langle -1,-1,-1 \rangle \cdot \langle 2u^2\cos(v), 2u^2\sin(v), u \rangle\, du\, dv$

$= \int_0^{2\pi}\int_0^1 \left(-2u^2\cos(v) - 2u^2\sin(v) - u \right) du\, dv$ $\rightarrow$ continue...

$= -\pi$

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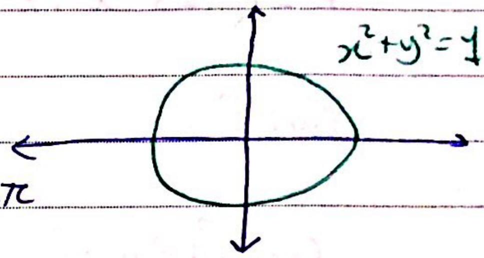
$\boxed{2}$ $\oint_C \vec{F} \cdot \vec{r}\,'(s)\,ds$



$C:\ \vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$

$0 \leqslant t \leqslant 2\pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0 \rangle$

$\vec{F} = \langle y, z, x \rangle \longrightarrow \vec{F}(\vec{r}(t)) = \langle \sin(t), 0, \cos(t) \rangle$

$\oint_C \vec{F} \cdot \vec{r}\,'(\underbrace{t}_{\text{dummy variable}})\,dt = \int_0^{2\pi} \langle \sin(t), 0, \cos(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle\,dt$

$= \int_0^{2\pi} (-\sin^2(t))\,dt = \frac{-1}{2} \int_0^{2\pi} (1 - \cos(2t))\,dt$

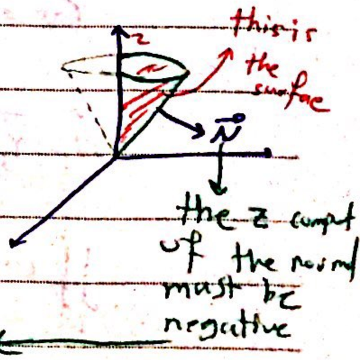
$= \frac{-1}{2} \left(t - \frac{\sin(2t)}{2}\right) \Big|_0^{2\pi} = -\pi$

$\boxed{\frac{1-8}{473}}$ Evaluate $\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\,dA$ directly

$\boxed{\frac{6}{673}}$ $\vec{F} = \langle z^2, x^2, y^2 \rangle$ $\quad S: z^2 = x^2 + y^2,\ y \geqslant 0,\ 0 \leqslant z \leqslant 2$

Sol: $S: \vec{r}(\overset{r}{u}, \overset{\theta}{v}) = \langle u\cos(v), u\sin(v), u \rangle$ $\quad 0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant \pi$

$\vec{r}_u = \langle \cos(v), \sin(v), 1 \rangle$



$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle -u\cos(v), -u\sin(v), u \rangle$

$\vec{N}^* = -\vec{N} = \langle u\cos(v), u\sin(v), -u \rangle$ $\quad 0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant \pi$

No.

$\vec{F} = \langle z^2, x^2, z^2 \rangle$

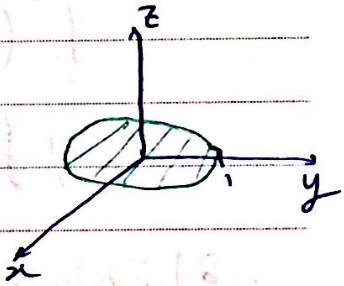
$\text{curl}\,\vec{F} = \ldots\ldots = \langle 2y, 2z, 2x \rangle$

$\text{curl}\,\vec{F}(\vec{r}(u,v)) = \langle 2u\sin(v), 2u, 2u\cos(v) \rangle$

$$\iint_S \text{curl}\,\vec{F}\cdot\vec{N}\,dA = \int_0^{\pi}\int_0^{2} \langle 2u\sin(v), 2u, 2\cos(v)\rangle \cdot \langle u\cos(v), u\sin(v), -u\rangle$$

$$= \int_0^{\pi}\int_0^{2} \left(2u^2\sin(v)\cos(v) + 2u^2\sin(v) - 2u\cos(v)\right) du\,dv$$ continue $\rightarrow$

$\boxed{\frac{8}{473}}$ $\vec{F} = \langle y^3, -x^3, 0 \rangle$ $\quad$ $S: x^2+y^2 \leq 9,\ z = 4$



sol: $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA =$

$\text{curl}\,\vec{F} = \ldots\ldots = \langle 0, 0, -3x^2-3y^2 \rangle$

$\vec{S}: \vec{r}(\overset{r}{u}, \overset{\theta}{v}) = \langle \overset{x}{u\cos(v)}, \overset{y}{u\sin(v)}, \overset{z}{4} \rangle$ $\qquad 0 \leq u \leq 3$

$\qquad\qquad 0 \leq v \leq 2\pi$

$\vec{r}_u = \langle \cos(v), \sin(v), 0 \rangle$

$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \ldots\ldots = \langle 0, 0, u \rangle$

$$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA = \int_0^{2\pi}\int_0^{3} \text{curl}\,\vec{F}(\vec{r}(u,v))\cdot\vec{N}\,du\,dv$$

$\text{curl}\,\vec{F}(\vec{r}(u,v)) = \langle 0, 0, \underset{-3(x^2+y^2)}{\underbrace{-3(u)^2}} \rangle$

followed

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$$=\int_0^{2\pi}\int_0^{3} \langle 0,0,-3u^2\rangle \cdot \langle 0,0,u\rangle \, du\, dv$$

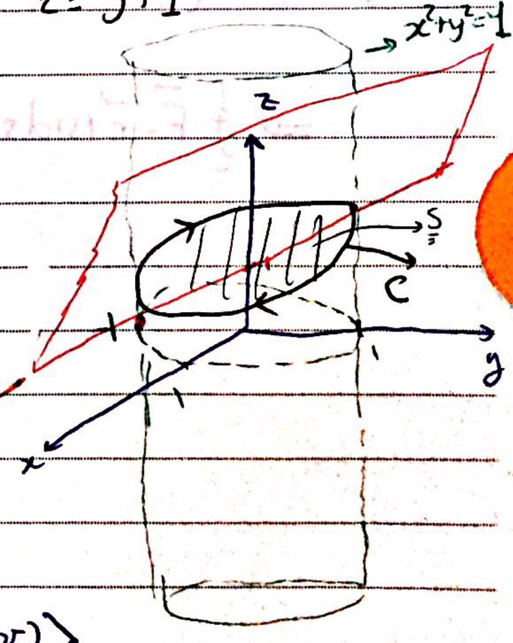
$$=\int_0^{2\pi}\int_0^{3} (-3u^3)\, du\, dv = (2\pi)\left(\frac{-3u^4}{4}\Big|_0^3\right)$$

$$=(2\pi)\left(\frac{-243}{4}\right)$$

11–18 / 473 | Calculate $\oint_C \vec{F}\cdot\vec{r}'(s)\,ds$ using Stokes's theorem

C: clockwise with respect as seen person standing at the origin.

12 / 473 | $\vec{F}=\langle \overset{F_1}{4z}, \overset{F_2}{-2x}, \overset{F_3}{2x}\rangle$ C: is the intersection of $x^2+y^2=1$ and $z=y+1$



Sol: $\text{Curl}\,\vec{F}=\langle 0, 2, -2\rangle$

$S: \vec{r}(u,v)=\langle \overset{x}{r\cos(\theta)}, \overset{y}{r\sin(\theta)}, \overset{z=y+1}{r\sin(\theta)+1}\rangle$ (r θ)

$=\langle u\cos(v), u\sin(v), r\sin(v)+1\rangle$

$0\le u\le 1$ & $0\le v\le 2\pi$

$\vec{r}_u=\langle \cos(v), \sin(v), \sin(v)\rangle$

$\vec{r}_v=\langle -u\sin(v), u\cos(v), u\cos(v)\rangle$

$\vec{N}=\vec{r}_u\times\vec{r}_v=\langle 0, -u, u\rangle$ → pointing to +ve side

$-\vec{N}=\langle 0, u, -u\rangle$

→ followed

No.

$\text{curl}\,\vec{F}(\vec{r}(u,v)) = \langle 0, 2, -2 \rangle$

$$\oint_C \vec{F}\cdot\vec{r}\,'(s)\,ds = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA$$

$$= \int_0^{2\pi}\int_0^1 \langle 0, 2, -2\rangle\cdot\langle 0, +u, -u\rangle\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 (2u + 2u)\,du\,dv = \int_0^{2\pi}\int_0^1 (4u)\,du\,dv$$

$$= (2\pi)\ \frac{4u^2}{2}\Big|_0^1 = (2\pi)(2) = 4\pi$$

16/473 $\vec{F} = \langle x^2, y^2, z^2\rangle$ C: the intersection of $x^2+y^2+z^2 = 4$ and $z = y^2$

Sol: $\text{curl}\,\vec{F} = \ldots\ldots = \langle 0, 0, 0\rangle$

$$\Rightarrow \oint_C \vec{F}\cdot\vec{r}\,'(s)\,ds = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA = \text{Zero}$$

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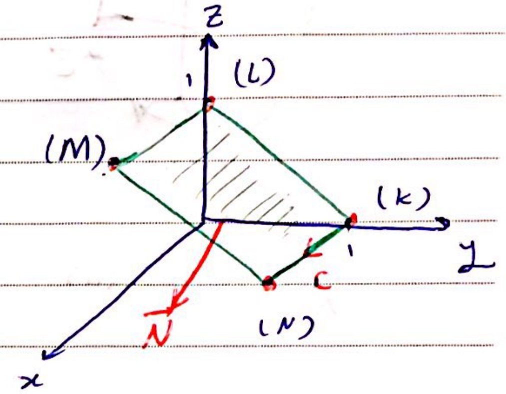
$\vec{F} = \langle \cos(\pi y), \sin(\pi x), 0 \rangle$

C: around the rectangle with vertices

K(0,1,0), L(0,0,1), M(1,0,1), N(1,1,0)

find $\oint_C \vec{F} \cdot \vec{r}'(s)\, ds$?



sol: $\nabla \times \vec{F} = \langle 0, 0, \pi\cos(\pi x) + \pi\sin(\pi y) \rangle$

$\vec{NM} = \langle 0, -1, 1 \rangle$

$\vec{NK} = \langle -1, 0, 0 \rangle$

$\vec{N} = \vec{NM} \times \vec{NK} = \langle 0, -1, 1 \rangle$

$\Rightarrow$ the equation of the plane with $\vec{N} = \langle \overset{a}{0}, \overset{b}{-1}, \overset{c}{-1} \rangle$ and the point $(\underset{x_0}{0}, \underset{y_0}{0}, \underset{z_0}{1})$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$\Rightarrow \quad y + z = 1$

$\vec{r}(u,v) = \langle u, v, 1 - v \rangle \quad 0 \leq u \leq 1, \; 0 \leq v \leq 1$

$\vec{r_u} = \langle 1, 0, 0 \rangle$, $\vec{r_v} = \langle 0, 1, -1 \rangle$, $\vec{N} = \langle 0, 1, 1 \rangle$

but, from figure the normal must be in the opposite direction, so $\vec{N}^* = -\vec{N} = \langle 0, -1, -1 \rangle$

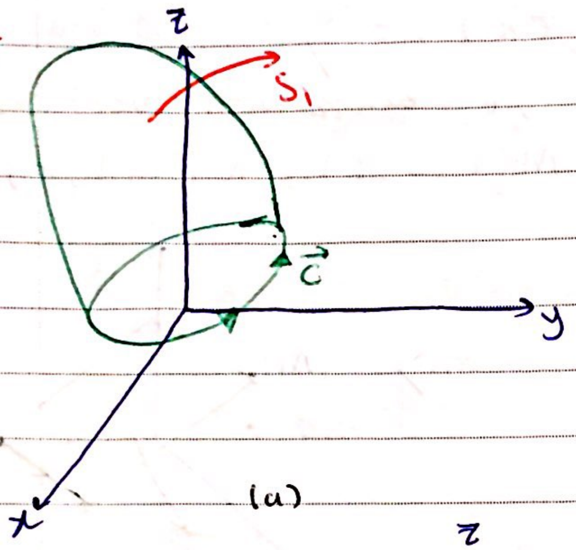
$$\Rightarrow \int_0^1 \int_0^1 \langle 0, 0, \pi\cos(\pi u) + \pi\sin(\pi v) \rangle \cdot \langle 0, -1, -1 \rangle \, du\, dv$$

$\Rightarrow$ continue

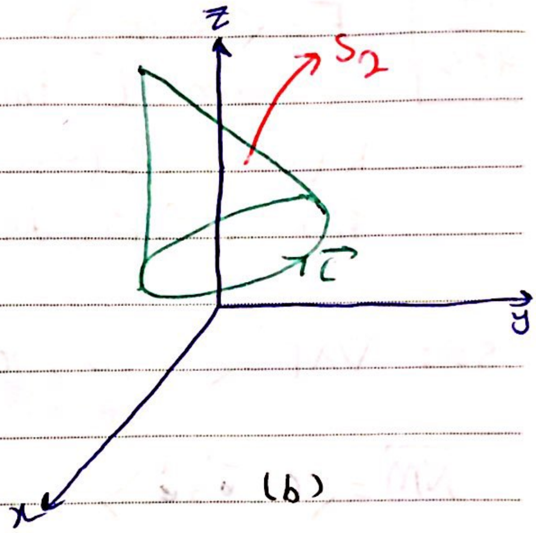
Followed $\Rightarrow$

No.

