# Instructor's Solutions Manual for
# Elements of Electromagnetics

International Fifth Edition

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# Contents

# CHAPTER 1

**P. E. 1.1**

(a) $$\boldsymbol{A}+\boldsymbol{B}=(1,0,3)+(5,2,-6)=(6,2,-3)$$

$$|\boldsymbol{A}+\boldsymbol{B}|=\sqrt{36+4+9}=\underline{\underline{7}}$$

(b) $$5\boldsymbol{A}-\boldsymbol{B}=(5,0,15)-(5,2,-6)=\underline{\underline{(0,-2,21)}}$$

(c) The component of $\boldsymbol{A}$ along $\mathbf{a}_y$ is $A_y=\underline{\underline{0}}$

(d) $$3\boldsymbol{A}+\boldsymbol{B}=(3,0,9)+(5,2,-6)=(8,2,3)$$

A unit vector parallel to this vector is

$$\boldsymbol{a}_{11}=\frac{(8,2,3)}{\sqrt{64+4+9}}$$
$$=\underline{\underline{\pm(0.9117\boldsymbol{a}_x+0.2279\boldsymbol{a}_y+0.3419\boldsymbol{a}_z)}}$$

**P. E. 1.2** (a) $\boldsymbol{r}_p=\underline{\underline{\boldsymbol{a}_x-3\boldsymbol{a}_y+5\boldsymbol{a}_z}}$

$$\boldsymbol{r}_R=\underline{\underline{3\boldsymbol{a}_y+8\boldsymbol{a}_z}}$$

(b) The distance vector is

$$\boldsymbol{r}_{QR}=\boldsymbol{r}_R-\boldsymbol{r}_Q=(0,3,8)-(2,4,6)=\underline{\underline{-2\boldsymbol{a}_x-\boldsymbol{a}_y+2\boldsymbol{a}_z}}$$

(c) The distance between Q and R is

$$|r_{QR}|=\sqrt{4+1+4}=\underline{\underline{3}}$$

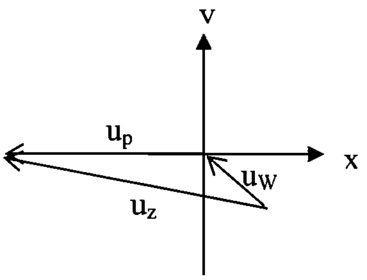
**P. E. 1.3** Consider the figure shown on the next page:

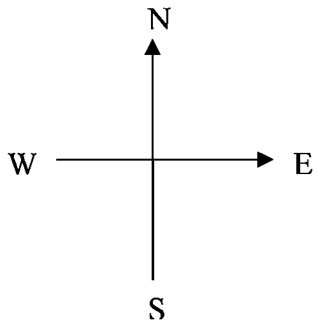
$$\boldsymbol{u}_Z=\boldsymbol{u}_P+\boldsymbol{u}_W=-350\boldsymbol{a}_x+\frac{40}{\sqrt{2}}(-\boldsymbol{a}_x+\boldsymbol{a}_y)$$
$$=-378.28\boldsymbol{a}_x+28.28a_y \text{ km/hr}$$

or

$$\boldsymbol{u}_z=379.3\angle 175.72^\circ \text{ km/hr}$$

Where $\mathbf{u}_p$ = velocity of the airplane in the absence of wind
$\mathbf{u}_w$ = wind velocity
$\mathbf{u}_z$ = observed velocity





**P. E. 1.4**

Using the dot product,

$$\cos\theta_{AB} = \frac{\mathbf{A} \bullet \mathbf{B}}{AB} = \frac{-13}{\sqrt{10}\sqrt{65}} = -\sqrt{\frac{13}{50}}$$

$$\underline{\underline{\theta_{AB} = 120.66^\circ}}$$

**P. E. 1.5**

$$\text{(a)}\ \boldsymbol{E}_F = (\boldsymbol{E} \cdot \boldsymbol{a}_F)\boldsymbol{a}_F = \frac{(\boldsymbol{E} \cdot \boldsymbol{F})\boldsymbol{F}}{|\boldsymbol{F}|^2} = \frac{-10(4,-10,5)}{141}$$

$$= \underline{\underline{-0.2837\boldsymbol{a}_x + 0.7092\boldsymbol{a}_y - 0.3546\boldsymbol{a}_z}}$$

$$\text{(b)}\ \boldsymbol{E} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55,16,-12)$$

$$\boldsymbol{a}_{E\times F} = \underline{\underline{\pm(0.9398,0.2734,-0.205)}}$$

**P. E. 1.6** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$ showing that ***a***, ***b***, and ***c*** form the sides of a triangle.

$\boldsymbol{a} \cdot \boldsymbol{b} = 0$,
hence it is a right angle triangle.

$$\text{Area} = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}|\boldsymbol{b} \times \boldsymbol{c}| = \frac{1}{2}|\boldsymbol{c} \times \boldsymbol{a}|$$

$$\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}\begin{vmatrix} 4 & 0 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \frac{1}{2}|(3,-17,12)|$$

$$\text{Area} = \frac{1}{2}\sqrt{9 + 289 + 144} = \underline{\underline{10.51}}$$

**P. E. 1.7**

(a) $$P_1P_2 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$= \sqrt{25+4+64} = \underline{\underline{9.644}}$$

(b) $$\boldsymbol{r_P} = \boldsymbol{r}_{P_1} + \lambda(\boldsymbol{r}_{P_2} - \boldsymbol{r}_{P_1})$$

$$= (1,2,-3) + \lambda(-5,-2,8)$$

$$= \underline{\underline{(1-5\lambda, 2-2\lambda, -3+8\lambda)}}.$$

(c) The shortest distance is

$$\boldsymbol{d} = \boldsymbol{P_1P_3}\sin\theta = \left|\boldsymbol{P_1P_3} \times \boldsymbol{a}_{P_1P_2}\right|$$

$$= \frac{1}{\sqrt{93}}\begin{vmatrix} 6 & -3 & 5 \\ -5 & -2 & 8 \end{vmatrix}$$

$$= \frac{1}{\sqrt{93}}\left|(-14,-73,-27)\right| = \underline{\underline{8.2}}$$

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**Prob.1.1**

$$\boldsymbol{r} = (-3,2,2)-(2,4,4) = (-5,-2,-2)$$

$$\boldsymbol{a}_r = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \frac{(-5,-2,-2)}{\sqrt{25+4+4}} = \underline{\underline{-0.8704\boldsymbol{a}_x - 0.3482\boldsymbol{a}_y - 0.3482\boldsymbol{a}_z}}$$

**Prob. 1.2**

$$\mathbf{r}_{OP} = 4\mathbf{a}_x - 5\mathbf{a}_y + \mathbf{a}_z$$

$$\mathbf{a}_{\mathbf{r}_{OP}} = \frac{\mathbf{r}_{OP}}{|\mathbf{r}_{OP}|} = \frac{4\mathbf{a}_x - 5\mathbf{a}_y + \mathbf{a}_z}{\sqrt{16+25+1}} = 0.6172\mathbf{a}_x - 0.7715\mathbf{a}_y + 0.1543\mathbf{a}_z$$

**Prob. 1.3**

$$\boldsymbol{A}\bullet\boldsymbol{B} = 2-0+20 = 22$$

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 2 & 0 & 5 \\ 1 & -3 & 4 \end{vmatrix} = (15,-3,-6)$$

$$|\boldsymbol{A}\times\boldsymbol{B}| + \boldsymbol{A}\bullet\boldsymbol{B} = \sqrt{15^2+3^2+6^2} + 22 = \underline{\underline{38.432}}$$

**Prob. 1.4**

$$\boldsymbol{r}_{MN} = \boldsymbol{r}_N - \boldsymbol{r}_M = (3,5,-1)-(1,-4,-2) = \underline{\underline{2\boldsymbol{a}_x + 9\boldsymbol{a}_y + \boldsymbol{a}_z}}$$

**Prob. 1.5**

(a) Let $\boldsymbol{C} = \boldsymbol{A} - 3\boldsymbol{B} = (4,-2,6)-(36,54,-24) = \underline{\underline{-32\boldsymbol{a}_x - 56\boldsymbol{a}_y + 30\boldsymbol{a}_z}}$

(b)

Let $\mathbf{D} = 2\boldsymbol{A} + 5\boldsymbol{B} = (8,-4,12)+(60,90,-40) = (68,86,-28)$

$|\boldsymbol{B}| = \sqrt{12^2 + 18^2 + 8^2} = 23.065$

$$\frac{\mathbf{D}}{|\boldsymbol{B}|} = \underline{\underline{2.9482\boldsymbol{a}_x + 3.7286\boldsymbol{a}_y - 1.214\boldsymbol{a}_z}}$$

(c ) $\boldsymbol{a}_x \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & 0 & 0 \\ 4 & -2 & 6 \end{vmatrix} = \underline{\underline{-6\boldsymbol{a}_y - 2\boldsymbol{a}_z}}$

(d)

$$\boldsymbol{B} \times \boldsymbol{a}_z = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 12 & 18 & -8 \\ 0 & 0 & 1 \end{vmatrix} = 18\boldsymbol{a}_x - 12\boldsymbol{a}_y$$

$(\boldsymbol{B} \times \boldsymbol{a}_z)\bullet\boldsymbol{a}_y = \underline{\underline{-12}}$

**Prob. 1.6**

(a)

$$\boldsymbol{B} \times \boldsymbol{C} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z$$

$\boldsymbol{A}\bullet(\boldsymbol{B} \times \boldsymbol{C}) = (1,0,-1)\bullet(1,-2,1) = 1 + 0 - 1 = \underline{\underline{0}}$

(b)

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z$$

$(\boldsymbol{A} \times \boldsymbol{B})\bullet\boldsymbol{C} = (1,-2,1)\bullet(0,1,2) = 0 - 2 + 2 = \underline{\underline{0}}$

(c ) $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \underline{\underline{-2\boldsymbol{a}_x - 2\boldsymbol{a}_y - 2\boldsymbol{a}_z}}$

(d) $$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ 1 & -2 & 1\\ 0 & 1 & 2\end{vmatrix}=\underline{\underline{-5\boldsymbol{a}_x-2\boldsymbol{a}_y+\boldsymbol{a}_z}}$$

**Prob.1.7**

(a) T = $\underline{(3, -2, 1)}$ and S = $\underline{(4, 6, 2)}$

(b) $\mathbf{r}_{TS}=\mathbf{r}_s-\mathbf{r}_t=(4, 6, 2)-(3, -2, 1)=\underline{a_x+8a_y+a_z}$

(c ) distance = $|\mathbf{r}_{TS}|=\sqrt{1+64+1}=\underline{\underline{8.124}}$ m

**Prob. 1.8**

(a) If **A** and **B** are parallel, **A=kB**, where k is a constant.

$$(\alpha,3,-2)=k(4,\beta,8)$$

Equating coefficients gives

$$-2=8k \longrightarrow k=-\frac{1}{4}$$

$$\alpha=4k=\underline{\underline{-1}}$$

$$3=\beta k \longrightarrow \beta=3/k=\underline{\underline{-12}}$$

This can also be solved using **A X B** = 0.

(b) If A and B are perpendicular to each other,

$$\boldsymbol{A}\bullet\boldsymbol{B}=0 \longrightarrow \underline{\underline{4\alpha+3\beta-16=0}}$$

**Prob. 1.9**

(a) $A\cdot B=\text{AB}\cos\theta_{AB}$

$A\times B=\text{AB}\sin\theta_{AB}a_n$

$$(A\cdot B)^2+|A\times B|^2=(AB)^2\left(\cos^2\theta_{AB}+\sin^2\theta_{AB}\right)=(\text{AB})^2$$

(b) $\boldsymbol{a}_x\cdot(\boldsymbol{a}_y\times\boldsymbol{a}_z)=\boldsymbol{a}_x\cdot\boldsymbol{a}_x=1$. Hence,

$$\frac{\boldsymbol{a}_y\times\boldsymbol{a}_z}{\boldsymbol{a}_x\cdot\boldsymbol{a}_y\times\boldsymbol{a}_z}=\frac{\boldsymbol{a}_x}{1}=\boldsymbol{a}_x$$

$$\frac{\boldsymbol{a}_z\times\boldsymbol{a}_x}{\boldsymbol{a}_x\cdot\boldsymbol{a}_y\times\boldsymbol{a}_z}=\frac{\boldsymbol{a}_y}{1}=\boldsymbol{a}_y$$

$$\frac{\boldsymbol{a}_x\times\boldsymbol{a}_y}{\boldsymbol{a}_x\cdot\boldsymbol{a}_y\times\boldsymbol{a}_z}=\frac{\boldsymbol{a}_z}{1}=\boldsymbol{a}_z$$

**Prob. 1.10**

(a) $\boldsymbol{P}+\boldsymbol{Q}=(6,2,0), \boldsymbol{P}+\boldsymbol{Q}-\boldsymbol{R}=(7,1,-2)$

$$|\boldsymbol{P}+\boldsymbol{Q}-\boldsymbol{R}|=\sqrt{49+1+4}=\sqrt{54}=\underline{\underline{7.3485}}$$

(b) $$\boldsymbol{P}.\boldsymbol{Q}\times\boldsymbol{R}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ 2 & -1 & -2\\ 4 & 3 & 2\\ -1 & 1 & 2\end{vmatrix}=2(6-2)+(8+2)-2(4+3)=8+10-14=\underline{\underline{4}}$$

$$\boldsymbol{Q}\times\boldsymbol{R}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ 4 & 3 & 2\\ -1 & 1 & 2\end{vmatrix}=(4,-10,7)$$

$$\boldsymbol{P}.\boldsymbol{Q}\times\boldsymbol{R}=(2,-1,-2)\cdot(4,-10,7)=8+10-14=\underline{\underline{4}}$$

(c) $$\boldsymbol{Q}\times\boldsymbol{P}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ 4 & 3 & 2\\ 2 & -1 & -2\end{vmatrix}=(-4,12,-10)$$

$$\mathbf{Q}\times\boldsymbol{P}\bullet\boldsymbol{R}=(-4,12,-10)\cdot(-1,1,2)=4+12-20=\underline{\underline{-4}}$$

or $$\boldsymbol{Q}\times\boldsymbol{P}\cdot\boldsymbol{R}=\boldsymbol{R}\cdot\boldsymbol{Q}\times\boldsymbol{P}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ -1 & 1 & 2\\ 4 & 3 & 2\\ 2 & -1 & -2\end{vmatrix}=-(-6+2)-(-8-4)+2(-4-6)=\underline{\underline{-4}}$$

(d) $$(\boldsymbol{P}\times\boldsymbol{Q})\cdot(\mathbf{Q}\times\boldsymbol{R})=(4,-12,10)\cdot(4,-10,7)=16+120+70=\underline{\underline{206}}$$

(e) $$(\boldsymbol{P}\times\mathbf{Q})\times(\mathbf{Q}\times\boldsymbol{R})=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ 4 & -12 & 10\\ 4 & -10 & 7\end{vmatrix}=\underline{\underline{16a_x+12a_y+8a_z}}$$

(f) $$\cos\theta_{PR}=\frac{\mathbf{P}\bullet\mathbf{R}}{|\boldsymbol{P}||\boldsymbol{R}|}=\frac{(-2-1-4)}{\sqrt{4+1+4}\sqrt{1+1+4}}=\frac{-7}{3\sqrt{6}}=-0.9526$$

$$\underline{\underline{\theta_{PR}=162.3^\circ}}$$

$$\text{(g) } \sin\theta_{PQ} = \frac{|P \times Q|}{|P||Q|} = \frac{\sqrt{16+144+100}}{3\sqrt{16+9+4}} = \frac{\sqrt{260}}{3\sqrt{29}} = 0.998$$

$$\theta_{PQ} = \underline{\underline{86.45^\circ}}$$

**Prob. 1.11**

(a)

$$\boldsymbol{A} \bullet \boldsymbol{B} = (4,-6,1) \bullet (2,0,5) = 8 - 0 + 5 = 13$$

$$|\boldsymbol{B}|^2 = 2^2 + 5^2 = 29$$

$$\boldsymbol{A} \bullet \boldsymbol{B} + 2|\boldsymbol{B}|^2 = 13 + 2 \times 29 = \underline{\underline{71}}$$

(b)

$$\boldsymbol{a}_\perp = \pm \frac{\boldsymbol{A} \times \boldsymbol{B}}{|\boldsymbol{A} \times \boldsymbol{B}|}$$

$$\text{Let } \boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 4 & -6 & 1 \\ 2 & 0 & 5 \end{vmatrix} = (-30,-18,12)$$

$$\boldsymbol{a}_\perp = \pm \frac{\boldsymbol{C}}{|\boldsymbol{C}|} = \pm \frac{(-30,-18,12)}{\sqrt{30^2+18^2+12^2}} = \underline{\underline{\pm(-0.8111\boldsymbol{a}_x - 0.4867\boldsymbol{a}_y + 0.3244\boldsymbol{a}_z)}}$$

**Prob. 1.12**

$$\boldsymbol{P} \bullet \boldsymbol{Q} = (2,-6,5) \bullet (0,3,1) = 0 - 18 + 5 = \underline{-13}$$

$$\boldsymbol{P} \times \boldsymbol{Q} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix} = \underline{\underline{-21\boldsymbol{a}_x \text{ - } 2\boldsymbol{a}_y + 6\boldsymbol{a}_z}}$$

$$\cos\theta_{PQ} = \frac{\boldsymbol{P} \cdot \boldsymbol{Q}}{PQ} = \frac{-13}{\sqrt{10}\sqrt{65}} = -0.51 \quad \longrightarrow \quad \theta_{PQ} = \underline{\underline{120.66^o}}$$

**Prob. 1.13**

If **A** and **B** are parallel, then **B** = k**A** and **A** x **B** = **0.** It is evident that k = -2 and that

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & -2 & 3 \\ -2 & 4 & -6 \end{vmatrix} = \boldsymbol{0}$$

as expected.

**Prob. 1.14**

(a) Using the fact that

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{A},$$

we get

$$\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{B})=-(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{A}=\underline{\underline{(\boldsymbol{B}\cdot\boldsymbol{A})\boldsymbol{A}-(\boldsymbol{A}\cdot\boldsymbol{A})\boldsymbol{B}}}$$

(b) $$\boldsymbol{A}\times\ (\boldsymbol{A}\times\ (\boldsymbol{A}\times\ \boldsymbol{B}))=\boldsymbol{A}\times\ \left[(\boldsymbol{A}\bullet\boldsymbol{B})\boldsymbol{A}-(\boldsymbol{A}\bullet\boldsymbol{A})\boldsymbol{B}\right]$$

$$=(\boldsymbol{A}\bullet\boldsymbol{B})\ (\boldsymbol{A}\times\ \boldsymbol{A})-(\boldsymbol{A}\bullet\boldsymbol{A})\ (\boldsymbol{A}\times\ \boldsymbol{B})$$

$$=\underline{\underline{-A^2(\boldsymbol{A}\times\ \boldsymbol{B})}}$$

since AxA = **0**

**Prob. 1.15**

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix},\qquad (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}}$$

Hence, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$

Also, each equals the volume of the parallelopiped formed by the three vectors as sides.

**Prob. 1.16**

(a)

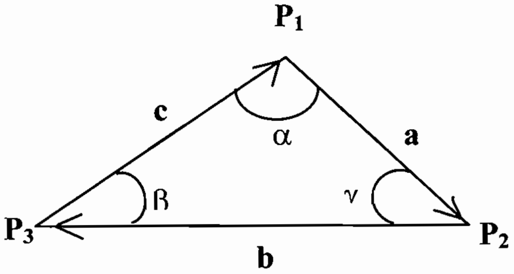
$$P_1P_2=r_{P_2}-r_{P_1}=(-6,0,-3)$$

$$P_1P_3=r_{P_3}-r_{P_1}=(1,5,-6)$$

$$P_1P_2\times P_1P_3=\begin{vmatrix}-6 & 0 & -3\\ 1 & 5 & -6\end{vmatrix}=(15,-39,-30)$$

Area of the triangle $=\dfrac{1}{2}\left|P_1P_2\times P_1P_3\right|=\dfrac{1}{2}\sqrt{15^2+39^2+30^2}=\underline{\underline{25.72}}$

(b)



$$\boldsymbol{a} = r_{P_2} - r_{P_1} = (-5,2,0)-(1,2,3) = (-6,0,-3)$$
$$\boldsymbol{b} = r_{P_3} - r_{P_2} = (2,7,-3)-(-5,2,0) = (7,5,-3)$$
$$\boldsymbol{c} = r_{P_1} - r_{P_3} = (1,2,3)-(2,7,-3) = (-1,-5,6)$$

Note that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=0$

$$\alpha = \text{angle at P}_1 = \cos^{-1}\left[\frac{(\boldsymbol{r}_{p2}-\boldsymbol{r}_{p1})\bullet(\boldsymbol{r}_{p3}-\boldsymbol{r}_{p1})}{|\boldsymbol{r}_{p2}-\boldsymbol{r}_{p1}||\boldsymbol{r}_{p3}-\boldsymbol{r}_{p1}|}\right]$$
$$= \cos^{-1}\frac{\boldsymbol{a}\bullet(-\boldsymbol{c})}{|\boldsymbol{a}||\boldsymbol{c}|} = \cos^{-1}\left[\frac{12}{\sqrt{45}\sqrt{62}}\right] = \quad 76.87^{o}$$

$$\beta = \text{angle at P}_3 = \cos^{-1}\left[\frac{(\boldsymbol{r}_{p1}-\boldsymbol{r}_{p3})\bullet(\boldsymbol{r}_{p2}-\boldsymbol{r}_{p3})}{|\boldsymbol{r}_{p1}-\boldsymbol{r}_{p3}||\boldsymbol{r}_{p2}-\boldsymbol{r}_{p3}|}\right]$$
$$= \cos^{-1}\frac{c\bullet -b}{|c||b|} = \cos^{-1}\left[\frac{60}{\sqrt{62}\sqrt{83}}\right] = \underline{\underline{45.81^{o}}}$$

$$\gamma = \text{angle at P}_2 = \cos^{-1}\left[\frac{(\boldsymbol{r}_{p1}-\boldsymbol{r}_{p2})\bullet(\boldsymbol{r}_{p3}-\boldsymbol{r}_{p2})}{|\boldsymbol{r}_{p1}-\boldsymbol{r}_{p2}||\boldsymbol{r}_{p3}-\boldsymbol{r}_{p2}|}\right]$$
$$= \cos^{-1}\frac{\boldsymbol{-a}\bullet\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \cos^{-1}\left[\frac{33}{\sqrt{45}\sqrt{83}}\right] = \underline{\underline{57.32^{o}}}$$

Note that $\alpha+\beta+\gamma=180^{o}$.

**Prob. 1.17**

Given $\boldsymbol{r}_P = (-1,4,8)$, $\boldsymbol{r}_Q = (2,-1,3)$, $\boldsymbol{r}_R = (-1,2,3)$

(a) $|PQ| = \sqrt{9+25+25} = \underline{\underline{7.6811}}$

(b) $\underline{\underline{\boldsymbol{PR} = -2\boldsymbol{a}_y - 5\boldsymbol{a}_z}}$

(c ) $\angle PQR = \cos^{-1}\left(\dfrac{\boldsymbol{QP}\bullet\boldsymbol{QR}}{|\boldsymbol{QP}||\boldsymbol{QR}|}\right) = \underline{\underline{42.57^o}}$

(d) Area of triangle PQR = $\underline{11.023}$

(e) Perimeter = $\underline{17.31}$

**Prob.1.18**

Let R be the midpoint of PQ.

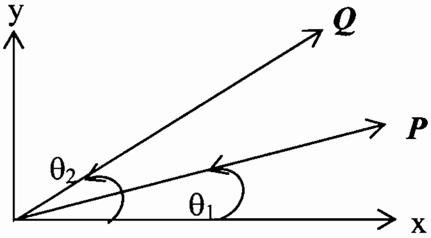
$$\boldsymbol{r}_R = \frac{1}{2}\{(2,4,-1)+(12,16,9)\} = (7,10,4)$$

$$OR = \sqrt{49+100+16} = \sqrt{165} = 12.845$$

$$t = \frac{OR}{v} = \frac{12.845}{300} = \underline{\underline{42.82 \text{ ms}}}$$

**Prob. 1.19**

(a) Let **P** and **Q** be as shown below:



$|\boldsymbol{P}| = \cos^2\theta_1 + \sin^2\theta_1 = 1, |\boldsymbol{Q}| = \cos^2\theta_2 + \sin^2\theta_2 = 1,$

Hence **P** and **Q** are unit vectors.

(b) $\boldsymbol{P}\cdot\boldsymbol{Q} = (1)(1)\cos(\theta_2 - \theta_1)$

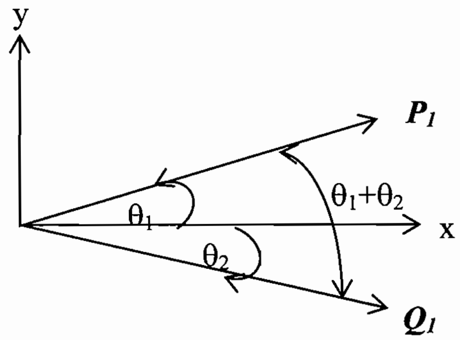
But $\boldsymbol{P}\cdot\boldsymbol{Q} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$. Thus,

$\underline{\underline{\cos(\theta_2-\theta_1) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}}$

Let $\boldsymbol{P}_1 = \boldsymbol{P} = \cos\theta_1\boldsymbol{a}_x + \sin\theta_1\boldsymbol{a}_y$ and

$\boldsymbol{Q}_1 = \cos\theta_2\boldsymbol{a}_x - \sin\theta_2\boldsymbol{a}_y$.

$P_1$ and $Q_1$ are unit vectors as shown below:



$$\boldsymbol{P}_1 \cdot \boldsymbol{Q}_1 = (1)(1)\cos(\theta_1 + \theta_2)$$

But $\boldsymbol{P}_1 \cdot \boldsymbol{Q}_1 = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$,

$$\underline{\underline{\cos(\theta_2 + \theta_1) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}}$$

Alternatively, we can obtain this formula from the previous one by replacing $\theta_2$ by $-\theta_2$ in $\boldsymbol{Q}$.

(c )

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}|(\cos\theta_1 - \cos\theta_2)a_x + (\sin\theta_1 - \sin\theta_2)a_y$$

$$= \frac{1}{2}\sqrt{\cos^2\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 + \sin^2\theta_2 - 2\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2}$$

$$= \frac{1}{2}\sqrt{2 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)} = \frac{1}{2}\sqrt{2 - 2\cos(\theta_2 - \theta_1)}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between **P** and **Q.**

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}\sqrt{2-2\cos\theta}$$

But $\cos 2A = 1 - 2\sin^2 A$.

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}\sqrt{2-2+4\sin^2\theta/2} = \sin\theta/2$$

Thus,

$$\underline{\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = |\sin\frac{\theta_2-\theta_1}{2}|}$$

**Prob. 1.20**

$$w = \frac{w(1,-2,2)}{3} = (1,-2,2), \quad r = r_p - r_o = (1,3,4) - (2,-3,1) = (-1,6,3)$$

$$u = w \times r = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & -2 & 2 \\ -1 & 6 & 3 \end{vmatrix} = (-18,-5,4)$$

$$\underline{\underline{u = -18a_x - 5a_y + 4a_z}}$$

**Prob. 1.21**

$$\text{(a) } T_s = T \cdot a_s = \frac{T \cdot S}{|S|} = \frac{(2,-6,3)\cdot(1,2,1)}{\sqrt{6}} = \frac{-7}{\sqrt{6}} = \underline{\underline{-2.8577}}$$

$$\text{(b) } \boldsymbol{S}_T = (\boldsymbol{S} \cdot \boldsymbol{a}_T)\boldsymbol{a}_T = \frac{(\boldsymbol{S} \cdot \boldsymbol{T})\boldsymbol{T}}{\boldsymbol{T}^2} = \frac{-7(2,-6,3)}{7^2}$$

$$= \underline{\underline{-0.2857\boldsymbol{a}_x + 0.8571\boldsymbol{a}_y - 0.4286\boldsymbol{a}_z}}$$

$$\text{(c) } \sin\theta_{TS} = \frac{|T \times S|}{|T||S|} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 2 & -6 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \frac{|(-12,1,10)|}{7\sqrt{6}} = \frac{\sqrt{245}}{7\sqrt{6}} = 0.9129$$

$$\Rightarrow \theta_{TS} = \underline{\underline{65.91^\circ}}$$

**Prob.1.22**

(a) $\boldsymbol{H}(1,3,-2) = 6\boldsymbol{a}_x + \boldsymbol{a}_y + 4\boldsymbol{a}_z$

$$\boldsymbol{a}_H = \frac{(6,1,4)}{\sqrt{36+1+16}} = \underline{\underline{0.8242\boldsymbol{a}_x + 0.1374\boldsymbol{a}_y + 0.5494\boldsymbol{a}_z}}$$

(b) $|\,\boldsymbol{H}\,| = 10 = \sqrt{4x^2y^2 + (x+z)^2 + z^4}$

or

$$\underline{\underline{100 = 4x^2y^2 + x^2 + 2xz + z^2 + z^4}}$$

**Prob. 1.23**

(a)

$$\boldsymbol{A}\bullet\boldsymbol{B} = (-3,1,2)\bullet(2,-5,1) = -9$$

$$|\boldsymbol{A}| = \sqrt{9+1+4} = \sqrt{14}$$

$$|\boldsymbol{B}| = \sqrt{4+25+1} = \sqrt{30}$$

$$\cos\theta_{AB} = \frac{\boldsymbol{A}\bullet\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|} = \frac{-9}{\sqrt{14\times 30}} = -0.4392$$

$$\theta_{AB} = 116.05^o$$

The minimum angle is $180^o - \theta_{AB} = \underline{\underline{63.95^o}}$

(b)

$$\boldsymbol{A}_C = (\boldsymbol{A}\bullet\boldsymbol{a}_C)\boldsymbol{a}_C = \frac{(\boldsymbol{A}\bullet\boldsymbol{C})\boldsymbol{C}}{|\boldsymbol{C}|^2} = \frac{\left[(-3,1,2)\bullet(0,1,4)\right](\boldsymbol{a}_y + 4a_z)}{1+16}$$

$$\underline{\underline{= 0.5294\boldsymbol{a}_y + 2.118\boldsymbol{a}_z}}$$

(c )

$$\boldsymbol{D} = \boldsymbol{A} + 2\boldsymbol{B} - 3\boldsymbol{C} = (-3,1,2) + (4,-10,2) - 3\boldsymbol{C}$$

$$= (1,-9,4) - (0,3,12) = \underline{\underline{\boldsymbol{a}_x - 12\boldsymbol{a}_y - 8\boldsymbol{a}_z}}$$

(d)

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ -3 & 1 & 2 \\ 2 & -5 & 1 \end{vmatrix} = 11\boldsymbol{a}_x + 7\boldsymbol{a}_y + 13\boldsymbol{a}_z$$

$$(\boldsymbol{A}\times\boldsymbol{B})\bullet\boldsymbol{C} = (11,7,13)\bullet(0,1,4) = \underline{\underline{59}}$$

**Prob. 1.24**

At point (1, 2,-4), x = 1, y = 2, z = -4. **A** = (2, -2, 16) and **B** = (3, 6, 1).

(a) $\boldsymbol{A}\bullet\boldsymbol{B} = 6 - 12 + 16 = \underline{\underline{10}}$

(b)
$$\boldsymbol{A}\bullet\boldsymbol{B} = AB\cos\theta_{AB} \quad \longrightarrow \quad \cos\theta_{AB} = \frac{10}{\sqrt{4+4+256}\sqrt{9+36+1}} = 0.09074$$

$$\theta_{AB} = \underline{\underline{84.79^o}}$$

(c ) $\boldsymbol{A}_B = (\boldsymbol{A}\bullet\boldsymbol{a}_B)\boldsymbol{a}_B = \dfrac{(\boldsymbol{A}\bullet\boldsymbol{B})\boldsymbol{B}}{|\boldsymbol{B}|^2} = \dfrac{10(3,6,1)}{46} = \underline{\underline{0.6521\boldsymbol{a}_x + 1.3043\boldsymbol{a}_y + 0.2174\boldsymbol{a}_z}}$

**Prob. 1.25**

$\boldsymbol{r}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = (-2,1,4) - (1,0,3) = (-3,1,1)$

At P, $\boldsymbol{H} = 0\boldsymbol{a}_x - 1\boldsymbol{a}_z = -\boldsymbol{a}_z$

The scalar component of $\boldsymbol{H}$ along $\boldsymbol{r}_{PQ}$ is

$$D = \boldsymbol{H} \bullet \boldsymbol{a}_{r_{PQ}} = \frac{\boldsymbol{H} \bullet \boldsymbol{r}_{PQ}}{|\boldsymbol{r}_{PQ}|} = \frac{-1}{\sqrt{9+1+1}} = \underline{\underline{-0.3015}}$$

**Prob. 1.26**

(a) At (1,2,3), $\boldsymbol{E} = (2,1,6)$

$$|\boldsymbol{E}| = \sqrt{4+1+36} = \sqrt{41} = \underline{\underline{6.403}}$$

(b) At (1,2,3), $\boldsymbol{F} = (2,-4,6)$

$$\boldsymbol{E}_F = (\boldsymbol{E} \cdot \boldsymbol{a}_F)\boldsymbol{a}_F = \frac{(\boldsymbol{E} \cdot \boldsymbol{F})\boldsymbol{F}}{|\boldsymbol{F}|^2} = \frac{36}{56}(2,-4,6)$$

$$= \underline{\underline{1.286\boldsymbol{a}_x - 2.571\boldsymbol{a}_y + 3.857\boldsymbol{a}_z}}$$

(c) At (0,1,-3), $\boldsymbol{E} = (0,1,-3)$, $\boldsymbol{F} = (0,-1,0)$

$$E \times F = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = (-3,0,0)$$

$$a_{E\times F} = \pm \frac{E \times F}{|E \times F|} = \underline{\underline{\pm a_x}}$$

## CHAPTER –Practice Examples

**P. E.**

(a) At P(1,3,5), x = 1, y = 3, z =5,

$\rho = \sqrt{x^2 + y^2} = \sqrt{10}$, z = 5, $\phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6$

$P(\rho,\phi,z) = P(\sqrt{10}, \tan^{-1} 3,5) = \underline{\underline{P(3.162,71.6^\circ,5)}}$

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$

$\theta = \tan^{-1} \sqrt{x^2 + y^2}/z = \tan^{-1} \sqrt{10}/5 = \tan^{-1} 0.6325 = 32.31^\circ$

$P(r,\theta,\varphi) = \underline{\underline{P(5.916,32.31^\circ,71.56^\circ)}}$

At T(0,-4,3), x = 0 y =-4, z =3;

$\rho = \sqrt{x^2 + y^2} = 4, z = 3, \varphi = \tan^{-1} y/x = \tan^{-1} - 4/0 = 270^\circ$

$\underline{\underline{T(\rho,\varphi,z) = T(4,270^\circ,3).}}$

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = 5, \theta = \tan^{-1} \rho/z = \tan^{-1} 4/3 = 53.13^\circ.$

$\underline{\underline{T(r,\theta,\varphi) = T(5,53.13^\circ,270^\circ).}}$

At S(-3-4-10), x =-3, y =-4, z =-10;

$\rho = \sqrt{x^2 + y^2} = 5,\ \phi = \tan^{-1} - 4/-3 = 233.1^\circ$

$\underline{\underline{S(\rho,\phi,z) = S(5,233.1,-10).}}$

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18.$

$\theta = \tan^{-1} \rho/z = \tan^{-1} \frac{5}{-10} = 153.43^\circ;$

$\underline{\underline{S(r,\theta,\phi) = S(11.18,153.43^\circ,233.1^\circ).}}$

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho\sin\phi$,

$$Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}; \quad Q_y = 0; \quad Q_z = -\frac{z\rho\sin\phi}{\sqrt{\rho^2 + z^2}};$$

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x \cos\phi = \frac{\rho\cos\phi}{\sqrt{\rho^2+z^2}}, \qquad Q_\phi = -Q_x \sin\phi = \frac{-\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

Hence,

$$\underline{\underline{\boldsymbol{Q} = \frac{\rho}{\sqrt{\rho^2+z^2}}(\cos\phi\boldsymbol{a}_\rho - \sin\phi\boldsymbol{a}_\phi - z\sin\phi\boldsymbol{a}_z).}}$$

In Spherical coordinates:

$$Q_x = \frac{r\sin\theta}{r} = \sin\theta;$$

$$Q_z = -r\sin\phi\sin\theta r\cos\theta\frac{1}{r} = -r\sin\theta\cos\theta\sin\phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x\sin\theta\cos\phi + Q_z\cos\theta = \sin^2\theta\cos\phi - r\sin\theta\cos^2\theta\sin\phi.$$

$$Q_\theta = Q_x\cos\theta\cos\phi - Q_z\sin\theta = \sin\theta\cos\theta\cos\phi + r\sin^2\theta\cos\theta\sin\phi.$$

$$Q_\phi = -Q_x\sin\phi = -\sin\theta\sin\phi.$$

$$\therefore\ \underline{\boldsymbol{Q}} = \sin\theta\left(\sin\theta\cos\phi - r\cos^2\theta\sin\phi\right)\boldsymbol{a}_r + \sin\theta\cos\theta(\cos\phi + r\sin\theta\sin\phi)\boldsymbol{a}_\theta - \sin\theta\sin\phi\boldsymbol{a}_\phi\ .$$

At T :

$$\bar{Q}(x,y,z) = \frac{4}{5}\bar{a}_x + \frac{12}{5}\bar{a}_z = 0.8\bar{a}_x + 2.4\bar{a}_z\ ;$$

$$\bar{Q}(\rho,\phi,z) = \frac{4}{5}(\cos 270°\bar{a}_\rho - \sin 270°\bar{a}_\phi - 3\sin 270°\bar{a}_z$$

$$= 0.8\bar{a}_\phi + 2.4\bar{a}_z\ ;$$

$$\bar{Q}(r,\theta,\phi) = \frac{4}{5}(0 - \frac{45}{25}(-1))\bar{a}_r + \frac{4}{5}(\frac{3}{5})(0 + \frac{20}{5}(-1))\bar{a}_\theta - \frac{4}{5}(-1)\bar{a}_\phi$$

$$= \frac{36}{25}\bar{a}_r - \frac{48}{25}\bar{a}_\theta + \frac{4}{5}\bar{a}_\phi = \underline{\underline{1.44\bar{a}_r - 1.92\bar{a}_\theta + 0.8\bar{a}_\phi}}\ ;$$

Note, that the magnitude of vector Q = 2.53 in all 3 cases above.

