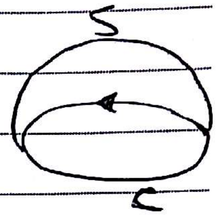
13/7

No.

Stoke's theorem:



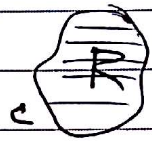
$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds = \oint_{C = \partial S} \vec{F} \cdot d\vec{r}$$

boundary of S

prove:

Green's theorem: $\vec{F} = f_1(x,y)\,\hat{i} + f_2(x,y)\,\hat{j}$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F}\ d$$



$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f_1 & f_2 & 0 \end{vmatrix} =$$

$$\oint_C (f_1\,dx + f_2\,dy) = \iint_R \left(\frac{df_2}{dx} - \frac{df_1}{dy} \right) dx\,dy$$

ex: $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ $\quad$ S: $z = f(x,y) = 1 - (x^2 + y^2)$

Verify stoke's theorem:

Sol: $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds = \oint_C \vec{F} \cdot d\vec{r}$

⇒

No.

L.H.S

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cancel{y} & \cancel{z} & x \end{vmatrix} = \cancel{\hat{i}(0-1)} \cancel{+ 0\hat{j}}$$

$$\hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1)$$

$$\nabla \times \vec{F} = -\hat{i} - \hat{j} - \hat{k}$$

$$\cancel{\hat{n} = \frac{\nabla f}{\|\nabla f\|}} \quad \cancel{\hat{n} = \frac{\nabla f(x,y)}{\|\nabla f(x,y)\|}}$$

$$\hat{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{2x\hat{i} + 2y\hat{j} + \hat{k}}{\sqrt{1 + 4x^2 + 4y^2}} \qquad g = z + x^2 + y^2 = 1$$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds = -\iint \left(\frac{2x}{\cancel{\sqrt{\ }}} + \frac{2y}{\cancel{\sqrt{\ }}} + \frac{1}{\cancel{\sqrt{\ }}} \right) \frac{dx\,dy}{\cancel{\frac{1}{\sqrt{\ }}}}$$

$$= -\iint (2x + 2y + 1)\, dx\,dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta + 2r\sin\theta + 1)\, r\,dr\,d\theta$$

No.

R.H.S

$$\oint_{C=\partial S} \vec{F}\cdot d\vec{r} = \oint y\,dx + \cancel{z\,dy} + \cancel{x\,dz}$$

on C → $z = 0 \rightarrow dz = 0$

XY plane [illegible]

$x = \cos\theta \rightarrow dx = -\sin\theta\, d\theta$

$y = \sin\theta$

$\theta \in [0, 2\pi]$

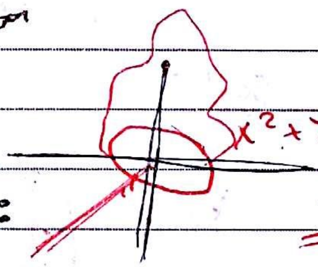
$$= \int_0^{2\pi} -\sin^2\theta \, d\theta$$

R.H.S & L.H.S are equal

ex: use Stoke's theorem

$$\vec{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$$

ds:



$x^2 + y^2 = 4$

$z = -3$

use Stoke's theorem to evaluate $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$

Sol $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \oint_C \vec{F}\cdot d\vec{r}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint y\,dx + x\overset{(-3)^3}{z^3}\,dy - \cancel{zy^3\,dz}$$

on C → $z = -3 \rightarrow dz = 0$

$$= \oint_C y\,dx - 27x\,dy$$

: $x^2 + y^2 = 4$

$x = 2\cos\theta$

$y = 2\sin\theta$

$\theta \in [0, 2\pi]$

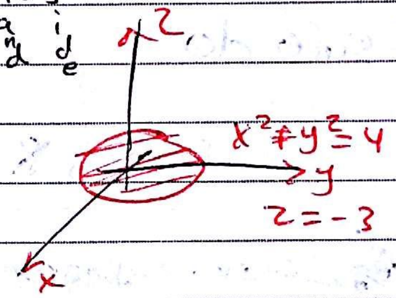
$dy = 2\cos\theta\, d\theta$ $dx = -2\sin\theta\, d\theta$

No.

$$= \int_C y\,dx - 27x\,dy$$

$$= \int_0^{2\pi} \left[-4\sin^2\theta - 27(4\cos^2\theta)\right] d\theta$$

now Solve previous example by $L_{eft}.H_{and}.S_{ide}$



Sol:

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} = \hat{i}\left(-3zy^2 - 3xz^2\right) - \hat{j}(0-0) + \hat{k}(z^3-1)$$

$$\boxed{\nabla\times\vec{F} = \hat{i}(-3zy^2 - 3xz^2) + \hat{k}(z^3-1)}$$

$\boxed{n = \hat{k}}$

$= \iint$

No.

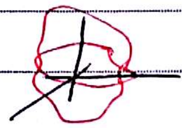
$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint (z^3-1)\,dx\,dy \quad : \; z=-3$$

$$= \iint -28\,dx\,dy = -28\,(4\pi)$$

مساحة الدائرة

* Divergence Theorem :

سطح مغلق

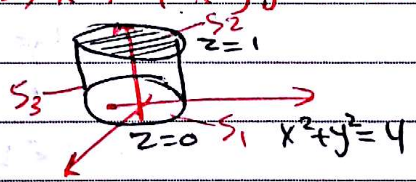


$$\cancel{\oiint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\times\vec{F}}$$

$$\boxed{\oiint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$$

ex: $\vec{F} = x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}$

S:



Verify Divergence theorem : ??

Sol: ① L.H.S

$$\nabla\cdot\vec{F} = 3x^2 + x^2 + x^2 = 5x^2$$

$$\therefore \iiint_V 5x^2\,dv = \iiint_G 5x^2\,dx\,dy\,dz$$

No.

$$\iiint_0 5x^2\,dx\,dy\,dz$$

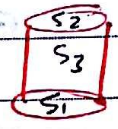
$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^1\int_0^{2\pi}\int_0^2 5r^3\cos^2\theta\,dr\,d\theta\,dz$$

② R.H.S



$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds + \iint_{S_3} \vec{F}\cdot\hat{n}\,ds$$

S_1	S_2	S_3
$z=0$	$z=1$	$z \in [0,1]$
$ds = dx\,dy$	$ds = dx\,dy$	$x = 2\cos\theta$
$\hat{n} = -\hat{k}$	$\hat{n} = \hat{k}$	$y = 2\sin\theta$
		$dx = -2\sin\theta\,d\theta$
		$dy = 2\cos\theta\,d\theta$

$$= -\cancel{\iint x^2 z\,dx\,dy} + \iint x^2\,dx\,dy +$$

$z=0$

~~zero~~

$dx\,dy = r\,dr\,d\theta$

$x = r\cos\theta$

$r \in [0,2]$

$\theta \in [0,2\pi]$

$g = x^2+y^2-4$

$\nabla g = 2x\hat{i} + 2y\hat{j}$

$\dfrac{\nabla g}{\|\nabla g\|} = \hat{n}$

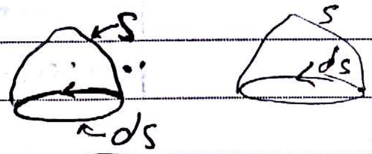
$\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = x\hat{i}+y\hat{j}+0\hat{k}$

$$= 0 + \int_0^{2\pi}\int_0^2 (r\cos\theta)^2\,r\,dr\,d\theta$$

17/7

No.

Stoke's theorem :



$$\boxed{\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds = \oint_{ds} \vec{F} \cdot \vec{dr}}$$

Divergence theorem :

$$\boxed{\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv}$$

Ex $\vec{F} = 7x\hat{i} + 0\hat{j} + -2\hat{k}$

$S: x^2+y^2+z^2=4$ verify ~~stokes~~ Divergence theorem ??