Note:-

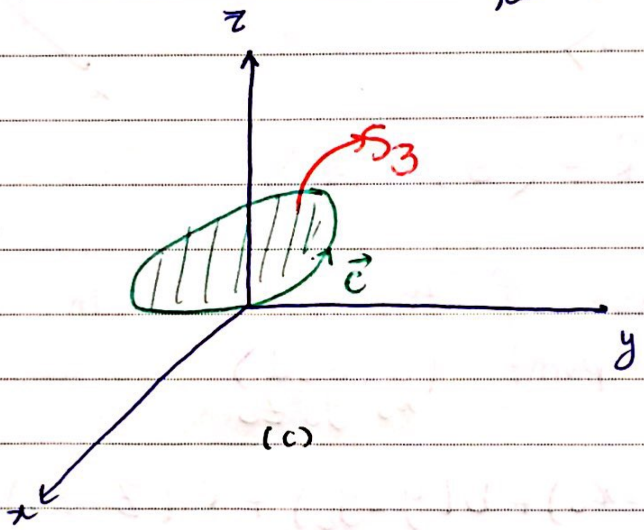


(a)



(b)

* All the $\vec{C}$'s for all graphs (a) & (b) & (c) are the same *



(c)

$$\Rightarrow \oint \vec{F} \cdot \vec{r}\,'(s)\, ds = \iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n}\, dA = \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}\, dA = \iint_{S_3} (\nabla \times \vec{F}) \cdot \vec{n}\, dA$$

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No.

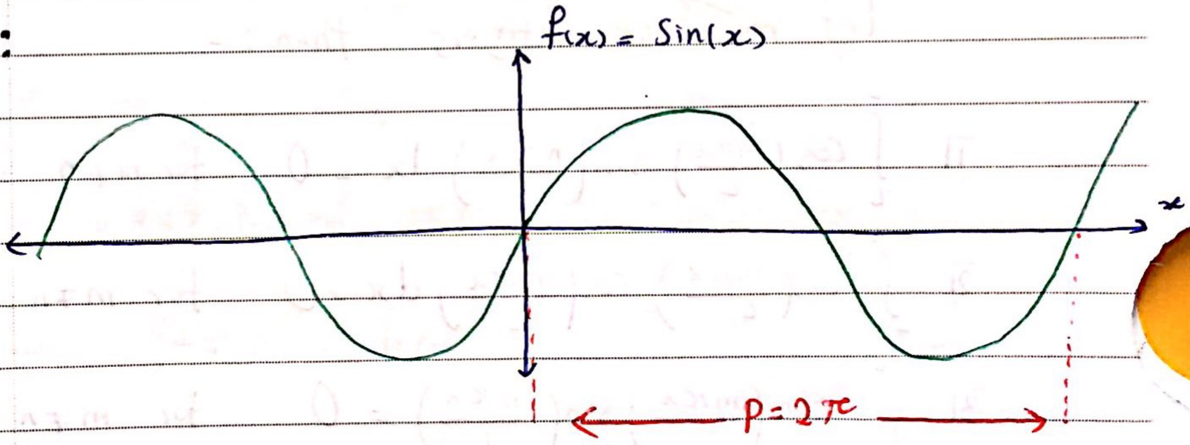
CHAPTER (11): Fourier Analysis:-

11.1 Fourier series
11.2 Arbitrary period Even & Odd Function :-

A function f is called a periodic function if $f(x)$ is defined for all real x except possibly at some points and if there is some positive number p, called a period of f such that :

$$f(x+p) = f(x) \quad , \text{for all } x.$$

Example:



$$f(x+p) = f(x+2\pi) = f(x) \quad , \quad x \in R$$

$$\sin(x+p) = \sin(x+2\pi) = \sin(x) \quad , \quad x \in R$$

In General :

$$f(x) = C\sin(Ax+B)+D \Rightarrow p = \frac{2\pi}{A}$$

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* suppose that f is a periodic function with period 2L and

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \quad \text{--- (1)}$$

Then

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \quad \text{--- (2)}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \dots \quad \text{--- (3)}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \dots \quad \text{--- (4)}$$

Theorem :- Orthogonality

$m, n = 1, 2, \dots$

Let m, n be integers, then :-

1] $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0$ for $m \neq n$ and $m = n$

2] $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0$ for $m \neq n$

3] $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = 0$ for $m \neq n$

Prove : (The second one)

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^{L} \left[\cos\left(\frac{m\pi x}{L} + \frac{n\pi x}{L}\right) + \cos\left(\frac{m\pi x}{L} - \frac{n\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2} \int_{-L}^{L} \cos\left(\frac{(m+n)x\pi}{L}\right) dx + \frac{1}{2} \int_{-L}^{L} \cos\left(\frac{(m-n)\pi x}{L}\right) dx$$

No.

$$= \frac{1}{2}\left.\frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{\frac{(m+n)\pi}{L}}\right]_{-L}^{L} + \frac{1}{2}\left.\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{\frac{(m-n)\pi}{L}}\right]_{-L}^{L}$$

$= 0$ (continue the previous calculations, you will get Zero!)

※ prove for equation (2) :-

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} a_0\,dx + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx}_{\text{Zero}} + b_n \underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)dx}_{\text{Zero}}\right]$$

$$\int_{-L}^{L} f(x)\,dx = a_0(2L) \Rightarrow a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

※ prove for equation (3) :-

Multiply (1) by $\cos\left(\frac{m\pi x}{L}\right)$ then integrate from $-L \to L$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) = a_0 \underbrace{\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)}_{\text{Zero}} + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_n \underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)}_{\text{Zero}}\right]$$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\right] \to \begin{matrix}\text{for } m\neq n\ \ = 0 \\ \text{for } m=n\end{matrix}$$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) = a_m \underbrace{\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}$$

for n's $\neq$ m $\Rightarrow$ the integral $= 0$

Just we deal with case $m=n$! →follow

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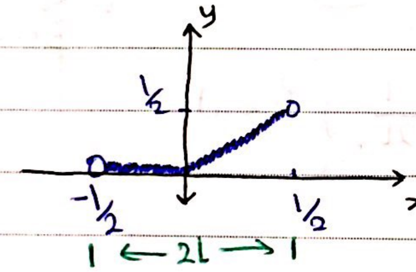
Theorem: Let f be a periodic function with period $2L$, and piecewise continous in the interval $-L \leq x \leq L$, furthermore, Let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval, Then the Fourier series:-

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] = \frac{f(x^-)+f(x^+)}{2}$$

13 (a) Find the Fourier series of:-



$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x \leq 0 \\ x, & 0 \leq x \leq \frac{1}{2} \end{cases}$$

$2L = 1 \Rightarrow L = \frac{1}{2}$

Sol: The Fourier serieas of f:

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2(\frac{1}{2})}\int_{-1/2}^{1/2} f(x)\,dx = \int_{-1/2}^{0} 0\,dx + \int_{0}^{1/2} x\,dx$$

$$= \frac{x^2}{2}\Big|_0^{1/2} = \frac{1}{8}$$

No.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad , \quad n=1,2,\ldots$$

$$a_n = 2\int_{-1/2}^{1/2} f(x)\cos(2n\pi x)\,dx = 2\left[\overbrace{\int_{-1/2}^{0} 0\cos(2n\pi x)\,dx}^{\text{Zero}} + \int_{0}^{1/2} x\cos(n\pi x)\,dx\right]$$

$$= 2\int_0^{1/2} x\cos(n\pi x)\,dx \Rightarrow \text{By parts}$$

$u = x$, $du = dx$

$dv = \cos(n\pi x)dx$

$v = \frac{\sin(2n\pi x)}{2n\pi}$

$$a_n = 2\left[\frac{x\sin(2n\pi x)}{2n\pi}\Big|_0^{1/2} - \int_0^{1/2}\frac{\sin(2n\pi x)}{2n\pi}dx\right]$$

$$= 2\left[\underbrace{\frac{\frac{1}{2}\sin(n\pi)}{2n\pi}}_{\text{Zero}} - 0 + \frac{\cos(2n\pi x)}{(2n\pi)^2}\Big|_0^{1/2}\right]$$

$$a_n = 2\left(\frac{\cos(n\pi)}{(2n\pi)^2} - \frac{1}{(2n\pi)^2}\right) = 2\left(\frac{\cos(n\pi) - 1}{(2n\pi)^2}\right)$$

$\Rightarrow$ note $\cos(n\pi) = (-1)^n$, $n = 1,2,3\ldots$

$$a_n = 2\left(\frac{(-1)^n - 1}{(2n\pi)^2}\right) \quad , \quad n = 1,2\ldots$$

$$a_n = \begin{cases} 2\left(\frac{-1-1}{(2n\pi)^2}\right) & , \ n = \text{odd} = 1,3,5\ldots \\ 2\left(\frac{1-1}{(2n\pi)^2}\right) & , \ n = \text{even} = 2,4,6\ldots \end{cases}$$

$$a_n = \begin{cases} \frac{-1}{(n\pi)^2} & , \ n = 1,3,5\ldots \\ 0 & , \ n = 2,4,6\ldots \end{cases}$$

$$a_{2n-1} = \frac{-1}{((2n-1)\pi)^2} \ , \ n = 1,2,3\ldots \quad \& \quad a_{2n} = 0 \ , \ n = 1,2\ldots$$

or $a_{2n+1} = \frac{-1}{((2n+1)\pi)^2}$, $n = 0,1,2,\ldots$

Followed $\Rightarrow$

No.

$$\Rightarrow b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \Rightarrow L = \frac{1}{2}$$

$$b_n = 2\left[\underbrace{\int_{-1/2}^{0} 0\sin(2n\pi x)\,dx}_{\text{Zero}} + \int_{0}^{1/2} x\sin(2n\pi x)\,dx\right]$$

$$= 2\int_{0}^{1/2} x\sin(2n\pi x)\,dx \Rightarrow \text{By parts}$$

$$u = x \qquad dv = \sin(2n\pi x)\,dx$$
$$du = dx \qquad v = \frac{-\cos(2n\pi x)}{2n\pi}$$

$$= 2\left[\frac{-x\cos(2n\pi x)}{2n\pi}\Bigg|_0^{1/2} + \int_0^{1/2}\frac{\cos(2n\pi x)}{2n\pi}dx\right]$$

$$= 2\left[\frac{-\frac{1}{2}\cos(n\pi)}{2n\pi} - 0 + \underbrace{\frac{\sin(2n\pi x)}{(2n\pi)^2}\Bigg|_0^{1/2}}_{\text{Zero}}\right]$$

$$b_n = 2\left[\frac{-1}{2}\,\frac{\cos(n\pi)}{2n\pi}\right] = \frac{-\cos(n\pi)}{2n\pi} = \frac{-(-1)^n}{2n\pi} = \frac{(-1)^{n+1}}{2n\pi}, \; n = 1, 2 \ldots$$

$\Rightarrow$ The fourier series of f:

$$a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(2n\pi x) + b_n\sin(2n\pi x)\right]$$

$$\frac{1}{8} + \sum_{n=1}^{\infty} a_n\cos(2n\pi x) + \sum_{n=1}^{\infty} b_n\sin(2n\pi x)$$

$$\frac{1}{8} + \sum_{n=1}^{\infty} a_{2n-1}\cos(2(2n-1)\pi x) + \sum_{n=1}^{\infty} b_n\sin(2n\pi x)$$

$$\frac{1}{8} + \sum_{n=1}^{\infty}\frac{-1}{((2n-1)\pi)^2}\cos(2(2n-1)\pi x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n\pi}\sin(2n\pi x)$$

$$= \frac{f(x^-) + f(x^+)}{2}$$

$\rightarrow$ followed

No.