**P.E.**

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z \sin\phi \\ 3\rho\cos\phi \\ \rho\cos\phi\sin\phi \end{bmatrix}$$

$$\bar{A} = (\rho z \cos\phi \sin\phi - 3\rho\cos\phi\sin\phi)\bar{a}_x + (\rho z \sin^2\phi + 3\rho\cos^2\phi)\bar{a}_y + \rho\cos\phi\sin\phi$$

But $\rho = \sqrt{x^2+y^2}$, $\tan\phi = \frac{y}{x}$, $\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$;

Substituting all this yields:

$$\bar{A} = \frac{1}{\sqrt{x^2+y^2}}[(xyz - 3xy)\bar{a}_x + (zy^2 + 3x^2)\bar{a}_y + xy\bar{a}_z].$$

---

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r^2 \\ 0 \\ \sin\theta \end{bmatrix}$$

Since $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = \frac{y}{z}$;

and $\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, $\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$;

and $\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$, $\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$;

$$B_x = r^2\sin\theta\cos\phi - \sin\theta\sin\phi = rx - \frac{y}{r} = \frac{1}{r}(r^2x - y).$$

$$B_y = r^2\sin\theta\sin\phi + \sin\theta\cos\phi = ry + \frac{x}{r} = \frac{1}{r}(r^2y + x).$$

$$B_z = r^2\cos\theta = rz = \frac{1}{r}(r^2z).$$

Hence,

$$B = \frac{1}{\sqrt{x^2+y^2+z^2}}[\{x(x^2+y^2+z^2) - y\}a_x + \{y(x^2+y^2+z^2) + x\}\bar{a}_y + z(x^2+y^2+z^2)\bar{a}_z]$$

**CHAPTER —Problems**

(a) $x = \rho\cos\phi = 5\cos 120^\circ = -2.5$

$y = \rho\sin\phi = 5\sin 120^\circ = 4.33$, $z = 0$

$P_1(x, y, z) = \underline{\underline{P_1(-2.5, 4.33, 0)}}$.

(b) $x = 1\cos 30^\circ = 0.866$, $y = 1\sin 30^\circ = 0.5$, $z = -10$

$P_2(x, y, z) = \underline{\underline{P_2(0.866, 0.5, -10)}}$.

(c) $x = r\sin\theta\cos\phi = 10\sin 135^\circ\cos 90^\circ = 0$

$y = r\sin\theta\sin\phi = 10\sin 135^\circ\sin 90^\circ = 7.071$

$z = r\cos\theta = 10\cos 135^\circ = -7.071$

$P_3(x, y, z) = \underline{\underline{P_3(0, 7.071, -7.071)}}$

(d) $x = 3\sin 30^\circ\cos 240^\circ = -0.75$

$y = 3\sin 30^\circ\sin 240^\circ = -1.299$

$z = r\cos\theta = 3\cos 30^\circ = 2.598$

$P_4(x, y, z) = \underline{\underline{P_4(-0.75, -1.299, 2.598)}}$.

**Prob.**

**(a)** Given P(1,-4,-3), convert to cylindrical and spherical values;

$$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-4)^2} = \sqrt{17} = 4.123.$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-4}{1} = 284.04°.$$

$$\therefore P(\rho,\phi,z) = \underline{\underline{(4.123,\ 284.04°,\ -3).}}$$

Spherical:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 16 + 9} = 5.099.$$

$$\theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{4.123}{-3} = 126.04°.$$

$$P(r,\theta,\phi) = \underline{\underline{P(5.099,\ 126.04°,\ 284.04°).}}$$

(b) $\rho = 3, \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{3} = 0^o$

$$Q(\rho,\phi,z) = \underline{\underline{Q(3, 0^o, 5)}}$$

$$r = \sqrt{9 + 0 + 25} = 5.831, \qquad \theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{3}{5} = 30.96^o$$

$$Q(r,\theta,\phi) = \underline{\underline{Q(5.831, 30.96^o, 0^o)}}$$

(c ) $\rho = \sqrt{4 + 36} = 6.325, \qquad \phi = \tan^{-1}\frac{6}{-2} = 108.4^o$

$$R(\rho,\phi,z) = \underline{\underline{R(6.325, 108.4^o, 0)}}$$

$$r = \rho = 6.325, \qquad \theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{6.325}{0} = 90^o$$

$$R(r,\theta,\phi) = \underline{\underline{R(6.325, 90^o, 108.4^o)}}$$

**Prob.**

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(a) $\vec{a}_x \cdot \vec{a}_\rho = (\cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi) \cdot \vec{a}_\rho = \cos\phi$

$\vec{a}_x \cdot \vec{a}_\phi = (\cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi) \cdot \vec{a}_\phi = -\sin\phi$

$\vec{a}_y \cdot \vec{a}_\rho = (\sin\phi \, \vec{a}_\rho + \cos\phi \, \vec{a}_\phi) \cdot \vec{a}_\rho = \sin\phi$

$\vec{a}_y \cdot \vec{a}_\phi = (\sin\phi \, \vec{a}_\rho + \cos\phi \, \vec{a}_\phi) \cdot \vec{a}_\phi = \cos\phi$

(b) Since $\vec{a}_\rho$, $\vec{a}_\phi$, and $\vec{a}_z$ are mutually orthogonal,

$\vec{a}_z \cdot \vec{a}_z = 1$, $\vec{a}_z \cdot \vec{a}_\rho = 0 = \vec{a}_z \cdot \vec{a}_\phi$.

Also, $\vec{a}_x \cdot \vec{a}_z = 0 = \vec{a}_y \cdot \vec{a}_z$. Hence

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{a}_x \cdot \vec{a}_\rho & \vec{a}_x \cdot \vec{a}_\phi & \vec{a}_x \cdot \vec{a}_z \\ \vec{a}_y \cdot \vec{a}_\rho & \vec{a}_y \cdot \vec{a}_\phi & \vec{a}_y \cdot \vec{a}_z \\ \vec{a}_z \cdot \vec{a}_\rho & \vec{a}_z \cdot \vec{a}_\phi & \vec{a}_z \cdot \vec{a}_z \end{bmatrix}$$

(c) In spherical system,

$\vec{a}_x = \sin\theta\cos\phi \, \vec{a}_r + \cos\theta\cos\phi \, \vec{a}_\theta - \sin\phi \, \vec{a}_\phi$

$\vec{a}_y = \sin\theta\sin\phi \, \vec{a}_r + \cos\theta\sin\phi \, \vec{a}_\theta + \cos\phi \, \vec{a}_\phi$

$\vec{a}_z = \cos\theta \, \vec{a}_r - \sin\theta \, \vec{a}_\theta$.

Hence $\vec{a}_x \cdot \vec{a}_r = \sin\theta\cos\phi$,

$\vec{a}_x \cdot \vec{a}_\theta = \cos\theta\cos\phi$,

$\vec{a}_y \cdot \vec{a}_r = \sin\theta\sin\phi$,

$\vec{a}_y \cdot \vec{a}_\theta = \cos\theta\sin\phi$,

$\vec{a}_z \cdot \vec{a}_r = \cos\theta$,

$\vec{a}_z \cdot \vec{a}_\theta = -\sin\theta$.

**Prob.**

**(a)**

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$

$$V = \underline{\underline{\rho z\cos\phi - \rho^2\sin\phi\cos\phi + \rho z\sin\phi}}$$

**(b)**

$$U = x^2 + y^2 + z^2 + y^2 + 2z^2$$

$$= r^2 + r^2\sin^2\theta\sin^2\phi + 2r^2\cos^2\theta$$

$$= \underline{\underline{r^2[1 + \sin^2\theta\sin^2\phi + 2\cos^2\theta]}}$$

**Prob.**

(a)
$$\begin{bmatrix} P_\rho \\ P_\phi \\ P_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y+z \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{P} = (y+z)\left(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi\right)$$

But $y = \rho\sin\phi, \quad z = z$

$$\vec{P} = \underline{\underline{(\rho\sin\phi + z)(\cos\phi\,\vec{a}_\rho - \sin\phi\,a_\phi)}}$$

$$\begin{bmatrix} P_r \\ P_\theta \\ P_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y+z \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{P} = (y+z)(\sin\theta\cos\phi\,\bar{a}_r + \cos\theta\cos\phi\,\bar{a}_\theta - \sin\phi\,\vec{a}_\phi)$$

But $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$$\vec{P} = r(\sin\theta\sin\phi + \cos\theta)(\sin\theta\cos\phi\,\bar{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi).$$

(b) $\bar{Q}(\rho,\phi,z) = (y\cos\phi + xz\sin\phi)\vec{a}_\rho + (-y\sin\phi + xz\cos\phi)\bar{a}_\phi + (x+y)\bar{a}_z.$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$,

$$\vec{Q} = \frac{1}{2}\rho\sin 2\phi(1+z)\vec{a}_\rho + \rho(z\cos^2\phi - \sin^2\phi)\vec{a}_\phi + \rho(\cos\phi + \sin\phi)\bar{a}_z.$$

$$\vec{Q}(r,\theta,\phi) = [y\sin\theta\cos\phi + xz\sin\theta\sin\phi + (x+y)\cos\theta]\vec{a}_r + [y\cos\theta\cos\phi + xz\cos\theta\sin\phi - (x+y)\sin\theta]\bar{a}_\theta + [-y\sin\phi + xz\cos\phi]\vec{a}_\phi$$

But $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$\vec{Q} = [\frac{1}{2}r\sin 2\phi\sin^2\theta + \frac{1}{2}r^2\sin^2\theta\cos\theta\sin 2\phi + \frac{1}{2}r(\cos\phi + \sin\phi)\cos 2\theta]\vec{a}_r$$

$$+\left[\frac{1}{4} r \sin 2\theta \sin 2\phi + \frac{1}{2} r^2 \cos^2\theta \sin\theta \sin 2\phi\right.$$

$$\left. - \frac{r}{2}(\cos\phi + \sin\phi) \sin^2\theta\right] \vec{a}_\theta$$

$$+ \underline{\underline{\left(-r\sin\theta \sin^2\phi + \frac{1}{2} r^2 \sin 2\theta \cos^2\phi\right) \vec{a}_\phi}} \cdot$$

(c) $\vec{T}(\rho,\phi,z) = \left[\left(\frac{x^2}{x^2+y^2} - y^2\right)\cos\phi + \left(\frac{xyz}{x^2+y^2} + xy\right)\sin\phi\right]\vec{a}_\rho$

$$+\left[\left(\frac{-x^2}{x^2+y^2} + y^2\right)\sin\phi + \left(\frac{xyz}{x^2+y^2} + xy\right)\cos\phi\right]\vec{a}_\phi + \vec{a}_z$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$,

$$\vec{T} = (\cos^3\phi - \rho^2 \sin^2\phi \cancel{\cos\phi} + \sin^2\phi\cos\phi + \rho^2 \cancel{\sin^2}\phi\cos\phi)\,\bar{\rho}$$

$$+ (-\sin\phi \cancel{\cos^2}\phi + \rho^2 \sin^3\phi + \sin\phi \cancel{\cos^2}\phi$$

$$+ \rho^2 \sin\phi \cos^2\phi)\, \vec{a}_\phi + \vec{a}_z$$

$$\vec{T}(\rho,\phi,z) = \underline{\underline{\cos\phi\, \vec{a}_\rho + \rho^2 \sin\phi\, \vec{a}_\phi + \vec{a}_z}} \cdot$$

$$\begin{bmatrix} T_r \\ T_\theta \\ T_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\theta\sin\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x^2}{x^2+y^2} - y^2 \\ \frac{xy}{x^2+y^2} + xy \\ 1 \end{bmatrix}$$

Since $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$;

$$T_r = \left( \frac{r^2 \sin^2\theta \cos^2\phi}{r^2 \sin^2\theta} - r^2 \sin^2\theta \sin^2\phi \right) \sin\theta \cos\phi$$

$$+ (\cos\phi \sin\phi + r^2 \sin^2\theta \sin\phi \cos\phi) \sin\theta \sin\phi + \cos\theta$$

$$= \sin\theta \cos\phi + \cos\theta ,$$

$$T_\theta = (\cos^2\phi - r^2 \sin^2\theta \sin^2\phi) \cos\theta \sin\phi$$

$$+ (\cos\phi \sin\phi + r^2 \sin^2\theta \sin\phi \cos\phi) \cos\theta \sin\phi - \sin\theta$$

$$= \cos\theta \cos\phi - \sin\theta$$

$$T_\phi = -(\cos^2\phi - r^2 \sin^2\theta \sin^2\phi) \sin\phi$$

$$+ (\cos\phi \sin\phi + r^2 \sin^2\theta \sin\phi \cos\phi) \cos\phi$$

$$= -\cos^2\phi \sin\phi + r^2 \sin^2\theta \sin^3\phi + \sin\phi \cos\phi$$

$$+ r^2 \sin^2\theta \sin\phi \cos^2\phi = r^2 \sin^2\theta \sin\phi$$

$$\vec{T}(r,\theta,\phi) = (\sin\theta \cos\phi + \cos\theta) \vec{a}_r$$

$$+ (\cos\theta \cos\phi - \sin\theta) \vec{a}_\theta$$

$$+ r^2 \sin^2\theta \sin\phi \, \vec{a}_\phi .$$

(d)
$$\begin{bmatrix} S_\rho \\ S_\phi \\ S_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{y}{x^2+y^2} \\ -\frac{x}{x^2+y^2} \\ 10 \end{bmatrix}$$

$$= \left( \cancel{\frac{\rho \sin\phi}{\rho^2}} \cos\phi - \cancel{\frac{\rho \cos\phi}{\rho^2}} \sin\phi \right) \hat{a}_\rho$$

$$+\left(-\frac{\rho\sin\phi}{\rho^2}\sin\phi-\frac{\rho\cos\phi}{\rho^2}\cos\phi\right)\vec{a}_\phi+10\vec{a}_z$$

$$\vec{S}(\rho,\phi,z)=\underline{\underline{-\frac{1}{\rho}\vec{a}_\phi+10\vec{a}_z}}\,.$$

$$\begin{bmatrix}S_r\\S_\theta\\S_\phi\end{bmatrix}=\begin{bmatrix}\sin^2\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\frac{y}{x^2+y^2}\\ \frac{-x}{x^2+y^2}\\ 10\end{bmatrix}$$

$$=\left(\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta}\sin\theta\cos\phi-\frac{r\sin\theta\cos\phi}{r^2\sin^2\theta}\sin\theta\cos\phi\right.$$

$$\left.+10\cos\theta\right)\vec{a}_r+\left(\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta}\cos\theta\cos\phi\right.$$

$$\left.-\frac{r\sin\theta\cos\phi}{r^2\sin^2\theta}\cos\theta\sin\phi-10\sin\theta\right)\vec{a}_\theta$$

$$+\left(-\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta}\sin\phi-\frac{r\sin\theta\cos\phi}{r^2\sin^2\theta}\cos\phi\right)\vec{a}_\phi$$

$$\bar{S}(r,\theta,\phi)=\underline{\underline{10\cos\theta\,\bar{a}_r-10\sin\theta\,\bar{a}_\theta-\frac{1}{r\sin\theta}\bar{a}_\phi}}\,.$$

**Prob.** (a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x}{\sqrt{\rho^2+z^2}} \\ \dfrac{y}{\sqrt{\rho^2+z^2}} \\ \dfrac{4}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho}{\sqrt{\rho^2+z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2+z^2}}[-\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2+z^2}};$$

$$\underline{\underline{\bar{F} = \frac{1}{\sqrt{\rho^2+z^2}}(\rho\,\bar{a}_\rho + 4\bar{a}_z).}}$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{x}{r} \\ \dfrac{y}{r} \\ \dfrac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r}\sin^2\theta\cos^2\phi + \frac{r}{r}\sin^2\theta\sin^2\phi + \frac{4}{r}\cos\theta \quad = \sin^2\theta + \frac{4}{r}\cos\theta;$$

$$F_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \frac{4}{r}\sin\theta = \sin\theta\cos\theta - \frac{4}{r}\sin\theta;$$

$$F_\phi = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0;$$

$$\therefore \underline{\underline{\bar{F} = (\sin^2\theta + \frac{4}{r}\cos\theta)\bar{a}_r + \sin\theta(\cos\theta - \frac{4}{r})\bar{a}_\theta.}}$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{y\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{z\rho^2}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho^3}{\sqrt{\rho^2+z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2+z^2}};$$

$$\underline{\underline{\bar{G} = \frac{\rho^2}{\sqrt{\rho^2+z^2}}(\rho\,\bar{a}_\rho + z\,\bar{a}_z).}}$$

Spherical :

$$\mathbf{G} = \frac{\rho^2}{r}(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) = \frac{r^2\sin^2\theta}{r} r\mathbf{a}_r = \underline{\underline{r^2\sin^2\theta\mathbf{a}_r}}$$

**Prob.**

(a) $\vec{A} = (x^2z^2\sin\phi + yz\sin\phi)\vec{a}_\rho + (yz\cos\phi - y^2z^2\cos\phi)\vec{a}_\phi$
$+ yx\,\vec{a}_z$

$= (x^2z^2 + yz)\sin\phi\,\vec{a}_\rho + yz(1 - yz)\cos\phi\,\vec{a}_\phi$
$+ xy\,\vec{a}_z$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$A_x = A_\rho \cos\phi - A_\phi \sin\phi = x^2 z^2 \cos\phi \sin\phi + \cancel{yz \sin\phi \cos\phi}$$
$$- \cancel{yz \sin\phi \cos\phi} + y^2 z^2 \sin\phi \cos\phi$$
$$= (x^2+y^2) z^2 \cos\phi \sin\phi = (x^2+y^2) \cdot z^2 \cdot \frac{xy}{x^2+y^2} = xyz^2$$

$$A_y = A_\rho \sin\phi + A_\phi \cos\phi$$
$$= x^2 z^2 \sin^2\phi + yz \sin^2\phi + yz \cos^2\phi - y^2 z^2 \cos^2\phi$$
$$= yz + z^2 \left( \cancel{x^2 \cdot \frac{y^2}{x^2+y^2}} - \cancel{y^2 \cdot \frac{x^2}{x^2+y^2}} \right) = yz$$

$$\vec{A}(x,y,z) = \underline{\underline{xyz^2 \bar{a}_x + yz \bar{a}_y + xy \bar{a}_z}}.$$

(b) $\vec{B} = r^3 (-\sin\phi \vec{a}_x + \cos\phi \vec{a}_y)$

$+ 4r [-\sin\theta \bar{a}_y + \cos\theta (\cos\phi \bar{a}_x + \sin\phi \vec{a}_y)] \cos\theta \sin$

$+ 6r^2 \sin\theta \cos\phi [\cos\theta \bar{a}_z + \sin\theta (\cos\phi \bar{a}_x + \sin\phi \vec{a}_y)]$

$= \vec{a}_x (-r^3 \sin\phi + 4r \cos^2\theta \sin\phi \cos\phi + 6r^2 \sin^2\theta \cos^2\phi)$

$+ \bar{a}_y (r^3 \cos\phi + 4r \cos^2\theta \sin^2\phi + 6r^2 \sin^2\theta \sin\phi \cos\phi)$

$+ \bar{a}_z (-4r \sin\theta + 6r^2 \sin\theta \cos\theta \cos\phi)$

But $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \dfrac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = \dfrac{y}{x}$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}.$$

$$\vec{B}(x,y,z) = \left[ 6x^2 + \frac{4xyz^2}{(x^2+y^2)(x^2+y^2+z^2)^{1/2}} - \right.$$

$$\left. - \frac{y(x^2+y^2+z^2)^{3/2}}{(x^2+y^2)^{1/2}} \right] \bar{a}_x$$

$$+ \left[ 6xyz + \frac{4y^2z^2}{(x^2+y^2)(x^2+y^2+z^2)^{1/2}} \right.$$

$$\left. + \frac{x(x^2+y^2+z^2)^{3/2}}{(x^2+y^2)^{3/2}} \right] \bar{a}_y$$

$$+ \left[ 6xz - 4(x^2+y^2)^{1/2} \right] \bar{a}_z .$$

**Prob.**

**(a)**

$$\begin{bmatrix} D_\rho \\ D_\phi \\ D_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix}$$

$$D_\rho = (x+z)\sin\phi = (\rho\cos\phi + z)\sin\phi$$

$$D_\phi = (x+z)\cos\phi = (\rho\cos\phi + z)\cos\phi$$

$$\bar{D} = \underline{\underline{(\rho\cos\phi + z)[\sin\phi\,\bar{a}_\rho + \cos\phi\,\bar{a}_\phi]}}$$

*Spherical*: Since $\mathrm{D_x} = D_z = 0$, we may leave out the first and third column of the transformation matrix. Thus,

$$\begin{bmatrix} D_r \\ D_\theta \\ D_\phi \end{bmatrix} = \begin{bmatrix} \ldots & \sin\theta\sin\phi & \ldots \\ \ldots & \cos\theta\sin\phi & \ldots \\ \ldots & \cos\phi & \ldots \end{bmatrix} \begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix}$$

$$D_r = (x+z)\sin\theta\sin\phi = r(\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi.$$

$$D_\theta = (x+z)\cos\theta\sin\phi = r(\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi.$$

$$D_\phi = (x+z)\cos\phi \qquad = r(\sin\theta\cos\phi + \cos\theta)\cos\phi.$$

$$\bar{D} = \underline{\underline{r(\sin\theta\cos\phi + \cos\theta)[\sin\theta\sin\phi\,\bar{a}_r + \cos\theta\sin\phi\,\bar{a}_\theta + \cos\phi\,\bar{a}_\phi]}}.$$

**(b)** Cylindrical:

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^2 - x^2 \\ xyz \\ x^2 - z^2 \end{bmatrix}$$

$$\begin{aligned} E_\rho &= (y^2 - x^2)\cos\phi + xyz\sin\phi \\ &= \rho^2(\sin^2\phi - \cos^2\phi)\cos\phi + \rho^2 z\cos\phi\sin^2\phi \\ &= -\rho^2\cos 2\phi\cos\phi + \rho^2 z\sin^2\phi\cos\phi. \end{aligned}$$

$$\begin{aligned} E_\phi &= -(y^2 - x^2)\sin\phi + xyz\cos\phi \\ &= \rho^2\cos 2\phi\sin\phi + \rho^2 z\sin\phi\cos^2\phi. \end{aligned}$$

$$E_z = x^2 - z^2 = \rho^2\cos^2\phi - z^2.$$

$$\bar{E} = \underline{\underline{\rho^2\cos\phi(z\sin^2\phi - \cos 2\phi)\bar{a}_\rho + \rho^2\sin\phi(z\cos^2\phi + \cos 2\phi)\bar{a}_\phi + (\rho^2\cos^2\phi - z^2)\bar{a}_z}}.$$

In spherical:

$$\begin{bmatrix} E_r \\ E_\theta \\ E_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y^2 - x^2 \\ xyz \\ x^2 - z^2 \end{bmatrix}$$

$$E_r = (y^2 - x^2)\sin\theta\cos\phi + xyz\sin\theta\sin\phi + (x^2 - z^2)\cos\theta;$$

$$\text{but } x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta;$$

$$E_r = r^2\sin^3\theta(\sin^2\phi - \cos^2\phi)\cos\phi + r^3\sin^3\theta\cos\theta\sin^2\phi\cos\phi$$
$$+ r^2(\sin^2\theta\cos^2\phi - \cos^2\theta)\cos\theta;$$

$$E_\theta = (y^2 - x^2)\cos\theta\cos\phi + xyz\cos\theta\sin\phi - (x^2 - z^2)\sin\theta;$$
$$= -r^2\sin^2\theta\cos 2\phi\cos\theta\cos\phi + r^3\sin^2\theta\cos^2\theta\sin^2\phi\cos\phi - r^2(\sin^2\theta\cos^2\phi - \cos^2\theta)\sin\theta;$$

$$E_\phi = (x^2 - y^2)\sin\phi + xyz\cos\phi$$
$$= r^2\sin^2\theta\cos 2\phi\sin\phi + r^3\sin^2\theta\cos^2\phi\sin\phi\cos\theta;$$

$$\bar{E} = [-r^2\sin^3\theta\cos 2\phi\cos\phi + r^3\sin^3\theta\cos\theta\sin^2\phi\cos\phi + r^2(\sin^2\theta\cos^2\phi - \cos^2\theta)\cos\theta]\bar{a}_r +$$
$$[-r^2\sin^2\theta\cos 2\phi\cos\theta\cos\phi + r^3\sin^2\theta\cos^2\theta\sin^2\phi\cos\phi - r^2\sin\theta(\sin^2\theta\cos^2\phi - \cos^2\theta)]\bar{a}_\theta +$$
$$+ \underline{\underline{[r^2\sin^2\theta\cos 2\phi\sin\phi + r^3\sin^2\theta\cos^2\phi\sin\phi\cos\theta]\bar{a}_\phi}}$$

**Prob.**

(a)

$$\begin{bmatrix} H_\rho \\ H_\phi \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$H_\rho = 3\cos\phi + 2\sin\phi, \qquad H_\phi = -3\sin\phi + 2\cos\phi, \qquad H_z = -4$$

$$\mathbf{H} = \underline{\underline{(3\cos\phi + 2\sin\phi)\mathbf{a}_\rho + (-3\sin\phi + 2\cos\phi)\mathbf{a}_\phi - 4\mathbf{a}_z}}$$

(b) At P, $\rho = 2, \quad \phi = 60°, \quad z = -1$

$$\mathbf{H} = (3\cos 60° + 2\sin 60°)\mathbf{a}_\rho + (-3\sin 60° + 2\cos 60°)\mathbf{a}_\phi - 4\mathbf{a}_z$$
$$= \underline{\underline{3.232\mathbf{a}_\rho - 1.598\mathbf{a}_\phi - 4\mathbf{a}_z}}$$

**Prob.**

(a)

$$\begin{bmatrix} H_\rho \\ H_\phi \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xy^2z \\ x^2yz \\ xyz^2 \end{bmatrix}$$

$$H_\rho = xy^2z\cos\phi \; + \; x^2yz\sin\phi \; = \rho^3z\cos^2\phi\,\sin^2\phi + \rho^3z\cos^2\phi\,\sin^2\phi .$$

$$= \frac{1}{2}\rho^3 z\sin^2 2\phi$$

$$H_\phi = -\,xy^2z\sin\phi + x^2yz\cos\phi = -\,\rho^3z\cos\phi\,\sin^3\phi + \rho^3z\cos\phi\,\sin\phi$$

$$= \rho^3z\cos\phi\,\sin\phi\,\cos 2\phi .$$

$$H_z = xyz^2 \; = \rho^2z^2\sin\phi\,\cos\phi .$$

$$\bar{H} = \underline{\underline{\frac{1}{2}\rho^3 z\sin^2 2\phi\,\bar{a}_\rho + \frac{1}{2}\rho^3 z\sin 2\phi\,\cos 2\phi\,\bar{a}_\phi + \frac{1}{2}\rho^3 z\sin 2\phi\,\bar{a}_z .}}$$

$$\begin{bmatrix} H_r \\ H_\theta \\ H_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} xy^2z \\ x^2yz \\ xyz^2 \end{bmatrix}$$

$x = r\sin\theta\,\cos\phi ,\qquad y = r\sin\theta\,\sin\phi ,\qquad z = r\cos\theta .$

$$H_r = xyz[y\sin\theta\cos\phi + x\sin\theta\sin\phi + z\cos\theta$$

$$= r^3\sin^2\theta\cos\theta\sin\phi[r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\sin\phi\cos\phi + r\cos^2\theta]$$

$$H_\theta = xyz[y\cos\theta\cos\phi + x\cos\theta\sin\phi - z\sin\theta]$$

$$= r^3\sin^2\theta\cos\theta\sin\phi\cos\phi\,[r\sin\theta\cos\theta\sin\phi\cos\phi + r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos\theta\sin\theta]$$

$$H_\phi = xyz[-y\sin\phi + x\cos\phi]$$

$$= r^3\sin^2\theta\cos\theta\sin\phi\cos\phi[-r\sin\theta\sin^2\phi + r\sin\theta\cos^2\phi]$$

$$= r^4\sin^3\theta\cos\theta\sin\phi\cos 2\phi.$$

$$\bar{H} = r^4\sin^2\theta\cos\theta\sin\phi\cos\phi[(\sin^2\theta\sin 2\phi + \cos^2\theta)\bar{a}_r +$$

$$\underline{\underline{(\sin\theta\cos\theta\sin 2\phi - \cos\theta\sin\theta)\bar{a}_\theta + \sin\theta\cos 2\phi\,\bar{a}_\phi].}}$$

(b)

$$At\,(3-45),\ \bar{H}(x,y,z) = -60(-4,3,5)$$

$$\left|\bar{H}(x,y,z)\right| = 424.3$$

This will help check $H(\rho,\phi,z)$ and $H(r,\theta,\phi)$

$$\rho = 5,\ z = 5,\ \phi = 360° - \tan^{-1}\frac{4}{3} = 306.87°$$

$$\bar{H} = \frac{1}{2}(125)(5)(-0.96)\bar{a}_\rho + \frac{1}{2}(125)(5)(-0.90)(-0.277))\bar{a}_\phi + \frac{1}{2}(25)(5)(-0.96)\bar{a}_z$$

$$= \underline{\underline{288\bar{a}_\rho + 84\bar{a}_\phi - 300\bar{a}_z}}$$

Spherical,

$$r = \sqrt{50} = 5\sqrt{2};\quad \sin\theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}};\quad \cos\theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\&\quad \sin\phi = -\frac{4}{5},\quad \cos\phi = \frac{3}{5}.$$

$$\therefore\ \bar{H} = 2500(\frac{1}{2})(\frac{1}{\sqrt{2}})(-\frac{12}{25})[\{\frac{1}{2}*2(-\frac{12}{28}) + \frac{1}{2}\}\bar{a}_r + \{\frac{1}{2}*2(-\frac{12}{25}) - \frac{1}{2}\}\bar{a}_\theta + \frac{1}{\sqrt{2}}\{\frac{9}{12} - \frac{16}{25}\}\bar{a}_\phi]$$

$$= \underline{\underline{-8.485\bar{a}_r + 415.8\bar{a}_\theta + 84\bar{a}_\phi.}}$$

**Prob.**

(a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1}\frac{\rho}{z}; \qquad \phi = \phi.$$

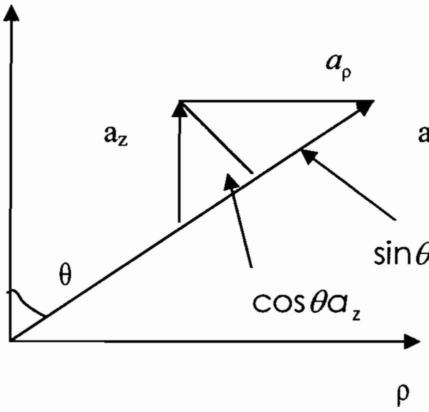
*or*

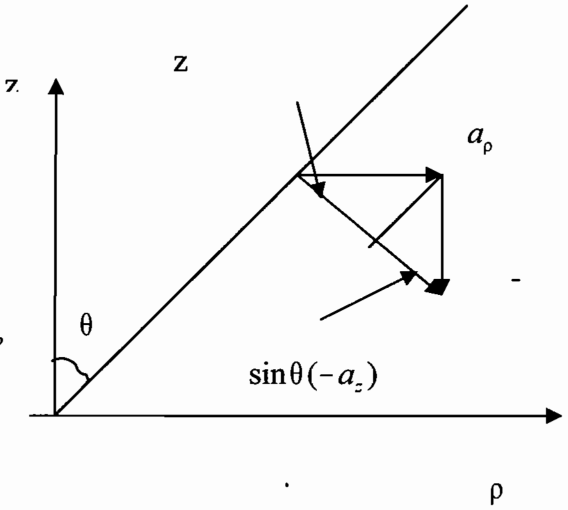
$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi}.$$

$$= r\sin\theta;$$

$$z = r\cos\theta; \qquad \phi = \phi.$$

(b) From the figures below,





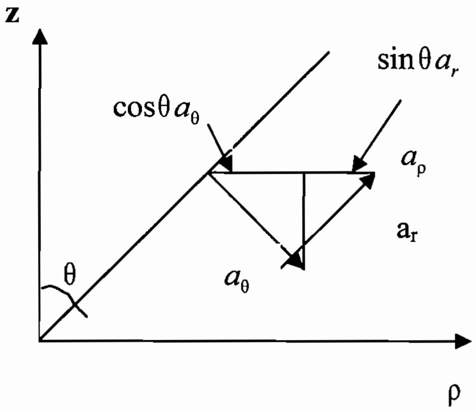
$$\bar{a}_r = \sin\theta\bar{a}_\rho + \cos\theta\bar{a}_z; \quad \bar{a}_\theta = \cos\theta\bar{a}_\rho - \sin\theta\bar{a}_z; \quad \bar{a}_\phi = \bar{a}_\phi.$$

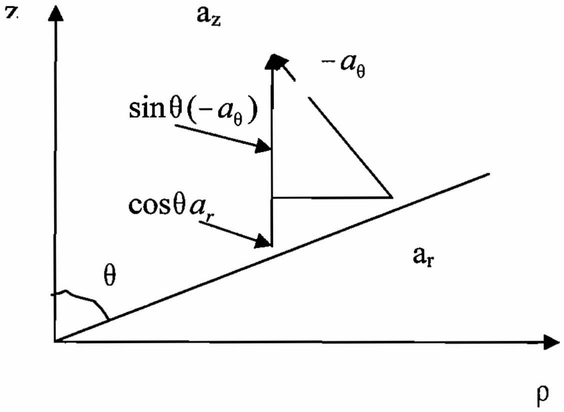
Hence,

$$\begin{bmatrix} \bar{a}_r \\ \bar{a}_\theta \\ \bar{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_\rho \\ \bar{a}_\phi \\ \bar{a}_z \end{bmatrix}$$

From the figures below,

$$\bar{a}_\rho = \cos\theta\bar{a}_\theta + \sin\theta\bar{a}_r; \quad \bar{a}_z = \cos\theta\bar{a}_r - \sin\theta\bar{a}_\theta; \quad \bar{a}_\phi = \bar{a}_\phi.$$





$$\begin{bmatrix} \bar{a}_\rho \\ \bar{a}_\phi \\ \bar{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_r \\ \bar{a}_\theta \\ \bar{a}_z \end{bmatrix}$$

**Prob.**

(a) $r = \sqrt{8^2 + (-15)^2 + 12^2} = 20.81$

$\theta = \tan^{-1} \sqrt{x^2+y^2}/z = \tan^{-1} \frac{17}{12} = 54.78°$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-15}{8} = 298^\circ$$

$$P(r, \theta, \phi) = \underline{\underline{P(20.81,\ 54.78^\circ,\ 298^\circ)}}.$$

(b) $$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2xy \\ -x^2 \\ 0 \end{bmatrix}$$

$F_\rho = 2xy\cos\phi - x^2\sin\phi$, $F_\phi = -2xy\sin\phi - x^2\cos\phi$,

$F_z = 0$.