Sol

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

L.H.S

$\iint_S \vec{F} \cdot \hat{n} ds$

parametric equations

$x = 2\cos\theta \sin\phi$

$y = 2\sin\theta \sin\phi \qquad \theta \in [0, 2\pi]$

$z = 2\cos\phi \qquad \phi \in [0, \pi]$

$\vec{r}(\theta, \phi) = 2\cos\theta \sin\phi\, \hat{i} + 2\sin\theta \sin\phi\, \hat{j} + 2\cos\phi\, \hat{k}$

$\hat{n} = \dfrac{\vec{r}_\theta \times \vec{r}_\phi}{|r_\theta \times r_\phi|}$: $\vec{r}_\theta = -2\sin\theta \sin\phi\, \hat{i} + 2\cos\theta \sin\phi\, \hat{j} + 0\hat{k}$

$\vec{r}_\phi = 2\cos\theta \cos\phi\, \hat{i} + 2\sin\theta \cos\phi\, \hat{j} - 2\sin\phi\, \hat{k}$

$\hat{n} = \vec{N} \cdot d\theta\, d\phi \qquad \vec{N} = \vec{r}_\theta \times r_\phi$

$\iint \vec{F}(\vec{r}(\theta, \phi)) \cdot \vec{N}\, d\theta\, d\phi$

No.

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix}$$

$$= \hat{i}(-4\cos\theta\sin^2\phi) - \hat{j}(4\sin\theta\sin^2\phi) + \cancel{\hat{k}(-4\sin^2\theta\sin^2\phi)}$$

$$+\hat{k}(4\sin^2\theta\sin\phi\cos\phi + 4\cos^2\theta\sin\phi\cos\phi)$$

$$= -4\cos\theta\sin^2\phi\,\hat{i} - 4\sin\theta\sin^2\phi\,\hat{j} + 4\sin\phi\cos\phi\,\hat{k}$$

$$\cancel{|\vec{r}_\theta \times \vec{r}_\phi| = 4\sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi}}$$

$$\cancel{= 4\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = 4\sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)}}$$

$$\cancel{= 4\sin\phi}$$

$$\cancel{\int_0^\pi\int_0^{2\pi} 4}$$

$$\vec{F}(\vec{r}(\theta,\phi)) = 14\cos\theta\sin\phi\,\hat{i} - 2\cos\phi\,\hat{k}$$

$$= \int_0^\pi\int_0^{2\pi} 4\left[14r^2\sin^3\phi + 2\sin\phi\cos^2\phi\right] d\theta\, d\phi = 64\pi$$

R.H.S

$$\iiint_V \nabla\cdot\vec{F}\,dv = 6\iiint_{\text{sphere}} dv$$

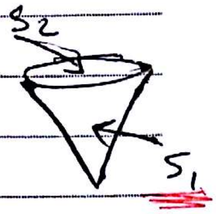
$$= 6 * \frac{4}{3}\pi(2)^3 = 64\pi$$

$$\boxed{\text{Volume of sphere} = \frac{4}{3}\pi r^3}$$

Scanned by CamScanner

No.

Ex: Verify the div. theorem ??



$\vec{F} = 7x\hat{i} - z\hat{k}$ Cone

$S: \sqrt{x^2+y^2} = z \quad , \quad 0 \leqslant z \leqslant 1$

& $x > 0$
$y > 0$

ع متكامله على السطح المغلق S_1

Parametric equations

Sol: L.H.S

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, ds$$

on S_1

$\vec{r} = \frac{\rho}{\sqrt{2}} \left(\cos\theta \hat{i} + \sin\theta \hat{j} + 1\hat{k} \right)$

$\vec{r}_\rho = \frac{1}{\sqrt{2}} \left(\cos\theta \hat{i} + \sin\theta \hat{j} + 1\hat{k} \right)$

$\vec{r}_\theta = \frac{\rho}{\sqrt{2}} \left(-\sin\theta \hat{i} + \cos\theta \hat{j} + 0\hat{k} \right)$

$$\begin{cases} x = \dfrac{\rho\cos\theta}{\sqrt{2}} \\ y = \dfrac{\rho\sin\theta}{\sqrt{2}} \\ z = \dfrac{\rho}{\sqrt{2}} \end{cases}$$

$\rho \in [0, \sqrt{2}]$

$\theta \in [0, \pi/4]$

$$\vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\cos\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\rho\sin\theta}{\sqrt{2}} & \frac{\rho\cos\theta}{\sqrt{2}} & 0 \end{vmatrix}$$

$$= \frac{\rho}{2} \left(\cos\theta \hat{i} - \sin\theta \hat{j} + \hat{k} \right)$$

$$\vec{F}\left(\vec{r}(\rho,\theta)\right) = \frac{7}{\sqrt{2}} \rho\cos\theta \hat{i} + 0\hat{j} - \frac{\rho}{\sqrt{2}} \hat{k}$$

$$\boxed{\iint_{S_1} \vec{F} \cdot \hat{n}\, ds = \int_0^{2\pi}\int_0^{\sqrt{2}} \frac{\rho}{2} \left(\frac{7}{\sqrt{2}} \rho\cos^2\theta - \frac{\rho}{\sqrt{2}} \right) d\rho\, d\theta} = 3\pi$$

No.

On S_2

$S = x^2+y^2=1 \rightarrow z\ [0,1]$

$\sqrt{x^2+y^2} = z$

$\hat{n} = \hat{k} \qquad \vec{F}\cdot\hat{n} = -z = -1 \qquad ds = dx\,dy$

$-\iint_{S_2} dx\,dy = -\pi \qquad \iint_{S=S_1+S_2} \vec{F}\cdot\hat{n}\,ds = \int_{S_1} \vec{F}\cdot\hat{n}\,ds - \pi$

~~R.H.~~ R.H.S

$\iiint_V \nabla\cdot\vec{F}\,dv = 6 * \text{Volume of the cone}$

$6 * \frac{1}{3}\pi * 1 =$

$\frac{1}{3} S \times h = 2\pi$

Surface Area height

No.

*special cases for Divergence theorem:

(Applications):-

1) $\vec{F} = f\nabla g$

$$\oiint_S f\nabla g \cdot \underbrace{\hat{n}\, ds}_{\text{normal derivative}} = \iiint_V f\nabla \cdot \nabla g \, dv$$

~~$= \iiint_V \nabla^2 g\, dv$~~

Green's first Identity

$$\boxed{\oiint_S f\frac{\partial g}{\partial n}\, ds = \iiint_V f\nabla^2 g\, dv}$$

2) $\vec{F} = f\nabla g$ $\qquad \nabla fg = f\nabla g + g\nabla f$

$$\oiint_S f\nabla g \cdot \hat{n}\, ds = \iiint_V \nabla \cdot f\nabla g \, dv$$

$$\boxed{\oiint_S f\frac{\partial g}{\partial n}\, ds = \iiint_V (f\nabla^2 g + \nabla f \cdot \nabla g)\, dv}$$

$\Rightarrow \vec{F} = g\nabla f$

$$\boxed{\oiint_S g\frac{\partial f}{\partial n}\, ds = \iiint_V (g\nabla^2 f + \nabla g \cdot \nabla f)\, dv}$$

Green's second Identity

$$\boxed{\oiint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) ds = \iiint_V (f\nabla^2 g - g\nabla^2 f)\, dv}$$

18/7

No.