(b) Show that $\sum_{n=1}^{\infty} \frac{1}{[(2n-1)\pi]^2} = \frac{1}{8}$ Method $\boxed{1}$

Sol: find $f(0)$

$$\frac{1}{8} + \sum_{n=1}^{\infty} \frac{-1}{((2n-1)\pi)^2} \underbrace{\cos\left(2(2n-1)\pi(0)\right)}_{1} + \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n\pi} \sin(2n\pi(0))}_{\text{Zero}} = \frac{f(0^+)+f(0^-)}{2}$$

$$\frac{1}{8} + \sum_{n=1}^{\infty} \frac{-1}{((2n-1)\pi)^2} = \frac{0+0}{2} \implies \sum_{n=1}^{\infty} \frac{1}{[(2n-1)\pi]^2} = \frac{1}{8}$$

(c) Show that $\sum_{n=1}^{\infty} \frac{(+1)}{[(2n-1)\pi]^2} = \frac{1}{8}$ Method $\boxed{2}$

Sol: take $x = \frac{1}{2}$

$$\frac{1}{8} + \sum_{n=1}^{\infty} \frac{-1}{[(2n-1)\pi]^2} \overbrace{\cos\left(2(2n-1)\frac{\pi}{2}\right)}^{-1} + \overbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n\pi} \sin\left(2n\frac{\pi}{2}\right)}^{\text{Zero}} = \frac{f(\frac{1}{2}^+)+f(\frac{1}{2}^-)}{2}$$

$$\frac{1}{8} + \sum_{n=1}^{\infty} \frac{-(-1)}{[(2n-1)\pi]^2} = \frac{\frac{1}{2}+0}{2} = \frac{1}{4}$$

$$\sum_{n=1}^{\infty} \frac{(+1)}{[(2n-1)\pi]^2} = \frac{1}{8} \implies \text{check}$$

Sunday | Dr. Ahmad Abdullah
6/11/2016 |

$f(x) = f(-x) \rightarrow$ even
$-f(x) = f(-x) \rightarrow$ odd

* The Fourier series of f (periodic with period $2L$) is given by

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots$$

* If f is even ; The Fourier series of f will be :-

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + \underbrace{\cancel{b_n}}_{\text{Zero}} \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$a_0 = \frac{1}{L} \int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2 \ldots$$

$b_n =$ Zero !

$$\Rightarrow a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) \right] = \frac{f(x^-) + f(x^+)}{2}$$

No.

*If f is odd, then the fourier series of f is given by :-

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) dx = \frac{f(x^-)+f(x^+)}{2}$$

$a_0 = a_n$ = Zero !

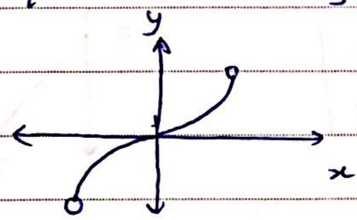
16/491 Find the Fourier series of

$f(x) = x|x|$, $-1 < x < 1$, period $= 2 = 2L$

Sol: $|x| = \begin{cases} x & , x>0 \\ -x & , x<0 \end{cases}$ $f(x) = x|x| = \begin{cases} x(x) & , 1>x>0 \\ x(-x) & , -1<x<0 \end{cases}$

$$f(x) = \begin{cases} +x^2 & , 0<x<1 \\ -x^2 & , -1<x<0 \end{cases}$$

odd function ⇒



or we know that $g(x) = x$ odd

& $h(x) = |x|$ even

odd x even = odd function.

⇒ The fourier series for odd function given by :-

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad , n = 1, 2, \dots$$

$$b_n = \frac{2}{1}\int_0^1 x^2 \sin\left(\frac{n\pi x}{1}\right) dx \quad , n = 1, 2, \dots$$ → continue by parts Two times

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No.

$$b_n = 2\left[x^2\left(\frac{-\cos(n\pi x)}{n\pi}\right)\right]_0^1 + \int_0^1 2x\left(\frac{\cos(n\pi x)}{n\pi}\right)dx$$

$-\cos(n\pi) = (-1)^{n+1}$

$$= 2\left[\frac{(-1)^{n+1}}{n\pi} + \underbrace{\frac{2x\sin(n\pi x)}{(n\pi)^2}\Big|_0^1}_{\text{zero}} - \int_0^1 \frac{\sin(n\pi x)}{(n\pi)^2}dx\right]$$

$$b_n = 2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2(-1)^n}{(n\pi)^3} - \frac{2}{(n\pi)^3}\right]$$

Half Range Expansion:-

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Find (a) the Fourier cosine series

(b) " " sine "

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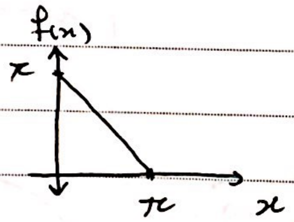
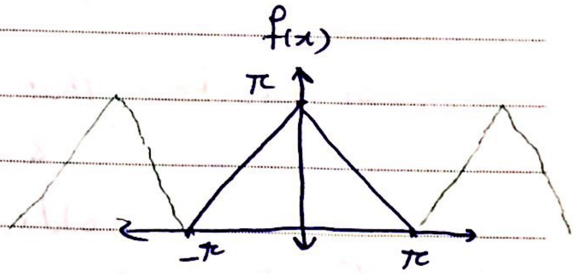


(a) even expansion:

$$f(x) = \begin{cases} x+\pi, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$



The Fourier Cosine series:- $a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

$L = \pi \Rightarrow a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)dx = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

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$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(nx)\,dx$$

$$a_n = \frac{2}{\pi}\left[(\pi - x)\frac{\sin(nx)}{n}\Big|_0^\pi - \int_0^\pi (-1)\frac{\sin(nx)}{n}\,dx \right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos(nx)}{n^2}\Big|_0^\pi \right] = \frac{-2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] =$$

$$\therefore \quad a_n = \begin{cases} 0 & , \text{ o even} \\ \frac{4}{\pi n^2} & , n \text{ is odd} \end{cases}$$

The Fourier Cosine series:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$a_{2n} = 0, \quad n = 1, 2, 3, \ldots$

$a_{2n+1} = \frac{4}{\pi(2n-1)^2}, \quad n = 1, 2, 3, \ldots$

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} a_{2n-1}\cos((2n-1)x)$$

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2}\cos((2n-1)x) = \frac{f(x^+) + f(x^-)}{2}$$

(2) Find $\sum_{n=1}^{\infty} \frac{(+1)}{(2n-1)^2}$

Take $x = 0 \Rightarrow \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2}\cos(0) = \frac{f(0^+) + f(0^-)}{2}$

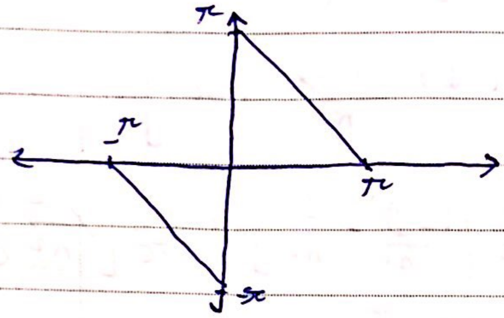
$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2} = \pi \Rightarrow \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2} = \frac{\pi}{2}$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2} \cdot \frac{\pi}{4}$$

→ followed

No.

(b) Find the odd expansion:-



$$f(x) = \begin{cases} -x-\pi, & -\pi < x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$$

The Fourier Sine series :-

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Sunday 13/11/2016 | Dr. Ahmad Abdullah
11.7 Fourier Integral

No.

11.7 Fourier Integral :

f is absolutely integrable on the x-axis if

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \quad \text{--- (1)}$$

The Fourier integral of f is given by

$$\int_{0}^{\infty} \left[A(w) \cos(wx) + B(w) \sin(wx) \right] dw \quad \text{--- (2)}$$

where :

$$\left.\begin{aligned} A(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(wv)\, dv \\ B(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(wv)\, dv \end{aligned}\right\} \quad \text{---- (3)}$$

Theorem 1: (Fourier Integral)

If f is piecewise continous in every finite interval and has a right hand derivative and a left hand derivative at every point and if the integral (1) exists, then $f(x)$ can be represented by a Fourier integral (2) with A & B given by (3)

Moreover :-

$$\int_{0}^{\infty} \left[A(w) \cos(wx) + B(w) \sin(wx) \right] dw = \frac{f(x^+) + f(x^-)}{2}$$

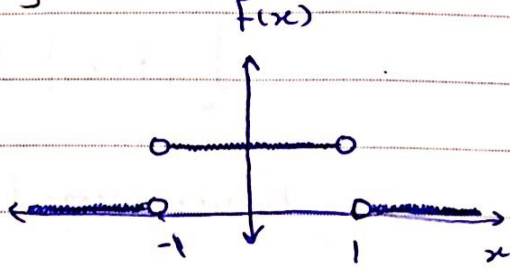
Followed.

No.

Example 2: (a) Find the Fourier integral of

$$f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

sol:



The Fourier integral of $\underline{f}$ is

$$\int_0^\infty \left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right] d\omega$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\, dv$

$$A(\omega) = \frac{1}{\pi}\int_{-1}^{1} (1)\cos(\omega v)\, dv = \frac{1}{\pi\omega} \sin(\omega v)\Big|_{-1}^{1}$$

$$A(\omega) = \frac{1}{\pi\omega}\left[\sin(\omega) - \sin(-\omega)\right] = \frac{2}{\pi\omega}\sin(\omega)$$

& $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)\, dv$

$$= \frac{1}{\pi}\int_{-1}^{1} (1)\sin(\omega v) = \frac{-1}{\pi\omega}\cos(\omega v)\Big|_{-1}^{1}$$

$$= \frac{-1}{\pi\omega}\left[\cos(\omega) - \cos(-\omega)\right] = \text{Zero !}$$

$$\int_0^\infty \left[\frac{2}{\pi\omega}\sin(\omega)\cos(\omega x) + \underline{0}\sin(\omega x)\right] d\omega$$

$$\int_0^\infty \left[\frac{2}{\pi\omega}\sin(\omega)\cos(\omega x)\right] d\omega$$

→ followed

No.

(b) evaluate $\int_0^\infty \frac{\sin(w)}{w} dw$

from Theorem 1

sol: $\int_0^\infty \left(\frac{2}{\pi w} \sin(w) \cos(wx) \right) dw = \frac{f(x^+) + f(x^-)}{2}$

Take $x = 0$

$$\int_0^\infty \left(\frac{2}{\pi w} \sin(w) \cos(0) \right) dw = \frac{f(0^+) + f(0^-)}{2} = \frac{1+1}{2}$$

$$\frac{2}{\pi} \int_0^\infty \left(\frac{\sin(w)}{w} \right) dw = 1 \Rightarrow \int_0^\infty \frac{\sin(w)}{w} = \frac{\pi}{2}$$

(c) evaluate $\int_0^\infty \frac{\sin(2w)}{w} dw$

sol: Take $x = 1$

$\sin(2w) = 2\cos(w)\sin(w)$

$$\frac{2}{\pi} \int_0^\infty \frac{\sin(w)\cos(w)}{w} dw = \frac{f(1^+) + f(1^-)}{2}$$

$$\frac{1}{\pi} \int_0^\infty \frac{\sin(2w)}{w} dw = \frac{0+1}{2}$$

$$\int_0^\infty \frac{\sin(2w)}{w} dw = \frac{\pi}{2}$$

Followed

No.