But $x = \rho\cos\phi$, $y = \rho\sin\phi$,

$$F_\rho = \rho^2(2\cos^2\phi\sin\phi - \cos^2\phi\sin\phi) = \rho^2\cos^2\phi\sin\phi$$

$$F_\phi = -\rho^2(2\cos\phi\sin^2\phi + \cos^3\phi) = -\rho^2\cos\phi(1+\sin^2\phi)$$

$$\vec{F} = \rho^2\cos\phi\left[\cos\phi\sin\phi\,\vec{a}_\rho - (1+\sin^2\phi)\,\vec{a}_\phi\right].$$

**Prob.**

Using eq. (

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2$$
$$= 10^2 + 5^2 - 2(5)(10)\cos(60^\circ - 30^\circ) + (-4-2)^2 = 74.4$$
$$d = \sqrt{74.4} = \underline{\underline{8.625}}$$

**Prob.**

(a) $d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$

(b) $$d^2 = 3^2 + 5^2 - 2(3)(5)\cos\pi + (-1-5)^2 = 100$$
$$d = \sqrt{100} = \underline{\underline{10}}$$

(c)

$$d^2 = 10^2 + 5^2 - 2(10)(5)\cos\frac{\pi}{4}\cos\frac{\pi}{6} - 2(10)(5)\sin\frac{\pi}{4}\sin\frac{\pi}{6}\cos(7\frac{\pi}{4} - \frac{3\pi}{4})$$

$$= 125 - 100(\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}) = 125 - 100\cos 75^\circ = 99.12$$

$$d = \sqrt{99.12} = \underline{\underline{9.956.}}$$

**Prob.**

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\underline{\underline{= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}}}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\underline{\underline{= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_\phi \end{bmatrix}}}$$

**Prob.**

At $P(3,-2,6)$, $x=3$, $y=-2$, $z=6$,

$$\rho = \sqrt{x^2+y^2} = \sqrt{13}, \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-2}{3}$$

$$r = \sqrt{x^2+y^2+z^2} = 7, \quad \theta = \tan^{-1}\rho/z = \tan^{-1}\frac{\sqrt{13}}{6}$$

(a) $$\vec{A} = \sqrt{13}\,\frac{3}{\sqrt{13}}\,\vec{a}_\rho + 6\left(\frac{-2}{\sqrt{13}}\right)\vec{a}_\phi - \sqrt{13}\,(36)\,\vec{a}_z$$

$$= \underline{\underline{3\vec{a}_\rho - 3.328\,\vec{a}_\phi - 129.8\,\vec{a}_z}}\,.$$

$$\vec{B} = 7\left(\frac{\sqrt{13}}{7}\right)\vec{a}_r + 49\left(\frac{6}{7}\right)\left(-\frac{2}{\sqrt{13}}\right)\vec{a}_\theta = \underline{\underline{3.606\,\vec{a}_r - 23.3\,\vec{a}_\theta}}\,.$$

(b) $$\vec{A}_B = (\vec{A}\cdot\vec{a}_B)\,\vec{a}_B = \frac{(\vec{A}\cdot\vec{B})\,\vec{B}}{B^2}$$

In spherical system,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{13}}{17} & 0 & \frac{6}{7} \\ \frac{6}{7} & 0 & -\frac{\sqrt{13}}{7} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ \frac{-12}{\sqrt{13}} \\ -36\sqrt{13} \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{13}}{7} - 2\,\frac{16\sqrt{13}}{7} \\ \frac{13}{7} + \frac{36\times 13}{7} \\ -\frac{12}{\sqrt{13}} \end{bmatrix}$$

$$\vec{A} = -109.71\,\vec{a}_r + 69.43\,\vec{a}_\theta - 3.328\,\vec{a}_\phi\,.$$

Hence

$$\vec{A}_B = \frac{-2013.33}{555.9}\left(3.606\,\bar{a}_r - 23.3\,\bar{a}_\theta\right)$$

$$= \underline{\underline{-13.06\,\bar{a}_r + 84.39\,\bar{a}_\theta}}.$$

(c) In cylindrical system,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{13}}{7} & \frac{6}{7} & 0 \\ 0 & 0 & 1 \\ \frac{6}{7} & -\frac{\sqrt{13}}{7} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{13} \\ -\frac{12 \times 49}{7\sqrt{13}} \\ 0 \end{bmatrix}$$

$$\vec{B} = \left(\frac{13}{7} - \frac{72}{\sqrt{13}}\right)\bar{a}_\rho + \left(\frac{6\sqrt{13}}{7} + 12\right)\bar{a}_z = -18.11\,\bar{a}_\rho + 15.09\,\bar{a}_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 3 & -3.328 & -129.8 \\ -18.11 & 0 & 15.09 \end{vmatrix}$$

$$= -52.92\,\bar{a}_\rho + 2396\,\bar{a}_\phi - 60.27\,\bar{a}_z$$

$$\hat{a}_{A\times B} = \underline{\underline{\pm\left(0.0221\,\bar{a}_\rho - 0.9994\,\bar{a}_\phi + 0.0251\,\bar{a}_z\right)}}.$$

**Prob.**

(a) $\mathbf{A} \bullet \mathbf{B} = (5,2,-1) \bullet (1,-3,4) = \underline{\underline{-5}}$

(b) $\mathbf{A} x \mathbf{B} = \begin{vmatrix} 5 & 2 & -1 \\ 1 & -3 & 4 \end{vmatrix} = \underline{\underline{5\mathbf{a}_\rho - 21\mathbf{a}_\phi - 17\mathbf{a}_z}}$

(c) $\cos\theta_{AB} = \frac{\mathbf{A} \bullet \mathbf{B}}{AB} = \frac{-5}{\sqrt{25+4+1}\sqrt{1+9+16}} = -0.179 \longrightarrow \theta_{AB} = \underline{\underline{100.31^\circ}}$

(d) $$\mathbf{a}_n = \frac{\mathbf{A}\mathrm{x}\mathbf{B}}{|\mathbf{A}\mathrm{x}\mathbf{B}|} = \frac{(5,-21,-17)}{\sqrt{5^2+21^2+17^2}} = \frac{(5,-21,-17)}{\sqrt{755}}$$

$$\underline{\underline{= 0.182\mathbf{a}_\rho - 0.7643\mathbf{a}_\phi - 0.6187\mathbf{a}_z}}$$

(e) $$\mathbf{A}_B = (\mathbf{A}\bullet\mathbf{a}_B)\mathbf{a}_B = \frac{(\mathbf{A}\bullet\mathbf{B})\mathbf{B}}{B^2} = \frac{-5\mathbf{B}}{26} = \underline{\underline{-0.1923\mathbf{a}_\rho + 0.5769\mathbf{a}_\phi - 0.7692\mathbf{a}_z}}$$

**Prob.**

$$\text{At } \mathrm{P}\left(8, 30^\circ, 60^\circ\right) = P(r,\theta,\phi),$$

$$x = r\sin\theta\cos\phi = 8\sin 30^\circ\cos 60^\circ = 2.$$

$$y = r\sin\theta\sin\phi = 8\sin 30^\circ\sin 60^\circ = 2\sqrt{3}$$

$$z = r\cos\theta = 8(\frac{1}{2}\sqrt{3}) = 4\sqrt{3}.$$

$$\bar{G} = 14\bar{a}_x + 8\sqrt{3}\,\bar{a}_y + (48+24)\bar{a}_z = (14,13.86,72);$$

$$\bar{a}_\phi = -\sin\phi\,\bar{a}_x + \cos\phi\,\bar{a}_y = -\frac{\sqrt{3}}{2}\bar{a}_x + \frac{1}{2}\bar{a}_y;$$

$$\bar{G}_\phi = (\bar{G}\bullet\bar{a}_\phi)\bar{a}_\phi = (-7\sqrt{3} + 4\sqrt{3})\frac{1}{2}(-\sqrt{3}\,\bar{a}_x + \bar{a}_y)$$

$$= \underline{\underline{4.5\,\bar{a}_x - 2.598\,\bar{a}_y.}}$$

**Prob.**

(a) At Q, $r = 10, \quad \theta = \pi/2, \quad \phi = \pi/3$

$$x = r\sin\theta\cos\phi = 10\sin\pi/2\cos\pi/3 = 10(1)(1/2) = 5$$

$$y = r\sin\theta\sin\phi = 10\sin\pi/2\sin\pi/3 = 10(1)(\sqrt{3}/2) = 8.66$$

$$z = r\cos\theta = 10\cos\pi/2 = 0$$

Q(x,y,z) = $\underline{Q(5,8.66,0)}$

(b) At P, $r = \sqrt{9+16+4} = 5.385$

$$\cos\theta = \frac{z}{r} = \frac{2}{5.385} = 0.3714 \longrightarrow \theta = 68.2^\circ$$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} = \frac{3}{5} = 0.6 \longrightarrow \phi = 306.87^\circ$$

$$P(r,\theta,\phi) = \underline{\underline{P(5.385, 68.2^\circ, 306.87^\circ)}}$$

(c) $$d^2 = (x_p - x_Q)^2 + (y_p - y_Q)^2 + (y_p - y_Q)^2 = 2^2 + 12.66^2 + 2^2 = 168.2756$$

$$d = \underline{\underline{12.97}}$$

**Prob.**

(a) An infinite line parallel to the z-axis.

(b) Point (2,-1,10).

(c) A circle of radius $r\sin\theta = 5$ , i.e. the intersection of a cone and a sphere.

(d) An infinite line parallel to the z-axis.

(e) A semi-infinite line parallel to the x-y plane.

(f) A semi-circle of radius 5 in the y-z plane

**Prob.**

(a) $F_z = F\cos\gamma = F\cos\theta \longrightarrow \gamma = \theta,$

$F_x = F\sin\theta\cos\phi = F\cos\alpha \longrightarrow \cos\alpha = \sin\theta\cos\phi,$

$F_y = F\sin\theta\sin\phi = F\cos\beta \longrightarrow \cos\beta = \sin\theta\sin\phi,$

Hence,

$$\underline{\underline{\alpha = \cos^{-1}(\sin\theta\cos\phi),\ \beta = \cos^{-1}(\sin\theta\sin\phi),\ \gamma = \theta}}$$

(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi$

$+ \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$

Alternatively,

$$\frac{\vec{F}\cdot\vec{F}}{|\vec{F}|^2} = 1 \longrightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

**Prob.**

(a) At $T, x = -3, y = 4, z = 1, \rho = 5, \cos\phi = -\frac{3}{5}$

$$\bar{A} = 0\bar{a}_\rho - 5(1)(-\frac{3}{5})\bar{a}_\phi + 25(1)\bar{a}_z$$

$$= \underline{\underline{3\bar{a}_\phi + 25\bar{a}_z}}$$

$r = \sqrt{26}, \qquad \sin\theta = \frac{5}{\sqrt{26},} \quad \cos\theta = \frac{1}{\sqrt{26}}$

$$\bar{B} = 26(\frac{-3}{5})\bar{a}_r + 2(\sqrt{26})\ \frac{5}{\sqrt{26}}\bar{a}_\phi$$

$$= \underline{\underline{-15.6\bar{a}_r + 10\bar{a}_\phi}}$$

(*b*) In cylindrical coordinates,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} -15.6 \\ 0 \\ 10 \end{bmatrix}$$

$$B_\rho = -15.6\sin\theta = -15.6(\frac{5}{\sqrt{26}}) = -15.3$$

$$B_\phi = 10, \quad B_z = -15.6\cos\theta = -3.059$$

$$\bar{B}(\rho,\phi,z) = (-15.3, 10, -3.059)$$

$$\bar{A}_B = (\bar{A}\bullet\bar{a}_B)\bar{a}_B = (\bar{A}\bullet\bar{B})\bar{B}\frac{1}{\left|\bar{B}\right|^2} = \frac{(30-76.485)(-15.3,10,-3.059)}{343.45}$$

$$= \underline{\underline{2.071\bar{a}_\rho - 1.354\bar{a}_\phi + 0.4141\bar{a}_z}}.$$

(*c*) In spherical coordinates,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 25 \end{bmatrix}$$

$$A_r = 25\cos\theta = \frac{25}{\sqrt{26}} = 4.903$$

$$A_\theta = -25\sin\theta = -25(\frac{5}{\underline{\sqrt{26}}}) = -24.51$$

$$A_\phi = 3$$

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ 4.903 & -24.51 & 3 \\ -15.6 & 0 & 10 \end{vmatrix} = -245.1\mathbf{a}_r - 95.83\mathbf{a}_\theta - 382.43\mathbf{a}_\phi$$

$$\bar{a}_{AxB} = \frac{\pm\bar{A}\times\bar{B}}{464.23} = \underline{\underline{\pm(0.528\bar{a}_r - 0.2064\bar{a}_\theta + 0.8238\bar{a}_\phi}}.$$

**Prob.**

At $P(0,2,-5)$, $\phi = 90^\circ$;

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$$\bar{B} = -\bar{a}_x - 5\bar{a}_y - 3\bar{a}_z$$

$$(a)\ \bar{A} + \bar{B} = (2,4,10) + (-1,-5,-3)$$

$$= \underline{\underline{\bar{a}_x - \bar{a}_y + 7\bar{a}_z.}}$$

$$(b)\ \cos\theta_{AB} = \frac{\bar{A} \bullet \bar{B}}{AB} = \frac{-52}{\sqrt{4200}}$$

$$\theta_{AB} = \cos^{-1}\left(\frac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.36^\circ.}}$$

$$(c)\ A_B = \bar{A} \bullet \bar{a}_B = \frac{\bar{A} \bullet \bar{B}}{B} = -\frac{52}{\sqrt{35}} = \underline{\underline{-8.789.}}$$

**Prob.**

$$\bar{G} = \cos^2\phi\,\mathbf{a}_x + \frac{2r\cos\theta\sin\phi}{r\sin\theta}\bar{a}_y + (1-\cos^2\phi)\bar{a}_z$$

$$= \cos^2\phi\,\bar{a}_x + 2\cot\theta\sin\phi\,\bar{a}_y + \sin^2\phi\,\bar{a}_z$$

$$\begin{pmatrix} G_r \\ G_\theta \\ G_\phi \end{pmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos^2\phi \\ 2\cot\theta\sin\phi \\ \sin^2\phi \end{bmatrix}$$

$$G_r = \sin\theta\cos^3\phi + 2\cos\theta\sin^2\phi + \cos\theta\sin^2\phi$$

$$= \sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi$$

$$G_\theta = -\cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi$$

$$G_\phi = -\sin\phi\cos^2\phi + 2\cot\theta\sin\phi\cos\phi$$

$$\bar{G} = [\sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi]\mathbf{a}_r$$

$$+[-\cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi]\mathbf{a}_\theta$$

$$\underline{\underline{+\sin\phi\cos\phi(2\cot\theta - \cos\phi)\mathbf{a}_\phi}}$$

## Chapter —Practice Examples

## P.E.

(a) $$DH = \int_{\phi=45^\circ}^{\phi=60^\circ} r\sin\theta\, d\phi \Big|_{r=3,\theta=90^\circ} = 3(1)[\frac{\pi}{3} - \frac{\pi}{4}] = \frac{\pi}{4} = \underline{\underline{0.7854.}}$$

(b) $$FG = \int_{\theta=60^\circ}^{\theta=90^\circ} r d\theta \Big|_{r=5} = 5(\frac{\pi}{2} - \frac{\pi}{3}) = \frac{5\pi}{6} = \underline{\underline{2.618.}}$$

(c)

$$AEHD = \int_{\theta=60^\circ}^{\theta=90^\circ} \int_{\phi=45^\circ}^{\phi=60^\circ} r^2 \sin\theta\, d\theta\, d\phi \Big|_{r=3} = 9(-\cos\theta)\Big|_{\theta=60^\circ}^{\theta=90^\circ} \phi\Big|_{\phi=45^\circ}^{\phi=60^\circ}$$

$$= 9(\frac{1}{2})(\frac{\pi}{12}) = \frac{3\pi}{8} = \underline{\underline{1.178.}}$$

(d)

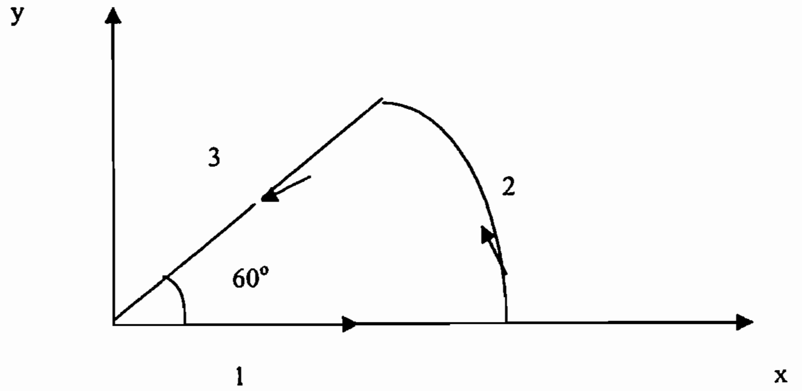
$$ABCD = \int_{r=3}^{r=5} \int_{\theta=60}^{\theta=90} r d\theta\, dr = \frac{r^2}{2}\Big|_{r=3}^{r=5} (\frac{\pi}{2} - \frac{\pi}{3}) = \frac{4\pi}{3} = \underline{\underline{4.189.}}$$

(e)

$$\text{Volume} = \int_{r=3}^{r=5} \int_{\phi=45^\circ}^{\phi=60^\circ} \int_{\theta=60}^{\theta=90} r^2 \sin\theta dr\, d\theta\, d\phi = \frac{r^3}{3}\Big|_{r=3}^{r=5} (-\cos\theta)\Big|_{\theta=60^\circ}^{\theta=90^\circ} \phi\Big|_{\phi=45^\circ}^{\phi=60^\circ} = \frac{1}{3}(98)(\frac{1}{2})\frac{\pi}{1}$$

$$= \underline{\underline{\frac{49\pi}{36} = 4.276.}}$$

## P.E.



$$\oint_L \bar{A}\bullet d\bar{l} = (\int_1 + \int_2 + \int_3)\bar{A}\bullet d\bar{l} = C_1 + C_2 + C_3$$

Along (1), $C_1 = \int \bar{A}\bullet d\bar{l} = \int_0^2 \rho\cos\phi\, d\rho\,|_{\phi=0} = \frac{\rho^2}{2}\Big|_0^2 = 2.$

Along (2), $d\bar{l} = \rho d\phi\, \bar{a}_\phi,\ \bar{A}\bullet d\bar{l} = 0,\ \ C_2 = 0$

Along (3), $C_3 = \int_2^0 \rho\cos\phi\, d\rho_{\phi=60^\circ} = -\frac{\rho^2}{2}\Big|_0^2 (\frac{1}{2}) = -1$

$$\oint_l \bar{A}\bullet d\bar{l} = C_1 + C_2 + C_3 = 2 + 0 - 1 = \underline{\underline{1}}$$

**P.E.**

(a) $$\nabla U = \frac{\partial U}{\partial x}\bar{a}_x + \frac{\partial U}{\partial y}\bar{a}_y + \frac{\partial U}{\partial z}\bar{a}_z$$

$$= \underline{\underline{y(2x+z)\bar{a}_x + x(x+z)\bar{a}_y + xy\,\bar{a}_z}}$$

(b) $$\nabla V = \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z$$

$$= \underline{\underline{(z\sin\phi + 2\rho)\bar{a}_\rho + (z\cos\phi - \frac{z^2}{\rho}\sin 2\phi)\bar{a}_\phi + (\rho s in\phi + 2z\cos^2\phi)\bar{a}_z}}$$

**(c)**

$$\nabla f = \frac{\partial f}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\bar{a}_\phi$$

$$= (\frac{\cos\theta\sin\phi}{r} + 2r\phi)\boldsymbol{a}_r - \frac{\sin\theta\sin\phi\ln r}{r}\boldsymbol{a}_\theta + \frac{(\cos\theta\cos\phi\ln r + r^2)}{r\sin\theta}\boldsymbol{a}_\phi$$

$$= \underline{\underline{\left(\frac{\cos\theta\sin\phi}{r} + 2r\phi\right)a_r - \frac{\sin\theta\sin\phi\ln r}{r}a_\theta + \left(\frac{\cot\theta\cos\phi\ln r}{r} + r\cos ec\theta\right)a_\phi}}$$

**P.E.**

$\nabla\Phi = (y+z)\bar{a}_x + (x+z)\bar{a}_y + (x+y)\bar{a}_z$

At $(1,2,3)$ $\nabla\Phi = \underline{\underline{(5,4,3)}}$

$\nabla\Phi \bullet \bar{a}_l = (5,4,3)\bullet\frac{(2,2,1)}{3} = \frac{21}{3} = \underline{\underline{7}},$

where $(2,2,1) = (3,4,4) - (1,2,3)$

**P. E.**

Let $f = x^2y + z - 3, \quad g = x\log z - y^2 + 4,$

$$\nabla f = 2xy\,\bar{a}_x + x^2\,\bar{a}_y + \bar{a}_z,$$

$$\nabla g = \log z\,\bar{a}_x - 2y\,\bar{a}_y + \frac{x}{z}\,\bar{a}_z$$

At P(−1,2,1),

$$\bar{n}_f = \pm\frac{\nabla f}{|\nabla f|} = -\frac{(-4\,\bar{a}_x + \bar{a}_y + \bar{a}_z)}{\sqrt{18}}, \qquad \bar{n}_g = \pm\frac{\nabla g}{|\nabla g|} = -\frac{(-4\,\bar{a}_y - \bar{a}_z)}{\sqrt{17}}$$

$$\cos\theta = \bar{n}_f\,.\,\bar{n}_g = \pm\frac{(-5)}{\sqrt{18\times 17}}$$

Take positive value to get acute angle.

$$\theta = \cos^{-1}\frac{5}{17.493} \quad \underline{\underline{= 73.39^\circ}}$$

**P.E.**

(*a*) $\nabla \bullet \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = \underline{\underline{4x.}}$

At $(1,-2,3)$, $\nabla \bullet \bar{A} = \underline{\underline{4.}}$

**(b)**

$$\nabla \bullet \bar{B} = \frac{1}{\rho}\frac{\partial}{\partial p}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial p}$$

$$= \frac{1}{\rho}2\rho z\sin\phi - \frac{1}{\rho}3\rho z^2\sin\phi = 2z\sin\phi - 3z^2\sin\phi$$

$$= \underline{\underline{(2-3z)z\sin\phi}}.$$

$At\,(5,\frac{\pi}{2},1), \quad \nabla \bullet \bar{B} = (2-3)(1) = \underline{\underline{-1.}}$

(c)

$$\nabla \bullet \bar{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2C_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(C_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}6r^2\cos\theta\cos\phi$$

$$= \underline{\underline{6\cos\theta\cos\phi}}$$

$At\,(1,\frac{\pi}{6},\frac{\pi}{3}), \quad \nabla \bullet \bar{C} = 6\cos\frac{\pi}{6}\cos\frac{\pi}{3} = \underline{\underline{2.598.}}$

**P.E.**

$$\Psi = \oint_S \bar{D} \bullet d\bar{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since $\bar{A}$ has no z-component

$$\Psi_c = \iint \rho^2 \cos^2 \phi \rho d\phi dz = \rho^3 \int_{\phi=0}^{\phi=2\pi} \cos^2 \phi d\phi \int_{z=0}^{z=1} dz \Bigg|_{\rho = 4}$$

$$= (4)^3 \pi(1) = 64\pi$$

$$\Psi = 0 + 0 + 64\pi = \underline{\underline{64\pi}}$$

By the divergence theorem,

$$\oint_S \bar{D} \bullet d\bar{S} = \oint_V \nabla \bullet \bar{D} dv$$

$$\nabla \bullet \bar{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \cos^2 \phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi} z \sin\phi + \frac{\partial A_z}{dz}$$

$$= 3\rho \cos^2 \phi + \frac{z}{\rho}\cos\phi.$$

$$\Psi = \int_V \nabla \bullet \bar{D} dv = \int_V (3\rho \cos^2 \phi + \frac{z}{\rho}\cos\phi)\rho d\phi dz d\rho$$

$$= 3\int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2 \phi d\phi \int_0^1 dz + \int_0^4 d\rho \int_0^{2\pi} \cos\phi d\phi \int_0^1 z dz$$

$$= 3(\frac{4^3}{3})\pi(1) = \underline{\underline{64\pi.}}$$

**P.E.**

(a)

$$\nabla \times \bar{A} = \bar{a}_x(1 - 0) + \bar{a}_y(y - 0) + \bar{a}_z(4y - z)$$

$$= \underline{\underline{\bar{a}_x + y\bar{a}_y + (4y - z)\bar{a}_z}}$$

At $(1,-2,3)$, $\nabla \times \bar{A} = \underline{\underline{\bar{a}_x - 2\bar{a}_y - 11\bar{a}_z}}$

(b)

$$\nabla \times \bar{B} = \bar{a}_\rho(0 - 6\rho z\cos\phi) + \bar{a}_\phi(\rho\sin\phi - 0) + \bar{a}_z\frac{1}{\rho}(6\rho z^2\cos\phi - \rho z\cos\phi)$$

$$= \underline{\underline{-6\rho z\cos\phi\,\bar{a}_\rho + \rho\sin\phi\,\bar{a}_\phi + (6z - 1)z\cos\phi\,\bar{a}_z}}$$

At $(5,\frac{\pi}{2},-1)$, $\nabla \times \bar{B} = \underline{\underline{5\bar{a}_\phi}}$.

(c)

$$\nabla\times\bar{C} = \bar{a}_r\frac{1}{r\sin\theta}(r^{-1/2}\cos\theta - 0) + \frac{\bar{a}_\theta}{r}(-\frac{2r\cos\theta\sin\phi}{\sin\theta} - \frac{3}{2}r^{1/2}) + \frac{\bar{a}_\phi}{r}(0 + 2r\sin\theta\cos\phi)$$

$$= \underline{\underline{r^{-1/2}\cot\theta\,\bar{a}_r - (2\cot\theta\sin\phi + \frac{3}{2}r^{-1/2})\bar{a}_\theta + 2\sin\theta\cos\phi\,\bar{a}_\phi}}$$

At $(1,\frac{\pi}{6},\frac{\pi}{3})$, $\nabla\times C = \underline{\underline{1.732\bar{a}_r - 4.5\bar{a}_\theta + 0.5\bar{a}_\phi}}$

**P.E.**

$$\oint_L \bar{A}\bullet d\bar{l} = \int_S (\nabla\times\bar{A})\bullet d\bar{S}$$

But $(\nabla\times\bar{A}) = \sin\phi\,\bar{a}_z + \frac{z\cos\phi}{\rho}\bar{a}_\rho$ and $d\bar{S} = \rho d\phi d\rho\,\bar{a}_z$

$$\int_S (\nabla\times\bar{A})\bullet d\bar{S} = \iint \rho\sin\phi\; d\phi\, d\rho$$

$$= \frac{\rho^2}{2}\Big|_0^2 (-\cos\phi)\Big|_0^{60^\circ}$$

$$= 2(-\frac{1}{2}+1) = \underline{\underline{1.}}$$

**P.E.**

$$\nabla \times \nabla V = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \end{vmatrix} =$$

$$= \left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial y \partial z}\right)\bar{a}_x + \left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x}\right)\bar{a}_y + \left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right)\bar{a}_z = 0$$

**P.E.**

(a)

$$\nabla^2 U = \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz) + \frac{\partial}{\partial z}(xy)$$

$$= \underline{\underline{2y.}}$$

**(b)**

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho(z\sin\phi + 2\rho) + \frac{1}{\rho^2}(-\rho z\sin\phi - 2z^2\frac{\partial}{\partial \rho}\sin\phi\cos\phi) + \frac{\partial}{\partial z}(\rho\sin\phi + 2z\cos^2\phi)$$

$$= \frac{1}{\rho}(z\sin\phi + 4\rho) - \frac{1}{\rho^2}(z\rho\sin\phi + 2z^2\cos 2\phi) + 2\cos^2\phi.$$

$$= \underline{\underline{4 + 2\cos^2\phi - \frac{2z^2}{\rho^2}\cos 2\phi.}}$$

**(c)**

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{1}{r}\cos\theta\sin\phi + 2r^3\phi] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}[-\sin^2\theta\sin\phi\ln r]$$

$$+ \frac{1}{r^2\sin^2\theta}[-\cos\theta\sin\phi\ln r]$$

$$= \underline{\underline{\frac{1}{r^2}\cos\theta\sin\phi(1 - 2\ln r - \csc^2\theta\ln r) + 6\phi}}$$

**CHAPTER —Problems**

**Prob.**

(a) $S = \int_{\rho=0}^{10}\int_{\phi=0}^{\pi} \rho \, d\phi \, d\rho = \pi \frac{\rho^2}{2}\Big|_0^{10} = 50\pi = \underline{\underline{157.1}}$

(b) $dS = \rho \, d\phi \, dz$, $S = \int_{z=0}^{4}\int_{\phi=\pi/4}^{\pi/2} \rho \, d\phi \, dz \Big|_{\rho=50}$

$= 50(4)(\pi/4) = 50\pi = \underline{\underline{157.1}}$

(c) $dS = r^2 \sin\theta \, d\theta \, d\phi$,

$$S = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi \Big|_{r=5}$$

$$= 25(2\pi)(-\cos\theta)\Big|_0^{\pi/2} = 50\pi = \underline{\underline{157.1}}$$

(d) $dS = r\sin\theta \, dr \, d\phi$,

$$S = \int_{r=0}^{10}\int_{\phi=0}^{2\pi} r\sin\theta \, dr \, d\phi \Big|_{\theta=30^\circ}$$

$$= \sin 30^\circ (2\pi)\left(\frac{100}{2}\right) = 50\pi = \underline{\underline{157.1}}$$

(e) $dS = r^2 \sin\theta \, d\theta \, d\phi$,

$$S = (8.26)^2 \int_0^{2\pi} d\phi \int_{30^\circ}^{60^\circ} \sin\theta \, d\theta = (2\pi)^2 (2\pi)(-0.5 + 0.866)$$

$$= 50\pi = \underline{\underline{157.1}}$$

**Prob.**

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\alpha} \int_{r=o}^{a} r^2 \sin\theta \, d\theta \, dr \, d\phi = \frac{2\pi a^3}{3}(1-\cos\alpha)$$

$$V(\alpha = \frac{\pi}{3}) = \frac{2\pi a^3}{3}(1-\frac{1}{2}) = \underline{\underline{\frac{\pi a^3}{3}}}$$

$$V(\alpha = \frac{\pi}{2}) = \frac{2\pi a^3}{3}(1-0) = \underline{\underline{\frac{2\pi a^3}{3}}}$$

**Prob.**

(a)

$dl = \rho d\phi; \qquad \rho = 3$

$$L = \int dl = 3\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\phi = 3(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{3\pi}{4} = \underline{\underline{2.356}}$$

(b)

$dl = r\sin\theta d\phi; \qquad r = 1, \quad \theta = 30°;$

$$L = \int dl = r\sin\theta \int_{0}^{\frac{\pi}{3}} d\phi = (1)\sin 30° [(\frac{\pi}{3}) - 0] = \underline{\underline{0.5236.}}$$

(c)

$dl = rd\theta$

$$L = \int dl = r\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta = 4(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{4\pi}{3} = \underline{\underline{4.189}}$$

**Prob.**

**(a)**

$$\int \bar{F}\bullet d\,\bar{l} = \int_{y=0}^{1} (x^2 - z^2)dy\Big|_{x=0,z=0} + \int_{x=0}^{x=2} 2xydx\Big|_{y=1,z=0} + \int_{z=0}^{z=3} (-3xz^2)dz\Big|_{x=2,y=1}$$

$$= 0 + 2(1)\frac{x^2}{2}\Big|_0^2 - 3(2)\frac{z^3}{3}\Big|_0^3$$

$$= 0 + 4 - 54 = \underline{\underline{-50}}$$

**(b)**

$$Let\ \ x = 2t.\ \ y = t,\ \ z = 3t$$

$$dx = 2dt,\ \ dy = dt,\ \ dz = 3dt;$$

$$\int \bar{F}\bullet d\,\bar{l} = \int_0^1 (8t^2 - 5t^2 - 162t^3)\ dt$$

$$= (t^3 - 40.5t^4)\Big|_0^1 = \underline{\underline{-39.5}}$$

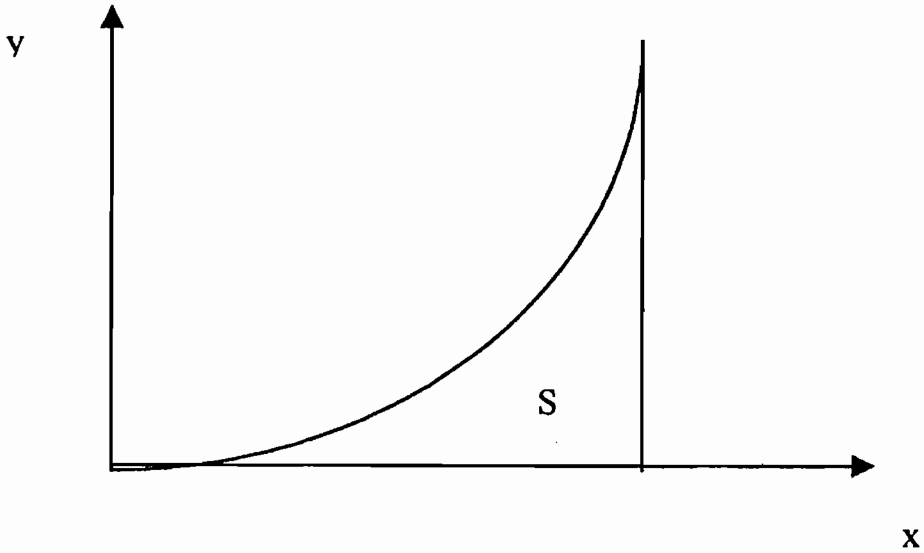
**Prob.**

$$\int_L \bar{H}\bullet d\bar{l} = \int (x^2dx + y^2\ dy)$$

But on $L$, $y = x^2 \quad dy = 2xdx$

$$\int_L \bar{H}\bullet d\bar{l} = \int_0^1 (x^2 + x^4.2x)dx = \frac{x^3}{3} + 2\frac{x^6}{6}\Big|_0^1 = \frac{1}{3} + \frac{1}{3} = \underline{\underline{0.6667}}$$

**Prob.**



$$\int \rho_s dS = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}(x^2+xy)dy\,dx$$

$$= \int_0^1 \left(x^2y+\frac{xy^2}{2}\right|_{y=0}^{x^2}\left.\vphantom{\frac{xy^2}{2}}\right)dx = \int_0^1 (x^4+\frac{x^5}{2})dx$$

$$= \frac{1}{5}+\frac{1}{12} = \frac{17}{60} = \underline{\underline{0.2833}}$$

**Prob.**

$$\int \bar{H}\bullet d\bar{l} = \int_{x=1}^{0}(x-y)dx\Big|_{y=0,z=0} + \int_{z=0}^{1}5yzdz\Big|_{x=0,y=0}$$

$$+\int (x^2+zy)dy+5yzdz\Big|_{x=0,z=1-y/2}$$

$$= \int_1^0 xdx+\int_0^2 (y-\frac{y^2}{2})dy+\int_1^0 (10z-10z^2)dz$$

$$= \underline{\underline{-1.5}}$$

**Prob.**

$$\Psi = \oint_s \vec{A}\cdot d\vec{S} = \int_v \nabla\cdot\vec{A}\,dv = \iiint (4z-y)\,dx\,dy\,dz$$

$$= 4\int_0^1\int_0^1\int_0^1 z\,dx\,dy\,dz - \int_0^1\int_0^1\int_0^1 y\,dx\,dy\,dz$$

$$= 3(1)(1)\left(\tfrac{1}{2}\right) = \underline{\underline{1.5}}$$

**Prob.**

(a) $\nabla \bullet A = \dfrac{\partial}{\partial x}(2xy) + \dfrac{\partial}{\partial y}(xz) + \dfrac{\partial}{\partial z}(-y) = 2y$

$$\int \nabla \bullet A dv = \iiint 2y dx dy dz = 2\int_0^2 dx \int_0^2 y dy \int_0^2 dz$$

$$= 2(2)\frac{y^2}{2}\Big|_0^2 (2) = 4(4-0) = \underline{\underline{16}}$$

(b) $\nabla \bullet A = 2y = 2\rho\sin\phi$

$$\int \nabla \bullet A dv = \iiint 2\rho\sin\phi\rho d\rho d\phi dz = 2\int_0^3 \rho^2 d\rho \int_0^{2\pi} \sin\phi d\phi \int_0^5 dz = \underline{\underline{0}}$$

since integrating sin $\phi$ over $0 < \phi < 2\pi$ is 0.