Revision:

Green's 1st Identity:

$$\boxed{\iint_S f\frac{\partial g}{\partial n}\,ds = \iiint_V f\nabla^2 g\,dv}$$

Green's 2nd Identity:

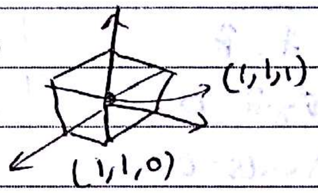
$$\boxed{\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)ds = \iiint_V \left(f\nabla^2 g - g\nabla^2 f\right)dv}$$

Ex: $f = 4y^2$

$g = x^2$

S: unit cube

Verify Green's 1st Identity



sol:

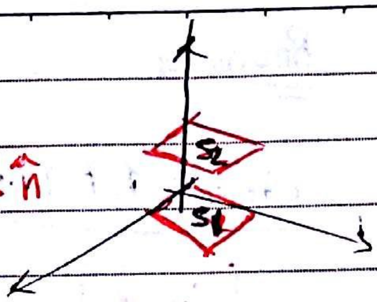
R.H.S

$$\iint_S f\frac{\partial g}{\partial n}\,ds = \iiint_V f\nabla^2 g\,dv$$

$\nabla^2 g = 2 + 0 + 0 = 2$ $\qquad f = 4y^2$

$$\int_0^1\int_0^1\int_0^1 8y^2\,dx\,dy\,dz = \frac{8}{3}$$

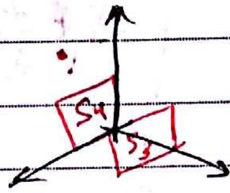
No.



L.H.S $\iint_S f \frac{\partial g}{\partial n}\, ds$

~~$\frac{dg}{dn} = \nabla g \cdot \hat{n}$~~ $\frac{\partial g}{\partial n} = \nabla g \cdot \hat{n}$

$\nabla g = 2x\hat{i} + 0\hat{j} + 0\hat{k}$



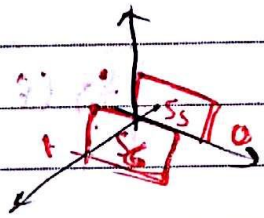
~~On S_1 & S_2~~

On S_1 ~~& S_2~~

$\hat{n} = -\hat{k}$

$\nabla g \cdot \hat{n} = 0$

$\iint_{S_1} 0\, ds = \text{zero}$



On S_2

$\hat{n} = \hat{k}$

$\nabla g \cdot \hat{n} = 0$

$\iint_{S_2} 0\, ds = 0$

On S_3

$\hat{n} = \hat{j}$

$\nabla g \cdot \hat{n} = 0$

$\iint_{S_3} 0\, ds = 0$

On S_4

$\hat{n} = -\hat{j}$

$\nabla g \cdot \hat{n} = 0$

$\iint_{S_4} 0\, ds = 0$

On S_5

$\hat{n} = -\hat{i}$

$\nabla g \cdot \hat{n} = -2x$

~~$\iint_{S_5} -2x\, dy\, dz$~~

$\int_0^1\int_0^1 4y^2(-2x)\, dy\, dz$

$\boxed{x = \text{zero}}$

$\iint_{S_5} \nabla g \cdot \hat{n}\, ds = 0$

On S_6

$\hat{n} = \hat{i}$

$\nabla g \cdot \hat{n} = 2x$

$\int_0^1\int_0^1 4y^2(+2x)\, dy\, dz$

$\boxed{x = 1}$

$= \int_0^1\int_0^1 8y^2\, dy\, dz = \frac{8}{3}$

$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} = 0 + 0 + 0 + 0 + 0 + \frac{8}{3} = \frac{8}{3}$

No.

Now Verify 2nd Identity :-

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) ds = \iiint_V \left(f\nabla^2 g - g\nabla^2 f\right) dv$$

$g = x^2 \qquad f = 4y^2$

R.H.S

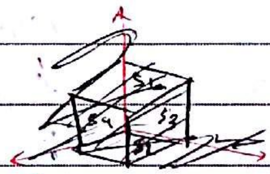
$\nabla^2 g = 2$

$\nabla^2 f = 8$

$\cancel{\iiint 2(4y^2)}$

$$\iiint_V \left(2(4y^2) - 8(x^2)\right) dv = \int_0^1\int_0^1\int_0^1 (8y^2 - 8x^2)\, dx\,dy\,dz$$

$= \frac{8}{3} - \frac{8}{3} = \text{zero}$

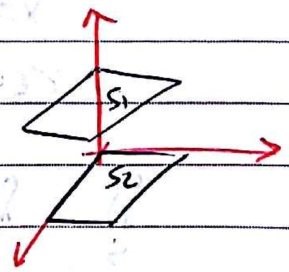


L.H.S

$f\frac{\partial g}{\partial n} = f\nabla g\cdot\hat{n} \qquad g\frac{\partial f}{\partial n} = g\nabla f\cdot\hat{n}$

$\nabla g = 2x\hat{i} \qquad \nabla f = 8y\hat{j}$

On S_1	On S_2
$\hat{n} = \hat{k}$	$\hat{n} = -\hat{k}$
$\nabla g\cdot\hat{n} = 0$	$\nabla g\cdot\hat{n} = 0$
$\nabla f\cdot\hat{n} = 0$	$\nabla f\cdot\hat{n} = 0$
$\cancel{\iint_{S_1} ds = \text{zero}}$	$\cancel{\iint_{S_2} ds = \text{zero}}$



$\iint_{S_1}(f\nabla g\cdot\hat{n} - g\nabla f\cdot\hat{n})\,ds = \text{zero} \qquad \iint_{S_2}(f\nabla g\cdot\hat{n} - g\nabla f\cdot\hat{n})\,ds = \text{zero}$

Scanned by CamScanner

No.

On S_3

$\hat{n} = -\hat{j}$

$\nabla g \cdot \hat{n} = 0$

$\nabla f \cdot \hat{n} =$

$= -8y$

$$\iint_{S_3} (f \nabla g \cdot \hat{n} - g \nabla f \cdot \hat{n})\, ds = \cancel{\int_0^1\int_0^1 8x^2 y \, dx dz} \xrightarrow{} \text{zero}$$

$y = 0$

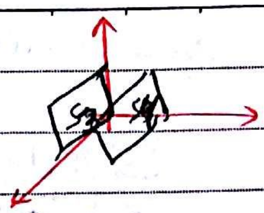
On S_4

$\hat{n} = \hat{j}$

$\nabla g \cdot \hat{n} = 0$

$\nabla f \cdot \hat{n} =$

$= 8y$



$$\iint_{S_4} (\underbrace{\cancel{f \nabla g \cdot \hat{n}}}_{\text{zero}} - g \nabla f \cdot \hat{n})\, ds$$

$$= \int_0^1\int_0^1 -8x^2 \, dx dz \qquad \boxed{y=1}$$

$$= -\frac{8}{3}$$

On S_5

~~From pre~~

$\hat{n} = -\hat{i}$

$\nabla g \cdot \hat{n} = -2x$

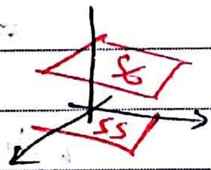
$\nabla f \cdot \hat{n} = 0$

$$\cancel{\int_0^1\int_0^1 -8y^2 x \, dy dz} \xrightarrow{} \text{zero}$$

$x = 0$

$= \text{zero}$

On S_6



$\hat{n} = \hat{i}$

$\nabla g \cdot \hat{n} = 2x$

$\nabla f \cdot \hat{n} = 0$

$$\int_0^1\int_0^1 8xy^2 \, dy dz$$

$x = 1$

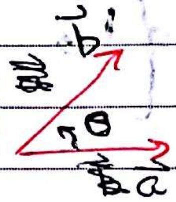
$$= \frac{8}{3}$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} = \text{zero}$$

*

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$



$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$= |\vec{a}|\,|\vec{b}|\cos\theta$

$\vec{a} \perp \vec{b} \longleftrightarrow \vec{a}\cdot\vec{b} =$ zero

diff ⇒ $\vec{a}$ & $\vec{b}$ are linearly independent if

$$c_1\vec{a} + c_2\vec{b} = 0 \Rightarrow c_1 = c_2 = 0$$

* Inner product

$\langle \vec{a}, \vec{b} \rangle$ $\qquad f, g : [a,b] \to \mathbb{C}$

$r : [a,b] \to \mathbb{R}^+$

$$\boxed{\langle f, g \rangle_r = \int_a^b f(x)\,\overline{g(x)}\,r(x)\,dx}$$ weight

$$\boxed{\langle f, g+h \rangle_r = \int_a^b \left[f(x)\,\overline{g(x)}\,r(x) + f(x)\,\overline{h(x)}\,r(x) \right] dx}$$

$\langle f, f \rangle_r = \|f\|_r^2$

$\langle f, g \rangle = \overline{\langle g, f \rangle}$

$\boxed{\overline{} : \text{complex conjugate}}$

No.