Notes : - If f is even function, The Fourier cosine integral of f is given by :-

$$\int_0^\infty A(\omega)\cos(\omega x)\,d\omega = \frac{f(x^+)+f(x^-)}{2}$$ → from theorem 1

where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\,dv .$$

- If f is odd function, The Fourier sine integral of f is given by :-

$$\int_0^\infty B(\omega)\sin(\omega x)\,d\omega = \frac{f(x^+)+f(x^-)}{2}$$

where

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)\,dv = \frac{2}{\pi}\int_0^\infty f(v)\sin(\omega v)\,dv$$

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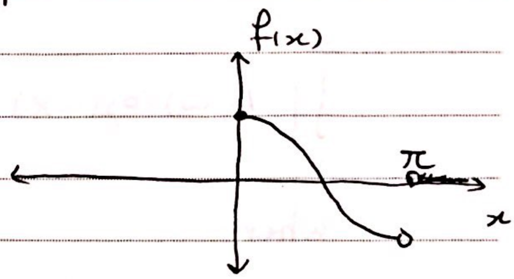
18/517 Fourier sine integral representation

$$f(x) = \begin{cases} \cos(x) & , 0 < x < \pi \\ 0 & , x > \pi \end{cases}$$



Sol:- $\int_0^\infty B(\omega) \sin(\omega x) \, d\omega$

where : $B(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \sin(\omega v) \, dv$

$= \frac{2}{\pi} \int_0^\pi \cos(v) \sin(\omega v) \, dv$

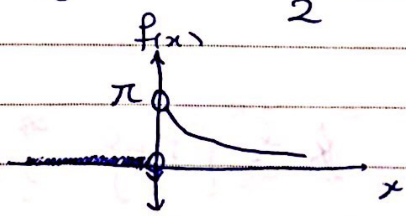
continue

1-6/517 Show that:

1/517
$$\int_0^\infty \frac{\cos(x\omega) + \omega \sin(x\omega)}{1 + \omega^2} \, d\omega = \begin{cases} 0 & , x < 0 \\ \pi/2 & , x = 0 \\ \pi e^{-x} & , x > 0 \end{cases}$$

Sol:
$$\int_0^\infty [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)] \, d\omega = \frac{f(x^-) + f(x^+)}{2}$$

take $f(x) = \begin{cases} 0 & , x < 0 \\ \pi e^{-x} & , x > 0 \end{cases}$



to check

→ $\frac{f(-2^-) + f(-2^+)}{2} = 0$ ✓

$\frac{f(0^-) + f(0^+)}{2} = \frac{\pi}{2}$ ✓ , $\frac{f(1^+) + f(1^-)}{2} = f(1) = \pi e^{-1}$ ✓

→ followed

No.

The Fourier integral of $\underline{f}$:-

$$\int_0^{\infty}\left[A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x)\right]d\omega \quad \text{....}(1)$$

where

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv=\frac{1}{\pi}\int_0^{\infty}\pi e^{-v}\cos(\omega v)\,dv$$

continue

$$=\text{-----}=\frac{1}{1+\omega^2}\quad \text{---}(2)$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)\,dv=\frac{1}{\pi}\int_0^{\infty}\pi e^{-v}\sin(\omega v)\,dv$$

continue

$$=\text{------}=\frac{\omega}{1+\omega^2}\quad \text{---}(3)$$

putting (2) & (3) into (1) :-

$$\int_0^{\infty}\left[\frac{1}{1+\omega^2}\cos(\omega x)+\frac{\omega}{1+\omega^2}\sin(\omega x)\right]d\omega=\int_0^{\infty}\frac{\cos(\omega x)+\omega\sin(\omega x)}{1+\omega^2}$$

$$\int_0^{\infty}\frac{\cos(\omega x)+\omega\sin(\omega x)}{1+\omega^2}=\frac{f(x^+)+f(x^-)}{2}=\begin{bmatrix}0 & , x<0\\ \pi/2 & , x=0\\ \pi e^{-x} & , x>0\end{bmatrix}$$

Note :- $\mathcal{L}\{f(t)\}=\int_0^{\infty}e^{-st}f(t)\,dt$

$$\mathcal{L}\{\cos(\omega v)\}=\int_0^{\infty}e^{-sv}\cos(\omega v)\,dv=\frac{s}{s^2+\omega^2}$$

& in our case $s=1 \rightarrow \frac{1}{1^2+\omega^2}=\frac{1}{1+\omega^2}$

$\rightarrow$ followed

No.

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$$\int_0^\infty \frac{\sin(w) - w\cos(w)}{w^2} \sin(wx)\, dw = \begin{cases} \frac{1}{2}\pi x , & 0 < x < 1 \\ \frac{1}{4}\pi , & x = 1 \\ 0 , & x > 0 \end{cases}$$

Sol: $f(x) = \begin{cases} \frac{1}{2}\pi x , & 0 < x < 1 \\ 0 , & x > 1 \end{cases}$

Fourier sine integral (no presence of $\cos(wx)$) of f:-

$$\int_0^\infty B(w) \sin(wx)\, dw \quad --- (1)$$

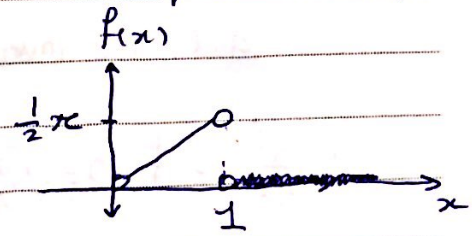
where $B(w) = \frac{1}{\pi} \int_0^\infty f(v) \sin(wv)\, dv$

$$= \frac{2}{\pi} \int_0^1 \frac{1}{2} \pi v \sin(wv)\, dv$$

$$= \_\_\_ \text{ by parts } = \frac{\sin(w) - w\cos(w)}{w^2} \quad -(2)$$

putting (2) into (1)

$$\int_0^\infty \frac{\sin(w) - w\cos(w)}{w^2} \sin(wx)\, dw = \frac{f(x^-) + f(x^+)}{2} = \begin{cases} \frac{1}{2}\pi x , & 0 < x < 1 \\ \frac{1}{4}\pi , & x = 1 \\ 0 , & x > 0 \end{cases}$$



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$$f(x) = \begin{cases} \cos(x) & , 0 < x < \pi \\ 0 & , x > \pi \end{cases}$$

Find the Fourier sine integral of f.

Sol: $\int_0^\infty B(w) \sin(wx)\, dw$

where $B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(wv)\, dv$

$= \frac{2}{\pi} \int_0^\pi \cos(v) \sin(wv)\, dv$

$= \ldots\ldots$ continue

11.8: Fourier Cosine and sine Transform:

The Fourier cosine transform of f is given by

$$\mathcal{F}(f) = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(wx)\, dx$$

and the inverse cosine transform is

$$\mathcal{F}_c^{-1} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos(wx)\, dw$$

No.

The Fourier Sine Transform of f

$$\mathcal{F}_s(f) = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, dx$$

and the inverse sine transform is

$$\mathcal{F}_s^{-1} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin(\omega x)\, d\omega$$

Example 1: $f(x) = \begin{bmatrix} k, & 0 < x < a \\ 0, & x > a \end{bmatrix}$

Find the Fourier cosine and sine transform of f.

Sol: $\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx = \sqrt{\frac{2}{\pi}} \int_a^a k \cos(\omega x)\, dx$

$$= \sqrt{\frac{2}{\pi}}\, k \left.\frac{\sin(\omega x)}{\omega}\right|_0^a = k\sqrt{\frac{2}{\pi}}\, \frac{\sin(\omega a)}{\omega}$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, dx \ldots\ldots = \sqrt{\frac{2}{\pi}}\, (k)\left(\frac{1 - \cos(\omega a)}{\omega}\right)$$

properties of Fourier cosine and sine transform:-

$$\mathcal{F}_c\{af + bg\} = a\,\mathcal{F}_c\{f\} + b\,\mathcal{F}_c\{g\}$$

$$\mathcal{F}_s\{af + bg\} = a\,\mathcal{F}_s\{f\} + b\,\mathcal{F}_s\{g\}$$

$\longrightarrow$ followed

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Theorem 1: Let $f(x)$ be continous and absolutely integrable on the x-axis. Let $f'(x)$ be piecewise continous on every finite interval and let

$$f(x) \to 0 \quad as \quad x \to \infty$$

Then

$$\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\}$$

Example 3: Find $\mathcal{F}_c\{e^{-ax}\}$ for $f(x) = e^{-ax}$, $a > 0$

Sol: * method $\boxed{1}$: Direct method $\mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos(\omega x)\, dx$

* method $\boxed{2}$ (using another property of theorem 1)

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, \omega f(0)$$

⇒ The conditions on f and f' and f'' respectively satisfy in theorem 1.

→ followed

No.

$f(x) = e^{-ax}$, $f'(x) = -a e^{-ax}$, $f''(x) = a^2 e^{-ax}$

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$\mathcal{F}_c\{a^2 e^{-ax}\} = -\omega^2 \mathcal{F}_c\{e^{-ax}\} - \sqrt{\frac{2}{\pi}}\,(-a e^{(-a)(0)})$$

$$a^2 \mathcal{F}_c\{e^{-ax}\} = -\omega^2 \mathcal{F}_c\{e^{-ax}\} + a\sqrt{\frac{2}{\pi}}$$

$$a^2 \mathcal{F}_c\{e^{-ax}\} + \omega^2 \mathcal{F}_c\{e^{-ax}\} = a\sqrt{\frac{2}{\pi}}$$

$$(a^2+\omega^2)\, \mathcal{F}_c\{e^{-ax}\} = a\sqrt{\frac{2}{\pi}}$$

$$\mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+\omega^2}$$

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11.9 Fourier Transform

The complex Fourier integral of f is :-

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) e^{i\omega(x-v)} \, dv \, d\omega$$

$\Rightarrow$ you can see the derivation on the book, but not included in exam.

The Fourier transform of f is :-

$$\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \quad \text{---- (6)}$$

and the inverse Fourier transform of $\hat{f}(\omega)$:-

$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega$$

Theorem 1: If $f(x)$ is absolutely integrable on the x-axis $(-\infty < x < \infty)$ and piecewise continous on every finite interval, then the Fourier transform $\hat{f}(\omega)$ of $f(x)$ is given by (6) exists.

→ Follow

No.

Ex: Find the Fourier transform of

$$\mathcal{F}(e^{-ax}) \qquad f(x) = \begin{cases} e^{-ax} & , x>0 \\ 0 & , x<0 \end{cases} \qquad a>0$$

Sol:
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+i\omega)x}\, dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right]_0^{\infty}$$

$$\Rightarrow \lim_{x\to\infty} e^{-(a+i\omega)x} = \lim_{x\to\infty} e^{-ax} e^{-i\omega x}$$

squeeze theorem ←
$$= \lim_{x\to\infty} \underset{\substack{e^{-ax}\to 0 \\ \text{as } x\to\infty}}{e^{-ax}} \underbrace{\left[\cos(-\omega x) + i\sin(-\omega x)\right]}_{\text{bounded}} = 0$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right]_0^{\infty} = 0 - \left(\frac{1}{\sqrt{2\pi}} \frac{1}{-(a+i\omega)} \right) = \frac{1}{\sqrt{2\pi}} \frac{1}{a+i\omega}$$

properties:-

$$\mathcal{F}(f'(x)) = i\omega\, \mathcal{F}(f(x))$$

→ conditions :-
① f is continuous on the x-axis
② $f(x) \to 0$ as $|x| \to \infty$
③ $f'(x)$ is absolutely integrable

$$\mathcal{F}(f''(x)) = -\omega^2\, \mathcal{F}(f(x))$$

No.

Ex: Find the Fourier transform of xe^{-x^2} from Table III

Sol: take $f(x) = \frac{-1}{2} e^{-x^2} \Rightarrow f'(x) = \left(\frac{-1}{2} e^{-x^2}\right)(-2x) = xe^{-x^2}$

$$\mathcal{F}(f'(x)) = i\omega \mathcal{F}\{f(x)\}$$

$$\mathcal{F}(xe^{-x^2}) = i\omega\, \mathcal{F}\left(\frac{-1}{2} e^{-x^2}\right) = (i\omega)\left(\frac{-1}{2}\right) \mathcal{F}\underbrace{\left(e^{-x^2}\right)}_{\text{from table}}$$

$$= \left(\frac{-i\omega}{2}\right)\underbrace{\left(\frac{1}{\sqrt{2}} e^{-\omega^2/4}\right)}_{\text{from tables}}$$

Chapter 12 : Partial Differential Equations (PDEs)

12.1 Basic Concepts of PDEs.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ 1-dimensional Heat Eqn.

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ Two dimensional Heat Eqn.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ 1-dimensional wave Eqn.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ 2-dimensional Laplace Eqn.

Ex: Solve the PDE $u_{xx} - u_x - 2u = 0$

Sol:

PDE	Reminder: ordinary ODE
$u_{xx} - u_x - 2u = 0$	$y'' - y' - 2y = 0$
$r^2 - r - 2 = 0$	$r^2 - r - 2 = 0$
$(r+1)(r-2) = 0$	$(r+1)(r-2) = 0$
$r = -1, 2$	$r = -1, 2$
$u(x,y) = f(y)e^{-x} + g(y)e^{2x}$	$y(x) = c_1 e^{-x} + c_2 e^{2x}$

where $f(y)$ and $g(y)$ are arbitrary functions of y.