(c) $\nabla \bullet A = 2y = 2r\sin\theta\sin\phi$

$$\int \nabla \bullet A dv = \iiint 2r\sin\theta\sin\phi r^2\sin\theta d\theta d\phi dr = 2\int_0^4 r^3 dr \int_0^{2\pi} \sin\phi d\phi \int_0^{\pi} \sin^2 d\theta = \underline{\underline{0}}$$

**Prob.**

(a) $\nabla U = \frac{\partial U}{\partial x}\vec{a}_x + \frac{\partial U}{\partial y}\vec{a}_y + \frac{\partial U}{\partial z}\vec{a}_z$

$= \underline{4z^2\,\vec{a}_x + 3z\,\vec{a}_y + (8xz + 3y)\,\vec{a}_z}$.

(b) $\nabla V = 2e^{(2x+3y)}\cos 5z\,\vec{a}_x + 3e^{(2x+3y)}\cos 5z\,\vec{a}_y$

$\underline{- 5e^{(2x+3y)}\sin 5z\,\vec{a}_z}$.

(c) $\nabla T = \frac{\partial T}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\vec{a}_\phi + \frac{\partial T}{\partial z}\vec{a}_z$

$= \underline{5e^{-2z}\sin\phi\,\vec{a}_\rho + 5e^{-2z}\cos\phi\,\vec{a}_\phi - 10\rho e^{-2z}\sin\phi\,\vec{a}_z}$

(d) $\nabla W = 2(z^2+1)\cos\phi\,\vec{a}_\rho - 2(z^2+1)\sin\phi\,\vec{a}_\phi$

$\underline{+ 4\rho z\cos\phi\,\vec{a}_z}$.

**Prob.**

(a) $\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$

$$= \underline{\underline{2e^{(2x+3y)}\cos 5z\,\bar{a}_z + 3e^{(2x+3y)}\cos 5z\,\bar{a}_y - 5e^{(2x+3y)}\sin 5z\,\bar{a}_z}}.$$

At $(0.1,-0.2,0.4)$

$$e^{(2x+3y)} = e^{0.2-0.6} = 0.6703,\ \cos 5z = \cos 2 = -0.4161,\quad \sin 5z = 0.9092$$

$$\nabla V = 2(0.6073)(-0.4161)\bar{a}_x + 3(0.6703)(-0.4161)\bar{a}_y - 5(0.6203)(0.9092)$$

$$= \underline{\underline{-0.5578\,\bar{a}_x - 0.8367\,\bar{a}_y - 3.047\,\bar{a}_z}}$$

(b)

$$\nabla T = \underline{\underline{5e^{-2z}\sin\phi\,\bar{a}_\rho + 5e^{-z}\cos\phi\,\bar{a}_\phi - 10\rho e^{-2z}\sin\phi\,\bar{a}_z}}$$

At $\quad (2,\frac{\pi}{3},0)$,

$$\nabla T = (5)(1)(0.5)\bar{a}_\rho + 5(1)(0.5)\bar{a}_\phi - 10(2)(1)(0.866)\bar{a}_z$$

$$= \underline{\underline{2.5\bar{a}_\rho + 2.5\bar{a}_\phi - 17.32\bar{a}_z}}$$

(c)

$$\nabla Q = \underline{\underline{\frac{-2\sin\theta\sin\phi}{r^3}\bar{a}_r + \frac{\cos\theta\sin\phi}{r^3}\bar{a}_0 + \frac{\cos\phi}{r^3}\bar{a}_\phi}}$$

At $(1,30^\circ,90^\circ)$,

$$\nabla Q = \frac{-2(0.5)(1)}{1}\bar{a}_r + \frac{(0.86)(1)}{1}\bar{a}_0 + 0 = \underline{\underline{-\bar{a}_r + 0.866\,\bar{a}_0}}$$

**Prob.**

$\nabla S = 2x\bar{a}_x + 2y\bar{a}_y - \bar{a}_z$

*At* $\quad (1,3,0)$,

$\nabla S = 2\bar{a}_x + 6\bar{a}_y - \bar{a}_z$ and $\qquad \bar{a}_n = \frac{\nabla S}{|\nabla S|} = \frac{(2,6,-1)}{\sqrt{4+36+1}}$

$$\bar{a}_n = \underline{\underline{0.3123\bar{a}_x + 0.937\bar{a}_y - 0.1562\bar{a}_z}}$$

**Prob.**

$\bar{\nabla} T = 2x\bar{a}_x + 2y\bar{a}_y - \bar{a}_z$

At $(1,1,2)$, $\bar{\nabla} T = (2,2,-1)$. The mosquito should move in the direction of

$\underline{\underline{2\bar{a}_x + 2\bar{a}_y - \bar{a}_z}}$

**Prob.**

(a) $\nabla \cdot \vec{A} = \underline{\underline{y e^{xy} + x \cos xy - 2x \cos zx \sin zx}}$.

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin xy & \cos^2 xz \end{vmatrix}$$

$$= (0-0)\vec{a}_x + (0 + 2z \cos xz \sin xz)\vec{a}_y + (y \cos xy - x e^{xy})\vec{a}_z$$

$$= \underline{\underline{z \sin 2xz\, \vec{a}_y + (y \cos xy - x e^{xy})\vec{a}_z}}.$$

(b) $\nabla \cdot \vec{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 z^2 \cos\phi) + 0 + \sin^2\phi$

$$= \underline{\underline{2z^2 \cos\phi + \sin^2\phi}}.$$

$$\nabla \times \vec{B} = \left(\frac{1}{\rho}\frac{\partial B_z}{\partial \phi} - 0\right)\vec{a}_\rho + \left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right)\vec{a}_\phi + \frac{1}{\rho}\left(0 - \frac{\partial B_\rho}{\partial \phi}\right)\vec{a}_z$$

$$= \underline{\underline{\frac{z \sin 2\phi}{\rho}\vec{a}_\rho + 2\rho z \cos\phi\, \vec{a}_\phi + z^2 \sin\phi\, \vec{a}_z}}$$

(c) $\nabla \cdot \vec{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 \cos\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(-\frac{1}{r}\sin^2\theta\right) + 0$

$$= \underline{\underline{3\cos\theta - \frac{2\cos\theta}{r^2}}}$$

$$\nabla \times \vec{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(2r^2\sin^2\theta) - 0\right]\bar{a}_r$$
$$+\frac{1}{r}\left[0 - \frac{\partial}{\partial r}(2r^3\sin\theta)\right]\bar{a}_\theta$$
$$+\frac{1}{r}\left(\frac{\partial}{\partial r}\cancelto{0}{(-\sin\theta)} + r\sin\theta\right]\bar{a}_\phi$$

$$= \underline{\underline{4r\cos\theta\,\vec{a}_r - 6r\sin\theta\,\bar{a}_\theta + \sin\theta\,\bar{a}_\phi}}$$

**Prob.**

$$\bar{\nabla}\bullet\bar{H} = k\bar{\nabla}\bullet\bar{\nabla}T = k\bar{\nabla}^2 T$$

$$\bar{\nabla}^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}\left(-\frac{\pi^2}{4} + \frac{\pi^2}{4}\right) = 0$$

Hence, $\bar{\nabla}\bullet\bar{H} = 0$

**Prob.**

**(a)**

$$\nabla\bullet(V\bar{A}) = \frac{\partial}{\partial x}(VA_x) + \frac{\partial}{\partial y}(VA_y) + \frac{\partial}{\partial z}(VA_z)$$

$$= \left(A_x\frac{\partial V}{\partial x} + V\frac{\partial A_x}{\partial x}\right) + \left(A_y\frac{\partial V}{\partial y} + V\frac{\partial A_y}{\partial y}\right) + \left(A_z\frac{\partial V}{\partial z} + V\frac{\partial A_z}{\partial z}\right)$$

$$= V\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + A_x\frac{\partial V}{\partial x} + A_y\frac{\partial V}{\partial y} + A_z\frac{\partial V}{\partial z}$$

$$= \underline{\underline{V\nabla\bullet\bar{A} + \bar{A}\bullet\nabla V}}$$

**(b)**

$$\nabla\bullet A = 2 + 3 - 4 = 1; \quad \nabla V = yz\bar{a}_x + xz\bar{a}_y + xy\bar{a}_z$$

$$\nabla\bullet(V\bar{A}) = V\nabla\bullet\bar{A} + \bar{A}\bullet\nabla V$$
$$= xyz + 2xyz + 3xyz - 4xyz = \underline{\underline{2xyz}}$$

**Prob.**

$$grad\ U = \frac{\partial U}{\partial x}\bar{a}_x + \frac{\partial U}{\partial y}\bar{a}_y + \frac{\partial U}{\partial z}\bar{a}_z$$

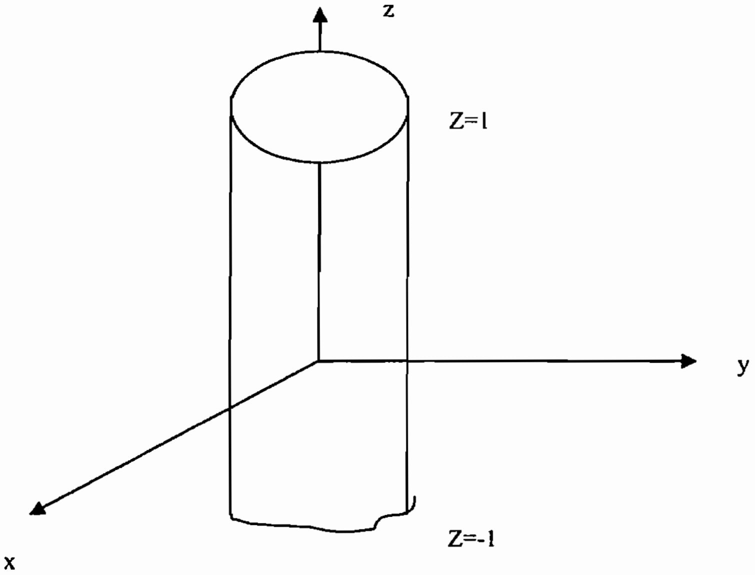
$$= (z - 2xy)\bar{a}_x + (2yz^2 - x^2)\bar{a}_y + (x + 2y^2z)\bar{a}_z$$

$$Div\ grad U = \nabla \bullet \nabla U = \frac{\partial}{\partial x}(z - 2xy) + \frac{\partial}{\partial y}(2yz^2 - x^2) + \frac{\partial}{\partial z}(x + 2y^2z)$$

$$= -2y + 2z^2 + 2y^2$$

$$= \underline{\underline{2(z^2 + y^2 - y)}}$$

**Prob.**



**(a)**

$$\oint \bar{D}\bullet d\bar{s} = [\iint_{z=-1} + \iint_{z=1} + \iint_{\rho=5}]\bar{D}\bullet d\bar{s}$$

$$= -\iint \rho^2 \cos^2\phi d\phi d\rho + \iint \rho^2 \cos^2\phi d\phi d\rho + \iint 2\rho^2 z^2 d\phi dz\Big|_{\rho=5}$$

$$= 2(5)^2 \int_0^{2\pi} d\phi \int_{-1}^{1} z^2 dz = +50(2\pi)(\frac{z^3}{3}\Big|_{-1}^{1})$$

$$= \frac{200\pi}{3} = \underline{\underline{209.44}}$$

(b) $\nabla \bullet \bar{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 z^2) = 4z^2$

$$\int \nabla \bullet D dv = \iiint 4z^2 \rho d\rho d\phi dz = 4\int_{-1}^{1} z^2 dz \int_0^5 \rho d\rho \int_0^{2\pi} d\phi$$

$$= 4x\frac{z^3}{3}\Big|_{-1}^{1} \frac{\rho^2}{2}\Big|_0^5 (2\pi) = \frac{200\pi}{3} = \underline{\underline{209.44}}$$

**Prob.**

We convert A to cylindrical coordinates; only the ρ-component is needed.

$$A_\rho = A_x \cos\phi + A_y \sin\phi = 2x\cos\phi - z^2 \sin\phi$$

But $x = \rho\cos\phi$,

$$A_\rho = 2\rho\cos^2\phi - z^2 \sin\phi$$

$$\Psi = \int_S A \cdot dS = \iint A_\rho \rho d\phi dz = \iint \left[2\rho^2 \cos^2\phi - \rho z^2 \sin\phi\right] d\phi dz$$

$$= 2(2)^2 \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\phi) d\phi \int_0^1 dz - 2\int_0^1 z^2 dz \int_0^{\pi/2} \sin\phi d\phi$$

$$= 4(\phi + \frac{1}{2}\sin 2\phi)\Big|_0^{\pi/2} - 2\frac{z^3}{3}\Big|_0^1 (-\cos\phi)\Big|_0^{\pi/2} = 2\pi - 2/3 = \underline{\underline{5.6165}}$$

**Prob.**

(a)

$$(\bar{\nabla}\bullet\bar{r})\bar{T}=3\bar{T}=\underline{\underline{6yz\,\bar{a}_y+3xy^2\,\bar{a}_y+3x^2yz\,\bar{a}_z}}$$

(b)

$$x\frac{\partial\bar{T}}{\partial x}+y\frac{\partial\bar{T}}{\partial y}+z\frac{\partial\bar{T}}{\partial z}=x\ (y^2\,\bar{a}_y+2xyz\,\bar{a}_z)+y(2z\,\bar{a}_x+2xy\,\bar{a}_y+x^2z\,\bar{a}_z)$$

$$+z(2y\,\bar{a}_x+x^2y\,\bar{a}_z)$$

$$=\underline{\underline{4yz\,\bar{a}_x+3xy^2\,\bar{a}_y+4x^2yz\,\bar{a}_z}}$$

(c)

$$\nabla\bullet\bar{r}(\bar{r}\bullet\bar{T})=3\,(2xyz+xy^3+x^2yz^2)$$

$$=\underline{\underline{6xyz+3xy^3+3x^2yz^2}}$$

(d)

$$(\bar{r}\bullet\nabla)\bar{r}^2=(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})(x^2+y^2+z^2)$$

$$=x(2x)+y(2y)+z(2z)$$

$$=\underline{\underline{2(x^2+y^2+z^2)=2r^2}}$$

**Prob.**

$$\oint_S \boldsymbol{H}\cdot d\boldsymbol{S} = \int_v \nabla\cdot\boldsymbol{H}dv$$

$$\oint_S \boldsymbol{H}\cdot d\boldsymbol{S} = -\iint 2xydydz\Big|_{x=0} + \iint 2xydydz\Big|_{x=1} - \iint (x^2+z^2)dxdz\Big|_{y=1}$$
$$+\iint (x^2+z^2)dxdz\Big|_{y=2} - \iint 2yzdxdy\Big|_{z=-1} + \iint 2yzdxdy\Big|_{z=3}$$

$$= 0 + 2\int_1^2 ydy\int_{-1}^3 dz + 2\int_0^1 dx\int_1^2 ydy + 6\int_0^1 dx\int_1^2 ydy$$
$$= 12 + 3 + 9 = \underline{\underline{24}}$$

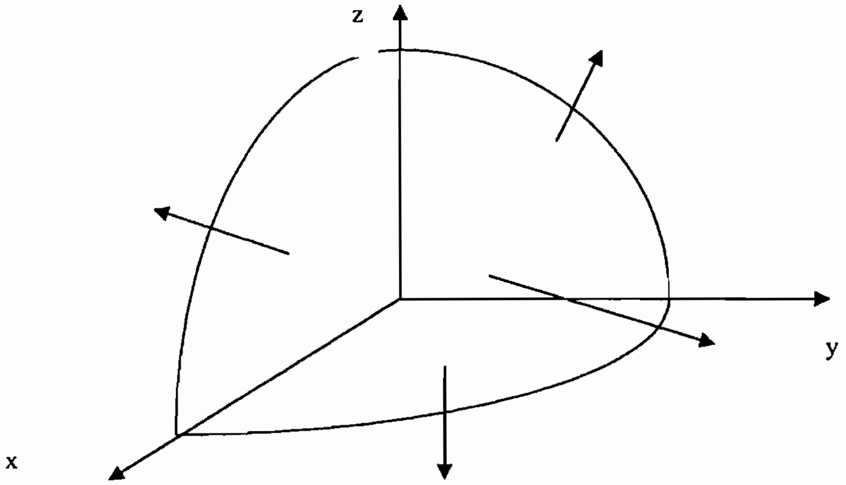
$$\nabla\cdot\boldsymbol{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 2y + 0 + 2y = 4y$$

$$\int_v \nabla\cdot Hdv = \iiint 4ydxdydz = 4\int_0^1 dx\int_1^2 ydy\int_{-1}^3 dz$$
$$= 4(1)\frac{y^2}{2}\Big|_1^2 (3+1) = \underline{\underline{24}}$$

**Prob.**

$$\nabla\bullet\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi)$$
$$= 4\ r\ + 2\cos\theta\ \cos\phi$$

$$\int\nabla\bullet\bar{A}\,dv = \iiint 4r^3\sin\theta\, d\theta\, d\phi\, dr + \iiint 2r^2\sin\theta\cos\theta\cos\phi\, d\theta\, d\phi\, dr$$
$$= 4\frac{r^4}{4}\Big|_0^3(-\cos\theta)\Big|_0^{\pi/2}(\frac{\pi}{2}) + \frac{2r^3}{3}\Big|_0^3(-\frac{\cos\theta}{2})\Big|_0^{\pi/2}\sin\phi\Big|_0^{\frac{\pi}{2}}$$
$$= 81(1)(\frac{\pi}{2}) + 18(0+\frac{1}{2})(1-0)$$
$$= \frac{81\pi}{2} + 9 = \underline{\underline{136.23}}$$



$$\int \bar{A} \bullet d\bar{S} = [\iint_{\phi=0} + \iint_{\phi=\pi/2} + \iint_{r=3} + \iint_{\theta=\pi/2}] \bar{A}.d\bar{S}$$

Since $\bar{A}$ has no $\phi -$component, the first two integrals on the right hand side vanish

$$\int \bar{A} \bullet d\bar{S} = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} r^4 \sin\theta \, d\theta \, d\phi \Big|_{r=3} + \int_{r=0}^{3} \int_{\phi=0}^{\pi/2} r^2 \sin^2\theta \cos\phi dr d\phi \Big|_{\theta=\pi/2}$$

$$= 81 \, (\frac{\pi}{2}) \, (-\cos\theta)\Big|_0^{\pi/2} + 9(1)\sin\phi\Big|_0^{\pi/2}$$

$$= \frac{81\pi}{2} + 9 \quad = \underline{\underline{136.23}}$$

**Prob.**

Transform $\bar{F}$ into cylindrical system.

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ z^2 - 1 \end{bmatrix}$$

$$F_\rho = x^2\cos\phi + y^2\sin\phi = \rho^2\cos^3\phi + \rho^2\sin^3\phi, F_z = z^2 - 1$$

$$F_\phi = -x^2\sin\phi + y^2\cos\phi = -\rho^2\cos^2\phi\sin\phi + \rho^2\sin^2\phi\cos\phi$$

$$\nabla\bullet\bar{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^3\cos^3\phi + \rho^3\sin^3\phi) + 2z - \rho\cos^3\phi - 2\rho\cos\phi\sin^2\phi$$

$$+ 2\rho\sin\phi\cos^2\phi - \rho\sin^3\phi$$

$$= 2\rho\cos^3\phi + 2\rho\sin^3\phi - 2\rho\cos\phi\sin^2\phi + 2\rho\cos^2\phi\sin\phi + 2z$$

$$\int\bar{F}\bullet d\bar{S} = \int\nabla\bullet\bar{F}\,dv$$

Due to the fact that we are integrating $\sin\phi$ and $\cos\phi$ over $0 < \phi < 2\pi$, all terms involving $\cos\phi$ *and* $\sin\phi$ will vanish. Hence,

$$\int\bar{F}d\bar{S} = \iiint 2z\ \rho d\rho d\phi dz = 2\int_0^{2\pi} d\phi \int_0^2 zdz \int_0^2 \rho d\rho$$

$$= 2(2\pi)\left(\frac{z^2}{2}\bigg|_0^2\right)\left(\frac{\rho^2}{2}\bigg|_0^2\right) = 16\pi$$

$$= \underline{\underline{50.26}}$$

**Prob.**

**(a)**

$$\oint \bar{A} \bullet d\bar{S} = \int_V \nabla \bullet \bar{A} dv, \quad \nabla \bullet \bar{A} = y + z + x$$

$$\oint \bar{A} \bullet d\bar{S} = \int_0^1 \int_0^1 \int_0^1 (x + y + z) dx dy dz$$

$$= 3 \int_0^1 x dy \int_0^1 dy \int_0^1 dz = 3 (\frac{x^2}{2} \Big|_0^1)(1)(1)$$

$$= \underline{\underline{1.5}}$$

**(b)**

$\nabla \bullet \bar{A} = 0.$ Hence, $\oint \bar{A} \bullet d\bar{S} = \underline{\underline{0}}$

**Prob.**

$$\int \rho_V \, dv = \oint_S \bar{A} \bullet d\bar{S} \qquad \text{(divergence theorem)}$$

where $\rho_V = \nabla \bullet \bar{A} = x^2 + y^2$. Assuming that

$$\frac{\partial A_x}{\partial x} = x^2 \longrightarrow A_x = \frac{x^3}{3} + C_1$$

$$\frac{\partial A_y}{\partial y} = y^2 \longrightarrow A_y = \frac{y^3}{3} + C_2$$

Hence,

$$\underline{\underline{\bar{A} = (\frac{x^3}{3} + C_1)\bar{a}_x + (\frac{y^3}{3} + C_2)\bar{a}_y}} \quad \text{(solution is not unique)}$$

**Prob.** .

Let $\psi = \oint \bar{F} \bullet d\bar{S} = \psi_t + \psi_b + \psi_o + \psi_i$

where $\psi_t, \psi_b, \psi_o, \psi_i$ are the fluxes through the top surface, bottom surface, outer surface ($\rho = 3$), and inner surface respectively.

For the top surface, $d\bar{S} = \rho\, d\phi\, d\rho\, \bar{a}_z, \qquad z = 5;$

$$\bar{F} \bullet d\bar{S} = \rho^2\, z\, d\phi\, dz. \text{ Hence:}$$

$$\psi_t = \int_{\rho=2}^{3} \int_{\phi=0}^{2\pi} \rho^2\, z\, d\phi\, dz\Big|_{z=5} = \frac{190\ \pi}{3}$$

For the bottom surface, $z = 0,\ d\bar{S} = \rho\, d\phi\, d\rho(-\bar{a}_z)$

$$\bar{F} \bullet d\bar{S} = \rho^2 z\, d\phi\, d\rho = 0. \text{ Hence, } \psi_b = 0.$$

For the outer curved surface, $\rho = 3,\ d\bar{S} = \rho\, d\phi\, dz\, \bar{a}_\rho$

$$\bar{F} \bullet d\bar{S} = \rho^3 \sin\phi\, d\phi\, dz. \text{ Hence,}$$

$$\psi_a = \int_{z=0}^{5} dz\, \rho^3 \int_{\phi=0}^{2\pi} \sin\phi\, d\phi\Big|_{\rho=3} = 0$$

For the inner curved surface, $\rho = 2,\ d\bar{S} = \rho\, d\phi\, dz(-\bar{a}_z)$

$$\bar{F} \bullet d\bar{S} = \rho^3 \sin\phi\, d\phi\, dz. \text{ Hence,}$$

$$\psi_a = \int_{z=0}^{5} dz\, \rho^3 \int_{\phi=0}^{2\pi} \sin\phi\, d\phi\Big|_{\rho=2} = 0$$

$$\psi = \frac{190\pi}{3} + 0 + 0 + 0 = \underline{\underline{\frac{190\ \pi}{3}}}$$

$$\psi = \oint \bar{F} \bullet d\bar{S} = \int \nabla \bullet \bar{F} dV$$

$$\nabla \bullet \bar{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(z\cos\phi) + \rho$$

$$= 3\rho \sin\phi - \frac{z}{\rho}\sin\phi + \rho$$

$$\int_V \nabla \bullet \bar{F}\,dv = \iiint (3\rho \sin\phi - \frac{z}{\rho}\sin\phi + \rho)\,\rho\,d\phi\,d\rho\,dz$$

$$= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_2^3 \rho^2 d\rho$$

$$= \underline{\underline{\frac{190\,\pi}{3}}}$$

**Prob.**

(a) $$\nabla \text{x} \mathbf{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & -xz \end{vmatrix} = z\boldsymbol{a}_y - x\boldsymbol{a}_z$$

$$\nabla \text{x} \mathbf{B} = \left(\frac{1}{\rho}2\rho z 2\sin\phi\cos\phi - 0\right)\boldsymbol{a}_\rho + (2\rho z - 2z\sin^2\phi)\boldsymbol{a}_\phi + \frac{1}{\rho}\left(2\rho\sin^2\phi - 0\right)\boldsymbol{a}$$

(b) $$= 4z\sin\phi\cos\phi\boldsymbol{a}_\rho + 2(\rho z - z\sin^2\phi)\boldsymbol{a}_\phi + 2\sin^2\phi\boldsymbol{a}_z$$

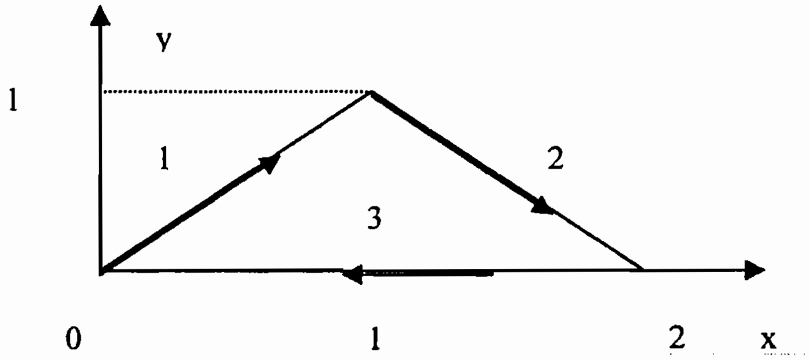
$$= \underline{\underline{2z\sin 2\phi\cos\phi\boldsymbol{a}_\rho + 2z(\rho - \sin^2\phi)\boldsymbol{a}_\phi + 2\sin^2\phi\boldsymbol{a}_z}}$$

$$\nabla \text{x} \mathbf{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(r\cos^2\theta\sin\theta\right]\boldsymbol{a}_r - \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos^2\theta\right]\boldsymbol{a}_\theta$$

(c ) $$= \frac{r}{r\sin\theta}\left[(2\cos\theta)(-\sin\theta)\sin\theta + \cos\theta(\cos^2\theta)\right]\boldsymbol{a}_r - \frac{\cos^2\theta}{r}(2r)\boldsymbol{a}_\theta$$

$$= \underline{\underline{\frac{(\cos^3\theta - 2\sin^2\theta\cos\theta)}{\sin\theta}\boldsymbol{a}_r - 2\cos^2\theta\boldsymbol{a}_\theta}}$$

**Prob.**



(a)

$$\oint_L \bar{F} \bullet d\bar{l} = (\int_1 + \int_2 + \int_3)\bar{F} \bullet d\bar{l}$$

For 1, $y = x \quad dy = dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_1 \bar{F} \bullet d\bar{l} = \int_0^1 x^3 dx - x dx = -\frac{1}{4}$$

For 2, $y = -x + 2, dy = -dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_2 \bar{F} \bullet d\bar{l} = \int_1^2 (-x^3 + 2x^2 - x + 2)dx = \frac{17}{12}$$

For 3,

$$\int_3 \bar{F} \bullet d\bar{l} = \int_2^0 x^2 y dx\Big|_{y=0} = 0$$

$$\oint_L \bar{F} \bullet d\bar{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \underline{\underline{\frac{7}{6}}}$$

(b)

$$\nabla \times \bar{F} = -x^2\,\bar{a}_z\ ; \qquad d\bar{S} = dxdy(-\bar{a}_z)$$

$$\int(\nabla \times \bar{F}) \bullet d\bar{S} = -\iint(-x^2)dxdy = \int_0^1\int_0^x x^2 dydx + \int_1^2 \int_{y=0}^{-x+2} x^2 dydx$$

$$= \int_0^1 x^2 y\Big|_0^x dx + \int_1^2 x^2 y\Big|_0^{-x+2} dx = \frac{x}{4}\Big|_0^1 + \int_1^2 x^2(-x+2)dx = \underline{\underline{\frac{7}{6}}}$$

(c) Yes

**Prob.**

$$\oint \mathbf{A} \bullet dl = \int_{\rho=2}^{1} \rho \sin\phi d\rho \Big|_{\phi = 0} + \int_{\phi=0}^{\pi/2} \rho^2 \rho d\phi d\rho \Big|_{\rho = 1} + \int_{\rho=1}^{2} \rho \sin\phi \Big|_{\phi = 90^\circ} + \int_{\phi=\pi/2}^{0} \rho^3 d\phi \Big|_{\rho = 2}$$

$$= \frac{\pi}{2} + \frac{1}{2}(4-1) + 8(-\frac{\pi}{2}) = \underline{\underline{-9.4956}}$$

**Prob.**

We need coordinate transformation to get $F_\phi$.

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2xy \\ y \\ 0 \end{bmatrix}$$

$$F_\phi = -2xy\sin\phi + y\cos\phi$$

But $x = \rho\cos\phi, \qquad y = \rho\sin\phi$

$$F_\phi = -2\rho^2\cos\phi\sin^2\phi + \rho\sin\phi\cos\phi$$

$$\oint_L \mathbf{F}\cdot d\mathbf{l} = \int F_y dy\Big|_{x=2} + \int F_x dx\Big|_{y=2} + \int F_\phi \rho d\phi\Big|_{\rho=2}$$

$$= \frac{y^2}{2}\Big|_0^2 + x^2 y\Big|_2^0 + \int_{\pi/2}^{0} (-16\cos\phi\sin^2\phi + 4\sin\phi\cos\phi)d\phi\Big|_{\rho=2}$$

$$= 2 - 8 - 16\int_{\pi/2}^{0} \sin^2\phi d(\sin\phi) + 4\int_{\pi/2}^{0} \sin\phi d(\sin\phi)$$

$$= -6 + 16/3 - 4/2 = \underline{\underline{-2.67}}$$

**Prob.**

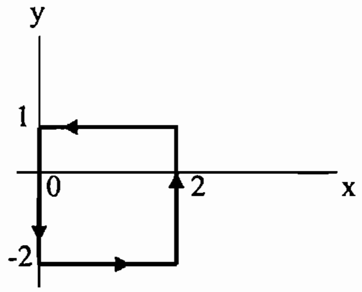
$$\nabla\cdot A = 8xe^{-y} + 8xe^{-y} = 16xe^{-y}$$

$$\nabla(\nabla\cdot\mathbf{A}) = 16e^{-y}\mathbf{a}_x - 16xe^{-y}\mathbf{a}_y$$

$$\nabla x\nabla(\nabla\cdot\mathbf{A}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16e^{-y} & -16xe^{-y} & 0 \end{vmatrix} = (-16e^{-y} + 16e^{-y})\mathbf{a}_z = \underline{\underline{0}}$$

Should be expected since $\nabla x\nabla V = 0$.

**Prob.**



$$\oint_L \mathbf{F}\cdot dl = \int_0^2 3y^2 z dx \Big|_{y=-2, z=1} + \int_{-2}^{1} 6x^2 y dy \Big|_{x=2, z=1}$$

$$+\int_2^0 3y^2 z dx \Big|_{y=1, z=1} + \int_1^{-2} 6x^2 y dy \Big|_{x=0, z=1}$$

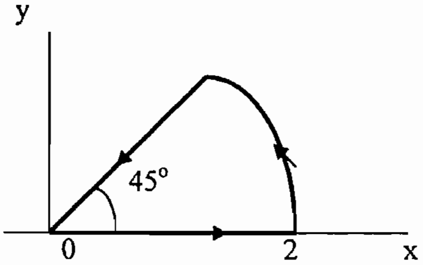
$$= 3(4)(1)(2) + 6(4)\frac{y^2}{2}\Big|_{-2}^{1} + 3(1)(1)(-2) + 0 = \underline{\underline{-18}}$$

$$\nabla x\mathbf{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y^2 z & 6x^2 y & 9xz^2 \end{vmatrix} = (12xy - 6yz)\boldsymbol{a}_z + \ldots$$

$$\int_S (\nabla xF)\cdot dS = \iint (12xy - 6yz)dxdy \Big|_{z=1} = 12\int_0^2 xdx \int_{-2}^{1} ydy - 6\int_{-2}^{1} ydy \int_0^2 dx$$

$$= 3x^2 \Big|_0^2 y^2 \Big|_{-2}^{1} - 3y^2 \Big|_{-2}^{1} (2) = 3(4)((1-4) - 6(1-4) = \underline{\underline{-18}}$$

**Prob.**



$$\oint \mathbf{F} \bullet dl = \int_0^2 2\rho z d\rho \Big|_{z=1} + \int_0^{\pi/4} 3z\sin\phi \, \rho d\phi \Big|_{\rho=2, z=1} + \int_2^0 2\rho z d\rho \Big|_{z=1}$$

$$= \rho^2 \Big|_0^2 + (-6\cos\phi)\Big|_0^{\pi/4} + \rho^2 \Big|_2^0 = (4-0) + 6(-\cos\pi/4 + 1) + (0-4) = \underline{\underline{1.757}}$$

$$\nabla x \mathbf{F} = \frac{1}{\rho}\left[3z\sin\phi - 0\right]\mathbf{a}_z + \ldots$$

$$\int (\nabla x F)\cdot dS = \int_{\rho=0}^{2} \int_{\phi=0}^{\pi/4} \frac{3z}{\rho}\sin\phi \, \rho d\phi d\rho \Big|_{z=} \quad = 3(2)(-\cos\phi)\Big|_0^{\pi/4}$$

$$= 6(-\cos\pi + 1) = \underline{\underline{1.757}}$$

**Prob.**

(a) $\nabla V = \underline{\underline{-\frac{\sin\theta\cos\phi}{r^2}\mathbf{a}_r + \frac{\cos\theta\cos\phi}{r^2}\mathbf{a}_\theta - \frac{\sin\phi}{r^2}\mathbf{a}_\phi}}$

(b) $\nabla x \nabla V = \underline{\underline{0}}$

(c)

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(-\sin\theta\cos\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\cos\theta\cos\phi}{r}) + \frac{1}{r^2\sin^2\theta}(-\frac{\sin\theta\cos\phi}{r})$$

$$= 0 + \frac{\cos\phi}{r^3\sin\theta}(1 - 2\sin^2\theta) - \frac{\cos\phi}{r^3\sin\theta}$$

$$= \underline{\underline{-\frac{2\sin\theta\cos\phi}{r^3}}}$$

**Prob.**

$$\bar{Q} = \frac{r}{r\sin\theta} r\sin\theta[(\cos\phi - \sin\phi)\bar{a}_x + (\cos\phi + \sin\phi)\bar{a}_y]$$

$$= r(\cos\phi - \sin\phi)\bar{a}_x + r(\cos\phi + \sin\phi)\bar{a}_y$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$\bar{Q} = r\sin\theta\bar{a}_r + r\cos\theta\bar{a}_\theta + r\bar{a}_\phi$$

(a)

$$d\bar{l} = \rho d\phi\bar{a}_\phi, \quad \rho = r\sin 30° = 2(\frac{1}{2}) = 1$$

$$z = r\cos 30° = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \bar{Q} \bullet d\bar{l} = \int_0^{2\pi} \sqrt{\rho^2 + z^2}\,\rho d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b)

$$\nabla \times \bar{Q} = \cot\theta\,\bar{a}_r - 2\bar{a}_\theta + \cos\theta\,\bar{a}_\phi$$

For $S_1$, $d\bar{S} = r^2\sin\theta\,d\theta\,d\phi\,\bar{a}_r$

$$\int_{S_1} (\nabla \times \bar{Q}) \bullet d\bar{S} = \int r^2 \sin\theta\cot\theta\,d\theta\,d\phi \Big|_{r=2}$$

$$= 4\int_0^{2\pi} d\phi \int_0^{30°} \cos\theta\,d\theta = \underline{\underline{4\pi}}$$

(c)

For $S_2$, $d\bar{S} = r\sin\theta\,d\theta\,dr\,\bar{a}_\theta$

$$\int_{S_2} (\nabla \times \bar{Q}) \bullet d\bar{S} = -2\int r\sin\theta\,d\phi\,dr \Big|_{\theta=30°}$$

$$= -2\sin 30 \int_0^2 r dr \int_0^{2\pi} d\phi$$

$$= \underline{\underline{-4\pi}}$$

(d)

For $S_1$, $d\bar{S} = r^2 \sin\theta d\phi d\theta \bar{a}_r$

$$\int_{S_1} \bar{Q} \bullet d\bar{S} = r^3 \int \sin^2\theta d\theta d\phi \Big|_{r=2}$$

$$= 8 \int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2\theta d\theta$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2}] = \underline{\underline{2.2767}}$$

(e)

For $S_2$, $d\bar{S} = r \sin\theta d\phi dr \bar{a}_\theta$

$$\int_{S_2} \bar{Q} \bullet d\bar{S} = \int r^2 \sin\theta \cos\theta d\phi dr \Big|_{\theta=30^\circ}$$

$$= \frac{4\pi\sqrt{3}}{3} = \underline{\underline{7.2552}}$$

(f)

$$\nabla \bullet \bar{Q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 \sin\theta) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 0$$

$$= 2\sin\theta + \cos\theta\cot\theta$$

$$\int \nabla \bullet \bar{Q} dV = \int (2\sin\theta + \cos\theta\cot\theta) r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{r^3}{3}\Big|_0^2 (2\pi) \int_0^{30} (1 + \sin^2\theta) d\theta$$

$$= \frac{4\pi}{3}(\pi - \frac{\sqrt{3}}{2}) = \underline{\underline{9.532}}$$

*Check*: $\int \nabla \bullet \bar{Q} dV = (\int_{S_1} + \int_{S_2} \bar{Q} \bullet d\bar{S}$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}]$$

$$= \frac{4\pi}{3}[\pi - \frac{\sqrt{3}}{2}] \quad \text{(It checks!)}$$

**Prob.**

Since $\bar{u} = \ddot{\omega} \times r$, $\nabla \times u = \nabla \times (\omega \times r)$. From Appendix B.10.