$f \perp g \leftrightarrow \langle f, g \rangle_r = \text{zero}$

~~on [-π, π]~~

on $[-L, L]$

$$\left\{ 1, \sin\frac{2n\pi}{L}, \cos\frac{2n\pi}{L} \right\}_{n=1} \qquad r(x) = 1$$

orthogonal ⇐ لما ال functions ال Inner product الها صفر

ex: $\left\langle 1, \sin\frac{2n\pi}{L}x \right\rangle$

$\left\langle 1, \cos\frac{2n\pi}{L}x \right\rangle$

$n \neq m \quad \left\langle \sin\frac{2n\pi}{L}x, \sin\frac{2m\pi}{L}x \right\rangle$

$n \neq m \quad \left\langle \cos\frac{2n\pi}{L}x, \cos\frac{2m\pi}{L}x \right\rangle$

$\left\langle \sin\frac{2n\pi}{L}x, \cos\frac{2m\pi}{L}x \right\rangle$

$$= \int_{-L}^{L} \sin\left(\frac{2n\pi}{L}x\right) dx = \frac{-L}{2n\pi} \cos\frac{2n\pi}{L}x \Big|_{-L}^{L}$$

$$= \frac{-L}{2n\pi}\left(\cos 2n\pi - \cos -2n\pi\right) \rightarrow \text{zero}$$

$= \text{zero}$

No.

Revision بعض المتطابقات

$\cos((a+b)x) = \cos ax \cos bx - \sin ax \sin bx$

$\cos((a-b)x) = \cos ax \cos bx + \sin ax \sin bx$

$\sin((a+b)x) = \sin ax \cos bx + \cos ax \sin bx$

$\sin((a-b)x) = \sin ax \cos bx - \cos ax \sin bx$

أسهل فترة للتعامل معها $[-\pi, \pi]$

19/17

Fourier Series

$\left\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$ is an

Orthogonal set of functions on $[-L, L]$

$$\int_{-L}^{L} f_n(x) f_m(x)\,dx = \langle f_n, f_m \rangle = 0$$

$$\|1\|^2 = \langle 1, 1 \rangle = \int_{-L}^{L} dx = 2L$$

$$\left\|\cos\frac{n\pi x}{L}\right\|^2 = \left\langle \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L} \right\rangle = \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \int_{-L}^{L} \frac{1}{2}\left(1 + \cos\frac{2n\pi x}{L}\right) dx = L + \underbrace{\frac{L}{n\pi}\sin\frac{2n\pi x}{L}\Bigg|_{-L}^{L}}_{\text{zero}}$$

$$= L$$

$$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$

$$f(x) \sim a_0/2 + a_1\cos\frac{\pi x}{L} + a_2\cos\frac{2\pi x}{L} + a_3\cos\frac{3\pi x}{L} + \cdots$$
$$+ b_1\sin\frac{\pi x}{L} + b_2\sin\frac{2\pi x}{L} + b_3\sin\frac{3\pi x}{L} + \cdots$$

$$= \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}}_{\text{Basic Fourier Series}}$$

No.

$\{f_n\}_{n=0}^{\infty}$ Orthogonal on $[a,b]$

$f : [a,b] \to \mathbb{R}$ $\qquad f(x) \sim \sum_{n=0}^{\infty} C_n f_n$

~~f(x)~~

$\cancel{f(x) = \sum C_n f_n}$

$$\langle f, f_n \rangle = \left\langle \sum_{n=0}^{\infty} C_n f_n , f_m \right\rangle$$

$$= \sum_{n=0}^{\infty} C_n \langle f_n , f_m \rangle = C_m \langle f_m , f_m \rangle$$

$$C_m = \frac{\langle f, f_m \rangle}{\langle f_m, f_m \rangle}$$

$\hookrightarrow$ Fourier constant

$$\boxed{\frac{a_0}{2} = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \cancel{\frac{\int f(x)dx}{2L}} \; \frac{1}{2L} * \int_{-L}^{L} f(x)dx}$$

$$\boxed{a_n = \frac{\left\langle f, \cos\left(\frac{n\pi x}{L}\right) \right\rangle}{\left\langle \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L} \right\rangle} = \frac{1}{L} * \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} \, dx}$$

$$\boxed{b_n = \frac{\left\langle f, \sin\frac{n\pi x}{L} \right\rangle}{\left\langle \sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L} \right\rangle} = \frac{1}{L} * \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L} \, dx}$$

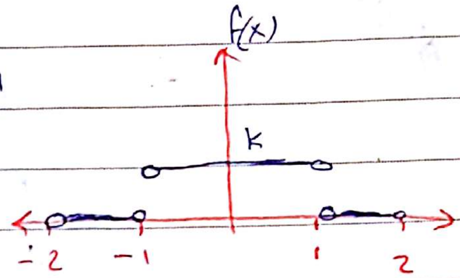
$$\frac{a_0}{\cancel{2}} = \frac{1}{\cancel{2}L} \int_{-L}^{L} f(x)dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L} \, dx$$

No.

Ex: Find the Basic Fourier Series for :

$$f(x)=\begin{cases}0 & , -2<x<-1\\ k & , -1<x<1\\ 0 & , 1<x<2\end{cases}$$



$x \in [-2,2] \leftarrow L=2$

Sol:

$$f(x)\sim \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)+\sum_{n=1}^{\infty} b_n \sin(n\pi x/2)$$

$$a_n=\frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = 2\left(\frac{1}{2}\int_{0}^{2} f(x)\cos\frac{n\pi x}{2}\,dx\right)$$

zero

$$=\int_{0}^{1} k\cos\frac{n\pi x}{2}\,dx+\int_{1}^{2} 0\cos = \frac{2}{n\pi}k\left(\sin\frac{n\pi}{2}\right)$$

$$b_n=\frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \text{zero} \qquad \sin\rightarrow \text{odd}$$

*

even * even = even
odd * even = odd $\Rightarrow$
odd * odd = even

$$\int_{-L}^{L}\text{even} = \text{Value}$$

$$\int_{-L}^{L}\text{odd} = \text{zero}$$

$$\cancel{a_n=\frac{2}{n\pi}k\left(\sin\frac{n\pi}{2}\right)=\frac{2k}{n\pi}\begin{cases} & n=\text{odd}\\ \text{zero} & n=2m\end{cases}}$$

$$a_n=\frac{2}{n\pi}k\left(\sin\left(\frac{n\pi}{2}\right)\right)=\frac{2k}{n\pi}\begin{cases}(-1)^m & , n=2m+1\\ 0 & , n=2m\end{cases}$$

$$a_0=\frac{1}{2}\int_{-2}^{2} f(x)\,dx=\frac{1}{2}\int_{-1}^{1} k\,dx=\frac{2k}{2}=k$$

$$\boxed{f(x)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^m}{2m+1}\cos\left(\frac{\pi x}{2}(2m+1)\right)}$$

أو $\cos\left(\left(\frac{\pi x}{2}\right)(2m+1)\right)$

Scanned by CamScanner

No.