→ followed

Ex: Solve $u_{xy} = -u_x$

Sol: Let $u_x = \omega$

$\omega_y = -\omega$

ODE

$\frac{dw}{dy} = -\omega$

$\int \frac{dw}{dy} = -\int dy$

$\ln|w| = -y + c_1$

$|w| = e^{-y+c_1}$

$|w| = \underbrace{e^{c_1}}_{c_2} e^{-y}$

$w = \pm c_2 e^{-y}$

$w = c_3 e^{-y}$, $c_3 = \pm c_2$

$\omega = f(x) e^{-y}$

$w = u_x$

$u_x = f(x) e^{-y}$

$u = \int f(x) e^{-y} dx + C(y)$

$g(x) = \int f(x)$ ← $u = e^{-y} g(x) + C(y)$

$u = e^{-y} g(x) + C(y)$ → where $g(x)$ is an arbitrary function of x

" $C(y)$ is an " function of y

$\frac{19}{543}$ Solve $u_y + y^2 u = y^2$

Sol: PDE | ODE

$y' + p(x)y = r(x)$

Linear in x

$\mu(x) = e^{\int p(x)dx}$

$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x) r(x) dx + c\right]$

let $u_y = y'$ & $y = x$ & $u = y$

$p(x) = x^2$

$r(x) = x^2$

$\mu(x) = e^{\int p(x)dx} = e^{\int x^2 dx} = e^{\frac{x^3}{3}}$

$u(x,y) = \frac{1}{e^{y^3/3}}\left[\int e^{y^3/3} y^2 dy + f(x)\right] \longleftarrow y(x) = \frac{1}{e^{x^3/3}}\left[\int e^{x^3/3} x^2 dx + c\right]$

23/543	solve $x^2 u_{xx} + 2x u_x - 2u = 0$

Sol:

PDE

$a = 2 \quad , \quad b = -2$

$r^2 + (a-1)r + b = 0$

$r^2 + (2-1)r - 2 = 0$

$r^2 + r - 2 = 0$

$(r-1)(r+2) = 0$

$r = -2, 1$

$u(x,y) = f(y)x^{-2} + g(y)x^{1}$

ODE

$x^2 y'' + axy' + by = 0$

$r^2 + (a-1)r + b = 0$

21/543	$u_{yy} + 6u_y + 13u = 4e^{3y}$

ODE

$y'' + 6y' + 13y = 4e^{3x}$

$y = y_h + y_p$

This Example not included in exam !

not included in second exam.

12.3 Solution by seperation of variables

Use of Fourier Series

EX: Consider the wave Eqn

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0<x<L, \quad t>0$$

with boundary conditions $u(0,t)=0$, $u(L,t)=0$ $t \geq 0$

and intial conditions $u(x,0)=f(x)$, $u_t(x,0)=g(x)$ $0 \leq x \leq L$

Sol: Separation of Variables: $u(x,t)=X(x)\,T(t)$

$u_x = X'T \Rightarrow u_{xx} = X''T$

$u_t = XT' \Rightarrow u_{tt} = XT''$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$XT'' = c^2 X'' T$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda, \quad \lambda \equiv \text{constant} \quad f(t) = g(x) = \text{constant}$$

$$X'' - \lambda X = 0, \quad T'' - c^2 \lambda T = 0$$

First we solve $X'' - \lambda X = 0$ but $u(0,t) = 0 = X(0)T(t)$

$$u(0,t) = T(t)X(0) = 0 \xrightarrow{T \neq 0} X(0) = 0$$

$$u(L,t) = T(t)X(L) = 0 \xrightarrow{T(t) \neq 0} X(L) = 0$$

$X'' - \lambda X = 0$, $X(0) = 0$, $X(L) = 0$

$\boxed{1}$ $\lambda > 0 \rightarrow \lambda = \alpha^2 > 0$

$X'' - \lambda X = 0 \rightarrow X'' - \alpha^2 X = 0 \rightarrow r^2 - \alpha^2 = 0$

$\rightarrow r = \pm \alpha$

$X(x) = C_1 e^{-\alpha x} + C_2 e^{\alpha x}$

$X(0) = 0 \rightarrow 0 = C_1 e^{-\alpha(0)} + C_2 e^{\alpha(0)} \rightarrow C_2 = -C_1$

$X(L) = 0 \rightarrow 0 = C_1 e^{-\alpha L} + C_2 e^{\alpha L} = C_1 e^{-\alpha L} - C_1 e^{\alpha L} = 0$

$\rightarrow C_1 \left(e^{-\alpha L} - e^{\alpha L} \right) = 0$

$C_1 = 0$ OR $e^{-\alpha L} - e^{\alpha L} = 0 \rightarrow e^{-\alpha L} = e^{\alpha L}$

$\rightarrow 1 = e^{2\alpha L}$ since $L \neq 0$

so $\alpha = 0$ "Impossible"

$\Rightarrow C_1 = 0 \Rightarrow C_2 = -C_1 = 0$

$\Rightarrow$ NO eigenvalues

$\rightarrow$ followed

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$\boxed{2}$ $\lambda = 0$

$$X'' - \lambda X = 0 \Rightarrow X'' = 0 \rightarrow X' = c_1 \rightarrow X(x) = c_1 x + c_2$$

$$X(0) = 0 \rightarrow 0 = c_1(0) + c_2 \longrightarrow c_2 = 0$$

$$X(L) = 0 \rightarrow 0 = c_1 L + c_2 \longrightarrow 0 = c_1 L \xrightarrow{L \neq 0} c_1 = 0$$

$\Rightarrow$ No eigen values

$\boxed{3}$ $\lambda < 0 \rightarrow \lambda = -\alpha^2 < 0$, $\alpha > 0$

$$X'' - \lambda X = 0 \rightarrow X'' + \alpha^2 X = 0 \rightarrow r^2 + \alpha^2 = 0$$
$$r^2 = -\alpha^2 \rightarrow r = \pm \alpha i$$

$$\Rightarrow X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

$$X(0) = 0 \rightarrow 0 = c_1 \cos(\alpha(0)) + c_2(\sin(\alpha(0)))$$
$$\rightarrow 0 = c_1 + 0 \rightarrow c_1 = 0$$

$$X(L) = 0 \rightarrow 0 = c_2 \sin(\alpha L) \xrightarrow{c_2 \neq 0} \sin(\alpha L) = 0$$
$$\alpha L = n\pi$$
$$\alpha_n = \frac{n\pi}{L}, \quad n = 1, 2, 3 \ldots$$

$$X(x) = c_2 \sin(\alpha x)$$

$\Rightarrow$ Eigen values: $\lambda_n = -\alpha_n^2 = -\left(\frac{n\pi}{L}\right)^2$

$\Rightarrow$ Eigen functions: $X_n(x) = \sin(\alpha_n x) = \sin\left(\frac{n\pi}{L} x\right)$, $n = 1, 2, 3 \ldots$

$y'' + y = 0 \rightarrow y_1 = \sin x$ $\quad y_3 = C_1 \sin x + C_2 \cos x$

$y_2 = \cos(x)$

$$\lambda_n = -\alpha_n^2 = -\left(\frac{n\pi}{L}\right)^2$$

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Next we solve $T'' - \lambda_n c^2 T = 0$

$$T'' + \left(\frac{n\pi}{L}\right)^2 c^2 T = 0 \rightarrow T'' + \left(\frac{cn\pi}{L}\right)^2 T = 0$$

$$r^2 + \left(\frac{cn\pi}{L}\right)^2 = 0 \rightarrow r = \pm \frac{cn\pi}{L} i$$

$$\rightarrow T_n(t) = A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)$$

$$\Rightarrow U_n(x,t) = X_n(x)\, T_n(t)$$

$$U_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi}{L} t\right) + B_n \sin\left(\frac{n\pi}{L} t\right)\right]$$

$n = 1, 2, 3, \ldots$

Superposition: " the summation of all n's solutions is a solution"

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

But $U(x,0) = f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n\right]$

$$f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \underbrace{\cos\left(\frac{n\pi(0)}{L}\right)}_{1} + B_n \underbrace{\sin\left(\frac{n\pi(0)}{L}\right)}_{0}\right]$$

$$f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) A_n$$

Fourier Sine Series :

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \ldots$$

→ Followed

$$U_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[-A_n\left(\frac{cn\pi}{L}\right)\sin\left(\frac{cn\pi t}{L}\right) + B_n\left(\frac{cn\pi}{L}\right)\cos\left(\frac{cn\pi t}{L}\right)\right]$$

$$U_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{-A_n cn\pi}{L}\sin\left(\frac{cn\pi t}{L}\right) + \frac{B_n cn\pi}{L}\cos\left(\frac{cn\pi t}{L}\right)\right]$$

$$U_t(x,0) = g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{B_n cn\pi}{L}\right]$$

$$g(x) = \sum_{n=1}^{\infty} \frac{B_n cn\pi}{L} \sin\left(\frac{n\pi x}{L}\right)$$

Again it is a Fourier Sine Series

$$B_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)dx = \dots\dots$$

---

for the same example but $u(x,0)=0$ & $U_t(x,0)=x$ $0<x<L$

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n=1,2,\dots$$

$$T_n(t) = A_n\cos\left(\frac{cn\pi t}{L}\right) + B_n\sin\left(\frac{cn\pi t}{L}\right)$$

$$u(x,0)=0 \longrightarrow X(x)\,T(0)=0 \xrightarrow{X\neq 0} T(0)=0$$

$$0 = A_n\cos\left(\frac{cn\pi(0)}{L}\right) + B_n\sin\left(\frac{cn\pi(0)}{L}\right) \longrightarrow A_n = 0$$

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$$T_n(t) = B_n \sin\left(\frac{cn\pi t}{L}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[B_n \sin\left(\frac{cn\pi t}{L}\right)\right]$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{B_n cn\pi}{L} \cos\left(\frac{cn\pi t}{L}\right)\right]$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \frac{B_n Cn\pi}{L} = x$$

it is Fourier series

$$B_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \ldots$$

EX: Solve the PDE

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0$$

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t \geq 0$$

$$u(x,0) = f(x), \quad u_t(x,0), \quad 0 \leq x \leq L$$

sol: saperation of variables

$$u(x,t) = X(x)\,T(t)$$

$$XT'' = c^2 X'' T$$

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda \quad \begin{cases} X'' - \lambda X = 0 \\ T'' - c^2\lambda T = 0 \end{cases}$$

First we solve : $X'' - \lambda X = 0$

$u_x(0,t) = 0$

$X'(0)T(t) = 0 \xrightarrow{T \neq 0} X'(0) = 0$

$u_x(L,t) = 0$

$X'(L)\,T(t) = 0 \xrightarrow{T \neq 0} X'(L) = 0$

I) $\lambda > 0 \to \lambda = \alpha^2 > 0$, $\alpha > 0$

$X'' - \lambda X = 0 \to X'' - \alpha^2 X = 0 \to r^2 - \alpha^2 = 0 \to r = \pm\alpha$

$X(x) = C_1 e^{-\alpha x} + C_2 e^{\alpha x} \to X'(x) = -\alpha C_1 e^{-\alpha x} + \alpha C_2 e^{\alpha x}$

But $X'(0) = 0 = -\alpha C_1 + \alpha C_2 = \alpha(C_2 - C_1) = 0 \xrightarrow{\alpha \neq 0} C_2 - C_1 = 0$

$\to C_2 = C_1$

& $X'(L) = 0 = -\alpha C_1 e^{-\alpha L} + \alpha C_2 e^{\alpha L} = 0 \overset{C_2 = C_1}{=} \alpha C_1 (e^{\alpha L} - e^{-\alpha L}) = 0$

But $(\alpha \,\&\, L) \neq 0$ and $e^{\alpha L} = e^{-\alpha L}$ can never equal except $(\alpha \text{ or } L) = 0$ which impossible!