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{A}(\nabla \bullet \bar{B}) - \bar{B}(\nabla \bullet \bar{A}) + (B \bullet \nabla)\bar{A} - (\bar{A} \bullet \nabla)\bar{B}$$

$$\nabla \times \bar{u} = \nabla \times (\bar{\omega} \times \bar{r})$$

$$\nabla \times (\bar{\omega} \times \bar{r}) = \bar{\omega}(\nabla \bullet \bar{r}) - \bar{r}(\nabla \bullet \bar{\omega}) + (r \bullet \nabla)\bar{\omega} - (\bar{\omega} \bullet \nabla)\bar{r}$$

$$= \bar{\omega}(3) - \bar{\omega} = 2\bar{\omega}$$

or $\bar{\omega} = \frac{1}{2}\nabla \times \bar{u}$.

Alternatively, let $x = r\cos\omega t$, $y = r\sin\omega t$

$$\bar{u} = \frac{\partial x}{\partial t}\bar{a}_x + \frac{\partial y}{\partial t}\bar{a}_y$$

$$= -\omega r \sin\omega t\, \bar{a}_x + \omega r \cos\omega t\, \bar{a}_y$$

$$= -\omega y \bar{a}_x + \omega x \bar{a}_y$$

$$\nabla \times \bar{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\, \bar{a}_z = 2\omega$$

*i.e.*, $\bar{\omega} = \frac{1}{2}\nabla \times \bar{u}$

Note that we have used the fact that $\nabla \bullet \overline{\omega} = 0$, $(\bar{r} \bullet \nabla)\overline{\omega} = 0$, $(\overline{\omega} \bullet \nabla)\bar{r} = \overline{\omega}$

**Prob.**

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x^2y & 3xy^2 & 0 \end{vmatrix} = (3y^2 - 5x^2)\,\hat{a}_z$$

$$\int \nabla \times \vec{A}\, dv = \iiint (3y^2 - 5x^2)\, dx\, dy\, dz\, \vec{a}_z$$

$$= \vec{a}_z \left[ 3\int_0^2 dx \int_{-1}^{1} y^2 dy \int_{-5}^{5} dz - 5\int_0^2 x^2 dx \int_{-1}^{1} dy \int_{-5}^{5} dz \right]$$

$$= \vec{a}_z \left[ 40 - \frac{800}{3} \right] = -226.67\,\vec{a}_z$$

$$\oint \vec{A} \times d\vec{s} = \int \vec{A} \times dy\,dz\,\vec{a}_x \Big|_{x=2} - \int \vec{A} \times dy\,dz\,\vec{a}_x \Big|_{x=0}$$

$$+ \int \vec{A} \times dx\,dz\,\vec{a}_y \Big|_{y=1} - \int \vec{A} \times dx\,dz\,\vec{a}_y \Big|_{y=-1}$$

$$+ \int \vec{A} \times dx\,dy\,\vec{a}_z \Big|_{z=5} - \int \vec{A} \times dx\,dy\,\vec{a}_z \Big|_{z=-5}$$

$$= -\iint 3xy^2\, dy\,dz \Big|_{x=2} \vec{a}_z + \iint 3\underset{\downarrow 0}{\cancel{x}}y^2\, dy\,dz \Big|_{x=0} \vec{a}_z$$

$$+ \iint 5x^2y\, dx\,dz \Big|_{y=1} \vec{a}_z - \iint 5x^2y\, dx\,dz \Big|_{y=-1} \vec{a}_z$$

$$+ \iint (3xy^2\,\vec{a}_x - 5x^2y\,\vec{a}_y)\, dx\,dy \Big|_{z=5}$$

$$-\iint \left(3xy^2\,\vec{a}_x - 5x^2y\,\vec{a}_y\right)dx\,dy\Big|_{z=-5}$$

$$= -6\int_{-1}^{1} y^2\,dy \int_{-5}^{5} dz\,\vec{a}_z + 2(5)\int_{0}^{2} x^2\,dx \int_{-5}^{5} dz\,\vec{a}_z$$

$$= \left(-40 + \frac{800}{3}\right)\vec{a}_z = 226.67\,\vec{a}_z.$$

Thus $\int \nabla \times \vec{A}\,dv = -\oint \vec{A}\times d\vec{S}.$

**Prob.**

(a) $$\frac{\partial V_1}{\partial x} = -3x(x^2+y^2+z^2)^{-5/2}$$

$$\frac{\partial^2 V_1}{\partial x^2} = 15x^2(x^2+y^2+z^2)^{-7/2} - 3(x^2+y^2+z^2)^{-5/2}$$

Hence,

$$\nabla^2 V_1 = 15(x^2+y^2+z^2)(x^2+y^2+z^2)^{-7/2} - 9(x^2+y^2+z^2)^{-5/2}$$

$$= \underline{\underline{6(x^2+y^2+z^2)^{-5/2}}}$$

(b) $$\frac{\partial V_2}{\partial x} = ze^{-y} + 2x\ln y, \quad \frac{\partial^2 V_2}{\partial x^2} = 2\ln y$$

$$\frac{\partial V_2}{\partial y} = -xze^{-y} + \frac{2x}{y}, \quad \frac{\partial^2 V_2}{\partial y^2} = xze^{-y} - \frac{2x}{y^2}.$$

$$\nabla^2 V_2 = \underline{\underline{2\ln y + xze^{-y} - \frac{2x}{y^2}}}.$$

(c) $$\nabla^2 V_3 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(20\rho^2 \sin 2\phi) - \frac{40\rho^2}{\rho^2}\sin 2\phi$$

$$= 40\sin 2\phi - 40\sin 2\phi = \underline{\underline{0}}.$$

(d) $$\nabla^2 V_4 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho z)(\cos\phi + \sin\phi) - \frac{\rho z}{\rho^2}(\cos\phi + \sin\phi)$$

$$= \frac{z}{\rho}(\cos\phi + \sin\phi) - \frac{z}{\rho}(\cos\phi + \sin\phi)$$

$$= \underline{\underline{0}}.$$

(e) $$\nabla^2 V_5 = \frac{1}{r^2}\frac{\partial}{\partial r}(10r^3)\sin\theta\cos\phi + \frac{5r^2\cos\phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$- \frac{5r^2\sin\theta\cos\phi}{r^2\sin^2\theta}$$

$$= 30\sin\theta\cos\phi - 10\sin\theta\cos\phi$$

$$= \underline{\underline{20\sin\theta\cos\phi}}\ .$$

(f) $$\nabla^2 V_6 = \frac{\sin\theta}{r^2}\frac{\partial}{\partial r}\left(-\frac{2}{r}\right) + \frac{1}{r^4\sin\theta}\left(1-2\sin^2\theta\right)$$

$$= \underline{\underline{\frac{1}{r^4\sin\theta}}}\ .$$

**Prob.**

(a)

$$U = x^3y^2e^{xz}$$

$$\nabla^2 U = \frac{\partial}{\partial x}(3x^2y^2e^{xz}) + \frac{\partial}{\partial y}(2x^2ye^{xz}) + \frac{\partial}{\partial z}(x^4y^2e^{xz})$$

$$= 6xy^2e^{xz} + 2x^2e^{xz} + x^5y^2e^{xz} \qquad \underline{\underline{= (6xy^2 + 2x^2 + x^5y^2)e^{xz}}}$$

At $(1,-1,1)$,

$$\nabla^2 U = e^1\ (6+2+1) = 9e \underline{\underline{= 24.46}}$$

(b)

$$V = \rho^2 z(\cos\phi + \sin\phi)$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}[2\rho^2 z(\cos\phi + \sin\phi)] - z(\cos\phi + \sin\phi) + 0$$

$$= 4z(\cos\phi + \sin\phi) - z(\cos\phi + \sin\phi)$$

$$= 3z(\cos\phi + \sin\phi)$$

At $(5, \frac{\pi}{6}, -2)$, $\nabla^2 V = -6(0.866 + 0.5) \underline{\underline{= -8.196}}$

(c)

$$W = e^{-r}\sin\theta\cos\phi$$

$$\nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}\ (-r^2e^{-r}\sin\theta\cos\phi) + \frac{e^{-r}}{r^2\sin\theta}\cos\phi\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$-\frac{e^{-r}\sin\theta\cos\phi}{r^2\sin^2\theta}$$

$$= \frac{1}{r^2}(-2re^{-r}\sin\theta\cos\phi) + e^{-r}\sin\theta\cos\phi$$

$$+ \frac{e^{-r}\cos\phi}{r^2\sin\theta}(\cos^2\theta - \sin^2\theta) - \frac{-e^{-r}\cos\theta}{r^2\sin\theta}$$

$$\underline{\underline{\nabla^2 W = e^{-r}\sin\theta\cos\phi\left(1 - \frac{4}{r}\right)}}$$

At $(1, 60°, 30°)$,

$$\nabla^2 W = e^{-1}\sin 60\cos 30(1-4) = -2.25e^{-1}\ = \underline{\underline{-0.8277}}$$

**Prob.**

(a)

$$V_1 = x^3 + y^3 + z^3$$

$$\nabla^2 V_1 = \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}\left(3y^2\right) + \frac{\partial}{\partial x}(3z^2)$$

$$= 6x + 6y + 6z = \underline{\underline{6(x + y + z)}}$$

(b)

$$V_2 = \rho z^2 \sin 2\phi$$

$$\nabla^2 V_2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho z^2 \sin 2\phi) - \frac{4z^2}{\rho}\sin 2\phi + \frac{\partial}{\partial z}(2\rho z \sin 2\phi)$$

$$= \frac{z}{\rho}\sin 2\phi - \frac{4z^2}{\rho}\sin 2\phi + 2\rho \sin 2\phi$$

$$= \underline{\underline{(\frac{-3z^2}{\rho} + 2\rho)\sin 2\phi}}$$

(c)

$$V_3 = r^2(1 + \cos\theta \sin\phi)$$

$$\nabla^2 V_3 = \frac{1}{r^2}\frac{\partial}{\partial r}[2r^3(1 + \cos\theta \sin\phi)] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(-\sin\theta \sin\phi\right)r^2$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(-\sin^2\theta \sin\phi)r^2 + \frac{1}{r^2 \sin^2\theta}r^2(-\cos\theta \sin\phi)$$

$$= 6(1 + \cos\theta \sin\phi) - \frac{2\sin\theta}{\sin\theta}\cos\theta \sin\phi - \frac{\cos\theta \sin\phi}{\sin^2\theta}$$

$$= \underline{\underline{6 + 4\cos\theta \sin\phi - \frac{\cos\theta \sin\phi}{\sin^2\theta}}}$$

**Prob.**

$$(a)\ \nabla V = \underline{\underline{(6xy + z)\bar{a}_x + 3x^2 \bar{a}_y + x\bar{a}_z}}$$

$$\nabla \bullet \nabla V = \underline{\underline{6y}}$$

$$\nabla \times \nabla V = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z & 3x^2 & x \end{vmatrix} = \underline{\underline{0}}$$

$$(b)\ \nabla V = \underline{\underline{z\cos\phi\, \bar{a}_\rho - z\sin\phi\, \bar{a}_\phi + \rho\cos\phi\, \bar{a}_z}}$$

$$\nabla \bullet \nabla V = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho z \cos\phi) + \frac{z}{\rho}\cos\theta + 0 = \frac{z}{\rho}\cos\phi - \frac{z}{\rho}\cos\phi = \underline{\underline{0}}$$

$$\nabla \times \nabla V = \underline{\underline{0}}$$

$$(c)\ \bar{V} V = \frac{1}{r^2}(24r^2)\cos\theta\sin\phi + \frac{4r\cos\phi}{r\sin\theta}(\cos^2\theta\sin^2\theta)$$

$$-\frac{4}{r^2\sin^2\theta}\cos\theta\sin\phi$$

$$= \underline{\underline{24r\cos\theta\sin\phi + \frac{4\cos\phi}{\sin\theta} - 8\cos\phi\sin\theta - \frac{4\cos\theta\sin\phi}{\sin^2\theta}}}$$

$$\nabla \times \nabla V = \underline{0}$$

**Prob.**

(a)

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2}$$

$$= \underline{\underline{2(y^2z^2 + x^2z^2 + x^2y^2)}}$$

(b)

$$\nabla^2 \bar{A} = \nabla^2 A_x \ \ \bar{a}_x + \nabla^2 A_y \bar{a}_y + \nabla^2 A_z \bar{a}_z$$

$$= (2y + 0 + 0)\bar{a}_x + (0 + 0 + 6xz)\bar{a}_y + (0 - 2z^2 - 2y^2)\bar{a}_z$$

$$= \underline{\underline{2y\bar{a}_x + 6xz\bar{a}_y - 2(y^2 + z^2)\bar{a}_z}}$$

(c)

$$grad\ div\ \ A = \nabla(\nabla \bullet \bar{A}) \ \ = \nabla(2xy + 0 - 2y^2z)$$

$$= \underline{\underline{2y\bar{a}_x + 2(x - 2yz)\bar{a}_y - 2y^2\bar{a}_z}}$$

(d)

$$curl\ \ curl\ \ \bar{A} = \nabla x \nabla x \bar{A} \ = \nabla(\nabla \bullet \bar{A}) - \nabla^2 \bar{A}$$

From parts (b) and (c),

$$\nabla x \nabla x \bar{A} \ = \ \underline{\underline{2(x - 2yz - 3xz)\bar{a}_y + 2z^2\bar{a}_z}}$$

**Prob.**

Let $\bar{A} = U\nabla V$ and apply Stokes' theorem.

$$\int_L U\nabla V \bullet d\bar{l} = \int \nabla X(U\nabla V) \bullet d\bar{S}$$

$$= \int_s (\nabla UX\nabla V) d\bar{S} + \int_s U\ (\nabla X\nabla V) \bullet d\bar{S}$$

But $\nabla X\nabla V = 0$. Hence,

$$\int_L U\nabla V \bullet d\bar{l} = \int_s (\nabla UX\nabla V) \bullet d\bar{S}$$

Similarly, we can show that

$$\int_L V\nabla U \bullet d\bar{l} = \int (\nabla VX\nabla U) \bullet d\bar{S} - \int (\nabla UX\nabla V) \bullet d\bar{S}$$

Thus, $$\int_L U\nabla V \bullet d\bar{l} = -\int_L V\nabla U \bullet d\bar{l}$$

## Chapter —Practice Examples

### P.E.

$$(a)\ \bar{F} = \frac{1x10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left[\begin{array}{r} \frac{5x10^{-9}[(1,-3,7)-(2,0,4)]}{[(1,-3,7)-2,0,4)]^{3}} \\ +\frac{(-2x10^{-9})[(1,-3,7)-(-3,0,5)]}{[(1,-3,7)-(-3,0,5)]^{3}} \end{array}\right]$$

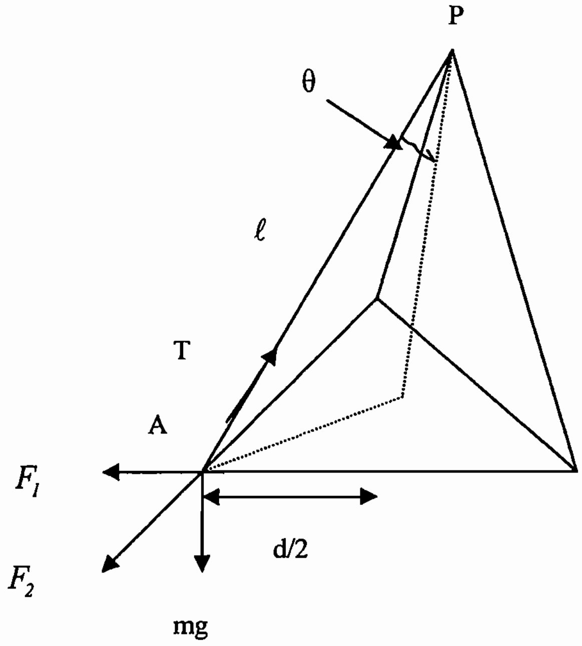
$$=[\frac{45(-1,-3,3)}{19^{3/2}}-\frac{18(4,-3,2)}{29^{3/2}}]\ \text{nN}$$

$$=\underline{\underline{-1.004\bar{a}_x-1.284\bar{a}_y+1.4\bar{a}_z\ \text{nN}}}$$

$$(b)\qquad \bar{E}=\frac{\bar{F}}{Q}=\underline{\underline{-1.004\bar{a}_x-1.284\bar{a}_y+1.4\bar{a}_z\ \text{V/m}}}$$

### P.E.

Let q be the charge on each sphere, i.e. q=Q/3. The free body diagram below helps us establish the relationship between various forces.



At point $A$,

$$T\sin\theta\cos 30° = F_1 + F_2\cos 60°$$

$$= \frac{q^2}{4\pi\varepsilon_0 d^2} + \frac{q^2}{4\pi\varepsilon_0 d^2}(\frac{1}{2})$$

$$= \frac{3q^2}{8\pi\varepsilon_0 d^2}$$

$$T\cos\theta = mg$$

Hence, $\tan\theta\cos 30° = \dfrac{3q^2}{8\pi\varepsilon_0 d^2 mg}$

But $\sin\theta = \dfrac{h}{l} = \dfrac{d}{\sqrt{3}\,l}\tan\theta = \dfrac{\frac{d}{\sqrt{3}}}{\sqrt{l^2 - \frac{d^2}{3}}}$

Thus, $\dfrac{\frac{d}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{\sqrt{l^2 - \frac{d^2}{3}}} = \dfrac{3q^2}{8\pi\varepsilon_0 d^2 mg}$

or $q^2 = \dfrac{4\pi\varepsilon_0 d^3 mg}{3\sqrt{l^2 - \frac{d^2}{3}}}$

but $q = \dfrac{Q}{3}$ $\longrightarrow$ $q^2 = \dfrac{Q}{9}$. Hence,

$$Q^2 = \underline{\underline{\frac{12\pi\varepsilon_0 d^3 mg}{\sqrt{l^2 - \frac{d^2}{3}}}}}$$

**P.E.**

$$e\bar{E} = m\frac{d^2\bar{l}}{dt^2}$$

$$eE_0(-2\bar{a}_x + \bar{a}_y) = m(\frac{d^2x}{dt^2}\bar{a}_x + \frac{d^2y}{dt^2}\bar{a}_y + \frac{d^2z}{dt^2}\bar{a}_z)$$

where $E_0 = 200$ kV / m

$$\frac{d^2z}{dt^2} = 0 \longrightarrow \qquad z = ct + c_2$$

$$m\frac{d^2x}{dt^2} = -2eE_0 \longrightarrow \qquad x = \frac{-2eE_0t^2}{2m} + c_3t + c_4$$

$$m\frac{d^2y}{dt^2} = eE_0 \longrightarrow \qquad y = \frac{eE_0t^2}{2m} + c_5t + c_6$$

At $t = 0$, $(x, y, z) = (0,0,0)$ $c_1 = 0 = c_4 = c_6$

Also, $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = (0,0,0)$

At $t = 0 \longrightarrow \qquad c_1 = 0 = c_3 = c_5$

Hence, $(x, y) = \frac{eE_0t^2}{2m}(-2,1)$

*i.e.* $2\,|y| = |x|$

Thus the largest value of is

80 cm = 0.8 m

**P.E.**
**(a)**

Consider an element of area $ds$ of the disk.

The contribution due to $ds = \rho d\phi d\rho$ is

$$dE = \frac{\rho_s ds}{4\pi\varepsilon_0 r^2} = \frac{\rho_s ds}{4\pi\varepsilon_0(\rho^2 + h^2)}$$

The sum of the contribution along $\rho$ gives zero.

$$E_z = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{h\rho\, d\rho\, d\phi}{(\rho^2 + h^2)^{3/2}} = \frac{h\rho_s}{2\varepsilon_0}\int_{\rho-0}^{a}\frac{\rho\, d\rho}{(\rho^2 + h^2)^{3/2}}$$

$$= \frac{h\rho_s}{4\varepsilon_0}\int_0^a(\rho^2 + h^2)^{-3/2}d(\rho^2) = \frac{h\rho_s}{4\varepsilon_0}(-2(\rho^2 + h^2)^{-1/2}\Big|_0^a$$

$$= \frac{\rho_s}{2\varepsilon_0}[1 - \frac{h}{(h^2 + a^2)^{1/2}}]$$

**(b)**

$$As\ a \longrightarrow \infty,$$

$$\bar{E} = \underline{\underline{\frac{\rho_s}{2\varepsilon_0}\bar{a}_z}}$$

**(c)** Let us recall that if $a/h << 1$ then $(1+a/h)^n$ can be approximated by $(1+na/h)$. Thus the expression for $E_z$ from (a) can be modified for $a<<h$ as follows.

$$E_z = \frac{\rho_s}{2\varepsilon_o}\left[1-\frac{1}{\sqrt{1+\frac{a^2}{h^2}}}\right] = \frac{\rho_s}{2\varepsilon_o}\left[1-\left(1+\frac{a^2}{h^2}\right)^{-\frac{1}{2}}\right] \xrightarrow[a\to 0,\ but\ \rho_s\pi a^2=Q]{} \frac{\rho_s}{2\varepsilon_o}\left[\frac{a^2}{2h^2}\right]$$

$$= \frac{\rho_s}{2\varepsilon_o}\left[\frac{\pi a^2}{2\pi h^2}\right] = \frac{Q}{4\pi\varepsilon_o h^2}$$

This is in keeping with original Coulomb's law.

**P.E.**

$$Q_S = \int \rho_s dS = \int_{-2}^{2}\int_{-2}^{2} 12|y|\, dx\, dy$$

$$= 12(4)\int_0^2 2y\, dy = \underline{\underline{192\ \text{mC}}}$$

$$\bar{E} = \int \frac{\rho_s dS}{4\pi\varepsilon r^2}\bar{a}_r = \int \frac{\rho_s dS|\bar{r}-\bar{r}'|}{4\pi\varepsilon_o|\bar{r}-\bar{r}'|^3}$$

where $\bar{r}-\bar{r}' = (0,0,10)-(x,y,z) = (-x,-y,10)$.

$$\bar{E} = \int_{x=-2}^{2} \int_{y=-2}^{2} \frac{12\,|y|10^{-3}(-x,-y,10)}{4\pi(\frac{10^{-9}}{36\pi})(x^2+y^2+100)^{3/2}}$$

$$= 108(10^6)[\int_{-2}^{2}|y| \int_{-2}^{2}\frac{-xdx\,dy\ \bar{a}_x}{(x^2+y^2+100)^{3/2}} + \int_{-2}^{2}\int_{-2}^{2}\frac{-y|y|\ dy\,dx\,\bar{a}_y}{(x^2+y^2+100)^{3/2}}$$

$$+10\bar{a}_z \int_{-2}^{2}\int_{-2}^{2}\frac{|y|\,dx\,dy}{(x^2+y^2+100)^{3/2}}]$$

$$\bar{E} = 108(10^7)\bar{a}_z \int_{-2}^{2} [2\int_{0}^{2}\frac{\frac{1}{2}d(y^2)}{(x^2+y^2+100)^{3/2}}]dx$$

$$= -216(10^7)\bar{a}_z \int_{-2}^{2} [\frac{1}{(x^2+104)^{1/2}} - \frac{1}{(x^2+100)^{1/2}}]dx$$

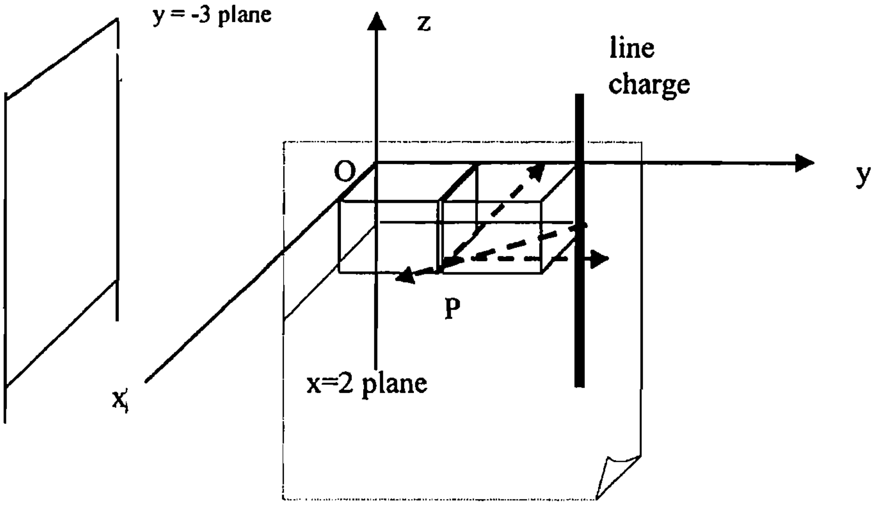
$$= -216(10^7)\bar{a}_z\ \ln|\frac{x+\sqrt{x^2+104}}{x+\sqrt{x^2+100}}|\,\Big|_{-2}^{2}$$

$$= -216(10^7)\bar{a}_z(\ln(\frac{2+\sqrt{108}}{2+\sqrt{104}}) - \ln(\frac{-2+\sqrt{108}}{-2+\sqrt{104}}))$$

$$= -216(10^7)\bar{a}_z(-7.6202\ (10^{-3}))$$

$$\bar{E} = \underline{\underline{16.46\,\bar{a}_z \quad \text{MV/m}}}$$

**P.E.**



$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\bar{a}_\rho$$

This expression, which represents the field due to a line charge,

is modified as follows. To get $\bar{a}_\rho$, consider the $z = -1$ plane. $\rho = \sqrt{2}$

$$\bar{a}_\rho = \bar{a}_x \cos 45° - \bar{a}_y \sin 45°$$

$$= \frac{1}{\sqrt{2}}(\bar{a}_x - \bar{a}_y)$$

$$\bar{E}_3 = \frac{10(10^{-9})}{2\pi(\frac{10^{-9}}{36\pi})}\frac{1}{2}(\bar{a}_x - \bar{a}_y)$$

$= 90\pi(\bar{a}_x - \bar{a}_y)$. Hence,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3$$

$$= -180\pi\bar{a}_x + 270\pi\bar{a}_y + 90\pi\bar{a}_x - 90\pi\bar{a}_y.$$

$$= -282.7\,\bar{a}_x + 565.5\bar{a}_y \text{ V/m}$$

**P.E.**

$$\bar{D} = \bar{D}_Q + \bar{D}_\rho = \frac{Q}{4\pi r^2}\bar{a}_r + \frac{\rho_s}{2}\bar{a}_n$$

$$= \frac{30x10^{-9}}{4\pi(5)^2}\frac{[(0,4,3)-(0,0,0)]}{5} + \frac{10x10^{-9}}{2}\bar{a}_y$$

$$= \frac{30}{500\pi}(0,4,3) + 5\bar{a}_y \text{ nC/m}^2$$

$$= \underline{\underline{5.076\bar{a}_y + 0.0573\bar{a}_z \text{ nC/m}^2}}$$

**P.E.**

(a) $\rho_v = \nabla \bullet \bar{D} = 4x$

$\rho_v(-1,0,3) = \underline{\underline{-4 \text{ C/m}^3}}$

(b) $\Psi = Q = \int \rho_v dv = \int_0^1\int_0^1\int_0^1 4xdxdydz$

$= 4(1)(1)(1/2) = \underline{\underline{2 \text{ C}}}$

(c) $Q = \Psi = \underline{\underline{2 \text{ C}}}$

**P.E.**

$$Q = \int \rho v dv = \psi = \oint \bar{D} \bullet d\bar{s}$$

For $0 \le r \le 10$,

$$D_r(4\pi r^2) = \iiint 2r\ (r^2)\sin\theta\, d\theta\, dr\, d\phi$$

$$D_r(4\pi r^2) = 4\pi(\frac{2r^4}{4}\Big|_0^r) = 2\pi r^4$$

$$D_r = \frac{r^2}{2} \qquad \bar{E} = \frac{r^2}{2\varepsilon_0}\bar{a}_r \text{ nV/m}$$

$$\bar{E}(r=2) = \frac{4(10^{-9})}{2(\frac{10^{-9}}{36\pi})}\bar{a}_r = 72\pi\,\bar{a}_r = \underline{\underline{226\ \bar{a}_r\ \text{V/m}}}$$

For $r \ge 10$,

$$D_r(4\pi r^2) = 2\pi r_0^{\ 4}, \qquad r_0 = 10\text{m}$$

$$D_r = \frac{r_0^{\ 4}}{2r^2} \longrightarrow \bar{E} = \frac{r_0^{\ 4}}{2\varepsilon_0 r^2}\bar{a}_r \text{ nV/m}$$

$$\bar{E}(r=12) = \frac{10^4(10^{-9})}{2(\frac{10^{-9}}{36\pi})(144)}\bar{a}_r = 1250\pi\,\bar{a}_r$$

$$= \underline{\underline{3.927\,\bar{a}_r \text{ kV/m}}}$$

**P.E.**

$$V(\bar{r}) = \sum_{k=1}^{3}\frac{Q_k}{4\pi\varepsilon_0\,|\bar{r}-\bar{r}_k|} + C$$

$At\ \ V(\infty) = 0, \quad C = 0$

$$|\bar{r}-\bar{r}_1| = |(-1,5,2)-(2,-1,3)| = \sqrt{46}$$

$$|\bar{r}-\bar{r}_2| = |(-1,5,2)-(0,4,-2)| = \sqrt{18}$$

$$|\bar{r}-\bar{r}_3| = |(-1,5,2)-(0,0,0)| = \sqrt{30}$$

$$V(-1,5,2) = \frac{10^{-6}}{4\pi(\frac{10^{-9}}{36\pi})}[\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}}]$$

$$= \underline{\underline{10.23\ k\text{V}}}$$

**P.E.**

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$

If $V(0,6,-8) = V(r=10) = 2;$

$$2 = \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})(10)} + C \qquad \longrightarrow \qquad C = -2.5$$

(a)

$$V_A = \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})|(-3,2,6) - (0,0,0)|} - 2.5$$

$$= \underline{\underline{3.929\ V}}$$

(b)

$$V_B = \frac{45}{\sqrt{7^2 + 1^2 + 5^2}} - 2.5 = \underline{\underline{2.696\,V}}$$

(c ) $V_{AB} = V_B - V_A = 2.696 - 3.929 = \underline{\underline{-\ 1.233\,V}}$

**P.E.**

(a)

$$\frac{-W}{Q} = \int \bar{E} \bullet d\,\bar{l} = \int (3x^2 + y)dx + xdy$$

$$= \int_0^2 (3x^2 + y)dx\ \Big|_{y=5} + \int_5^{-1} x\,dy\ \Big|_{x=2}$$

$$= 18 - 12 = 6 \text{ kV}$$

$$W = -6Q = \underline{\underline{12 \text{ mJ}}}$$

**(b)**

$$dy = -3\,dx$$

$$-\frac{W}{Q} = \int E \bullet dl = \int_0^2 (3x^2 + 5 - 3x)dx + x(-3)dx$$

$$= \int_0^2 (3\text{x}^2 - 6x + 5)dx = 8 - 12 + 10 = 6$$

$$W = \underline{\underline{12 \text{ mJ}}}$$

**P.E.**

(a)

$$(0,0,10) \longrightarrow (r=10, \theta=0, \phi=0)$$

$$V = \frac{100\cos 0}{4\pi\varepsilon_0(10^2)}(10^{-12}) = \frac{10^{-12}}{4\pi(\frac{10^{-9}}{36\pi})} = \underline{\underline{9\ \text{mV}}}$$

$$\bar{E} = \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})10^3}[2\cos 0\,\bar{a}_r + \sin 0\,\bar{a}_\theta]$$

$$= \underline{\underline{1.8\,\bar{a}_r\ \text{mV/m}}}$$

**(b)**

At $(1, \frac{\pi}{3}, \frac{\pi}{2})$,

$$V = \frac{100\cos\frac{\pi}{3}(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})(1)^2} = \underline{\underline{0.45\ V}}$$

$$\bar{E} = \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})(1)^2}(2\cos\frac{\pi}{3}\bar{a}_r + \sin\frac{\pi}{3}\bar{a}_\theta)$$

$$= \underline{\underline{0.9\,\bar{a}_r + 0.7794\,\bar{a}_\theta\ \ \text{V/m}}}$$

**P.E.**

After $Q_1$, $W_1 = 0$

After $Q_2$, $W_2 = Q_2 V_{21} = \dfrac{Q_2\, Q_1}{4\pi\varepsilon_0 |(1,0,0) - (0,0,0)|}$

$$= \frac{1(-2)(10^{-18})}{4\pi(10^{-9})\dfrac{1}{36\pi}} = \underline{\underline{-18 \text{ nJ}}}$$

After $Q_3$,

$$W_3 = Q_3(V_{31} + V_{32}) + Q_2 V_{21}$$

$$= 3(9)(10^{-9})\left\{\frac{1}{|(0,0,-1) - (0,0,0)|} + \frac{-2}{|(0,0,-1) - (1,0,0)|}\right\} - 18 \text{nJ}$$

$$= 27(1 - \frac{2}{\sqrt{2}}) - 18$$

$$= \underline{\underline{-29.18 \text{ nJ}}}$$

After $Q_4$,

$$W_4 = Q_4(V_{41} + V_{42} + V_{43}) + Q_3(V_{31} + V_{32}) + Q_2 V_{21}$$

$$= -4(9)(10^{-9})\left\{\frac{1}{|(0,0,1) - (0,0,0)|} + \frac{-2}{|(0,0,1) - (1,0,0)} + \frac{3}{|(0,0,1) - (0,0,-1)}\right\} + W_3$$

$$= -36(1 - \frac{2}{\sqrt{2}} + \frac{3}{2}) + W_3$$

$$= -39.09 - 29.18\ nJ = -68.27 \text{ nJ}$$

## CHAPTER —Problems

**Prob.**

(a) See Text.