Ex: $f(x) = x+\pi \; ; \; -\pi < x < \pi$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx = \cancel{\frac{1}{\pi}\left[\frac{1}{2}x^2\Big|_{-\pi}^{\pi} + \pi(\pi+\pi)\right]}$$

2π

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}(x+\pi)\sin nx + \frac{1}{n^2}\cos nx\right]\Bigg|_{-\pi}^{\pi}$$

zero

$$= \frac{2}{n^2\pi}\left[\cancel{\cos n\pi - \cos(-n\pi)}\right] = \text{zero}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx =$$

$$= \frac{1}{\pi}\left[-(x+\pi)\frac{\cos nx}{n} + \cancel{\frac{1}{n^2}\sin nx}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left(-2\pi\frac{\cos n\pi}{n}\right) = \frac{-2}{n}\cos n\pi$$

$$= \frac{-2(-1)^n}{n}$$

$$= \frac{2}{n}(-1)^{n+1}$$

2017

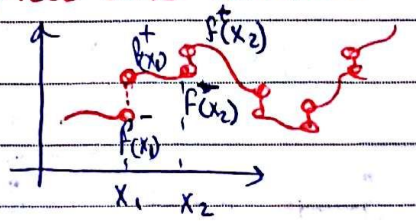
No.

Basic Fourier Series

$f: [-L, L] \to \mathbb{R}$

PWC → piece wise continuos

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$



$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx}$$

Special cases:

* f is even $\longleftrightarrow$ $f(-x) = f(x)$: $b_n = 0$ ~~$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\frac{n\pi x}{L}\,dx$~~

~~* f is odd $\longleftrightarrow$ $f(x) = -f(-x)$~~

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}$$

* f is odd $\longleftrightarrow$ $f(-x) = -f(x)$: $a_n = 0$

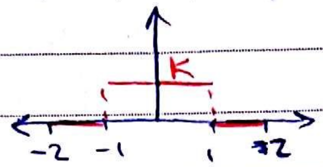
$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}$$

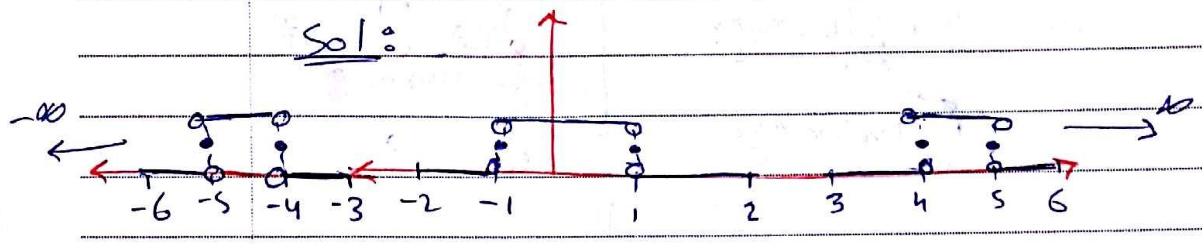
$$f(x) = \frac{f(x)^+ + f(x)^-}{2} \; : \; f(x) \text{ continuos or not}$$

No.

periodic extension of f is denoted by $\tilde{f}$



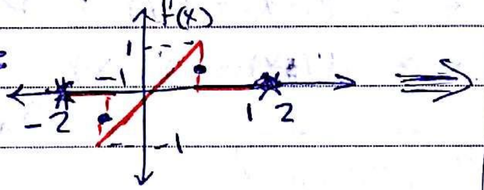
Sol:



$$\frac{\tilde{f}(x)^+ + \tilde{f}(x)^-}{2}$$ ← قيمة ال Series

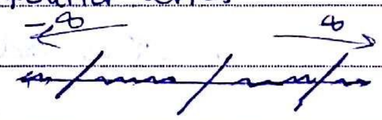
$$f(a)\Big|_{x=a}$$ ← قيمة الاقتران

ex:



$\Longrightarrow$

Fourier series



copy paste

find Basic Fourier Series

sol:

f is odd $\boxed{f(-x) = -f(x)}$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{L}\,dx = \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 \text{zero}\,dx$$ ← Integration by parts

$$\cancel{f(x) = b_n = \frac{\sin n\pi x}{2}\Big|_0^1 - \frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^1}$$

$$\cancel{= 1 - \frac{2}{n\pi}} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

~~udv~~ = uv - v du

No. ~~u dv - v du~~

cont. ex $\quad b_n = \int_0^1 x \sin\frac{n\pi x}{2}\,dx$ $\quad$ by parts

$$\cancel{-\frac{2}{n\pi} x\cos\frac{n\pi x}{2}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{n\pi x}{2}}$$

$$b_n = \frac{-2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \begin{cases} (-1)^m \times \frac{-4}{n\pi^2}, & n = 2m+1 \\ (-1)^m \times \frac{2}{n\pi}, & n = 2m \end{cases}$$

~~f(x) =~~

$$f(x) = \frac{1}{\pi}\sum_{m=0}^{\infty} b_{2m} \sin m\pi x + \frac{4}{\pi^2}\sum_{m=0}^{\infty}\left(b_{2m+1} \sin\frac{2m+1}{2}\right)\frac{4}{\pi^2}$$

$$b_{2m} \to \frac{(-1)^{m+1}}{m} \qquad b_{2m+1} \to \frac{(-1)^m}{(2m+1)^2}$$

~~6/8 Second~~ ~~Final~~

No.

~~Steer~~

Sturm - Liouville Differential equation

$$(P(x)\, y')' + (\lambda r(x) + q(x))\, y = 0 \qquad x \in [a,b]$$

$y \equiv 0$ is the trivial solution

the eigenvalues: $\lambda_1 < \lambda_2 < \lambda_3 < \dots \lambda_n < \dots$ are the eigen values

the corresponding eigen functions: $y_1, y_2, \dots, y_n, \dots$ are orthogonal with the weight functions

$\langle y_n, y_m \rangle = 0$

~~Separate Separated boundary equations~~

(i) $a_1\, y(a) + a_2\, y'(a)$
$b_1\, y(b) + b_2\, y'(b)$
(separated boundary conditions)

(ii) $y(a) = y(b)$
$y'(a) = y'(b)$
(periodic boundary conditions)

ex: Find the eigen Values & the corresponding eigen functions for the following SLDE

ex1
$$y'' + \lambda y = 0 \quad , x \in [-L, L]$$
$y(-L) = 0$
$y(L) = 0$
??

ex2
$y'' + \lambda y = 0$
$y(-L) = y(L)$
$y'(-L) = y'(L)$
??

No.

Sol ex2:

$y = e^{mx}$

$m^2 + \lambda = 0$

$\boxed{m = \mp\sqrt{\lambda}}$

① $\lambda = 0$

$m = 0$

$y = C_1 + C_2 x$

$y_0 = C_1 = 1$

② $\lambda < 0$

$m = \mp\alpha, \quad \lambda = \alpha^2$

$y = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$

$= C_1 \sinh\alpha x + C_2 \cosh\alpha x$

$y(L) = C_1 \sinh\alpha L + C_2 \cosh\alpha L$

$y(-L) = -C_1 \sinh\alpha L + C_2 \cosh\alpha L$

$y(L) = y(-L)$

$2C_1 \sin\alpha L = 0 \Rightarrow C_1 = 0, \quad \sinh\alpha L \neq 0$

$C_2 = 0$

$y = 0$ (trivial solution)

③ $\lambda > 0$,

$\lambda = -\alpha^2$

$m = \pm\alpha i$

$y = C_1 \sin\alpha x + C_2 \cos\alpha x$

$y(L) = C_1 \sin\alpha L + C_2 \cos\alpha L$

$y(-L) = -C_1 \sin\alpha L + C_2 \cos\alpha L$

$y(L) = y(-L)$

$2C_1 \sin\alpha L = 0 \Rightarrow \alpha L = n\pi, \quad \boxed{\alpha = \frac{n\pi}{L}}$

No.