$\Rightarrow C_1 = 0 = C_2$ $\therefore$ NO eigen values

2) $\lambda = 0 \Rightarrow X'' - \lambda X = 0 \rightarrow X'' = 0$

$\rightarrow X' = C_1 \rightarrow X = C_1 x + C_2$

But. $X'(0) = 0 \rightarrow 0 = C_1$

$X'(L) = 0 \rightarrow 0 = C_1$

take $C_2 = 1$

Eigen values : $\lambda = 0$, Eigen functions : $X(x) = C_2 = 1$

3) $\lambda < 0 \rightarrow \lambda = -\alpha^2 < 0 , \alpha > 0$

$X'' - \lambda X = 0 \rightarrow X'' + \alpha^2 X = 0 \rightarrow r^2 + \alpha^2 = 0 \rightarrow r = \pm \alpha i$

$\rightarrow X(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$

$\rightarrow X'(x) = -C_1 \alpha \sin(\alpha x) + C_2 \alpha \cos(\alpha x)$

But $X'(0) = 0 \Rightarrow 0 = C_2 \alpha \xrightarrow{\alpha \neq 0} C_2 = 0$

$X'(L) = 0 \Rightarrow 0 = -C_1 \alpha \sin(\alpha L) \xrightarrow{(\alpha \& C_1) \neq 0} \sin(\alpha L) = 0$

$\rightarrow \alpha L = n\pi , n = 1, 2, 3, \ldots \rightarrow \alpha_n = \frac{n\pi}{L}$

Eigen values : $\lambda_n = -\alpha_n^2 = -\left(\frac{n\pi}{L}\right)^2 , n = 1, 2, 3, \ldots$

Eigen vectors : $X_n(x) = \cos(\alpha_n x) = \cos\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, \ldots$

$\rightarrow$ follow

$\boxed{2}$ & $\boxed{3}$ $\Rightarrow$ Eigenvalues : $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \dots$

Eigenfunctions : $X_n(x) = \cos\left(\frac{n\pi}{L}\right)$, $n = 0, 1, 2, \dots$

second we solve : $T'' - \lambda c^2 T = 0$

$\longrightarrow T_n'' + \left(\frac{n\pi}{L}\right)^2 c^2 T_n = 0 \longrightarrow r^2 + \left(\frac{n\pi c}{L}\right)^2 = 0$

$\longrightarrow r = \pm\left(\frac{n\pi c}{L}\right) i$

$T_n(t) = A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)$, $n = 0, 1, 2, \dots$

But $U_t(x, 0) = 0 \Rightarrow X(x) T'(0) = 0 \xrightarrow{X \neq 0} T'(0) = 0$

$\rightarrow T_n'(t) = -A_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) + B_n\left(\frac{n\pi c}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$

$T_n'(0) = 0 = \frac{B_n n\pi c}{L} \longrightarrow B_n = 0$

$\Rightarrow T_n(t) = A_n \cos\left(\frac{n\pi c t}{L}\right)$, $n = 0, 1, 2, \dots$

$\Rightarrow U_n(x, t) = X_n(x) T_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$, $n = 0, 1, 2, \dots$

superposition :-

$$U(x, t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$

$\longrightarrow$ followed

But $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

↑ Fourier Cosine Series

$$\Rightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx \;\; \text{-----} \;\; \text{continue}$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \;\; \text{---} \;\; \text{continue}$$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right)$$

12.6 Heat Equation :-

EX: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $0 < x < L$, $t > 0$

Boundary conditions : $u_x(0,t) = 0$, $u(L,0) = 0$, $t \geq 0$

Initial conditions : $u(x,0) = f(x)$ $\quad 0 \leq x \leq L$

→ followed

Sol: Seperation of variables

$$u(x,t) = X(x)\,T(t)$$

$$XT' = c^2 X'' T \longrightarrow \frac{T'}{cT} = \frac{X''}{X} = \lambda$$

$$X'' - \lambda X = 0 \qquad \& \qquad T' - \lambda c^2 T = 0$$

Solve first : $X'' - \lambda X = 0$

But $u_x(0,t) = 0$ | $u(L,t) = 0$

$X'(0)\,T(x) \xrightarrow{T \neq 0} X'(0) = 0$ | $X(L)\,T(x) \xrightarrow{T\neq 0} X(L) = 0$

Π $\lambda > 0 \longrightarrow \lambda = \alpha^2 > 0$, $\alpha > 0$

$$X'' - \lambda X = 0 \longrightarrow X'' - \alpha^2 X \longrightarrow r^2 - \alpha^2 = 0 \longrightarrow r = \pm\alpha$$

$$X(x) = C_1 e^{-\alpha x} + C_2 e^{\alpha x} \qquad \& \qquad X'(x) = -\alpha C_1 e^{-\alpha x} + \alpha C_2 e^{\alpha x}$$

$$\longrightarrow X'(0) = 0 \longrightarrow 0 = -\alpha_1 C_1 + \alpha C_2 \longrightarrow \alpha(C_2 - C_1) = 0$$

$$\xrightarrow{\alpha \neq 0} C_2 = C_1$$

$$\longrightarrow X(L) = 0 \longrightarrow 0 = C_1 e^{-\alpha L} + C_1 e^{\alpha L} \longrightarrow C_1\left(e^{\alpha L} + e^{-\alpha L}\right) = 0$$

But $e^{\alpha L} + e^{-\alpha L} \neq 0 \longrightarrow C_1 = 0$ $\therefore$ No eigenvalues

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2] $\lambda = 0$

$$X'' - \lambda X = 0 \longrightarrow X'' = 0 \longrightarrow X' = c_1 \longrightarrow X = c_1 x + c_2$$

But $X'(0) = 0 \longrightarrow 0 = c_1$

$$X(L) = 0 \longrightarrow 0 = c_1 L + c_2 \longrightarrow c_2 = 0$$

3] $\lambda < 0 \longrightarrow \lambda = -\alpha^2 < 0 \quad , \alpha > 0$

$$X'' - \lambda X = 0 \longrightarrow X'' + \alpha^2 X = 0 \longrightarrow r^2 + \alpha^2 = 0 \longrightarrow r = \pm \alpha i$$

$$\Rightarrow X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

$$\Rightarrow X'(x) = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x)$$

But $X'(0) = 0 \longrightarrow 0 = c_2 \alpha \xrightarrow{\alpha > 0} c_2 = 0$

$$X(L) = 0 \longrightarrow 0 = c_1 \cos(\alpha L) \xrightarrow{c_1 \neq 0} \cos(\alpha L) = 0$$

$$\Rightarrow \alpha L = \frac{(2n-1)\pi}{2} \quad , n = 1, 2, 3 \ldots$$

$$\alpha_n = \frac{(2n-1)\pi}{2L} \quad , n = 1, 2, 3 \ldots$$

Eigenvalues : $\lambda_n = -\left(\frac{(2n-1)\pi}{2L}\right)^2 \quad , n = 1, 2, 3, \ldots$

Eigen functions : $X_n(x) = \cos(\alpha_n x) = \cos\left(\frac{(2n-1)\pi x}{2L}\right) \quad , n = 1, 2 \ldots$

→ followed

Second we solve $T_n' - c^2 \lambda_n T_n = 0$

$$T_n' - c^2\left(-\left(\frac{(2n-1)\pi}{2L}\right)^2\right)T_n = 0 \longrightarrow T_n' + c^2\left(\frac{(2n-1)\pi}{2L}\right)^2 T_n = 0$$

$$\rightarrow r + \left(\frac{c(2n-1)\pi}{2L}\right)^2 = 0 \rightarrow r = -\left(\frac{c(2n-1)\pi}{2L}\right)^2$$

$$T_n(t) = e^{-\left(\frac{c(2n-1)\pi}{2L}\right)^2 t} \quad , n = 1, 2, \dots$$

$$\Rightarrow U_n(x,t) = X_n(x)\, T_n(t) = \left[\cos\left(\frac{(2n-1)\pi x}{2L}\right)\right] e^{-\left(\frac{c(2n-1)\pi}{2L}\right)^2 t} \quad , n = 1, 2, \dots$$

Superposition:-

$$U(x,t) = \sum_{n=1}^{\infty} A_n \left[\cos\left(\frac{(2n-1)\pi x}{2L}\right)\right]\left[e^{-\left(\frac{c(2n-1)\pi}{2L}\right)^2 t}\right]$$

we add it to satisfy the linear combination

But $U(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \cos\left[\frac{(2n-1)\pi x}{2L}\right]$

Q: is $g_n(x) = \cos\left(\frac{(2n-1)\pi}{2L} x\right)$ an orthogonal set of functions for $0 \leq x \leq L$??

$$\xrightarrow{m \neq n} \int_0^L \cos\left(\frac{(2n-1)\pi x}{2L}\right) \cos\left(\frac{(2m-1)\pi x}{2L}\right) dx$$

..... continue the integration you will get zero!

So the set of functions are orthogonal

— followed!

$$f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi x}{2L}\right)$$

multiply by $\cos\left(\frac{(2m-1)\pi x}{2L}\right)$ then integrate

$$\int_0^L f(x) \cos\left(\frac{(2m-1)\pi x}{2L}\right) = \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{(2n-1)\pi x}{2L}\right) \cos\left(\frac{(2m-1)\pi x}{2L}\right)$$

$$\int_0^L f(x) \cos\left(\frac{(2m-1)\pi x}{2L}\right) dx = A_m \int_0^L \left(\cos\left(\frac{(2m-1)\pi x}{2L}\right)\right)^2 dx$$

$$\int_0^L f(x) \cos\left(\frac{(2m-1)\pi x}{2L}\right) dx = A_m \left[\frac{1}{2}\int_0^L \left(1 + \cos\left(\frac{(2m-1)\pi x}{L}\right)\right) dx\right]$$

$$= A_m \left[\frac{1}{2}\left(L + \underbrace{\frac{\sin\left(\frac{(2m-1)\pi x}{L}\right)}{\frac{(2m-1)\pi}{L}}}_{\text{zero}}\right)\right]_0^L$$

$$\int_0^l f(x) \cos\left(\frac{(2m-1)\pi x}{2L}\right) dx = A_m \left(\frac{L}{2}\right)$$

$$A_m = \frac{2}{L}\int_0^l f(x) \cos\left(\frac{(2m-1)\pi x}{2L}\right) dx$$

~ ~ ~ ~ ~

return to example

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi x}{2L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{(2n-1)\pi x}{2L}\right) dx \quad , n = 1, 2, \dots$$

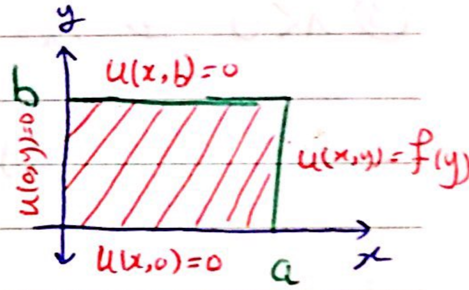
Ex: solve the Laplace Equation :-

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ; \quad 0 < x < a \ , \quad 0 < y < b$$

$$u(x,0) = 0 \ , \ u(x,b) = 0 \ , \quad 0 \leq x \leq a$$

$$u(0,y) = 0 \ , \ u(a,y) = f(y) \ , \quad 0 \leq y \leq b$$

Sol: By separation of variables :-



$$u(x,y) = X(x)\, Y(y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow X''Y + XY'' = 0$$

$$\Rightarrow X''Y = -XY'' \Rightarrow \frac{-X''}{X} = \frac{Y''}{Y} = \lambda \ , \quad \lambda \equiv \text{constant}$$

$$X'' + \lambda X = 0 \quad \& \quad \underbrace{Y'' - \lambda Y = 0}$$

↳ start with this equation since we have $u(x,0)$ & $u(x,b) = 0$

$$\Rightarrow \begin{array}{l} u(x,0) = 0 \\ X(x)Y(0) = 0 \xrightarrow{X \neq 0} Y(0) = 0 \end{array} \Bigg\{ \begin{array}{l} u(x,b) = 0 \\ X(x)Y(b) = 0 \xrightarrow{X \neq 0} Y(b) = 0 \end{array}$$

Followed

※ First we solve:

$$Y'' - \lambda Y = 0 \quad , \quad Y(0) = 0 \quad , \quad Y(b) = 0$$

(As we did before) $\Rightarrow$ ① $\lambda > 0$, No eigenvalues

② $\lambda = 0$, No eigenvalue.