(b) $$\vec{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{[(0,0,4)-(-2,6,1)]}{|(0,0,4)-(-2,6,1)|^3} + \frac{Q_1 Q_3}{4\pi\varepsilon_0} \frac{[(0,0,4)-(3,-4,-8)]}{|(0,0,4)-(3,-4,-8)|^3}$$

$$= 9\left[\frac{(4,-12,6)}{7^3} + \frac{(9,-12,-36)}{13^3}\right] \text{kN}$$

$$\vec{F}_1 = \underline{\underline{142\,\bar{a}_x - 364\,\bar{a}_y + 10\,\bar{a}_z \text{ N.}}}$$

**Prob.**

**(a)** $$Q = \int \rho_L dl = \int_0^5 12x^2 dx = 4x^3 \Big|_0^5 mC = \underline{\underline{0.5C}}$$

**(b)**

$$Q = \int \rho_S dS = \int_{z=0}^{4} \int_{\phi=0}^{2\pi} \rho z^2 \rho \, d\phi \, dz \Big|_{\rho=3} = 9(2\pi)\frac{z^3}{3}\Big|_0^4 nC$$

$$= \underline{\underline{1.206\ \mu C}}$$

(c)

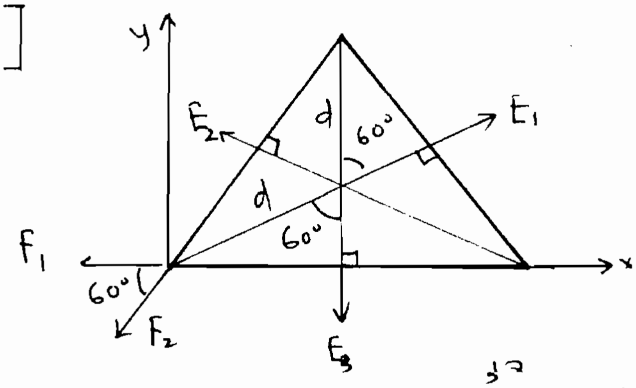
$$Q = \int \rho_V dV = \iiint \frac{10}{r\sin\theta}\ r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 10\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{4} r dr = 10(2\pi)(\pi)\frac{4^2}{2}$$

$$= \underline{\underline{1579.1\ C}}$$

**Prob.**

]



$^{12}$

(a) $$|\vec{F_1}| = |\vec{F_2}| = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2} = \frac{100\times10^{-18}}{4\pi\times\frac{10^{-9}}{36\pi}\cdot100\times10^{-4}}$$

$$= 9\times10^{-5}\ N$$

$$F_x = F_1 + F_2\cos 60^\circ = \frac{3}{2}F_1$$

$$F_y = F_2 \sin 60^\circ = \frac{\sqrt{3}}{2}F_1$$

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{3}F_1 = 9\sqrt{3}\cdot10^{-5}$$

$$F = \underline{\underline{156\ \mu N}}.$$

(b) $$E_1 = E_2 = E_3 = \frac{Q}{4\pi\epsilon_0 d^2}$$

$$E_x = E_1 \sin 60' - E_2 \sin 60' = 0$$

$$E_y = E_1 \cos 60' + E_2 \cos 60^\circ - E_3 = 0$$

Hence $\underline{\underline{\vec{E} = 0}}$ at the center.

**Prob.**

**(a)**

$$\bar{E}(5,0,6) = \frac{Q_1}{4\pi\varepsilon_0}\frac{[(5,0,6)-(4,0,-3)]}{|(5,0,6)-(4,0,-3)|^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{[(5,0,6)-(2,0,1)]}{|(5,0,6)-(2,0,1)|^3}$$

$$= \frac{Q_1}{4\pi\varepsilon_0}\frac{(1,0,9)}{(\sqrt{82})^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{(3,0,5)}{(34)^{3/2}}$$

If $\bar{E}_z = 0$, then

$$\frac{9Q_1}{4\pi\varepsilon_0}\frac{1}{(82)^{3/2}} + \frac{5Q_2}{4\pi\varepsilon_0}\frac{1}{(34)^{3/2}} = 0$$

$$\bar{Q}_1 = -\frac{5}{9}Q_2(\frac{82}{34})^{3/2} = -\frac{5}{9}4(\frac{82}{34})^{3/2} \text{ nC}$$

$$= \underline{\underline{-8.3232 \text{ nC}}}$$

**(b)**

$$\bar{F}(5,0,6) = q\,\bar{E}(5,0,6)$$

If $F_x = 0$, then

$$\frac{qQ_1}{4\pi\varepsilon_0(82)^{3/2}} + \frac{3qQ_2}{4\pi\varepsilon_0(34)^{3/2}} = 0$$

$$Q_1 = -3Q_2(\frac{82}{34})^{3/2} = -12(\frac{82}{34})^{3/2} \ nC$$

$$Q_1 = \underline{\underline{-44.945 \ nC}}$$

**Prob.**

$$\boldsymbol{E}_R = \int \frac{\rho_L dl\boldsymbol{R}}{4\pi\varepsilon_o R^3}, \qquad \boldsymbol{R} = -2\boldsymbol{a}_\rho, \qquad \rho_L = \frac{Q_L}{2\pi} = \frac{10}{2\pi}\ \text{nC/m}$$

But $\boldsymbol{a}_\rho = \cos\phi \boldsymbol{a}_x + \sin\phi \boldsymbol{a}_y$

Due to symmetry, contributions along y add up, while contributions along x cancel out.

$$E_R = \frac{\rho_L}{4\pi\varepsilon_o R^3}\int_0^\pi (-2\sin\phi)\boldsymbol{a}_y 2d\phi = \frac{\rho_L(2)}{4\pi\varepsilon_o 2^3}(-1-1)2\boldsymbol{a}_y = -\frac{\rho_L}{4\pi\varepsilon_o}\boldsymbol{a}_y = -\frac{Q_L}{8\pi^2\varepsilon_o}\boldsymbol{a}_y, \qquad \rho_L = \frac{Q_L}{2\pi}$$

$$\boldsymbol{E}_Q = \frac{Q\boldsymbol{R}}{4\pi\varepsilon_o R^3} = \frac{Q[(0,0,0)-(0,-4,0)}{4\pi\varepsilon_o |(0,0,0)-(0,-4,0)|^3} = \frac{Q\boldsymbol{a}_y}{64\pi\varepsilon_o}$$

$$E = E_R + E_Q = -\frac{Q_L}{8\pi^2\varepsilon_o} + \frac{Q}{64\pi\varepsilon_o} = 0 \quad \longrightarrow \quad Q = 8\frac{Q_L}{\pi}$$

$$Q = \frac{8x10nC}{\pi} = \underline{\underline{25.47\ \text{nC}}}$$

**Prob.**

Let $\rho_s = k\rho$

$$Q = \int k\rho\, dS = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} k\rho \cdot \rho\, d\phi\, d\rho = \frac{2\pi k a^3}{3}.$$

Hence $k = \dfrac{3Q}{2\pi a^3}$ and $\rho_s = \dfrac{3Q\rho}{2\pi a^3}$

Due to symmetry, $\vec{E}$ has only z-component.

$$dE_z = \frac{\rho_s\, dS \cos\alpha}{4\pi\epsilon_o r} = \frac{\rho_s\, dS}{4\pi\epsilon_o(\rho^2+h^2)} \cdot \frac{h}{(\rho^2+h^2)^{1/2}}$$

$$E_z = \frac{3Q}{2\pi a^3} \cdot \frac{1}{4\pi\epsilon_o}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{\rho h \cdot \rho\, d\phi\, d\rho}{(\rho^2+h^2)^{3/2}}$$

$$= \frac{3Qh}{4\pi\epsilon_o a^3}\int_0^a \frac{\rho^2\, d\rho}{(\rho^2+h^2)^{3/2}}.$$

Let $\rho = h\tan\theta$, $d\rho = h\sec^2\theta d\theta$.

$$\int_0^a \frac{\rho^2 d\rho}{(\rho^2+h^2)^{3/2}} = \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int (\sec\theta - \cos\theta) d\theta$$

$$= \ln(\sec\theta + \tan\theta) - \sin\theta$$

Hence,

$$E_z = \frac{3Qh}{4\pi\epsilon_0 a^3}\left[\ln\left(\frac{\sqrt{\rho^2+h^2}+\rho}{h}\right) - \frac{\rho}{\sqrt{\rho^2+h^2}}\right]\Bigg|_0^a$$

or

$$\vec{E} = \frac{3Qh}{4\pi\epsilon_0 a^3}\left[\ln\frac{a+\sqrt{a^2+h^2}}{h} - \frac{a}{\sqrt{a^2+h^2}}\right]\vec{a}_z$$

as required.

**Prob.**

$$V = \int_S \frac{\rho_S dS}{4\pi\varepsilon_0 r}; \qquad \rho_S = \frac{1}{\rho}; \quad dS = \rho\, d\phi\, d\rho; \quad r = \sqrt{\rho^2 + h^2}$$

$$V = \frac{1}{4\pi\varepsilon_0}\iint \frac{\frac{1}{\rho}(\rho\, d\phi\, d\rho)}{(\rho^2 + h^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_{\rho=0}^{a} \frac{d\rho}{(\rho^2 + h^2)}$$

$$= \frac{2\pi}{4\pi\varepsilon_0}\ln(\rho + \sqrt{\rho^2 + h^2})\Big|_{\rho=0}^{a} = \frac{1}{2\varepsilon_0}[\ln(a + \sqrt{\rho^2 + h^2}) - \ln h]$$

$$= \underline{\underline{\frac{1}{2\varepsilon_0}\ln\frac{a + \sqrt{\rho^2 + h^2}}{h}}}$$

**Prob.**

(a)

Due to symmetry, $\bar{E}$ has only $z$ – component given by

$$dE_z = dE\cos\alpha$$

$$= \frac{\rho_S dx\,dy}{4\pi\varepsilon_0(x^2+y^2+h^2)}\frac{h}{(x^2+y^2+h^2)^{1/2}}$$

$$E_z = \frac{\rho_S h}{4\pi\varepsilon_0}\int_{-a}^{a}\int_{-b}^{b}\frac{dxdy}{(x^2+y^2+h^2)^{3/2}}$$

$$= \frac{\rho_S h}{\pi\varepsilon_0}\int_{0}^{a}\int_{0}^{b}\frac{dxdy}{(x^2+y^2+h^2)^{3/2}}$$

$$= \frac{\rho_S h}{\pi\varepsilon_0}\int_{0}^{a}\frac{y\,dx}{(x^2+h^2)(x^2+y^2+h^2)^{1/2}}\Bigg|_0^b$$

$$= \frac{\rho_S h}{\pi\varepsilon_0}\int_{0}^{a}\frac{b\,dx}{(x^2+h^2)(x^2+b^2+h^2)^{1/2}}$$

By changing variables, we finally obtain

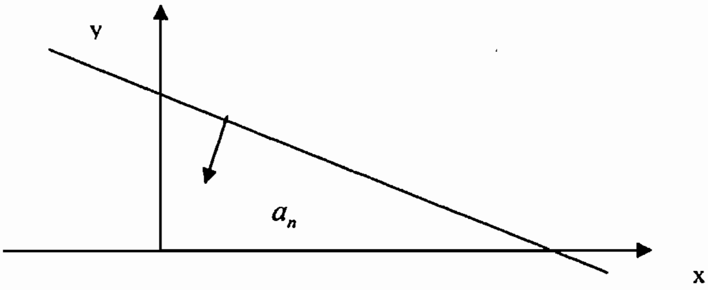
$$E_z = \frac{\rho_s}{\pi\ \varepsilon_0}\tan^{-1}\left\{\frac{ab}{h(a^2+b^2+h^2)^{1/2}} -\right\}\bar{a}_z$$

(b) $Q = \int \rho_s dS = \rho_s(2a)(2b) = 10^{-5}(4)(10) = \underline{\underline{0.4\text{ mC}}}$

$$\boldsymbol{E} = \frac{10^{-5}}{\pi\times\dfrac{10^{-9}}{36\pi}}\tan^{-1}\left[\frac{10}{10(4+25+100)^{1/2}}\right]\boldsymbol{a}_z = 36\times10^4(0.0878\text{ radians})\boldsymbol{a}_z$$

$$= \underline{\underline{31.61\boldsymbol{a}_z\text{ kV/m}}}$$

**Prob.**



Let $f(x,y) = x + 2y - 5$; $\nabla f = \bar{a}_x + 2\bar{a}_y$

$$\bar{a}_n = \pm \frac{\nabla f}{|\nabla f|} = \pm \frac{(\bar{a}_x + 2\bar{a}_y)}{\sqrt{5}}$$

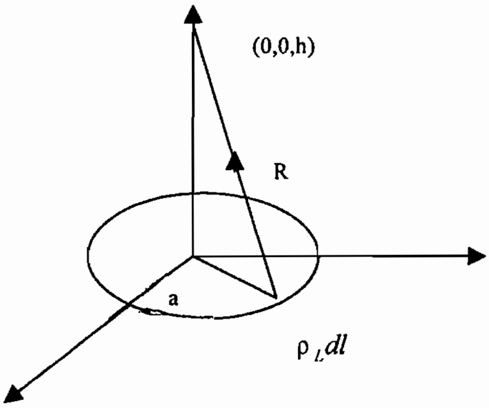
Since point $(-1,0,1)$ is below the plane,

$$\bar{a}_n = -\frac{(\bar{a}_x + 2\bar{a}_y)}{\sqrt{5}} \ .$$

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\bar{a}_n = \frac{6(10^{-9})}{2(10^{-9}/36\pi)}\left(-\frac{(\bar{a}_x + 2\bar{a}_y)}{\sqrt{5}}\right)$$

$$= \underline{\underline{-151.7\,\bar{a}_x - 303.5\,\bar{a}_y \ \ \text{V/m}}}$$

**Prob.**
**(a)**



$$\bar{D} = \int \frac{\rho_L dl\,\bar{R}}{4\pi\,R^3}, \qquad \bar{R} = -a\bar{a}_\rho + h\bar{a}_z$$

$$\bar{D} = \frac{\rho_L}{4\pi}\int_{\phi=0}^{\phi=2\pi} \frac{ad\phi(-a\bar{a}_\rho + h\bar{a}_z}{(a^2+h^2)^{3/2}}$$

Due to symmetry, the ρ component varies.

$$\bar{D} = \frac{\rho_L a(2\pi\,h)\bar{a}_z}{4\pi\,(a^2+h^2)^{3/2}} = \frac{\rho_L\,ah\bar{a}_z}{2(a^2+h^2)^{3/2}}$$

$a = 2,\ h = 3,\ \rho_L = 5\mu C/m$

Since the ring is placed in $x = 0$, $\bar{a}_z$ becomes $\bar{a}_x$.

$$\bar{D} = \frac{2(6)(5)\bar{a}_x}{2(4+9)^{3/2}} = \underline{\underline{0.64\,\bar{a}_x\ \mu C/m^2}}$$

**(b)**

$$\bar{D}_Q = \frac{Q}{4\pi}\frac{[(3,0,0)-(0,-3,0)]}{|(3,0,0)-(0,-3,0)|^3} + \frac{Q}{4\pi}\frac{[(3,0,0)-(0,3,0)]}{|(3,0,0)-(0,3,0)|^3}$$

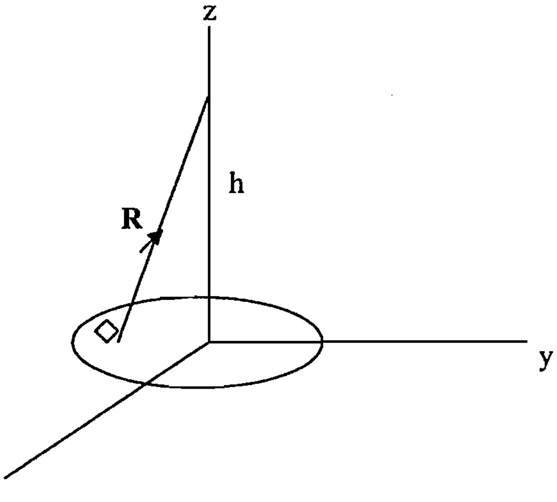
$$= \frac{Q(3,3,0)}{4\pi(18)^{3/2}} + \frac{Q(3,-3,0)}{4\pi(18)^{3/2}} = \frac{6Q}{4\pi(18)^{3/2}}$$

$$\bar{D} = \bar{D}_R + \bar{D}_Q = 0$$

$$0.64(10^{-6}) + \frac{6Q}{4\pi(18)^{3/2}} = 0$$

$$\therefore\ Q = -0.64(4\pi)(18^{3/2})10^{-6}\frac{1}{6} = \underline{\underline{-102.4\mu C}}$$

**Prob.**



$$E = \int_S \frac{\rho_s dS\boldsymbol{R}}{4\pi\varepsilon_o R^3}$$

$$\boldsymbol{R} = -\rho \boldsymbol{a}_\rho + h\boldsymbol{a}_z, \qquad dS = \rho d\phi d\rho$$

$$\boldsymbol{E} = \int\int \frac{\rho_s(-\rho\boldsymbol{a}_\rho + h\boldsymbol{a}_z)\rho d\phi d\rho}{4\pi\varepsilon_o(\rho^2+h^2)^{3/2}}$$

The ρ-component vanishes due to symmetry.

$$E_z = \frac{\rho_s h}{4\pi\varepsilon_o}\int_0^{2\pi} d\phi \int_0^a \frac{\rho d\rho}{(\rho^2+h^2)^{3/2}}$$

Let $u = (\rho^2 + h^2), du = 2\rho d\rho$

$$E_z = \frac{\rho_s h}{4\pi\varepsilon_o}(2\pi)\frac{1}{2}\int u^{-3/2}du = \frac{\rho_s h}{4\varepsilon_o}(2u^{-1/2}) = \frac{\rho_s}{2\varepsilon_0}(1 - \frac{h}{\sqrt{a^2+h^2}})$$

At the origin, h=1 so that

$$E = \frac{\rho_s}{2\varepsilon_0}(1 - \frac{1}{\sqrt{2}})\boldsymbol{a}_z$$

$$F = qE = \frac{q\rho_s}{2\varepsilon_0}(1 - \frac{1}{\sqrt{2}})\boldsymbol{a}_z = \frac{30\times10^{-6}\times200x10^{-12}}{2\times\frac{10^{-9}}{36\pi}}(1 - \frac{1}{\sqrt{2}})\boldsymbol{a}_z = \underline{\underline{99.24\boldsymbol{a}_z\ \mu\text{N}}}$$

**Prob.**

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r + \frac{\rho_L}{2\pi\varepsilon_0\rho}\bar{a}_\rho + \frac{\rho_S}{2\varepsilon_0}\bar{a}_n$$

$$= \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})}\left\{\frac{(1,1,1)-(4,1,-3)}{|(1,1,1)-(4,1,-3)|^3}\right\} + \frac{2(10^{-9})}{2\pi(\frac{10^{-9}}{36\pi})}\left\{\frac{(1,1,1)-(1,0,0)}{|(1,1,1)-(1,0,0)|^2}\right\} + \frac{5(10^{-9})}{2\pi(\frac{10^{-9}}{36\pi})}\bar{a}_z$$

$$= (-0.0216,0,0.0288) + (0,18,18) - 90\pi(0,0,1)$$

$$= \underline{\underline{-0.0216\bar{a}_x + 18\bar{a}_y - 264.7\bar{a}_z \text{ V/m}}}$$

**Prob.**

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_L}{2\pi\epsilon_0\rho}\vec{a}_\rho + \frac{\rho_S}{2\epsilon_0}\vec{a}_n$$

$$= \frac{30\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}}\frac{[(0,0,0)-(0,1,-3)]}{|(0,0,0)-(0,1,-3)|^2} + \frac{20\times10^{-9}(-\vec{a}_x)}{2\times10^{-9}/36\pi}$$

$$= \underline{\underline{-1131\vec{a}_x - 54\vec{a}_y + 162\vec{a}_z \text{ V|m.}}}$$

**Prob.**

(a) At P(5,-1,4),

$$\boldsymbol{E} = \sum_{k=1}^{3}\frac{\rho_{sk}}{2\varepsilon_o}\boldsymbol{a}_{nk} = \frac{10x10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(\boldsymbol{a}_x) + \frac{-20x10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(\boldsymbol{a}_y) + \frac{30x10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(-\boldsymbol{a}_z)$$

$$= 36\pi(5,-10,-15)\times10^3 = \underline{\underline{565.5\boldsymbol{a}_x - 1131\boldsymbol{a}_y - 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

(b) At R(0,-2,1)

$$\boldsymbol{E} = 36\pi\left[5(-\boldsymbol{a}_x) - 10(\boldsymbol{a}_y) + 15(-\boldsymbol{a}_z)\right]\times10^3 = \underline{\underline{-565.5\boldsymbol{a}_x - 1131\boldsymbol{a}_y - 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

(c) At Q(3,-4,10),

$$\boldsymbol{E} = 36\pi\left[5\boldsymbol{a}_x - 10(-\boldsymbol{a}_y) + 15\boldsymbol{a}_z\right]\times10^3 = \underline{\underline{565.5\boldsymbol{a}_x + 1131\boldsymbol{a}_y + 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

**Prob.**

from Gauss's law, $\oint \vec{D}\cdot d\vec{S} = Q = \Psi$. Since the faces are equidistant from the center, they must have the flux $\Psi_1$ thru them. Hence

$$6\Psi_1 = Q \longrightarrow \Psi_1 = \frac{Q}{6} = \underline{\underline{10\ \text{mC}}}.$$

**Prob.**

$$\boldsymbol{E} = \sum_{k=1}^{2} \frac{\rho_{Lk}}{2\pi\varepsilon_o \rho_k}\boldsymbol{a}_{\rho k} = \frac{10x10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}(2)}(\boldsymbol{a}_y) - \frac{10x10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}(2)}(-\boldsymbol{a}_y)$$

$$= 36(5/2)(1+1)\boldsymbol{a}_y = \underline{\underline{180\boldsymbol{a}_y\ \text{V/m}}}$$

**Prob.**

$$\vec{D} = \sum_{k=1}^{3} \frac{Q_k(\vec{r}-\vec{r}_k)}{4\pi|\vec{r}-\vec{r}_k|^3}$$

$$= \frac{1}{4\pi}\left[\frac{Q_1(0,0,-1)}{|(0,0,-1)|^3} + \frac{Q_2(6,-8,0)}{|(6,-8,0)|^3} + \frac{Q_3(0,-4,3)}{|(0,-4,3)|^3}\right]$$

$$\vec{D} = \underline{\underline{-2.39\vec{a}_x - 8.276\vec{a}_y - 314.5\,\vec{a}_z\ \text{nC}/\text{m}^2}}.$$

**Prob.**

**(a)**

$$\rho_V = \nabla \bullet D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 8y + 0 = \underline{\underline{8y \ \ \mathrm{C/m^3}}}$$

**(b)**

$$\rho_V = \nabla \bullet \bar{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(4\rho^2 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(2\rho\cos\phi) + \frac{\partial}{\partial z}(2z^2)$$

$$= 8\sin\phi - 2\sin\phi + 4z = \underline{\underline{6\sin\phi + 4z \ \ \mathrm{C/m^3}}}$$

**(c)**

$$\rho_V = \nabla \bullet \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{2}{r}\cos\theta\right) + \frac{1}{r^4 \sin\theta}\frac{\partial}{\partial \theta}(\sin^2\theta)$$

$$= \frac{-2}{r^4}\cos\theta + \frac{1}{r^4 \sin\theta}(2\sin\theta\cos\theta) = \underline{\underline{0 \ \mathrm{C/m^3}}}$$

**Prob.**

Let $Q_1$ be located at the origin. At the spherical surface of radius $r$,

$$Q_1 = \oint \vec{D}\cdot d\vec{S} = \epsilon E_r \cdot 4\pi r^2$$

or $$\vec{E} = \frac{Q_1}{4\pi\epsilon r^2}\, \vec{a}_r$$

by Gauss's law. If a second charge $Q_2$ is placed on the spherical surface, $Q_2$ experiences a force

$$\vec{F} = Q_2 \vec{E} = \frac{Q_1 Q_2}{4\pi\epsilon r^2}\, \vec{a}_r$$

which is Coulomb's law.

**Prob.**

for $r > a = 10\,\text{cm}$,

$$Q_T = \oint \vec{D} \cdot d\vec{S} = Q + \int \rho_v \, dv$$

If $\vec{D} = 0$ for $r > a$, then

$$Q = -\int \rho_v \, dv = -\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{0.1} 10^{-2} r^3 r^2 \sin\theta \, d\theta \, d\phi$$

$$= -4\pi \, 10^{-2} \frac{r^6}{6} \Big|_0^{0.1} = \underline{\underline{-21\,nC}}.$$

**Prob.**

$$\rho_v = \nabla \cdot D = \nabla \cdot \varepsilon_o E = \frac{\varepsilon_o}{r^2} \frac{\partial}{\partial r}(1 - \cos 3r) = \underline{\underline{\frac{3\varepsilon_o \sin 3r}{r^2} \text{ C/m}^3}}$$

**Prob.**

(a) $\rho_v = \nabla \bullet \boldsymbol{D} = \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z} = \underline{\underline{2y \text{ C/m}^3}}$

(b) $\Psi = \int \boldsymbol{D} \cdot d\boldsymbol{S} = \iint x^2 dxdz \Big|_{y=1} = \int_0^1 x^2 \, dx \int_0^1 dz = \underline{\underline{\frac{1}{3}}} \text{ C}$

(c ) $Q = \int_v \rho_v dv = \iiint 2y dxdydz = 2\int_0^1 dx \int_0^1 ydy \int_0^1 dz = \underline{\underline{1 \text{ C}}}$

**Prob.**

Let $Q_1 = 5\mu C$, $Q_2 = -3\mu C$, $Q_3 = 2\mu C$, and $Q_4 = 10\mu C$.
The corresponding distances of the charge from the origin are:

$$r_1 = \sqrt{12^2 + 5^2} = 13m,$$

$$r_2 = \sqrt{3^2 + 4^2} = 5m,$$

$$r_3 = \sqrt{2^2 + 6^2 + 3^2} = 7\,m,$$

$$r_4 = \sqrt{3^2} = 3\,m.$$

According to Gauss's law, $\Psi = Q_{enc}$.

(a) For $r = 1 < r_1, r_2, r_3, r_4$, $Q_{enc} = 0$

$$\underline{\underline{\Psi = 0}}.$$

(b) for $r = 10 > r_2, r_3, r_4$,

$$Q_{enc} = Q_2 + Q_3 + Q_4 = -3 + 2 + 10 = 9\mu C$$

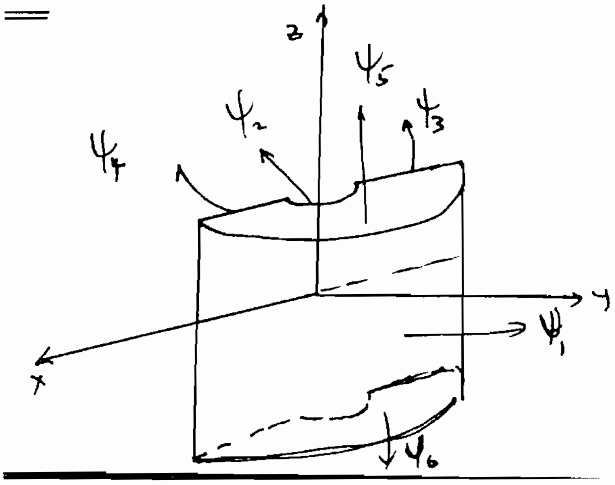
$$\Psi = \underline{\underline{9\mu C}}.$$

(c) For $r = 15 > r_1, r_2, r_3, r_4$

$$Q_{enc} = Q_1 + Q_2 + Q_3 + Q_4 = 14\mu C$$

$$\Psi = \underline{\underline{14\mu C}}.$$

**Prob.**



Method 1

Let $\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6$

where $\Psi_n$'s are shown on the previous page.

$$\Psi = \iint 2z^2 \sin\frac{\phi}{2}\, \rho\, d\phi\, dz \Big|_{\rho=4} - \iint 2z^2 \sin\frac{\phi}{2}\, \rho\, d\phi\, dz \Big|_{\rho=1}$$

$$+ \iint z^2 \cos\frac{\phi}{2}\, d\rho\, dz \Big|_{\phi=\pi} - \iint z^2 \cos\frac{\phi}{2}\, d\rho\, dz \Big|_{\phi=0}$$

$$+ \iint 4z\rho \sin\frac{\phi}{2}\, \rho\, d\phi\, d\rho \Big|_{z=1} - \iint 4z\rho \sin\frac{\phi}{2}\, \rho\, d\phi\, d\rho \Big|_{z=-2}$$

$$= 6\int_0^\pi \sin\frac{\phi}{2} \int_{-2}^1 z^2\, dz - \int_1^4 d\rho \int_{-2}^1 z^2\, dz + 12\int_1^4 \rho^2\, d\rho \int_0^\pi \sin\frac{\phi}{2}\, d\phi$$

$$= 4(9) - 9 + 8(63) = 531 \text{ mC}$$

$$\Psi = \underline{\underline{0.531 \text{ C}}}.$$

Method 2

$$\nabla \cdot \vec{D} = \frac{2z^2}{\rho} \sin\phi/2 - \frac{z^2}{2\rho} \sin\frac{\phi}{2} + 4\rho \sin\phi/2$$

$$= \sin\frac{\phi}{2}\left(4\rho + \frac{3}{2}z^2\right)$$

$$\Psi = \int \nabla \cdot \vec{D}\, dv = \int_0^\pi \sin\frac{\phi}{2}\, d\phi \iint \left(\frac{3z^2}{2\rho} + 4\rho\right) \rho\, d\rho\, dz$$

$$= 2\left[4\int_{-2}^1 dz \int_1^4 \rho^2\, d\rho + \frac{3}{2}\int_1^4 d\rho \int_{-2}^1 z^2\, dz\right]$$

$$\Psi = 2\left[4(63) + \frac{3}{2}(9)\right] = 531 \text{ mC}$$

$$= \underline{\underline{0.531 \text{ C}}}.$$

**Prob.**

(a)

$$\psi = Q_{enc}$$

For $r = 1.5$m,

$$Q_{enc} = \int \rho_{s1} ds = \rho_{s1} \int ds = \rho_{s1}(4\pi R^2\ )$$

$$= 2(10^{-6})4\pi(1^2) = 8\pi(10^{-6})$$

$$\psi = Q_{enc} = \underline{\underline{25.13\quad \mu\text{C}}}$$

For $r = 2.5$m,

$$Q_{enc} = \rho_{s1}(4\pi R_1^{\ 2}) + \rho_{s2}(\ 4\pi R_2^{\ 2})$$

$$= \ 8\pi(10^{-6}) + (-4)10^{-6}(4\pi 2^2)$$

$$= \ (8\pi - 64\pi)10^{-6}$$

$$\psi = Q_{enc} \ = \underline{\underline{-175.93\quad \mu\text{C}}}$$

(b)

$$\psi = Q_{enc}, \qquad \int \bar{D} \bullet d\bar{S} = Q_{enc}$$

$$D_r(\ 4\pi r^2) = Q_{enc}$$

$$D_r = \frac{Q_{enc}}{4\pi r^2}$$

For $r = 0.5$, $\quad Q_{enc} = 0 \quad \longrightarrow \quad \underline{\underline{\bar{D} = 0}}$

For r = 2.5, $\quad Q_{enc} = -175.93\mu C = -56\pi(10^{-6})$

$$D_r = -\frac{56\pi(10^{-6})}{4\pi(25)} = \underline{\underline{-2.24\bar{a}_r\,\mu\text{C}/\text{m}^2}}$$

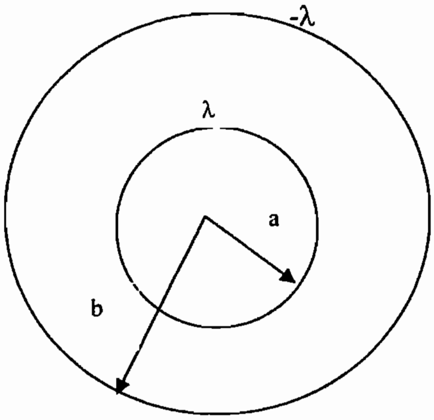
For $r = 3.5$, $\quad Q_{enc} = \rho_{si}4\pi R_1^{\ 2} + \rho_{s2}\ 4\pi R_2^{\ 2} + \rho_{s3}4\pi R_3^{\ 2}$

$$= -56\pi + 5(4\pi(3^3)\ )\quad \mu\text{C}$$

$$= 124\pi \quad \mu\text{C}$$

$$D_r \ = \frac{124\pi}{4\pi(3-5)^2}\mu\text{C}/\text{m}^2 = \underline{\underline{2.531\bar{a}_r \quad \mu\text{C}/\text{m}^2}}$$

**Prob.**



For $\rho < a.$ $E = 0$

For a <ρ <b, consider a Gaussian surface in the form of a cylinder or radius ρ and length L. Since λ is the charge per unit length,

$$\int_S D \cdot dS = \lambda L$$

$$\varepsilon_o E_\rho \iint \rho d\phi dz = \varepsilon_o E_\rho 2\pi\rho L = \lambda L$$

$$E_\rho = \frac{\lambda}{2\pi\rho\varepsilon_o}$$

For ρ>b,

$$\varepsilon_o E_\rho 2\pi\rho L = \lambda L - \lambda L = 0$$

Hence,

$$\underline{\underline{\boldsymbol{E} = \begin{cases} \dfrac{\lambda}{2\pi\rho\varepsilon_o}\boldsymbol{a}_\rho, & a < \rho < b \\ 0, & \text{otherwise} \end{cases}}}$$

**Prob.**

For $\rho < 1$, $Q_{enc.} = 0 \quad \longrightarrow \qquad \bar{D} = 0$

For $1 < \rho < 2$,

$$Q_{enc.} = \int_{\phi=0}^{2\pi} \int_{\rho=1} \int_{z=0}^{L} 12\rho \, d\phi \, d\rho \, dz$$

$$= 12(2\pi) \, L \frac{\rho^3}{3} \Big|_1^\rho = 8\pi L(\rho^3 - 1)$$

$$\psi = \int \bar{D} \bullet d\bar{S} = D_\rho \int_{\rho=0}^{2} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_\rho (2 \, \pi \rho L)$$

Hence,

$$8\pi \, L \, (\rho^3 - 1) = D_\rho (2\pi \rho \, L)$$

$$D_\rho = \frac{8 \, (\rho^3 - 1)}{2\rho}$$

For $\rho > 2$, $\qquad \psi = D_\rho (2\pi \rho \, L)$

$$Q_{enc} = 8\pi \, L\rho^3 \Big|_1^2 = 56 \, \pi \, L$$

$$56 \, \pi \, L = D_\rho (2\pi \rho \, L)$$

$$D_\rho = \frac{28}{\rho}$$

$$\text{Thus,} \quad D_\rho = \begin{cases} 0, \ \rho < 1, & 1 < \rho < 2 \\ \dfrac{8(\rho^3 - 1)}{2\rho}, & \rho > 2 \\ \dfrac{28}{\rho} \end{cases}$$

**Prob.**

**(a)**

$\psi = Q_{enc}$ at $r = 2$

$$Q_{enc} = \int \rho_V \, dV = \iiint \frac{10}{r^2} r^2 \sin\theta \, d\theta \, dr \, d\phi$$

$$= 10 \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$= 10(2)(2\pi)(2) = (80\pi) \text{ mC}$$

Thus, $\psi = \underline{\underline{251.3 \text{ mC}}}$

At $r = 6$;

$$Q_{enc.} = 10 \int_{r=0}^{4} dr \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$= 10(4)(2\pi)(2) = 160\pi$$