$y'(L) = y'(-L)$

$y' = C_1 \alpha \cos \alpha x - C_2 \alpha \sin \alpha x$

$y'(L) = C_1 \alpha \cos \alpha L - C_2 \alpha \sin \alpha L$

$y'(-L) = C_1 \alpha \cos \alpha L + C_2 \alpha \sin \alpha L$

$2C_2 \alpha \sin \alpha L = 0 \rightarrow \alpha L = n\pi \rightarrow \boxed{\alpha = \frac{n\pi}{L}}$

$$\boxed{y_n = \cos \frac{n\pi}{L}}$$

$\left\{1, \sin \frac{n\pi}{L} x, \cos \frac{n\pi}{L} x\right\}_{n=1}^{\infty}$ eigen functions

$\cancel{\left\{0, \left(\frac{n\pi}{L}\right)^2, -\left(\frac{n\pi}{L}\right)^2\right\}}$

$\dots < -\left(\frac{3\pi}{L}\right)^2 < -\left(\frac{2\pi}{L}\right)^2 < -\left(\frac{\pi}{L}\right)^2 < 0$ eigen Value

Sol for ex1: $y = e^{mx}$

$m^2 + \lambda = 0$

$m = \mp \sqrt{-\lambda}$

① $\lambda = 0$

$m = 0 \quad y = C_1 + C_2 x$

$0 = y(L) = C_1 + C_2 L$

$0 = y(-L) = C_1 - C_2 L$

$2C_1 = 0 \rightarrow C_1 = 0$

$2C_2 L = 0 \rightarrow C_2 = 0$

② $\lambda < 0$

$\lambda = \alpha^2, \ m = \mp \alpha$

$y = C_1 \sinh \alpha x + C_2 \cosh \alpha x$

$0 = y(L) = C_1 \sinh \alpha L + C_2 \cosh \alpha L$

$0 = y(-L) = -C_1 \sinh \alpha L + C_2 \cosh \alpha L$

$2C_2 \cosh \alpha L = 0 \rightarrow C_2 = 0 \quad \cosh \neq 0$

$2C_1 \sinh \alpha L = 0 \rightarrow C_1 = 0 \quad \sinh \alpha L = 0$ at only zero

Scanned by CamScanner

No.

(3) $\lambda > 0$ $\lambda = \alpha^2$

$m = \mp \alpha i$

$y = C_1 \sin\alpha x + C_2 \cos\alpha x$

$y(L) = y(-L)$

$y(L) = 0 = C_1 \sin\alpha L + C_2 \cos\alpha L$

$y(-L) = 0 = -C_1 \sin\alpha L + C_2 \cos\alpha L$ الأعداد الفردية

$\Rightarrow 2C_2 \cos\alpha L = 0 \rightarrow \alpha L = (2n+1)\frac{\pi}{2}$

$\boxed{y_n = \cos\left(\frac{(2n+1)}{2}\pi x\right)}$ $\boxed{\alpha = (2n+1)\frac{\pi}{2L}}$

$\Rightarrow 2C_1 \sin\alpha L = 0 \rightarrow \alpha L = n\pi \Rightarrow \alpha = \frac{n\pi}{L}$

$\boxed{y_n = \sin\left(\frac{n\pi}{L}x\right)}$

No.

$(P(x)y')' + (\lambda\, r(x) + q(x))\, y = 0$

ex: $x^2y'' + 3xy' + \lambda y = 0$, $[1, e]$

$y(1) = 0$, $y(e) = 0$??

$/x^2$ $\quad y'' + \frac{3}{x}y' + \frac{\lambda}{x^2}y = 0$ $\quad P = e^{\int \frac{3}{x}dx}$

$= e^{3\ln x}$

$* x^3$ $\quad x^3y'' + 3x^2y' + \lambda x y = 0$ $\quad = e^{\ln x^3}$

$= x^3$

$P(x) = x^3$, $r(x) = x$, $q(x) = 0$

$y = x^m \rightarrow m(m-1) + 3m + \lambda = 0$

$m^2 + 2m + \lambda = 0$

$\boxed{m = \frac{-2 \mp \sqrt{4 - 4\lambda}}{2}}$

① $4 - 4\lambda = 0$

$\lambda = 1$, $m = -1$ $\quad \cancel{y = c_1x^{-1} + c_2\ln x}$

$y = c_1x^{-1} + c_2x^{-1}\ln x$

$y(1) = c_1 = 0 \rightarrow \boxed{c_1 = 0}$

$y(e) = 0 = c_2 e^{-1} \ln e \Rightarrow \boxed{c_2 = 0}$

$\boxed{y_1 = 0}$

No.

② $\cancel{4-4\lambda = 0}$

② $\cancel{4-4\lambda < 0}$ ② $4-4\lambda > 0$ $\lambda = \alpha^2$

$\cancel{1 < \lambda}$ $1 > \alpha^2$

$1 > \alpha$

$\cancel{m = -1 \mp}$

$m = -1 \mp \alpha \rightarrow \beta_1, \beta_2$

$y = C_1 x^{\beta_1} + C_2 x^{\beta_2}$

$y(1) = C_1 + C_2 = 0 \rightarrow C_1 = -C_2$

$y = C_1 x^{\beta_1} - C_1 x^{\beta_2} = C_1 (x^{\beta_1} - x^{\beta_2}) \equiv 0$

$y(e) = 0 = C_1 (e^{\beta_1} - e^{\beta_2})$

③ $1 - \lambda = -\alpha^2$

$m = -1 \mp \alpha i$

$y = C_1 x^{-1} \sin \alpha \ln x + C_2 x^{-1} \cos \alpha \ln x$

$y(1) = 0 + C_2 \cos \alpha = 0 \rightarrow \boxed{\alpha_n = \frac{2n+1}{2} \pi}$

$y_n = x^{-1} \cos \left(\frac{2n+1}{2} \pi x \right)$

$y(e) = C_1 e^{-1} \sin \alpha + C_2 e^{-1} \cos \alpha$

$\sin \alpha = 0 \rightarrow \boxed{\alpha_n = n\pi}$

$\boxed{y_n = x^{-1} \sin (n\pi \ln x)}$

$\langle y_n, y_m \rangle = \int_1^e (x^{-1} \sin n\pi \ln x)(\cancel{x^{-1}} \cos m\pi \ln x) \cancel{x} \, dx$

subsitution $\ln x = z \rightarrow \ln 1 = 0$

$\frac{1}{x} = dz$ $\ln e = 1$

No.

$$\int_0^1 \sin n\pi z \cos m\pi z \, dz$$

متعامدة

25/7

No.

Fourier Transform:

$f : [0, L] \to \mathbb{R}$ → P.W.C

$$\boxed{F_s(n) = \int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}$$

finite fourier sine transform

$S_n(f)$

$$\boxed{f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s(n)\sin\left(\frac{n\pi}{L}x\right)}$$

Inverse

مهمة

$$\boxed{F_c(n) = \int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx}$$

finite fourier cosine transform

$C_n(f)$

Inverse

$$\boxed{f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi x}{L}\right) + \frac{F_c(0)}{L}}$$

اله مهمة

~~$f(x)$ $F_c(n)$ $F_s(n)$~~

Scanned by CamScanner

No.

ex $f(x) = x \qquad x \in [0, \pi]$

$F_c(n)$ ∗ $F_s(n)$??