③ $\lambda < 0 \Rightarrow \lambda_n = -\left(\frac{n\pi}{b}\right)^2$, $n = 1, 2, \ldots$ "Eigenvalues"

$Y_n(y) = \sin\left(\frac{n\pi y}{b}\right)$, $n = 1, 2, 3 \ldots$ "Eigenfunctions"

※ Next we solve:

$$X_n'' + \lambda_n X_n = 0 \longrightarrow X_n'' - \left(\frac{n\pi}{b}\right)^2 X = 0 \longrightarrow r^2 = \left(\frac{n\pi}{b}\right)^2$$

$$\longrightarrow r = \pm \frac{n\pi}{b} \Rightarrow \text{So } X_n(x) = c_1 e^{\frac{n\pi x}{b}} + c_2 e^{\frac{-n\pi x}{b}}$$

where $u(0, y) = 0$

$$X(0) Y(y) = 0 \xrightarrow{Y \neq 0} X(0) = 0$$

$$\Rightarrow X(0) = 0 = C_1 + C_2 \Rightarrow C_1 = -C_2$$

$\frac{e^x - e^{-x}}{2} = \sinh(x)$

$$X_n(x) = C_1 e^{\frac{n\pi x}{b}} - C_1 e^{\frac{-n\pi x}{b}} = 2C_1\left(\frac{e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}}{2}\right)$$

$$\Rightarrow X_n(x) = \sinh\left(\frac{n\pi x}{b}\right) , \quad n = 1, 2, 3 \ldots$$

$$U_n(x,y) = X_n(x)\,Y_n(y) = \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right), n = 1, 2 \ldots$$

Superposition :-

$$U(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

* now using $U(a,y) = f(y)$

$$\Rightarrow U(a,y) = \sum_{n=1}^{\infty} \underbrace{B_n \sinh\left(\frac{na\pi}{b}\right)}_{C_n \text{ "function of } \underline{n} \text{ Just"}} \sin\left(\frac{n\pi y}{b}\right) = f(y)$$

Remember Fourier Series :-

$$f(y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{L}\right)$$

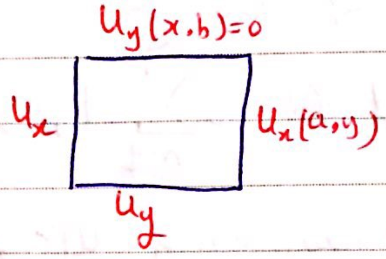
$$C_n = \frac{2}{L}\int_0^L f(y) \sin\left(\frac{n\pi y}{L}\right) dy$$

So $\quad B_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right) dy$

$B_n = \ldots\ldots\ldots$

Notes: ① This type of problems "the last one" when all the boundaries of u are defined called :-
"Dirichlet problem"

② If the Boundaries were the problem called :-
"Neumann problem"



③ Any change of one or more of the boundaries conditions will produce a different problem.

④ If the four conditions weren't homogenuous "not constants"
⇒ Here it look like 4 problems togather, so we have to solve them one by one.
such that $u(x,b) = h(x)$ & make the rest zeros and so on ...

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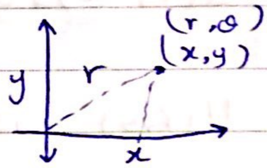
Laplace Equation :-

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$

In polar coordinates

$x = r\cos(\theta)$ $\qquad r = \sqrt{x^2+y^2}$

$y = r\sin(\theta)$ $\qquad \tan\frac{y}{x} = \theta$



$r = r(x,y)$, $r \geqslant 0$, $\theta = \theta(x,y)$

$\Rightarrow u_x = u_r r_x + u_\theta \theta_x \qquad \rightsquigarrow r_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{r\cos(\theta)}{r} = \frac{\cos(\theta)}{1}$

$u_y = u_r r_y + u_\theta \theta_y$

$u_{xx} = (u_x)_x = (u_r r_x + u_\theta \theta_x)_x = \ldots\ldots$

$u_{yy} = \ldots\ldots$

$\longrightarrow$ Followed

Ex: Solve the PDE (Dirichlet problem on a circle)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$, on the circular region $r<a$,

$u(a,\theta) = f(\theta)$, $0 \leq \theta \leq 2\pi$

u is periodic with period 2π "with respect to θ"

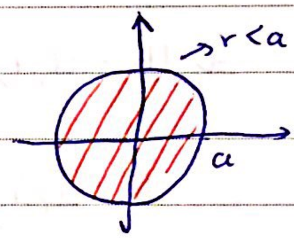
$$\begin{cases} u(r,-\pi) = u(r,\pi) , \quad u_\theta(r,-\pi) = u_\theta(r,\pi) , \quad r \leq a \end{cases}$$

$u(r,\theta)$ is bounded for $r \leq a$

$\downarrow$

$u(r,\theta) \leq M$ for some $M>0$

Sol: Separation of variables:-



$$u(r,\theta) = R(r)\Theta(\theta)$$ big Θ

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\Theta'' = 0$$

$$\left(R'' + \frac{1}{r}R'\right)\Theta = \frac{-1}{r^2}R\Theta''$$

$$\frac{\Theta''}{\Theta} = -\frac{r^2R'' + rR'}{R} = \lambda$$

$$\Theta'' - \lambda\Theta = 0 \quad \& \quad r^2R'' + rR' + \lambda R = 0$$

→ Followed

※ First we solve "$\Theta'' - \lambda \Theta = 0$"

$U(r,-\pi) = U(r,\pi)$, $R(r)\Theta(-\pi) = R(r)\Theta(\pi)$

$\xrightarrow{R \neq 0} \Theta(-\pi) = \Theta(\pi)$

$U_\theta(r,-\pi) = U_\theta(r,\pi)$

, $R(r)\Theta'(-\pi) = R(r)\Theta'(\pi) \xrightarrow{R \neq 0} \Theta'(\pi) = \Theta'(-\pi)$

$\boxed{1}$ $\lambda > 0$, $\lambda = \alpha^2 > 0$, $\alpha > 0$

$\Theta'' - \lambda\Theta = 0 \longrightarrow \Theta'' - \alpha^2\Theta = 0 \longrightarrow r^2 - \alpha^2 = 0 \longrightarrow r = \pm\alpha$

$\longrightarrow \Theta(\theta) = c_1 e^{-\alpha\theta} + c_2 e^{\alpha\theta}$

$\Theta(-\pi) = \Theta(\pi) \longrightarrow c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = c_1 e^{-\alpha\pi} + c_2 e^{\alpha\pi}$ .... (1)

$\Theta'(\theta) = -\alpha c_1 e^{-\alpha\theta} + \alpha c_2 e^{\alpha\theta}$

$\Theta'(-\pi) = \Theta'(\pi) \longrightarrow -\alpha c_1 e^{-\alpha\pi} + \alpha c_2 e^{\alpha\pi} = -\alpha c_1 e^{\alpha\pi} + \alpha c_2 e^{-\alpha\pi}$

$\xrightarrow{\div\alpha} -c_1 e^{-\alpha\pi} + c_2 e^{\alpha\pi} = -c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi}$ .. (2)

$\longrightarrow 2c_2 e^{-\alpha\pi} = 2c_2 e^{\alpha\pi} \longrightarrow 2c_2\left(e^{-\alpha\pi} - e^{\alpha\pi}\right) = 0$

But $e^{-\alpha\pi} \neq e^{\alpha\pi}$ $(\alpha > 0)$ $\longrightarrow e^{-\alpha\pi} - e^{\alpha\pi} \neq 0 \longrightarrow c_2 = 0$

$\xrightarrow{(1)} c_1 e^{\alpha\pi} = c_1 e^{-\alpha\pi} \longrightarrow c_1 = 0$

$\Rightarrow C_1 = C_2 = 0$ No Eigenvalues

2\ $\lambda = 0$

$\Theta'' - \lambda \Theta = 0 \longrightarrow \Theta'' = 0 \longrightarrow \Theta'(\theta) = C_1 \longrightarrow \Theta(\theta) = C_1\theta + C_2$

$\Theta(-\pi) = \Theta(\pi) \longrightarrow C_1\pi + \cancel{C_2} = -C_1\pi + \cancel{C_2}$

$\longrightarrow C_1\pi = -C_1\pi \longrightarrow 2C_1\pi = 0 \longrightarrow C_1 = 0$

$\Theta(\theta) = C_2$, But $\Theta'(-\pi) = \Theta(\pi)$

$\longrightarrow \Theta'(\theta) = 0 \xrightarrow{\Theta'(-\pi) = \Theta'(\pi)} 0 = 0$

Eigenvalues : $\lambda_0 = 0$

Eigen functions: $\Theta_0(\theta) = 1$

$r^2 R'' + rR' + \lambda R = 0 \xrightarrow{\lambda = 0} r^2 R'' + \underset{\sim}{r}R' = 0$

$a = 1$ & $b = 0$

$\longrightarrow m^2 - (1 - 1)m + 0 = 0 \longrightarrow m^2 = 0$, $m = 0, 0$

$R(r) = C_1 + C_2 \ln(r)$ "repeated root"

since $U(r,\theta)$ is bounded , as $r \to 0^+ \Rightarrow \ln(r) \to -\infty$ "unbounded"

so $R_0(r) = C_1$, take $C_1 = 1$

$\rightarrow$ followed

$\Rightarrow U_0(r,\theta) = R_0(r)\,\Theta_0(\theta) = (1)(1) = 1$

3] $\lambda < 0$, $\lambda = -\alpha^2 < 0$, $\alpha > 0$

$\Theta'' - \lambda\Theta = 0 \longrightarrow \Theta'' + \alpha\Theta = 0 \longrightarrow r^2 + \alpha^2 = 0 \longrightarrow r = \pm\alpha i$

$\Theta(\theta) = C_1\cos(\alpha\theta) + C_2\sin(\alpha\theta)$

$\Theta(-\pi) = \Theta(\pi) \Rightarrow \underbrace{C_1\cos(\alpha(-\pi))}_{\cos(-x)=\cos(x)} + \overbrace{C_2\sin(\alpha(-\pi))}^{\sin(-x)=-\sin(x)} = C_1\cos(\alpha\pi) + C_2\sin(\alpha\pi)$

$\Rightarrow \cancel{C_1\cos(\alpha\pi)} - C_2\sin(\alpha\pi) = \cancel{C_1\cos(\alpha\pi)} + C_2\sin(\alpha\pi)$

$\Rightarrow 2C_2\sin(\alpha\pi) = 0 \xrightarrow{C_2 \neq 0} \sin(\alpha\pi) = 0$

$\alpha\pi = n\pi$

$\alpha_n = n$, $n = 1, 2, \dots$

But $\Theta'(\theta) = -\alpha C_1\sin(\alpha\theta) + \alpha C_2\cos(\alpha\theta)$

$\Theta'(-\pi) = \Theta(\pi) \longrightarrow -\alpha C_1\sin(-\pi\alpha) + \cancel{\alpha C_2\cos(-\alpha\pi)} = -\alpha C_1\sin(\alpha\pi) + \cancel{\alpha C_2\cos(\alpha\pi)}$

$\longrightarrow 2\alpha C_1\sin(\alpha\pi) = 0 \xrightarrow{C_1 \neq 0} \sin(\alpha\pi) = 0$

$\alpha_n = n$, $n = 1, 2, \dots$

Eigenvalues: $\lambda = -\alpha_n^2 = -n^2$, $n = 1, 2, \dots$

Eigenfunctions: $\overset{*}{\Theta}(\theta) = \sin(\alpha_n\theta) = \sin(n\theta)$

: $\overset{**}{\Theta}_n(\theta) = \cos(\alpha_n\theta) = \cos(n\theta)$, $n = 1, 2, \dots$

$\longrightarrow$ followed

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$r^2 R_n'' + r R_n' - n^2 R = 0 \Rightarrow m(m-1) + m - n^2 = 0 \Rightarrow m = \pm n$

$R(r) = C_1 r^n + C_2 r^{-n}$, $r \to 0^+$ $r^{-n} \to +\infty \Rightarrow R_n(r) = r^n$, $n = 1, 2 \ldots$

* r^{-n} unbounded.