$\psi = \underline{\underline{502.6 \text{ mC}}}$

**(b)**

$$\psi = Q_{enc}$$

But $\psi = \oint \bar{D} \bullet d\bar{S} = D_r \oint dS = D_r (4\pi r^2)$

At $r = 1$,

$$Q_{enc} = 10 \int_{r=0}^{1} dr \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$Q_{enc} = 10(1)(2\pi)(2) = 40\pi$$

Thus,

$$D_r = \frac{Q_{enc}}{4\pi r^2} = \frac{40\pi}{4\pi(1)} = 10$$

$\bar{D} = \underline{\underline{10\,\bar{a}_r \ \ nC/m^2}}$

At $r = 5$, $Q_{enc} = 160\pi$

$$D_r = \frac{Q_{enc}}{4\pi r^2} = \frac{160\pi}{4\pi(5)^2} = 1.6$$

$= \underline{\underline{1.6\,\bar{a}_r \text{ nC/m}^3}}$

**Prob.**

(a) For r<a,

$$\oint_S \boldsymbol{D}\cdot dS = \int_v \rho_v dv$$

$$\varepsilon_o E_r 4\pi r^2 = \rho_o \iiint \frac{1}{r} r^2 \sin\theta d\theta d\phi dr = \rho_o \int_0^r r dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \rho_o 4\pi \frac{r^2}{2}$$

$$E_r = \frac{\rho_o}{2\varepsilon_o}$$

For r > a,

$$\varepsilon_o E_r 4\pi r^2 = \rho_o \int_0^a r dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \rho_o 4\pi \frac{a^2}{2}$$

$$E_r = \frac{\rho_o a^2}{2\varepsilon_o r^2}$$

Hence,

$$\boldsymbol{E} = \begin{cases} \dfrac{\rho_o}{2\varepsilon_o}\boldsymbol{a}_r, & r < a \\ \dfrac{\rho_o a^2}{2\varepsilon_o r^2}\boldsymbol{a}_r, & r > a \end{cases}$$

(b)

$$Q = \int_v \rho_v dv = \rho_o (4\pi) \int_0^a r dr = \underline{\underline{2\pi a^2 \rho_o}}$$

**Prob.**

$$\rho_v = \nabla\cdot\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(10r\right) = \frac{10}{r^2}\ nC/m^3$$

$$Q = \int \rho_v \, dv = \iiint \frac{10}{r^2} \cdot r^2 \sin\theta \, d\theta \, dr \, d\phi$$

$$= 10 \int_0^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta = 10(2)(2\pi)(2)$$

$$= 80\pi = \underline{\underline{251.3 \text{ nC}}}$$

or $$Q = \oint \bar{D} \cdot d\bar{S} = \iint \frac{10}{r} \cdot r^2 \sin\theta \, d\theta \, d\phi \Big|_{r=2}$$

$$= 10(2) \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 20(2)(2\pi) = 80\pi$$

$$= \underline{\underline{251.3 \text{ nC}}}$$

**Prob.**

**(a)**

From $A$ to $B$, $d\bar{l} = r d\theta \, \bar{a}_\theta$,

$$W_{AB} = -Q \int_{\theta=30°}^{90°} 10 r \cos\theta \, r d\theta \Big|_{r=5} = \underline{\underline{-1250 \text{ nJ}}}$$

**(b)**

From $A$ to $C$, $d\bar{l} = dr \, \bar{a}_r$,

$$W_{AC} = -Q \int_{r=5}^{10} 20 r \sin\theta \, dr \Big|_{\theta=30°} = \underline{\underline{-3750 \text{ nJ}}}$$

**(c)**

From $A$ to $D$, $d\bar{l} = r \sin\theta \, d\phi \, \bar{a}_\phi$,

$$W_{AD} = -Q \int 0(r \sin\theta) d\phi = \underline{\underline{0 \text{ J}}}$$

**(d)**

$$W_{AE} = W_{AD} + W_{DF} + W_{FE}$$

where $F$ is $(10,30,60)$. Hence,

$$W_{AE} = -Q \left\{ \int_{r=5}^{10} 20 r \sin\theta \, dr \Big|_{\theta=30°} + 10 \int_{\theta=30}^{90} 10 r \cos\theta \, r d\theta \Big|_{r=10} \right\}$$

$$= -100[\frac{75}{2} + \frac{100}{2}] \text{ nJ} = \underline{\underline{-8750 \text{ nJ}}}$$

**Prob.**

$$V = \int_S \frac{\rho_S dS}{4\pi\varepsilon_0 r}; \qquad \rho_S = \frac{1}{\rho}; \quad dS = \rho\, d\phi\, d\rho; \quad r = \sqrt{\rho^2 + h^2}$$

$$V = \frac{1}{4\pi\varepsilon_0} \iint \frac{\frac{1}{\rho}(\rho d\phi d\rho)}{(\rho^2 + h^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_{\rho=0}^{a} \frac{d\rho}{(\rho^2 + h^2)}$$

$$= \frac{2\pi}{4\pi\varepsilon_0} \ln(\rho + \sqrt{\rho^2 + h^2})\Big|_{\rho=0}^{a} = \frac{1}{2\varepsilon_0}[\ln(a + \sqrt{a^2 + h^2}) - \ln h]$$

$$= \underline{\underline{\frac{1}{2\varepsilon_0} \ln \frac{a + \sqrt{a^2 + h^2}}{h}}}$$

**Prob.**

$$V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0 |\vec{r} - \vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\vec{r} - \vec{r}_2|}$$

(a) $|\vec{r} - \vec{r}_1| = |(0,1,0) - (0,0,0)| = 1,$

$|\vec{r} - \vec{r}_2| = |(0,1,0) - (0,0,1)| = \sqrt{2}$

$$V = \frac{10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{3}{1} - \frac{2}{\sqrt{2}}\right] = \underline{\underline{14.27\,V.}}$$

(b) $|\vec{r} - \vec{r}_1| = |(1,1,1) - (0,0,0)| = \sqrt{3}$

$|\vec{r} - \vec{r}_2| = |(1,1,1) - (0,0,1)| = \sqrt{2}$

$$V = \frac{10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{3}{\sqrt{3}} - \frac{2}{\sqrt{2}}\right] = \underline{\underline{2.86\,V}}$$

**Prob.**

$$V = -\int \vec{E}\cdot d\vec{l} = -\int \frac{\rho_L}{2\pi\epsilon_0 \rho}\, d\rho = -\frac{\rho_L}{2\pi\epsilon_0} \ln \rho + C$$

$$V_A - V_P = -\frac{\rho_L}{2\pi\epsilon_0} \ln \frac{\rho_A}{\rho_P}$$

where $\rho_A = \sqrt{x_A^2 + z_A^2} = 5,$

$\rho_P = \sqrt{x_P^2 + z_P^2} = 6.$

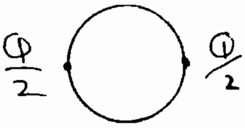
(a) $V_A - 5 = -\dfrac{10\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}} \ln \dfrac{5}{6}$

$V_A = 32.82 + 5 = \underline{\underline{37.82\text{ V}}}$

(b) $5 - V_P = -180 \ln 5/6$

$V_P = 5 - 32.82 = \underline{\underline{-27.82\text{ V}}}$

**Prob.**

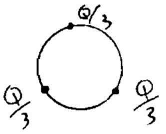


(a) $V = \dfrac{2\left(\frac{Q}{2}\right)}{4\pi\epsilon_0 r} = \dfrac{Q}{4\pi\epsilon_0 r}$

$= \dfrac{60\times10^{-6}}{4\pi\times\frac{10^{-9}}{36\pi}\cdot 4} = 1\times \underline{\underline{15\text{ kV}}}$

(b)

$V = \dfrac{3\left(\frac{Q}{3}\right)}{4\pi\epsilon_0 r} = \underline{\underline{15\text{kV}}}.$



(c) $V = \displaystyle\int \frac{\rho_L\, dl}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r} = \underline{\underline{15\text{ kV}}}.$

**Prob.**

(a)

$$\bar{E} = -(\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z)$$

$$= -2xy(z+3)\bar{a}_x - x^2(z+3)\bar{a}_y - x^2 y\bar{a}_z$$

At $(3,4,-6)$, $x=3,\ y=4,\ z=-6$,

$$\bar{E} = -2(3)(4)(-3)\bar{a}_x - 9(-3)\bar{a}_y - 9(4)\bar{a}_z$$

$$= \underline{\underline{72\bar{a}_x + 27\bar{a}_y - 36\bar{a}_z \text{ V/m}}}$$

(b)

$$\rho_V = \nabla \bullet \bar{D} = \varepsilon_0 \nabla \bullet \bar{E} = -\varepsilon_0 (2y)(z+3)$$

$$\psi = Q_{enc} = \int \rho_V dV = -2\varepsilon_0 \iiint y(z+3)dx\,dy\,dz$$

$$= -2\varepsilon_0 \int_0^1 dx \int_0^1 y\,dy \int_0^1 (z+3)dz = -2\varepsilon_0 (1)(1/2)(\frac{z^2}{2} + 3z)\Big|_0^1$$

$$= -\varepsilon_0(\frac{1}{2} + 3) = \frac{-7}{2}(\frac{10^{-9}}{36\pi})$$

$$Q_{enc} = \underline{\underline{-30.95\ pC}}$$

**Prob.**

$$V = \int \frac{\rho_s dS}{4\pi\epsilon_0 R} = \frac{\rho_s}{4\pi\epsilon_0}\int_{d=0}^{2\pi}\int_{\rho=0}^{a} \frac{\rho\, d\phi\, d\rho}{(h^2+\rho^2)^{\frac{1}{2}}}$$

$$= \frac{\rho_s}{4\pi\epsilon_0}(2\pi)\int_0^a (h^2+\rho^2)^{-\frac{1}{2}}\frac{1}{2}d(\rho^2)$$

$$= \frac{\rho_s}{4\epsilon_0}\, 2\,(h^2+\rho^2)^{\frac{1}{2}}\Big|_0^a$$

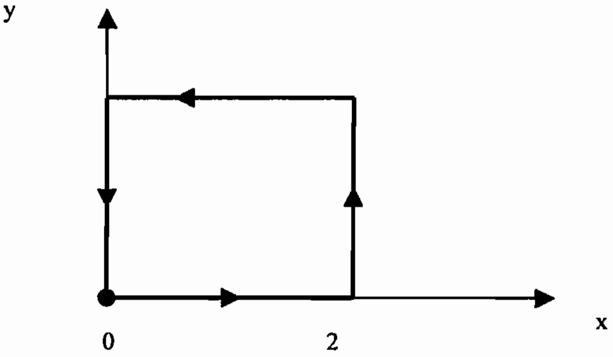
$$V = \frac{\rho_s}{2\epsilon_0}\left[\sqrt{h^2+a^2} - h\right].$$

**Prob.**

**(a)**

$$\nabla \times \bar{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x-x)\bar{a}_x + (y-y)\bar{a}_y + (z-z)\bar{a}_z \underline{\underline{= \ 0}}$$



**(b)**

$$\oint \bar{E} \bullet d\bar{l} = \int_{x=0}^{2} yz\,dx \Big|_{\substack{y=0\\z=1}} + \int_{y=0}^{2} xz\,dy \Big|_{\substack{z=1\\x=2}} + \int_{x=2}^{0} yz\,dx \Big|_{\substack{y=2\\z=1}} + \int_{y=2}^{0} xz\,dy \Big|_{\substack{x=0\\z=1}}$$

$$= 2y\Big|_0^2 + 2x\Big|_0^2 = 4 - 4 \underline{\underline{= \ 0}}$$

**Prob.**

$$\bar{E} = \begin{cases} \dfrac{\rho_0 a^3}{4\varepsilon_0 r^2}\bar{a}_r , & r > a \\ \dfrac{\rho_0 r^2}{4\varepsilon_0 a}\ \bar{a}_r , & r < a \end{cases}$$

Since $V = -\int \bar{E} \bullet d\bar{l} = -\int E\,dr$ ,

$$V = \begin{cases} \dfrac{-\rho_o r^3}{12\varepsilon_0 a} + C_1, & r < a \\ \dfrac{\rho_o a^3}{4\varepsilon_0 r} + C_2 , & r > a \end{cases}$$

But $V(\infty)=0 \quad \longrightarrow \quad C_2=0;$

$$V(r=a)=\frac{\rho_0 a^2}{4\varepsilon_0}=\frac{-\rho_0 a^2}{12\varepsilon_0}+C_1 \quad \longrightarrow \quad C_1=\frac{\rho_0 a^2}{3C_0}$$

Thus,
$$V=\begin{cases}\dfrac{-\rho_0 r^3}{12\varepsilon_0 a}+\dfrac{\rho_0 a^2}{3\varepsilon_0}, & r<a\\[2ex] \dfrac{\rho_0 a^2}{4\varepsilon_0 r}, & r>a\end{cases}$$

**Prob.**

(a) For $r \geq a,$,

$$\oint_S \mathbf{D}.d\mathbf{S}=\int_v \rho_v dv$$

$$\varepsilon_o E_r 4\pi r^2=\rho_o \iiint(1-\frac{r}{a})^2 r^2 \sin\theta d\theta d\phi dr$$

$$=4\pi\rho_o\int_0^a(1-\frac{2r}{a}+\frac{r^2}{a^2})r^2 dr=4\pi\rho_o\left(\frac{a^3}{3}-\frac{a^3}{2}+\frac{a^3}{5}\right)$$

$$E_r=\frac{\rho_o a^3}{30\varepsilon_o r^2}$$

$$V=-\int Edl=-\int E_r dr=\frac{\rho_o a^3}{30\varepsilon_o r}+c_1$$

But $V(\infty)=0 \quad \longrightarrow \quad c_1=0$

$$V=\frac{\rho_o a^3}{30\varepsilon_o r}$$

(b) $\oint_S \mathbf{D}.d\mathbf{S}=\int_v \rho_v dv$

$$\varepsilon_o E_r 4\pi r^2=\rho_o \iiint(1-\frac{r}{a})^2 r^2 \sin\theta d\theta d\phi dr$$

$$=4\pi\rho_o\int_0^r(1-\frac{2r}{a}+\frac{r^2}{a^2})r^2 dr=4\pi\rho_o\left(\frac{r^3}{3}-\frac{r^4}{2a}+\frac{r^5}{5a^2}\right)$$

$$E_r=\frac{\rho_o}{\varepsilon_o}\left[\frac{r}{3}-\frac{r^2}{2a}+\frac{r^3}{5a^2}\right]$$

$$V=-\int Edl=-\int E_r dr=\frac{\rho_o}{\varepsilon_o}\left[\frac{r^2}{6}-\frac{r^3}{6a}+\frac{r^4}{20a^2}\right]+c_2$$

$$V(r=a)=\frac{\rho_o a^2}{30\varepsilon_o}=\frac{\rho_o}{\varepsilon_o}\left(\frac{a^2}{6}-\frac{a^2}{6}+\frac{a^2}{20}\right)+c_2=\frac{\rho_o a^2}{20\varepsilon_o}+c_2$$

$$c_2=-\frac{\rho_o a^2}{60\varepsilon_o}$$

$$\underline{\underline{V=\frac{\rho_o}{\varepsilon_o}\left(\frac{r^2}{6}-\frac{r^3}{6a}+\frac{r^4}{20a^2}\right)-\frac{\rho_o a^2}{60\varepsilon_o}}}$$

(c) $Q=\int_v \rho_v dv=4\pi\rho_o\frac{a^3}{30}=\underline{\underline{\frac{2\pi\rho_o a^3}{15}}}$

**Prob.**

(a)

$$W_{AB}=q\int \bar{E}\bullet d\bar{l},\qquad d\bar{l}=d\rho\,\bar{a}_\rho$$

$$\frac{-W_{AB}}{q}=\int(z+1)\sin\phi\,d\rho\,\Big|_{\phi=0,z=0}=0$$

$$W_{AB}=0$$

(b)

$$\frac{-W_{BC}}{q}=\int_{\phi=0}^{30}(z+1)\cos\phi\,\rho\,d\phi\Big|_{\rho=4,z=0}=4\sin\phi\,\Big|_{0}^{30^\circ}=2$$

$$W_{BC}=-2q=\underline{\underline{-8\ \text{nJ}}}$$

(c)

$$\frac{-W_{CD}}{q}=\int_{z=0}^{-2}\rho\sin\phi\,dz\,\Big|_{\substack{\phi=30^\circ\\ \rho=4}}=4\sin 30^\circ\,(z\,\Big|_{0}^{-2})=-4$$

$$W_{CD}=4q=\underline{\underline{16\ \text{nJ}}}$$

(d)

$$W_{AD}=W_{AB}+W_{BC}+W_{CD}=0-8+16=\underline{\underline{8\ \text{nJ}}}$$

**Prob.**

**(a)**

$$Q_{enc} = 4\pi\rho_0\left(a^2\frac{r^3}{3} - \frac{r^5}{5}\right)\Big|_0^2$$

$$Q_{enc} = \frac{8\pi}{15}\rho_0$$

$$\psi = \int \bar{D}\bullet d\bar{S} = \varepsilon_0 E_r(4\pi r^2)$$

$$\psi = Q_{enc}\ :$$

$$\varepsilon_0 E_r(4\pi r^2) = \frac{8\pi}{15}\rho_0$$

$$E_r = \frac{2\rho_0}{15\varepsilon_0 r^2} \quad \text{or}$$

$$\bar{E} = \underline{\underline{\frac{2\rho_0}{15\varepsilon_0 r^2}\bar{a}_r}}$$

$$V = \int \bar{E}\bullet d\bar{l} = -\frac{2\rho_0}{15\varepsilon_0}\int r^{-2}dr = \frac{2\rho_0}{15\varepsilon_0 r} + C_1$$

Since $V(r->0) = 0$, $C_1 = 0$;

$$V = \underline{\underline{\frac{2\rho_0}{15\varepsilon_0 r}}}$$

**(b)**

For $r \leq a$,

$$Q_{enc} = \rho_0(4\pi)\left(\frac{a^2r^3}{3} - \frac{r^5}{5}\right)\Big|_0^r = 4\pi\rho_0\left(\frac{a^2r^3}{3} - \frac{r^5}{5}\right)$$

$$E_r = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{\rho_0}{\varepsilon_0}\left(\frac{a^2r}{3} - \frac{r^3}{5}\right)$$

$$\bar{E} = \underline{\underline{\frac{\rho_0}{\varepsilon_0}\left(\frac{a^2r^3}{3} - \frac{r^5}{5}\right)}}$$

$$V = -\int \bar{E}\bullet d\bar{l} = -\frac{\rho_0}{\varepsilon_0}\left(\frac{a^2r^2}{6} - \frac{r^4}{20}\right) + C_2$$

$$= \frac{\rho_0}{\varepsilon_0}\left(\frac{r^4}{20} - \frac{a^2r^2}{6}\right) + C_2$$

Since $V(r = a^+) = V(r = a^-)$,

$$\frac{2\rho_0}{15\varepsilon_0 a} = \frac{\rho_0}{\varepsilon_0}\left(\frac{a^4}{20} - \frac{a^4}{6}\right) + C_2 \quad \longrightarrow \quad C_2 = \frac{2\rho_0}{15\varepsilon_0 a} + \frac{7\rho_0 a^4}{60\varepsilon_0}$$

$$\underline{\underline{V = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^4}{20} - \frac{a^2 r^2}{6}\right) + \frac{2\rho_0}{15\varepsilon_0} + \frac{7\rho_0 a^4}{60\varepsilon_0}}}$$

**(c)**

The total charge is found in part (a) as

$$Q = \frac{8\pi\rho_0}{15}$$

**(d)**

For $r \geq a, \bar{E}$ decays to zero with no maxima.

For $r \leq a$,

$$E_r = \frac{\rho_o}{\varepsilon_0}\left(\frac{a^2 r}{3} - \frac{r^3}{5}\right)$$

$$\frac{\partial E_r}{dr} = \frac{\rho_0}{\varepsilon_0}\left(\frac{a^2}{3} - \frac{3r^2}{5}\right) = 0 \quad \longrightarrow \quad r = \frac{a\sqrt{5}}{3}$$

$$r = \underline{\underline{0.7453\,a}}$$

**Prob.**

(a)

$$\bar{E} = -\nabla V = -(2x\bar{a}_x + 4y\bar{a}_y + 8z\bar{a}_z)$$

$$= \underline{\underline{-2x\bar{a}_x - 4y\bar{a}_y - 8z\bar{a}_z \text{ V/m}}}$$

(b)

$$-\bar{E} = \frac{\partial V}{dx}\bar{a}_x + \frac{\partial V}{dy}\bar{a}_y + \frac{\partial V}{dz}\bar{a}_z$$

$$= \cos(x^2 + y^2 + z^2)^{1/2}[2x\bar{a}_x + 2y\bar{a}_y + 2z\bar{a}_z](\frac{1}{2})(x^2 + y^2 + z^2)^{-1/2}$$

$$\underline{\underline{= -\frac{(x\bar{a}_x + y\bar{a}_y + z\bar{a}_z)}{\sqrt{(x^2 + y^2 + z^2)}}\cos(x^2 + y^2 + z^2)^{1/2} \text{ V/m}}}$$

Alternatively, $V = \sin(x^2 + y^2 + z^2)^{1/2}$ can be converted into spherical coordinates as $V = \sin(r)$

Then $\overline{E} = -\cos r \quad \bar{a}_r$

(c )

$$-\bar{E} = \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z$$

$$= 2\rho(z+1)\sin\phi\,\bar{a}_\rho + \rho(z+1)\cos\phi\,\bar{a}_\rho + \rho^2\sin\phi\,\bar{a}_z$$

$$\underline{\underline{\bar{E} = -2\rho(z+1)\sin\phi\,\bar{a}_\rho - \rho(z+1)\cos\phi\,\bar{a}_\rho - \rho^2\sin\phi\,\bar{a}_z}}$$

(d)

$$-\bar{E} = \frac{\partial V}{\partial}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$$

$$-\bar{E} = -e^{-r}\sin\theta\cos 2\phi\,\bar{a}_r + \frac{1}{r}e^{-r}\cos\theta\cos 2\phi\,\bar{a}_\theta + \frac{e^{-r}}{r}(-2\sin 2\phi)\bar{a}_\phi$$

$$\underline{\underline{\bar{E} = e^{-r}\sin\theta\cos 2\phi\,\bar{a}_r - \frac{1}{r}e^{-r}\cos\theta\cos 2\phi\,\bar{a}_\theta + \frac{2e^{-r}}{r}(\sin 2\phi)\bar{a}_\phi}} \quad \text{V/m}$$

**Prob.**

(a)

$m\frac{d^2y}{dt^2} = eE$; divide by $m$, and integrate once, one obtains:

$$u = \frac{dy}{dt} = \frac{eEt}{m} + c_0$$

$$y = \frac{eEt^2}{2m} + c_0 t + c_1 \qquad (1)$$

"From rest" implies $c_1 = 0 = c_0$

At $t = t_0, \quad y = d, \quad E = \frac{V}{d} \; or \; V = Ed.$

Substituting this in (1) yields:

$$t^2 = \frac{2md}{eE}$$

Hence:

$$u = \frac{eE}{m}\sqrt{\frac{2md}{eE}} = \sqrt{\frac{2eEd}{m}} = \sqrt{\frac{2eV}{m}}$$

that is, $u \; \alpha \; \sqrt{V}$

or $\quad u = k\sqrt{V}$

(b)

$$k = \sqrt{\frac{2e}{m}} = \sqrt{\frac{2\ (1.603)\ 10^{-19}}{9.1066\ \ (10^{-31})}}$$

$$= \underline{\underline{5.933 \times 10^{5}}}$$

(c)

$$V = \frac{u^2\ m}{2e} = \frac{9(10^{16})\ \frac{1}{100}}{2(1.76)\ (10^{11})} = \underline{\underline{2.557\ k\ V}}$$

**Prob.**

$$V = \frac{p\cos\theta}{4\pi\,\varepsilon_0\, r^2} = \frac{k\cos\theta}{r^2}$$

$At\ \ (0,1\ \text{nm}),\quad \theta = 0,\quad r = 1\ \text{nm},\quad V = 9;$

that is, $\quad 9 = \dfrac{k(1)}{1(10^{-18})},\quad \therefore\ k = 9(10^{-18})$

$$V = 9(10^{-18})\ \frac{\cos\theta}{r^2}$$

At $(1,1)$ nm, $\quad r = \sqrt{2}$ nm, $\quad \theta = 45°,$

$$V = \frac{9(10^{-18})\cos 45°}{10^{-18}(\sqrt{2})^2} = \frac{9}{2\sqrt{2}} = \underline{\underline{3.182\ \ V}}$$

**Prob.**

(a) $\dfrac{dr}{E_r} = \dfrac{r\,d\theta}{E_\theta} = \dfrac{r\sin\theta\,d\phi}{E_\phi}$. Hence

$$\frac{dr}{2\cos\theta} = \frac{r\,d\theta}{\sin\theta} \longrightarrow \frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}\,d\theta$$

Integrating both sides,

$$\ln r = 2\int \frac{d(\sin\theta)}{\sin\theta} = 2\ln\sin\theta + \ln c$$

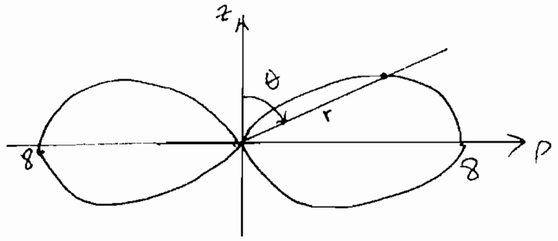
or $\quad r = c\sin^2\theta$

At $(4, \pi/4, \pi/2)$, $\quad r = 1,\ \theta = \pi/4,$

$$4 = c \cdot \tfrac{1}{2} \longrightarrow c = 8$$

$$r = 8\sin^2\theta.$$

The flux line is as sketched on the next page.



Since $\vec{E}$ is tangential to the line,

$$\vec{E}(1, \pi/4, \pi/2) = \frac{(2\vec{a}_r + \vec{a}_\theta)}{4^3\sqrt{2}}$$

A unit vector along $\vec{E}$ is

$$\vec{a} = \frac{1}{\sqrt{5}}(2\vec{a}_r + \hat{a}_\theta) = \underline{\underline{0.8944\vec{a}_r + 0.4472\vec{a}_\theta}}$$

(b) The equipotential line is $V=0$, i.e.

$$0 = V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2}$$

At an arbitrary point $(x, y, 0)$,

$$0 = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{-3}{|(x,y)-(1,0)|} + \frac{4}{|(x,y)-(-1,0)|}\right]$$

or

$$\frac{3}{\sqrt{(x-1)^2+y^2}} = \frac{4}{\sqrt{(x+1)^2+y^2}}$$

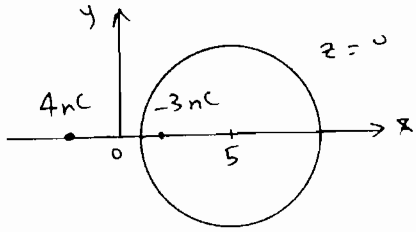
$$16(x-1)^2 + 16y^2 = 9(x+1)^2 + 9y^2$$

or

$$(x-5)^2 + y^2 = (4.86)^2$$

---

i.e. a circle of radius 4.86 centered at (5,0). Thus the equipotential line is sketched below



**Prob.**

Check whether $\nabla x\boldsymbol{E} = 0$.

(a) $$\nabla x\boldsymbol{E}_1 = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 5x^3 & 15x^2y & 0 \end{vmatrix} = 30xy\boldsymbol{a}_z \neq 0$$

Hence, $\mathbf{E}_1$ is <u>not</u> an E-field.

(b)

$$\nabla x\boldsymbol{E}_2 = \left[\frac{1}{\rho}\rho^2\cos\phi - \rho\cos\right]\boldsymbol{a}_\rho + (2\rho\sin\phi - 2\rho\sin\phi)\boldsymbol{a}_\phi + \frac{1}{\rho}[2\rho(z+1)\cos\phi - 2\rho(z+1)\cos\phi]\boldsymbol{a}_z = 0$$

Hence, $\mathbf{E}_2$ is an E-field.

(c )

$$\nabla x\boldsymbol{E}_3 = 0\boldsymbol{a}_r + \frac{1}{r}\left[-\frac{1}{\sin\theta}\frac{5}{r}\sin\theta\sin\phi - 0\right]\boldsymbol{a}_\theta + \frac{1}{r}\left[0 - \frac{5}{r}\cos\theta\cos\phi\right]\boldsymbol{a}_\phi \neq 0$$

Hence, $\mathbf{E}_3$ is not an E-field.

**Prob.**

Let us choose the following paths of two segments.

$(2,1,-1) \rightarrow (5,1,-1) \rightarrow (5,1,2)$

$$W = -q\int \boldsymbol{E} \bullet d\boldsymbol{l}$$

$$-\frac{W}{q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int_{x=2}^{5} 2xyzdx\Bigg|_{z=-1,\, y=1} + \int_{z=-1}^{2} x^2ydz\Bigg|_{x=5,\, y=1}$$

$$= 2(1)(-1)\frac{x^2}{2}\Bigg|_{2}^{5} + (5)^2(1)z\Bigg|_{-1}^{2} = -21 + 75 = 54$$

$$W = -54q = \underline{\underline{-108\ \mu\text{J}}}$$

**Prob.**

The dipole is oriented along $y-$axis.

$$V = \frac{\bar{p}\bullet\bar{r}}{4\pi\varepsilon_0 r^2}; \qquad \bar{p}\bullet\bar{r} = Qd\ \bar{a}_y\bullet\bar{a}_r = Qd\sin\theta\sin\phi$$

$$V = \frac{Qd\sin\theta\sin\phi}{4\pi\varepsilon_0 r^2}$$

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}_r - \frac{1}{r}\frac{\partial V}{\partial\theta}\bar{a}_\theta - \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\bar{a}_\phi$$

$$= \frac{Qd}{4\pi\varepsilon_0}\left\{\frac{2\sin\theta\sin\phi}{r^3}\bar{a}_r - \frac{\cos\theta\sin\phi}{r^3}\bar{a}_\theta - \frac{\cos\phi}{r^3}\bar{a}_\phi\right\}$$

$$\bar{E} = \underline{\underline{\frac{Qd}{4\pi\varepsilon_0 r^3}(2\sin\theta\sin\phi\,\bar{a}_r - \cos\theta\sin\phi\,\bar{a}_\theta - \cos\phi\,\bar{a}_\phi)}}$$

**Prob.**

$$\bar{E} = -\nabla V = -(\frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z)$$

$$\bar{E} = -(2\rho z \sin\phi\, \bar{a}_\rho + \rho z\cos\phi\, \bar{a}_\phi + \rho^2 \sin\phi\, \bar{a}_z)$$

$$W = \frac{1}{2}\varepsilon_0 \int |\bar{E}|^2 dv = \iiint (4\rho^2 z^2 \sin^2\phi + \rho^2 z^2 \cos^2\phi + \rho^4 \sin^2\phi)\rho d\phi dz d\rho$$

$$\frac{2W}{\varepsilon_0} = 4\int_1^4 \rho^3 d\rho \int_{-2}^{2} z^2 dz \int_0^{\pi/3} \sin^2\phi d\phi + \int_1^4 \rho^3 d\rho \int_{-2}^{2} z^2 dz \int_0^{\pi/3} \cos^2\phi d\phi$$

$$+ \int_1^4 \rho^5 d\rho \int_{-2}^{2} dz \int_0^{\pi/3} \sin^2\phi d\phi$$

But $\int_0^{\pi/3} \cos^2\phi d\phi = \frac{1}{2}\int_0^{\pi/3} [1+\cos 2\phi] d\phi = \frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3} = 0.7401$

$$\int_0^{\pi/3} \sin^2\phi d\phi = \frac{1}{2}\int_0^{\pi/3} (1-\cos 2\phi) d\phi = \frac{\pi}{6} - \frac{1}{4}\sin\frac{\pi}{3} = 0.3071$$

$$\frac{2W}{\varepsilon_0} = \frac{4}{4}\rho^4 \Big|_1^4 \frac{2z^3}{3}\Big|_0^2 (0.3071) + \frac{\rho^4}{4}\Big|_1^4 \frac{2z^3}{3}\Big|_0^2 (0.7401) + \frac{\rho^6}{6}\Big|_1^4 (4)(0.3071)$$

$$= 255(\frac{16}{3})(0.3071) + \frac{255}{4}(\frac{16}{3})(0.7041) + \frac{4096}{6}(0.3071)$$

$$= 417.67 + 239.394 + 838.59 = 1507.67$$

$$W = \frac{1507.67}{2}\ (\frac{10^{-9}}{36\pi})$$

$$= \underline{\underline{6.665 \quad \text{nJ}}}$$

# Chapter —Practice Examples

## P.E.

dS $= \rho d\phi dz a_\rho$

$$I = \int J \bullet dS = \int_{\phi=0}^{2\pi} \int_{z=1}^{5} 10z \sin^2 \phi \rho dz d\phi \big|_{\rho=2} = 10(2)\frac{z^2}{2}\bigg|_1^5 \int_0^{2\pi} \frac{1}{2}(1-\cos 2\phi) d\phi = 240\pi$$

$\underline{\underline{I = 754 \text{ A}}}$

## P.E.

$I = \rho_s wu = 0.5x10^{-6} x0.1x10 = 0.5\mu$ A

$V = IR = 10^{14} x0.5x10^{-6} = \underline{\underline{50 \text{ MV}}}$

## P.E.

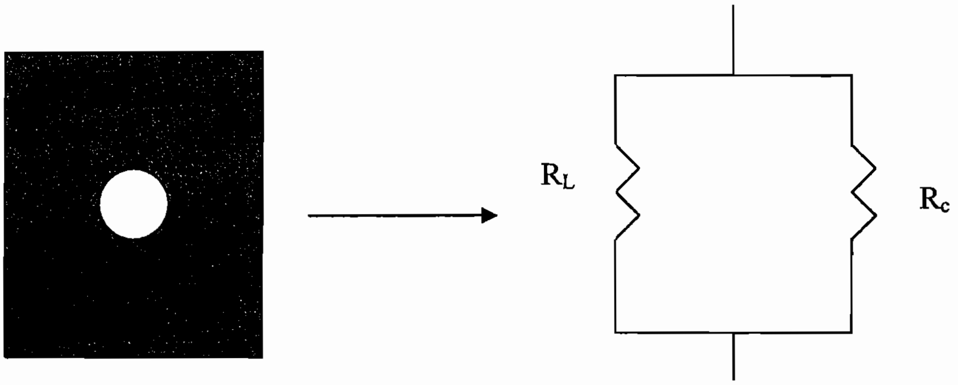
$\sigma = 5.8x10^7$ S/m

$$J = \sigma E \quad \longrightarrow \quad E = \frac{J}{\sigma} = \frac{8x10^6}{5.8x10^7} = \underline{\underline{0.138 \text{ V/m}}}$$

$$J = \rho_v u \quad \longrightarrow \quad u = \frac{J}{\rho_{..}} = \frac{8x10^6}{1.81x10^{10}} = \underline{\underline{4.42x10^{-4} \text{ m/s}}}$$

## P.E.

The composite bar can be modeled as a parallel combination of resistors as shown below.



For the lead, $R_L = \frac{l}{\sigma_L S_L}$, $S_L = d^2 - \pi r^2 = 9 - \frac{\pi}{4}$ cm$^2$

$$R_L = 0.974 \text{ m}\Omega$$

For copper, $R_c = \frac{l}{\sigma_c S_c}$, $S_c = \pi r^2 = \frac{\pi}{4}$ cm$^2$

$$R_c = \frac{4}{5.8x10^7 x\frac{\pi}{4}x10^{-4}} = 0.8781 \text{ m}\Omega$$

$$R = \frac{R_L R_c}{R_L + R_c} = \frac{0.974x0.8781}{0.974+0.8781} = \underline{\underline{461.7\mu\Omega}}$$

## P.E.

$$F = \frac{\rho_s^{\,2} S}{2\varepsilon_o} \longrightarrow \rho_s^{\,2} = \frac{2\varepsilon_o F}{S}$$

But $\rho_s = \varepsilon_o E = \varepsilon_o \frac{V_d}{d}$. Hence

$$\rho_s^{\,2} = \frac{2\varepsilon_o F}{S} = \frac{\varepsilon_o^{\,2} V_d^{\,2}}{d^2} \longrightarrow V_d^{\,2} = \frac{2Fd^2}{\varepsilon_o S}$$

i.e.

$$V_d = V_1 - V_2 = \sqrt{\frac{2Fd^2}{\varepsilon_o S}}$$

as required.