解.

$F_s(n) = \int_0^{\pi} x \sin\left(\frac{n \cancel{\pi} x}{\cancel{L}}\right) dx \qquad L = \pi$

$= \int_0^{\pi} \underset{u}{x} \underbrace{\sin(nx)\, dx}_{dv}$ by parts

$u.v - \int v\, du$

$-\frac{x}{n} \cos nx + \frac{1}{n}\int \cos nx\, dx$

$= \left(-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right) \Big|_0^{\pi}$

$= -\frac{\pi}{n} \cos n\pi + \frac{1}{n^2}(\underbrace{\sin n\pi - \sin 0}_{\text{zero}})$

$= -\frac{\pi}{n} (-1)^n = \frac{(-1)^{n+1}}{n} \pi$

Given $F_s(n) = \frac{(-1)^{n+1}}{n} \pi$ find $f(x)$
$f : [0, \pi]$

$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \pi \sin(nx)$

$F_c(n) = \int_0^{\pi} \underset{u}{x} \underbrace{\cos(nx)\, dx}_{dv}$ by parts

$U.V - \int V du$

$\frac{1}{n} x \sin nx - \frac{1}{n}\int \sin nx\, dx$

$= \frac{1}{n} x \sin nx + \frac{1}{n^2} \cos nx \Big|_0^{\pi} = \frac{1}{n}(\underbrace{\pi \sin n\pi}_{\text{zero}})$

$+ \frac{1}{n^2}(\cos n\pi - \underbrace{\cos 0}_{1})$

No.

$$= \frac{1}{n^2}(\cos n\pi - 1) = \boxed{\overset{F_c(n)}{\frac{1}{n^2}\left((-1)^n - 1\right)}} = \begin{cases} 0 & , n = 2m \\ \frac{-2}{n^2} & , n = 2m+1 \end{cases}$$

$$F_c(0) = \int_0^\pi x\,dx = \frac{1}{2}x^2\Big|_0^\pi = \frac{1}{2}\pi^2$$

$$\boxed{\frac{F_c(0)}{L} = \frac{1}{2}\pi}$$

Given $F_c(n)$ find $f(x)$

$$\boxed{f(x) = \frac{1}{2}\pi - \frac{4}{\pi}\sum_{n=0}^{\infty}{}' \frac{1}{(2n+1)^2}\cos(2n+1)x}$$

$$S_n[f'] = \int_0^L f'(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$dv = f'(x)$ $\quad$ $u = \sin\left(\frac{n\pi}{L}x\right)$

$V = f(x)$ $\quad$ $du = \frac{n\pi}{L}\cos\frac{n\pi}{L}x\,dx$

$$= \underbrace{f(x)\sin\left(\frac{n\pi}{L}x\right)\Big|_0^L}_{\text{zero}} - \int_0^L f(x)\cdot\frac{n\pi}{L}\cos\frac{n\pi}{L}x\,dx$$

$$\boxed{S_n(f') = -\frac{n\pi}{L}C_n(f)}$$

$$\boxed{C_n(f') = (-1)^n f(L) - f(0) + \frac{n\pi}{L}S_n(f)}$$

No.

$$S_n(f'') = S_n\left[(f')'\right] = -\frac{n\pi}{L} C_n(f')$$

$$\boxed{S_n(f'') = -\frac{n\pi}{L}\left((-1)^n f(L) - f(0) + \frac{n\pi}{L} S_n(f)\right)}$$

$$C_n(f'') = (-1)^n f'(L) - f'(0) + \frac{n\pi}{L} S_n(f')$$

$$\boxed{C_n(f'') = (-1)^n f'(L) - f'(0) - \left(\frac{n\pi}{L}\right)^2 C_n(f)}$$

ex $y'' + y = x \qquad y(0) = 0 \qquad y(\pi) = 0$??

solve by fourier Transform

sol: $S_n[y'' + y] = S_n[x]$

$$S_n[y''] + S_n[y] = S_n[x]$$

$$-\left(\frac{n\pi}{L}\right)^2 S_n[y] + S_n[y] = \frac{(-1)^{n+1}}{n}\pi$$

$$S_n[y]\left(1 - \left(\frac{n\pi}{L}\right)^2\right) = \frac{(-1)^{n+1}}{n}\pi$$

$$\boxed{y = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\frac{(-1)^{n+1}}{n}\pi}{1 - \left(\frac{n\pi}{L}\right)^2}}$$

27/7

No.

$f : [-L, L] \xrightarrow{PWC} \mathbb{R}$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$f : R \xrightarrow{P.W.C} \mathbb{R}$

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \longrightarrow \lim_{x \to \infty} f(x) = 0$$

Integral Representation of f

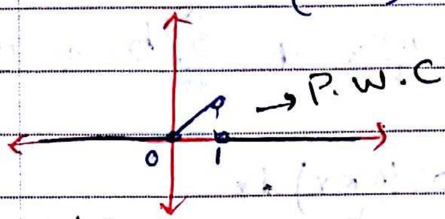
$$f(x) \sim \int_{0}^{\infty} (A_\alpha \cos \alpha x + B_\alpha \sin \alpha x) \, d\alpha$$

$$A_\alpha = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx$$

$$B_\alpha = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx$$

No.

ex: $f(x) = \begin{cases} x, & 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$



Sol:

$$f(x) \sim \int_0^{\infty} (A_\alpha \sin\alpha x + B_\alpha \cos\alpha x)\, d\alpha$$ تكامل series

$\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{2} < \infty$

$\Rightarrow f$: Absolutely Integrable

$$A_\alpha = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos\alpha x \, dx$$

$$= \frac{1}{\pi} \int_0^1 \underbrace{x}_{u} \underbrace{\cos\alpha x\, dx}_{dv} \qquad u.v - \int v\,du$$

$$= \frac{1}{\pi}\left(\frac{1}{\alpha} x \sin\alpha x \Big|_0^1 - \frac{1}{\alpha} \int_0^1 \sin\alpha x\, dx \right) \qquad \frac{\cos\alpha x}{\alpha}$$

$$= \frac{1}{\pi}\left(\frac{1}{\alpha}\sin\alpha + \frac{1}{\alpha^2} (\cos\alpha - 1) \right)$$

$$\boxed{A_\alpha = \frac{1}{\pi} \, \frac{\alpha \sin\alpha + \cos\alpha - 1}{\alpha^2}}$$

$$B_\alpha = \frac{1}{\pi} \int_0^1 \underbrace{x}_{u} \underbrace{\sin\alpha x\, dx}_{dv}$$

$$= \frac{1}{\pi}\left(\frac{-1}{\alpha} x \cos\alpha x \Big|_0^1 + \frac{1}{\alpha} \int_0^1 \cos\alpha x\, dx \right)$$

$$= \frac{1}{\pi}\left(\frac{-1}{\alpha} (\cos\alpha) + \frac{1}{\alpha^2} \sin\alpha \right)$$

$$\boxed{B_\alpha = \frac{1}{\pi} \, \frac{-\alpha \cos\alpha + \sin\alpha}{\alpha^2}}$$

No.