$U_n(r,\theta) = R_n(r)\,\Theta_n(\theta) \rightarrow U_n^*(r,\theta) = r^n \sin(n\theta)$

$U_n^{**}(r,\theta) = r^n \cos(n\theta)$

⇒ Superposition

$$U(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right]$$

But $U(a,\theta) = f(\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta)\right]$

* this is Fourier series with $\boxed{L = \pi}$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(\theta)\,d\theta \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

& $(a^n) A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$ & $B_n(a^n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$

$\frac{5}{591}$ take $U(2,\theta) = f(\theta) = 220$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Sol: $A_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 220\,d\theta = 110$

a in this example = 2

$$(a)\ A_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(220)\cos(n\theta)\,d\theta = \frac{220}{\pi n}\,\sin(n\theta)\Big|_{-\pi/2}^{\pi/2}$$

$$A_n 2^n = \frac{220}{\pi n}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-\pi n}{2}\right)\right] = \frac{220}{\pi n}\left[2\sin\left(\frac{n\pi}{2}\right)\right]$$

$$A_n 2^n = \frac{2(220)}{\pi n}\sin\left(\frac{n\pi}{2}\right) \Rightarrow A_n = \frac{(2)(220)}{\pi n\, 2^n}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{2(220)}{2^n \pi n}, \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n = 2, 4, 6, \ldots \\ 1, & n = 1, 5, 9 \ldots \\ -1, & n = 3, 7, 11 \ldots \end{cases}$$

$$A_{(2n-1)} = \frac{(220)(2)}{\pi n 2^n}\sin\left(\frac{(2n-1)\pi}{2}\right), \quad A(2n) = 0$$

6 / 591 take $u(2,\theta) = 400\cos^2(\theta) = f(\theta)$

sol: $a = 2$, $f(\theta) = 400\cos^2(\theta) = 200 + 200\cos(2\theta)$

$$f(\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n 2^n\cos(n\theta) + B_n 2^n\sin(n\theta)\right] = 200 + 200\cos(2\theta)$$

$\Rightarrow$ we can compare them to see that @ $n=2$

$$A_0 + A_1 2^1\cos(\theta) + B_1 2^1\sin(\theta) + A_2 2^2\cos(2\theta) + B_2 2^2\sin(2\theta) + \ldots$$

$A_0 = 200$ & $A_1 = 0 = B_1$ & $B_2 = 0$, $A_2(4) = 200$

$\rightarrow A_2 = 50$.

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No. 113

12.7 Heat Equation: Modleling very long Bars

EX: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = f(x)$ $\quad -\infty < x < \infty$ $\quad$ & $u(x,t)$ is bounded

Sol: seperation of variables

$u(x,t) = X(x)\,T(t) \Rightarrow XT' = c^2 X''T \Rightarrow \frac{T'}{c^2T} = \frac{X''}{X} = \lambda$

$X'' - \lambda X = 0$, $T' - c^2\lambda T = 0$

$\boxed{1}$ $\lambda > 0 \rightarrow \lambda = \alpha^2 > 0$, $\alpha > 0$

$X'' - \lambda X = 0 \rightarrow X'' - \alpha^2 X = 0 \rightarrow r^2 - \alpha^2 = 0 \rightarrow r = \pm\alpha$

$X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$

But $x \rightarrow -\infty \Rightarrow e^{-\alpha x} \rightarrow +\infty$ $\quad$ & $x \rightarrow +\infty \Rightarrow e^{\alpha x} \rightarrow \infty$

unbounded "no eigen values"

$\boxed{2}$ $\lambda = 0 \rightarrow X'' - \lambda X = 0 \rightarrow X'' = 0 \rightarrow X(x) = \underbrace{c_1 x}_{\times \text{(unbounded)}} + c_2$

$X(x) = c_2$, take $c_2 = 1 \rightarrow X(x) = 1$

followed

$\boxed{3}$ $\lambda < 0 \rightarrow \lambda = -\alpha^2 < 0 \quad , \alpha > 0$

$$X'' - \lambda X = 0 \rightarrow X'' + \alpha^2 X = 0 \rightarrow r^2 + \alpha^2 = 0 \rightarrow r = \pm \alpha i$$

$$X_\alpha(x) = \underset{A(\alpha)}{C_1} \cos(\alpha x) + \overset{B(\alpha)}{C_2} \sin(\alpha x)$$

all of them are bounded

$$\Rightarrow T' - c^2 \lambda T = 0$$

$$T' = c^2 \lambda T \rightarrow \int \frac{T'}{T} dt = c^2 \lambda \int dt \rightarrow \ln|T| = c^2 \lambda t + c_2$$

$$T(t) = \pm e^{c_2} e^{c^2 \lambda t} \rightarrow T_\alpha(t) = c_3 e^{c^2 \lambda t} \quad , \text{take } C_3 = 1$$

$\lambda = -\alpha^2$

$$U_\alpha(x,t) = X_\alpha(x) T_\alpha(t)$$

$$= [A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)] e^{-c^2 \alpha^2 t}$$

* Super position:-

$$U(x,t) = \int_0^\infty \left[(A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)) e^{-c^2 \alpha^2 t} \right] d\alpha \quad (1)$$

But $U(x,0) = f(x) = \int_0^\infty (A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)) \, d\alpha$

Fourier integral $\Rightarrow A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\alpha x) \, dx \quad (2)$

$$B(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\alpha x) \, dx \quad (3)$$

the condition

Q : if $u_t(x,0) = f(x)$?

Sol: from previous example

$$u(x,t) = \int_0^\infty \left[(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\, e^{-c^2\alpha^2 t} \right] d\alpha$$

$$u_t(x,t) = \int_0^\infty \left[(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)) - c^2\alpha^2 e^{-c^2\alpha^2 t} \right] d\alpha$$

$$u_t(x,t) = \int_0^\infty \left[(-c^2\alpha^2 A(\alpha)\cos(\alpha x) - c^2\alpha^2 B(\alpha)\sin(\alpha x))\, e^{-c^2\alpha^2 t} \right] d\alpha$$

$$u_t(x,0) = \int_0^\infty \left[-c^2\alpha^2 A(\alpha)\cos(\alpha x) - c^2\alpha^2 B(\alpha)\sin(\alpha x) \right] d\alpha$$

Fourier Integral : $-c^2\alpha^2 A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx$

$$-c^2\alpha^2 B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx$$

* we can show that

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^\infty e^{-c^2\alpha^2 t} \cos(\alpha x - \alpha v)\,d\alpha \right] dv$$

prove :- substituting (2) & (3) into (1)

$$u(x,t) = \int_0^\infty \left\{ \left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\alpha v)\,dv \right]\cos(\alpha x) + \left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\alpha v)\,dv \right]\sin(\alpha x) \right\} e^{-c^2\alpha^2 t}\, d\alpha$$

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$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left\{ \left(\int_{-\infty}^{\infty} f(v)\left[\cos(\alpha v)\cos(\alpha x) + \sin(\alpha v)\sin(\alpha x)\right] dv \right) e^{-c^2\alpha^2 t} \right\} d\alpha$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \left\{ \int_0^\infty \underbrace{\left[\cos(\alpha v)\cos(\alpha x) + \sin(\alpha v)\sin(\alpha x)\right]}_{\cos(\alpha v - \alpha x)} e^{-c^2\alpha^2 t} d\alpha \right\} dv$$

$\cos(\alpha v - \alpha x)$

$\cos(\alpha x - \alpha v) \rightarrow$ even function

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \left[\int_0^\infty e^{-c^2\alpha^2 t} \cos(\alpha x - \alpha v)\, d\alpha \right] dv \rightarrow \text{proved !}$$

Another Method : Fourier Transform

Ex: $u_t = c^2 u_{xx} \quad -\infty < x < \infty \quad , t > 0$

$u(x,0) = f(x) \quad -\infty < x < \infty$

Sol: $\xrightarrow[\text{with respect to } x \text{ not } t]{\mathcal{F}} \mathcal{F}\{u_t\} = \mathcal{F}\{c^2 u_{xx}\}$ $\xrightarrow{\text{Remember}}$ $\mathcal{F}\{f''(x)\} = -\omega^2 \mathcal{F}\{f(x)\}$

$\mathcal{F}\{u_{xx}(x,t)\} = -\omega^2 \mathcal{F}\{u(x,t)\}$

$= -\omega^2 U(\omega,t)$

$$\mathcal{F}\{u_t(x,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_t(x,t) e^{-i\omega x} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(u(x,t)) e^{-i\omega x} dx$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx \right]$$

$$= \frac{\partial}{\partial t}\mathcal{F}\{u(x,t)\} = \frac{\partial}{\partial t} U(\omega,t)$$

$\rightarrow$ Followed

$\xrightarrow[\text{with respect to } x]{\mathcal{F}}$ $\mathcal{F}[u_t] = \mathcal{F}[c^2 u_{xx}]$

$$U_t(\omega,t) = c^2(-\omega^2 U(\omega,t))$$

$$\frac{dU}{dt} = c^2(-\omega^2 U(\omega,t)) \longrightarrow \text{seprable}$$

$$\frac{dU}{U} = -c^2\omega^2\,dt \longrightarrow \ln|U| = -c^2\omega^2 t$$

PDE | ODE

$$\frac{dy}{dt} = -c^2\omega^2 y$$

$$\frac{dy}{y} = -c^2\omega^2\,dt$$

$$\ln|y| = -c^2\omega^2 t + c_1$$

$$|y| = e^{-c^2\omega^2 t + c_1} = e^{c_1} e^{-c^2\omega^2 t}$$

$$y = \pm e^{c_1} e^{-c^2\omega^2 t} = c_2 e^{-c^2\omega^2 t}$$

$$\boxed{U(\omega,t) = H(\omega) e^{-c^2\omega^2 t}}$$

But $U(x,0) = f(x) \longrightarrow \mathcal{F}[U(x,0)] = \mathcal{F}[f(x)]$

$\longrightarrow U(\omega,0) = \hat{f}(\omega)$ where $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx$

$$U(\omega,0) = \hat{f}(\omega) = H(\omega) e^{-c^2\omega^2(0)} = H(\omega)$$

$\xrightarrow{\text{so}}$ $\boxed{U(\omega,t) = \hat{f}(\omega) e^{-c^2\omega^2 t}}$

$$\xrightarrow{\mathcal{F}^{-1}} \quad u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-c^2\omega^2 t}\, e^{i\omega x}\, d\omega$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \right) e^{-c^2\omega^2 t}\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \quad \cdots\cdots$$

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Semi-infinite.

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Ex: $u_t = c^2 u_{xx} \qquad 0 < x < \infty \quad , \quad t > 0$

$u(x,0) = f(x) \qquad 0 < x < \infty$

$u(0,t) = 0$

Sol: Using Fourier Sine Transform with respect to x

Remember ⟹
$$\begin{cases} \mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\,\omega f(0) \quad \checkmark \text{ from condition} \\ \mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0) \end{cases}$$

$$\mathcal{F}_s\left\{\frac{\partial u}{\partial t}\right\} = \mathcal{F}_s\left\{c^2 \frac{\partial^2 u}{\partial x^2}\right\}$$

$$\frac{\partial}{\partial t}\hat{u}_s(\omega,t) = c^2\left[-\omega^2 \hat{u}_s(\omega,t) + \sqrt{\frac{2}{\pi}}\,\omega\, \underbrace{u(0,t)}_{\text{Zero}}\right]$$

$$\frac{\partial}{\partial t}\hat{u}_s(\omega,t) = -c^2\omega^2 \hat{u}_s(\omega,t)$$

pdE	OdE
	$\frac{dy}{dt} = -c^2\omega^2 y \longrightarrow \int \frac{dy}{y} = -c^2\omega^2 \int dt$
$\hat{u}_s(\omega,t) = k(\omega)e^{-c^2\omega^2 t}$ ⟵	$y(t) = k\,e^{-c^2\omega^2 t}$

follow

But $u(x,0) = f(x)$

$$\mathcal{F}_s\{u(x,0)\} = \mathcal{F}_s\{f(x)\}$$

$$\hat{u}_s(\omega,0) = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(\omega x)\,dx$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p)\sin(\omega p)\,dp$$ replace each x by p

$$\hat{u}_s(\omega,0) = K(\omega)\, e^{-c^2\omega^2(0)} = K(\omega) = \hat{f}_s(\omega)$$

$$\Rightarrow \hat{u}_s(\omega,t) = \hat{f}_s(\omega,t)\, e^{-c^2\omega^2 t}$$

$$\xrightarrow{\mathcal{F}_s^{-1}} u(x,t) = \mathcal{F}_s^{-1}\left[\hat{f}_s(\omega,t)\, e^{-c^2\omega^2 t}\right]$$

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega)\, e^{-c^2\omega^2 t} \sin(\omega x)\, d\omega$$

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \int_0^\infty f(p)\sin(\omega p)\,dp\right) e^{-c^2\omega^2 t} \sin(\omega x)\,d\omega$$

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p)\sin(\omega p)\, e^{-c^2\omega^2 t} \sin(\omega x)\,dp\,d\omega$$

* Note: We can solve this problem again using separation of variables. Try it!

Ex: $u_t = c^2 u_{xx} \quad 0 < x < \infty \quad , \quad t > 0$

$u(x,0) = f(x) \quad , \quad 0 \leq x < \infty$

$u_x(0,t) = 0 \quad , \quad t \geq 0$

sol: Here we use Fourier cosine Transform

then continue .....

انتهى ...

والحمد لله الذي بنعمته تتم الصالحات

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