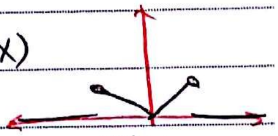
$$f(x) \cong \frac{1}{\pi}\int_0^\infty \left(\left(\frac{\alpha\sin\alpha + (\cos\alpha - 1)}{\alpha^2}\right)\cos\alpha x + \left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\right)\sin\alpha x\right)d\alpha$$

~~(I) If~~

(I) If f is even $\Rightarrow$ $f(-x) = f(x)$

$B_\alpha = 0, \ \forall \alpha$

$A_\alpha = \frac{2}{\pi}\int_0^\infty f(x)\cos\alpha x \, dx$

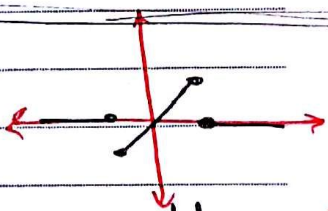


even

(II) If f is odd $\Rightarrow$ $f(-x) = -f(x)$

$A_\alpha = 0, \ \forall \alpha$

$B_\alpha = \frac{2}{\pi}\int_0^\infty f(x)\sin\alpha x \, dx$



odd

31/7

No.

$f:[-L,L]$ Fourier series representation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

Finite Fourier cosine Transform

$$F_c(n) = \int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = C_n[F]$$

$$f(x) = \frac{F_c(n)}{L} + \frac{2}{L}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi}{L}x$$

$$C_n(f'') = (-1)^n f'(L) - f'(0) + \left(\frac{n\pi}{L}\right)^2 C_n[f]$$

Finite Fourier Sine Transform

$$F_s(n) = \int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = S_n[F]$$

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi}{L}x$$

$$S_n[f''] = \left(-\frac{n\pi}{L}\right)\left((-1)^n f(L) - f(0)\right) - \left(\frac{n\pi}{L}\right)^2 S_n[F]$$

Fourier Integral Rep. of f

$$f: R \xrightarrow[A.I]{P.W.C} R \longrightarrow \int_{-\infty}^{\infty} |f(t)|\,dt < \infty$$

$$f(x) \sim \int_0^{\infty} (A\alpha \cos\alpha x + B\alpha \sin\alpha x)\,d\alpha$$

$$A\alpha = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$$

$$B\alpha = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$$

Fourier Sine & cosine Transform

$f : (0,\infty) \xrightarrow[A.I]{P.W.C} \mathbb{R}$

$$\int_0^\infty f(x)\cos\alpha x\, dx = C_\alpha[f] = F_c[\alpha] = \hat{f}_c(w)$$

$$f(x) = \frac{2}{\pi} \cdot \int_0^\infty F_c(\alpha)\cos\alpha x\, d\alpha$$

$$\int_0^\infty f(x)\sin\alpha x\, dx = S_\alpha[F] = F_s(\alpha) = \hat{f}_s(w)$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\, d\alpha$$

$$C_\alpha[f'] = \int_0^\infty f'(x)\cos\alpha x\, dx$$

$u.v - \int v\, du$

$u = \cos\alpha x \qquad dv = f'(x)\,dx$

$du = -\alpha\sin\alpha x \qquad v = f(x)\,dx$

$$C_\alpha[f'] = f(x)\cos\alpha x\Big|_0^\infty + \alpha\underbrace{\int_0^\infty \sin\alpha x\, f(x)\,dx}_{S_\alpha[f]}$$

$$\boxed{C_\alpha[f'] = -f(0) + \alpha S_\alpha[f]}$$

No.

$$S_\alpha[f'] = \int_0^\infty \underbrace{f'(x)}_{dv} \underbrace{\sin\alpha x}_{u}\, dx$$

$u.v - \int v\,du$

$$S_\alpha[f'] = f(x)\sin\alpha x\Big|_0^\infty - \alpha \underbrace{\int_0^\infty f(x)\cos\alpha x\,dx}_{C_\alpha[f]}$$

$$\boxed{S_\alpha[f'] = \cancel{f(x)} - \alpha C_\alpha[f]}$$

$$C_\alpha[f''] = -f'(0) + \alpha \underline{S_\alpha[f']} \longrightarrow -\alpha C_\alpha(f)$$

$$\boxed{C_\alpha[f''] = -f'(0) - \alpha^2 C_\alpha[f]}$$

$$S_\alpha[f''] = -\alpha\, C_\alpha[f']$$
$$= -\alpha\,(-f(0) + \alpha\, S_\alpha[f])$$

$$\boxed{S_\alpha[f''] = \alpha f(0) - \alpha^2 S_\alpha[f]}$$

No.

ex $C_\alpha[f]$, $S_\alpha[f]$??

$$f(x) = \begin{cases} x, & 0<x<1 \\ 0; & x>1 \end{cases}$$

Sol

~~$S_\alpha[f] = \int_0^\infty$~~

$$C_\alpha[f] = \int_0^\infty f(x)\cos\alpha x\,dx = \int_0^1 \underbrace{x}_{u}\underbrace{\cos\alpha x\,dx}_{dv}$$

$$= \frac{1}{\alpha}x\sin\alpha x\Big|_0^1 - \frac{1}{\alpha}\int_0^1 \sin\alpha x\,dx$$

$$C_\alpha[f] = \frac{1}{\alpha}\sin\alpha + \frac{1}{\alpha}\left(\frac{\cos\alpha x}{\alpha}\Big|_0^1\right)$$

$$\boxed{C_\alpha[f] = \frac{1}{\alpha}\sin\alpha + \frac{1}{\alpha^2}(\cos\alpha - 1)}$$

Power unit

$$S_\alpha[f] = \int_0^\infty f(x)\sin\alpha x\,dx$$

$$= \int_0^1 \underbrace{x}_{u}\underbrace{\sin\alpha x\,dx}_{dv}$$

$$-\frac{1}{\alpha}x\cos\alpha x\Big|_0^1 + \frac{1}{\alpha}\int_0^1 \cos\alpha x\,dx$$

$$\boxed{S_\alpha[f] = -\frac{1}{\alpha}\cos\alpha + \frac{1}{\alpha^2}\sin\alpha}$$

No.

ex $y'' + y = f = \begin{cases} x, & 0<x<1 \\ 0, & x>1 \end{cases}$, $x \in [0, \infty)$

$\boxed{y'(0) = 0}$ → we use cosine Transform to Solve.

Sol:

$$C_\alpha(y'' + y) = C_\alpha(f)$$

$$C_\alpha[y''] + C_\alpha[y] = C_\alpha[f]$$

$$C_\alpha[f] = \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2}$$

from prev. ex

$$\underbrace{-\cancel{y'(0)}}_{\text{zero}} - \alpha^2 C_\alpha[y] + C_\alpha[y] = C_\alpha[f]$$

$$C_\alpha[y]\,(1-\alpha^2) = \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2}$$

$$C_\alpha[y] = \left(\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2}\right) * \frac{1}{(1-\alpha^2)}$$

$$y = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\,dx$$

$$\boxed{y = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2}\right)\frac{1}{(1-\alpha^2)}\cos\alpha x\,dx}$$

No.

ex : $y'' + y = \begin{cases} x & , 0 < x < 1 \\ 0 & , x > 1 \end{cases} \qquad x \in [0, \infty]$

$y(0) = 0 \rightarrow$ use Sine Transform

sol

$$S_\alpha\{y'' + y\} = S_\alpha[f]$$

$$S_\alpha[y''] + S_\alpha[y] = S_\alpha[F]$$

$$\overset{\text{zero}}{\cancel{\alpha\, y(0)}} - \alpha^2 S_\alpha[y] + S_\alpha[y] = -\frac{1}{\alpha}\cos\alpha + \frac{1}{\alpha^2}\sin\alpha$$

~~Ex prev. ex~~

$$S_\alpha[y] = \left(-\frac{1}{\alpha}\cos\alpha + \frac{1}{\alpha^2}\sin\alpha\right) * \frac{1}{1-\alpha^2}$$

$$\boxed{y = \frac{2}{\pi}\int_0^\infty \left(-\frac{1}{\alpha}\cos\alpha + \frac{1}{\alpha^2}\sin\alpha\right) * \left(\frac{1}{1-\alpha^2}\right) \sin\alpha x \, d\alpha}$$

Power unit

نهاية مادة السكنر