**P.E.**

(a) Since $a_n = a_x$,

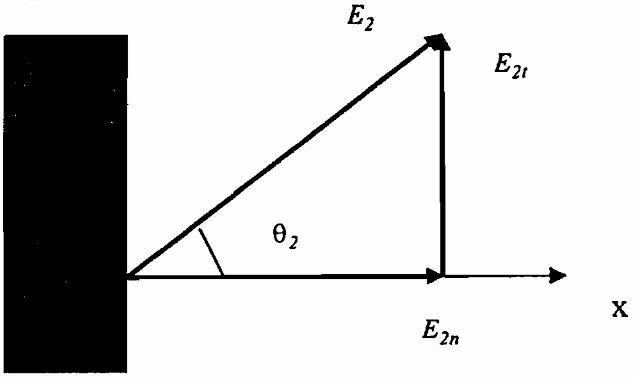
$$D_{1n} = 12a_x, \qquad D_{1t} = -10a_x + 4a_z, \qquad D_{2n} = D_{1n} = 12a_x$$

$$E_{2t} = E_{1t} \longrightarrow D_{2t} = \frac{\varepsilon_2 D_{1t}}{\varepsilon_1} = \frac{1}{2.5}(-10a_y + 4a_z) = -4a_y + 1.6a_z$$

$$D_2 = D_{2n} + D_{2t} = \underline{\underline{12a_x - 4a_y + 1.6a_z}} \text{ nC/m}^2.$$

$$\tan\theta_2 = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{(-4)^2 + (1.6)^2}}{12} = 0.359 \longrightarrow \underline{\underline{\theta_2 = 19.75^o}}$$

(b) $E_{1t} = E_{2t} = E_2 \sin\theta_2 = 12\sin 60^o = 10.392$



$$E_{1n} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} E_{2n} = \frac{1}{2.5} 12\cos 60^o = 2.4$$

$$E_1 = \sqrt{E_{1t}^{\ 2} + E_{1n}^{\ 2}} = \underline{\underline{10.67}}$$

$$\tan\theta_1 = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\tan\theta_2 = \frac{2.5}{1}\tan 60^o = 4.33 \longrightarrow \underline{\underline{\theta_1 = 77^o}}$$

Note that $\theta_1 > \theta_2$.

**P.E.**

$$D = \varepsilon_o E = \frac{10^{-9}}{36\pi}(60,20,-30)x10^{-3} = \underline{\underline{0.531a_x + 0.177a_y - 0.265a_z}} \text{ pC/m}^2$$

$$\rho_s = D_n = |D| = \frac{10^{-9}}{36\pi}(10)\sqrt{36 + 4 + 9}(10^{-3}) = \underline{\underline{0.619}} \text{ pC/m}^2$$

## CHAPTER —Problems

**Prob.**

$$I = -\frac{dQ}{dt} = 3\text{x}10^{-4}e^{-3t}$$

I(t=0) = 0.3 mA, I(t=2.5) = 0.3 $e^{-7.5}$ = $\underline{\underline{166\text{ nA}}}$

**Prob.**

$$I = \int J \bullet dS, \quad dS = r\sin\theta d\phi dr a_\theta$$

$$I = -\int_{r=0}^{2}\int_{\phi=0}^{2\pi} r^3\sin^2\theta d\phi dr \Big|_{\theta=30^\circ} = -(\sin 30^\circ)^2 \left.\frac{r^4}{4}\right|_0^2 (2\pi) = -2\pi = \underline{\underline{-6.283}}\text{ A}$$

**Prob.**

. ........ (a) $\nabla^2 V = -\rho_v/\varepsilon$

$$\nabla^2 V = \frac{\partial}{\partial x}(2xy^2z) + \frac{\partial}{\partial y}(2x^2yz) = 6xyz$$

$$\rho_v = -8xyz(2\varepsilon_o) = \underline{\underline{-16xyz\varepsilon_o}}$$

(b) $J = \rho_v u = -16xy^2z\varepsilon_o(10^4)a_y$

$$I = \int J \bullet dS = -16(10^4)\ \frac{10^{-9}}{36\pi}\int_0^{0.5} xdx \int_0^{0.5} zdz = -16(36\pi)(10^{-5})\left(\left.\frac{x^2}{2}\right|_0^{0.5}\right)^2$$

$$I = -4(36\pi)(10^{-5})(0.5)^2 = \underline{\underline{-1.131}}\text{ mA}$$

**Prob.**

Let a=1.6 mm be the radius of cross-section. $dS = \rho d\phi d\rho a_z$

$$I = \int J \bullet dS = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{500}{\rho}\rho d\phi d\rho = 500(2\pi a) = 1000\pi\text{x}1.6\text{x}10^{-3} = 1.6\pi = \underline{\underline{5.026}}\text{ A}$$

**Prob.**

$$Q = It = \underline{\underline{12C}}$$

$$n = \frac{Q}{e} = \frac{12}{1.6 \times 10^{-19}} = \underline{\underline{7.5 \times 10^{19} \text{ electrons}}}$$

**Prob.**

(a) $R = \dfrac{l}{\sigma S} = \dfrac{8x10^{-2}}{3x10^{4}x\pi x25x10^{-6}} = \dfrac{8}{75\pi} = \underline{\underline{33.95\text{m}\Omega}}$

(b) $I = V / R = 9x\dfrac{75\pi}{8} = \underline{\underline{265.1}}$ A

(c ) P = IV = $\underline{\underline{2.386\ \text{kW}}}$

**Prob.**

$$R = \frac{l}{\sigma S} \quad \longrightarrow \quad \sigma = \frac{l}{RS} = \frac{2x10^{-2}}{10^{6}x\pi(4x10^{-3})^{2}} = \underline{\underline{3.978x10^{-4}\ \text{S/m}}}$$

**Prob.**

$$P = I^2 R = I^2 \frac{\rho L}{S} = \left(\frac{I}{S}\right)^2 \rho L S = J^2 \rho L S$$

But $\rho_m = \dfrac{M}{V} = \dfrac{M}{LS}$, hence

$$P = J^2 \rho \frac{M}{\rho_m} = \frac{(0.8 \times 10^6)^2 \times 0.176 \times 10^{-6} \times 1}{8.9 \times 10^{-3} \times 10^6}$$

$$= \underline{\underline{1.26\ W}}$$

**Prob.**

$R = \dfrac{l}{\sigma S}$, $\quad S = \pi r^2 = \pi d^2 / 4, d = 0.4\text{mm}, \quad l = N2\pi R_t$, where $R_t$ is the radius of the turn.

$$R = \frac{2x150x\pi(6.5)x10^{-3}}{5.8x10^{7}x\pi\dfrac{(0.4)^2}{4}x10^{-6}} = \underline{\underline{0.84\Omega}}$$

**Prob.**

(a) $S_i = \pi r_i^2 = \pi(1.5)^2 x10^{-4} = 7.068x10^{-4}$

$S_o = \pi(r_o^2 - r_i^2) = \pi(4-2.25)x10^{-4} = 5.498x10^{-4}$

$$R_l = \frac{\rho_l l}{S_l} = \frac{11.8\times10^{-8}\times10}{7.068X10^{-4}} = 16.69\times10^{-4}$$

$$R_o = \frac{\rho_o l}{S_o} = \frac{1.77\times10^{-8}\times10}{5.498X10^{-4}} = 3.219\times10^{-4}$$

$$R = R_i // R_o = \frac{R_i R_o}{R_i + R_o} = \frac{16.69\times3.219\times10^{-4}}{16.69+3.219} = \underline{\underline{0.27\text{m}\Omega}}$$

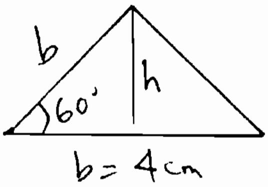
(b) $V = I_i R_i = I_o R_o \quad \longrightarrow \quad \frac{I_i}{I_o} = \frac{R_o}{R_i} = \frac{0.3219}{1.669} = 0.1929$

$I_i + I_o = 1.1929 I_o = 60$ A

$I_o = \underline{\underline{50.3\text{ A}}}$ (copper), $I_i = \underline{\underline{9.7\text{ A}}}$ (steel)

(c) $R = \frac{10x1.77x10^{-8}}{1.75\pi x10^{-4}} = \underline{\underline{0.322\text{m}\Omega}}$

**Prob.**



$h = b\sin60°, \quad S = \frac{1}{2}hb = \frac{1}{2}b^2\sin60°$

$$R = \frac{l}{\sigma S} = \frac{2l}{\sigma b^2 \sin60°} = \frac{2\times3}{10^{-15}\times16\times10^{-4}\sin60°}$$

$$= \underline{\underline{4.33\times10^{18}\,\Omega}}$$

**Prob.**

$$R = \frac{l}{\sigma S} = \frac{l}{\sigma \pi (b^2 - a^2)} = \frac{2}{3x10^4 \pi (25-9)x10^{-4}} = \underline{\underline{13.26 \text{ m}\Omega}}$$

**Prob.**

(a) $\frac{Q}{t} = I = \underline{\underline{10 \mu C/s}}.$

(b) $\rho_s = \frac{I}{wu} = \frac{10 \times 10^{-6}}{25 \times 0.5} = \underline{\underline{0.8 \mu C/m^2}}.$

**Prob.**

(a) $R = \frac{\rho l}{S} \longrightarrow \rho = RS/l = \frac{4.04}{10^3}\frac{\pi d^2}{4} = 2.855x10^{-8}$

$\sigma = 1/\rho = \underline{\underline{3.5x10^7}}$ S/m (Aluminum)

(b) $J = I/S = \frac{40}{\frac{\pi}{4}x9x10^{-6}} = \underline{\underline{5.66x10^6 \text{A/m}^2}}$

or

$$J = \sigma E = 3.5x0.1616x10^7 = 5.66x10^6 \text{A/m}^2$$

**Prob.**

(a) $\vec{E}_{2t} = \vec{E}_{1t} = -300\vec{a}_x + 50\vec{a}_y, \quad \vec{E}_{1n} = 70\vec{a}_z$

$\vec{D}_{2n} = \vec{D}_{1n} \longrightarrow \epsilon_2 \vec{E}_{2n} = \epsilon_1 \vec{E}_{1n}$

$\vec{E}_{2n} = \frac{\epsilon_1}{\epsilon_2}\vec{E}_{1n} = \frac{2.5}{4}(70\vec{a}_z) = 43.75\vec{a}_z$

$\vec{E}_2 = -30\vec{a}_x + 50\vec{a}_y + 43.75\vec{a}_z.$

$\vec{D}_2 = \epsilon_o \epsilon_{r2} \vec{E}_2 = \frac{4 \times 10^{-9}}{36\pi}(-30, 50, 43.75)$

$= \underline{\underline{-1.061\vec{a}_x + 1.768\vec{a}_y + 1.547\vec{a}_z \text{ nC/m}^2}}.$

(b) $\vec{P}_2 = \epsilon_0 \chi_{e2} \vec{E}_2 = \frac{3\times10^{-9}}{36\pi}(-30, 50, 43.75)$

$= 0.7958\vec{a}_x + 1.326\vec{a}_y + 1.161\vec{a}_z$ nC/m².

(c) $\vec{E}_1 \cdot \vec{a}_z = E_1 \cos\theta_n$

$$\cos\theta_n = \frac{70}{\sqrt{30^2+50^2+70^2}} = 0.7683$$

$\theta_n = 39.79^\circ$.

**Prob.**

' (a) $E_{2t} = E_{1t} = -300a_x + 50a_y, \quad E_{1n} = 70a_z$

$$D_{2n} = D_{1n} \longrightarrow \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{2.5}{4}(70a_z) = 43.75a_z$$

$$E_2 = -30a_x + 50a_y + 43.75a_z$$

$$D_2 = \varepsilon_o \varepsilon_r E_2 = 4x\frac{10^{-9}}{36\pi}(-30,50,43.75) = -1.061a_x + 1.768a_y + 1.547a_z \text{ nC/m}^2$$

(b) $P_2 = \varepsilon_o \chi_{e2} E_2 = 3x\frac{10^{-9}}{36\pi}(-30,50,43.75) = 0.7958a_x + 1.326a_y + 1.161a_z \text{ nC/m}^2$

(c ) $E_1 \bullet a_z = E_1 \cos\theta_n$

$$\cos\theta_n = \frac{70}{\sqrt{30^2+50^2+70^2}} = 0.7683 \longrightarrow \theta_n = 39.79^o$$

**Prob.**

**1** (a)

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_o \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}$$

$$D_x = 50\varepsilon_o, \quad D_y = 50\varepsilon_o, \quad D_z = 20\varepsilon_o$$

$$D = \underline{\underline{0.442a_x + 0.442a_y + 0.1768a_z}} \text{ nC/m}^2$$

(b)

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_o \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ -30 \end{bmatrix}$$

$$D_x = 30\varepsilon_o, \quad D_y = 60\varepsilon_o, \quad D_z = 90\varepsilon_o$$

$$D = \underline{\underline{0.2653a_x + 0.5305a_y + 0.7958a_z}} \text{ nC/m}^2$$

**Prob.**

(a)

$$\rho_v = \begin{cases} \rho_o, & 0 < r < a \\ 0, & r > a \end{cases}$$

For r < a, $\varepsilon E_r(4\pi r^2) = \rho_o \dfrac{4\pi r^3}{3} \longrightarrow E_r = \dfrac{\rho_o r}{3\varepsilon}$

$$V = -\int E \bullet dl = -\frac{\rho_o r^2}{6\varepsilon} + c_1$$

For r > a, $\varepsilon_o E_r(4\pi r^2) = \rho_o \dfrac{4\pi a^3}{3} \longrightarrow E_r = \dfrac{\rho_o a^3}{3\varepsilon_o r^2}$

$$V = -\int E \bullet dl = \frac{\rho_o a^3}{3\varepsilon_o r} + c_2$$

As $r \longrightarrow \infty$, $V = 0$ and $c_2 = 0$

At r = a, $V(a^+) = V(a^-)$

$$-\frac{\rho_o a^2}{6\varepsilon_o \varepsilon_r} + c_1 = \frac{\rho_o a^2}{3\varepsilon_o} \longrightarrow c_1 = \frac{\rho_o a}{6\varepsilon_o a}(2\varepsilon_r + 1)$$

$$V(r=0) = c_1 = \underline{\underline{\frac{\rho_o(2\varepsilon_r + 1)}{6\varepsilon_o a}}}$$

(b) $V(r = a) = \underline{\underline{\dfrac{\rho_o a^2}{3\varepsilon_o}}}$

**Prob.**

$$\rho_{pv} = -\nabla \bullet P = \underline{\underline{0}}, \qquad \rho_{ps} = P \bullet a_n = \underline{\underline{5\sin\alpha y}}$$

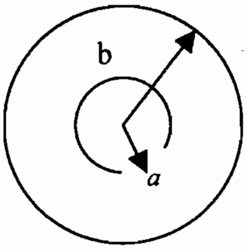
**Prob.**

$$F_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_o r^2}, \qquad F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_o \varepsilon_r r^2}$$

$$\frac{F_1}{F_2} = \varepsilon_r = 4.5 / 2 = 2.25$$

$\varepsilon_r = 2.25,$ polystrene

**Prob.**



For 0 < r < a,

$$D = \frac{Q}{4\pi r^2} a_r \quad \longrightarrow \quad E = \frac{Q}{4\pi\varepsilon_o r^2} a_r, \quad P = 0$$

For a < r < b,

$$D = \frac{Q}{4\pi r^2} a_r \quad \longrightarrow \quad E = \frac{Q}{4\pi\varepsilon\varepsilon_r r^2} a_r, \quad P = \chi_e \varepsilon_o E = \frac{\varepsilon_r - 1}{\varepsilon_r} \frac{Q}{4\pi\ r^2} a_r$$

For r > b,

$$D = \frac{Q}{4\pi r^2} a_r \quad \longrightarrow \quad E = \frac{Q}{4\pi\varepsilon_o r^2} a_r, \quad P = 0$$

Thus,

$$D = \frac{Q}{4\pi\varepsilon r^2} a_r, \quad r > 0$$

$$E = \begin{cases} \dfrac{Q}{4\pi\varepsilon\varepsilon_r r^2} a_r, & a < r < b \\ \dfrac{Q}{4\pi\varepsilon_o r^2} a_r, & \text{otherwise} \end{cases}$$

$$P = \begin{cases} \dfrac{\varepsilon_r - 1}{\varepsilon_r}\dfrac{Q}{4\pi\, r^2}a_r, \quad a < r < b \\ 0, \text{ otherwise} \end{cases}$$

**Prob.**

(a) $\vec{P} = \chi_e \epsilon_0 \vec{E} = (\epsilon_r - 1)\epsilon_0 \vec{E} = (\epsilon - \epsilon_0)\bar{E}$

$\vec{D} = \epsilon \vec{E} = \dfrac{\epsilon}{\epsilon - \epsilon_0}\vec{P} = \dfrac{\epsilon_r}{\epsilon_r - 1}\vec{P}$.

(b) $\epsilon_r = 1 + \chi_e = \underline{\underline{3.4}}$

$E = \dfrac{D}{\epsilon_0 \epsilon_r} = \dfrac{300 \times 10^{-6}}{3.4} 36\pi \times 10^{9}$

$= \underline{\underline{9.979 \text{ mV/m}}}.$

$P = \epsilon_0 \chi_e E = \epsilon_0 \chi_e \dfrac{D}{\epsilon_0 \epsilon_1} = \dfrac{\chi_e}{\epsilon_r} D$

$= \dfrac{2.4}{3.4}(300) = \underline{\underline{211.8\ \mu C/m^2}}.$

**Prob.**

$$\text{(a)} \quad \nabla \bullet J = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{100}{\rho}\right) = -\frac{100}{\rho^3}$$

$$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet J = -\frac{100}{\rho^3} \quad \longrightarrow \quad \frac{\partial \rho_v}{\partial t} = \underline{\underline{\frac{100}{\rho^3}\ \text{C/m}^3.s}}$$

$$\text{(b)} \quad I = \int J \bullet dS = \iint \frac{100}{\rho^2}\rho\, d\phi\, dz\Big|_{\rho=2} = \frac{100}{2}\int_0^{2\pi} d\phi \int_0^{1} dz = 100\pi = \underline{\underline{314.16\text{ A}}}\ \text{A}$$

**Prob.**

(a) $\vec{J} = 10^4(9+16)\vec{a}_z = \underline{\underline{250\vec{a}_z \ kA/m^2}}.$

(b) $\frac{\partial \rho_v}{\partial t} = -\nabla\cdot\vec{J} = \underline{\underline{0}}.$

(c) $$I = \int \vec{J}\cdot d\vec{S} = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{0.005} 10^4 \rho^2 \rho \, d\phi \, d\rho$$

$$= 10^4(2\pi)\frac{\rho^4}{4}\Big|_0^{0.005}$$

$$= \underline{\underline{9.817 \mu A}}.$$

**Prob.**

(a) $$I = \int \vec{J}.d\vec{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{3} \frac{e^{-10^3 t}}{\rho^2}\rho \, d\phi \, dz$$

$$= (2\pi)(3)\frac{e^{-10}}{2} = \underline{\underline{428 \mu A}}.$$

(b) $$\frac{\partial \rho_v}{\partial t} = -\nabla\cdot\vec{J} = -\frac{e^{-10^3 t}}{\rho}\frac{d}{d\rho}\left(\frac{1}{\rho}\right) = \frac{e^{10^3 t}}{\rho^3}$$

$$\rho_v = \frac{-10^{-3}e^{-10^3 t}}{\rho^3} + c(t)$$

If we let $\rho_v \to 0$ as $t \to \infty$, $c(t) = 0$.

$$\rho_v(\rho=2, t=10 ms) = -\frac{10^{-3}e^{-10}}{8} = \underline{\underline{-5.67 \ nC/m^3}}.$$

**Prob.**

(a)

$$I = \int J \bullet dS = \iint \frac{5e^{-10^4 t}}{r} r^2 \sin\theta d\theta d\phi \Big|_{r=2} = (2)(5)e^{-10^4 t} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 40\pi e^{-10^4 t}$$

At t=0.1 ms, $I = 40\pi e^{-1} = \underline{\underline{46.23}}$ A

(b) $-\frac{\partial \rho_v}{\partial t} = \nabla \bullet J \longrightarrow \rho_v = -\int \nabla \bullet J \partial t$

$$\nabla \bullet J = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 J_r) = \frac{5}{r^2} e^{-10^4 t}$$

$$\rho_v = -\int \frac{5}{r^2} e^{-10^4 t} dt = \frac{5}{10^4 r^2} e^{-10^4 t}$$

At t=0.1 ms and r = 2m,

$$\rho_v = \frac{5}{10^4 (2)^2} e^{-1} = \underline{\underline{45.98 \ \mu\mathrm{C/m^3}}}$$

**Prob.**

$$T = \frac{\epsilon}{\sigma} \longrightarrow \sigma = \frac{\epsilon}{T} = \frac{6 \times 10^{-9}}{53 \times 36\pi} = \underline{\underline{10^{-12} \ \mho/m.}}$$

Glass or Porcelain.

**Prob.**

(a) $$\frac{\varepsilon}{\sigma} = \frac{3.1x\dfrac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{2.741x10^{4}}}\ \text{s}$$

(b) $$\frac{\varepsilon}{\sigma} = \frac{6x\dfrac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{5.305x10^{4}}}\ \text{s}$$

(c) $$\frac{\varepsilon}{\sigma} = \frac{80x\dfrac{10^{-9}}{36\pi}}{10^{-4}} = \underline{\underline{7.07}}\ \mu\text{s}$$

**Prob.**

(a) $$T = \frac{\varepsilon_0 \varepsilon_r}{\sigma} = \frac{2.55 \times 10^{-9}}{10^{-16} \times 36\pi} = 2.255 \times 10^{5}\ \text{s}$$

$$= 62.64 \text{ hours} = \underline{\underline{2.61 \text{ days}}}.$$

(b) $$T = \frac{\varepsilon_0 \varepsilon_r}{\sigma} = \frac{20 \times 10^{-9}}{10^{-4} \times 36\pi} = \underline{\underline{1.76\ \mu\text{s}}}$$

(c) $$T = \frac{10^{-9}}{1.6 \times 10^{7} \times 36\pi} = \underline{\underline{5.53 \times 10^{-19}\ \text{s}}}.$$

**Prob.**

Since $\frac{\partial \rho_v}{\partial t} = 0$, $\nabla \bullet J = 0$ must hold.

(a) $\nabla \bullet J = 6x^2y + 0 - 6x^2y = 0 \longrightarrow$ This is <u>possible</u>.

(b) $\nabla \bullet J = y + (z+1) \neq 0 \longrightarrow$ This is <u>not possible</u>.

(c) $\nabla \bullet J = \frac{1}{\rho}\frac{\partial}{\partial \rho}(z^2) + \cos\phi \neq 0 \longrightarrow$ This is <u>not possible.</u>

(d) $\nabla \bullet J = \frac{1}{r^2}\frac{\partial}{\partial r}(\sin\theta) = 0 \longrightarrow$ This is <u>possible</u>.

**Prob.**

$$Q = Q_o e^{-t/T_r} \quad \longrightarrow \quad \frac{1}{3}Q_o = Q_o e^{-t_1/T_r} \quad \longrightarrow \quad e^{t_1/T_r} = 3$$

(a) But $T_r = \dfrac{\varepsilon_r \varepsilon_o}{\sigma}, \quad \varepsilon_r = \dfrac{\sigma T_r}{\varepsilon_o} = \dfrac{10^{-4} x 18.2 x 10^{-6}}{\dfrac{10^{-9}}{36\pi}} = \underline{\underline{205.8}}$

(b) $$T_r = \frac{t_1}{\ln 3} = \frac{20\mu s}{\ln 3} = \underline{\underline{18.2\mu s}}$$

(c ) $\dfrac{Q}{Q_o} = e^{-t_o/T_r} = e^{-30/18.3} = 0.1923$ i.e. $\underline{\underline{19.23\%}}$

**Prob.**

$$T_r = \frac{\varepsilon}{\sigma} = \frac{2.5x10^{-9}}{5x10^{-6} x 36\pi} = 4.42 \ \mu s$$

$$\rho_{vo} = \frac{Q}{V} = \frac{1}{\dfrac{4\pi}{3} x 10^{-6} x 8} = \underline{\underline{29.84 \text{ kC/m}^3}}$$

$$\rho_v = \rho_{vo} e^{-t/T_r} = 29.84 e^{-2/4.42} = \underline{\underline{18.98 \text{ kC/m}^3}}$$

**Prob.**

$$\boldsymbol{E}_{1n} = 8\boldsymbol{a}_z, \qquad \boldsymbol{E}_{1t} = 6\boldsymbol{a}_x - 12\boldsymbol{a}_y = \boldsymbol{E}_{2t}$$

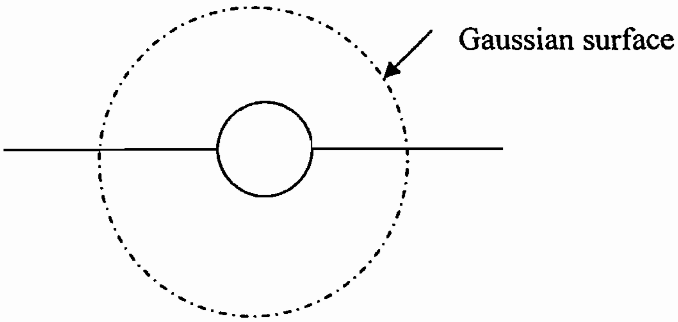
$$\boldsymbol{D}_{1n} = \boldsymbol{D}_{2n} \quad \longrightarrow \quad \varepsilon_2 \boldsymbol{E}_{2n} = \varepsilon_1 \boldsymbol{E}_{1n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{5}{2}(8) = 20$$

Hence,

$$\boldsymbol{E}_2 = \boldsymbol{E}_{2n} + \boldsymbol{E}_{2t} = \underline{\underline{6\boldsymbol{a}_x - 12\boldsymbol{a}_y + 20\boldsymbol{a}_z \ \text{ V/m}}}$$

**Prob.**



$$Q = \int D.dS = \varepsilon_1 E_r \frac{4\pi r^2}{2} + \varepsilon_2 E_r \frac{4\pi r^2}{2} = 2\pi r^2(\varepsilon_1 + \varepsilon_2)E_r$$

$$E_r = \begin{cases} \dfrac{Q}{2\pi(\varepsilon_1+\varepsilon_2)r^2}, & r > a \\ 0, \quad r < a \end{cases}$$

**Prob.**

(a) $P_l = \varepsilon_o \chi_{el} E_l = 2x\dfrac{10^{-9}}{36\pi}(10,-6,12) = \underline{\underline{0.1768a_x - 0.1061a_y + 0.2122a_z \text{ nC/m}^2}}$

(b) $E_{ln} = -6a_x, \quad E_{2t} = E_{lt} = 10a_x + 12a_z$

$$D_{2n} = D_{ln} \quad \longrightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_l E_{ln}$$

or $\quad E_{2n} = \dfrac{\varepsilon_l}{\varepsilon_2} E_{ln} = \dfrac{3\varepsilon_o}{4.5\varepsilon_o}(-6a_z) = -4a_y$

$$E_2 = \underline{\underline{10a_x - 4a_y + 12a_z \text{ V/m}}}$$

$$\tan\theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{10^2+12^2}}{4} = 3.905 \quad \longrightarrow \quad \underline{\underline{\theta_2 = 75.64^o}}$$

(c) $w_E = \dfrac{1}{2} D \bullet E = \dfrac{1}{2}\varepsilon|E|^2$

$$w_{E1} = \frac{1}{2}\varepsilon_1 |E_1|^2 = \frac{1}{2}x3x\frac{10^{-9}}{36\pi}(10^2+6^2+12^2) = \underline{\underline{3.7136 \text{ nJ/m}^3}}$$

$$w_{E2} = \frac{1}{2}\varepsilon_2 |E_2|^2 = \frac{1}{2}x4.5x\frac{10^{-9}}{36\pi}(10^2+4^2+12^2) = \underline{\underline{5.1725 \text{ nJ/m}^3}}$$

**Prob.**

(a) $D_{2n} = 12a_\rho = D_{1n}, \qquad D_{2t} = -6a_\phi + 9a_z$

$$E_{2t} = E_{2t} \qquad \longrightarrow \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

$$D_{1t} = \frac{\varepsilon_1}{\varepsilon_2} D_{2t} = \frac{3.5\varepsilon_o}{1.5\varepsilon_o}(-6a_\phi + 9a_z) = -14a_\phi + 21a_z$$

$$\underline{\underline{D_1 = 12a_\rho - 14a_\phi + 21a_z \ \text{nC}/\text{m}^2}}$$

$$E_1 = D_1 / \varepsilon_1 = \frac{(12,-14,21)x10^{-9}}{3.5x\dfrac{10^{-9}}{36\pi}} = \underline{\underline{387.8a_\rho - 452.4a_\phi + 678.6a_z \ \text{V}/\text{m}}}$$

(b) $P_2 = \varepsilon_o \chi_{e2} E_2 = 0.5\varepsilon_o \dfrac{D_2}{\varepsilon_2} = \dfrac{0.5\varepsilon_o}{1.5\varepsilon_o}(12,-6,9) = \underline{\underline{4a_\rho - 2a_\phi + 3a_z \ \text{nC}/\text{m}^2}}$

$\rho_{v2} = \nabla \bullet P_2 = 0$

(c) $w_{E1} = \dfrac{1}{2} D_1 \bullet E_1 = \dfrac{1}{2}\dfrac{D_1 \bullet D_1}{\varepsilon_o \varepsilon_{r1}} = \dfrac{1}{2}\dfrac{(12^2 + 14^2 + 21^2)x10^{-18}}{3.5x\dfrac{10^{-9}}{36\pi}} = \underline{\underline{12.62 \ \mu\text{J}/\text{m}^2}}$

$$w_{E2} = \frac{1}{2}\frac{D_2 \bullet D_2}{\varepsilon_o \varepsilon_{r2}} = \frac{1}{2}\frac{(12^2 + 6^2 + 9^2)x10^{-18}}{1.5x\dfrac{10^{-9}}{36\pi}} = \underline{\underline{9.839 \ \mu\text{J}/\text{m}^2}}$$

**Prob.**

$$f(x,y)= 4x +3y -10=0$$

$$\nabla f = 4\boldsymbol{a}_x + 3\boldsymbol{a}_y \quad \longrightarrow \quad \boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{4\boldsymbol{a}_x + 3\boldsymbol{a}_y}{5} = 0.8\boldsymbol{a}_x + 0.6\boldsymbol{a}_y$$

$$\boldsymbol{D}_{1n} = (\boldsymbol{D}_1 \cdot \boldsymbol{a}_n)\boldsymbol{a}_n = (1.6 - 2.4)\boldsymbol{a}_n = -0.64\boldsymbol{a}_x - 0.48\boldsymbol{a}_y$$

$$\boldsymbol{D}_{1t} = \boldsymbol{D}_1 - \boldsymbol{D}_{1n} = 2.64\boldsymbol{a}_x - 3.52\boldsymbol{a}_y + 6.5\boldsymbol{a}_z$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} = -0.64\boldsymbol{a}_x - 0.48\boldsymbol{a}_y$$

$$E_{2t} = E_{1t} \quad \longrightarrow \quad \frac{D_{2t}}{\varepsilon_1} = \frac{D_{1t}}{\varepsilon_2}$$

$$\boldsymbol{D}_{2t} = \frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{D}_{1t} = \frac{1}{2.5}(2.64, -3.52, 6.5) = (1.056, -1.408, 2.6)$$

$$\boldsymbol{D}_2 = \boldsymbol{D}_{2n} + \boldsymbol{D}_{2t} = \underline{\underline{0.416\boldsymbol{a}_x - 1.888\boldsymbol{a}_y + 2.6\boldsymbol{a}_z \text{ nC/m}^2}}$$

$$\boldsymbol{D}_2 \bullet a_n = |\boldsymbol{D}_2| \cos\theta_2 \quad \longrightarrow \quad \cos\theta_2 = \frac{D_{2n}}{D} = \frac{0.8}{3.24} = 0.2469$$

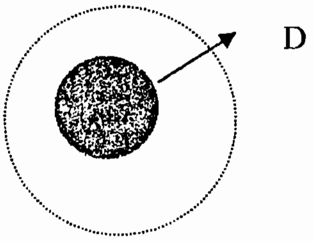
$$\theta_2 = \underline{\underline{75.71^o}}$$

**Prob.**

$$D_n = \rho_s = \frac{Q}{4\pi a^2} = \frac{12x10^{-9}}{4\pi x25x10^{-4}} = \frac{1200}{\pi} \text{ nC/m}^2$$

$$\underline{\underline{|D| = 381.97 \text{ nC/m}^2}}$$

Using Gauss's law,



$$D_r 4\pi r^2 = Q \quad \longrightarrow \quad D_r = \frac{Q}{4\pi r^2}$$

$$D = \frac{Q}{4\pi r^2} a_r = \frac{12}{4\pi r^2} a_r \text{ nC/m}^2 = \underline{\underline{\frac{0.955}{r^2} a_r \text{ nC/m}^2}}$$

(c ) $W = \frac{1}{2\varepsilon_o} \int |D|^2 dv = \frac{Q^2}{2\varepsilon_o 16\pi^2} \iiint \frac{1}{r^4} r^2 \sin\theta \, d\theta \, d\phi \, dr = \frac{Q^2}{32\pi^2 \varepsilon_o} 4\pi \int_a^\infty \frac{dr}{r^2}$

$$= \frac{Q^2}{8\pi\varepsilon_o a} = \frac{144x10^{-18}}{8\pi x \frac{10^{-9}}{36\pi} x5x10^{-2}} = \underline{\underline{12.96 \ \mu\text{J}}}$$

# Chapter —Practice Examples

## P.E.

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \longrightarrow \frac{d^2V}{dx^2} = -\frac{\rho_o x}{\varepsilon a}$$

$$V = -\frac{\rho_o x^3}{6\varepsilon a} + Ax + B$$

$$\boldsymbol{E} = -\frac{dV}{dx}\boldsymbol{a}_x = \left(\frac{\rho_o x^2}{2\varepsilon a} - A\right)\boldsymbol{a}_x$$

If **E** = 0 at x =0, then

$$0 = 0 - A \longrightarrow A = 0$$

If V= 0 at x =a, then

$$0 = -\frac{\rho_o a^3}{6\varepsilon a} + B \longrightarrow B = \frac{\rho_o a^2}{6\varepsilon}$$

Thus

$$\underline{\underline{V = \frac{\rho_o}{6\varepsilon a}(a^3 - x^3)}}, \qquad \underline{\underline{\boldsymbol{E} = \frac{\rho_o x^2}{2\varepsilon a}\boldsymbol{a}_x}}$$

## P.E.

$$V_1 = A_1 x + B_1, \quad V_2 = A_2 x + B_2$$

$$V_1(x = d) = V_o = A_1 d + B_1 \longrightarrow B_1 = V_o - A_1 d$$

$$V_1(x = 0) = 0 = 0 + B_2 \longrightarrow B_2 = 0$$

$$V_1(x = a) = V_2(x = a) \longrightarrow aA_1 + B_1 = A_2 a$$

$$D_{1n} = D_{2n} \longrightarrow \varepsilon_1 A_1 = \varepsilon_2 A_2 \longrightarrow A_2 = \frac{\varepsilon_1}{\varepsilon_2} A_1$$

$$A_1 a + V_o - A_1 d = \frac{\varepsilon_1}{\varepsilon_2} a A_1 \longrightarrow V_o = A_1\left(-a + d + \frac{\varepsilon_1}{\varepsilon_2} a\right)$$

or

$$A_1 = \frac{V_o}{d - a + \varepsilon_1 a / \varepsilon_2}, \quad A_2 = \frac{\varepsilon_1}{\varepsilon_2} A_1 \frac{\varepsilon_1 V_o}{\varepsilon_2 d - \varepsilon_2 a + \varepsilon_1 a}$$

Hence

$$\boldsymbol{E}_1 = -A_1 \boldsymbol{a}_x = \underline{\underline{\frac{-V_o a_x}{d - a + \varepsilon_1 a / \varepsilon_2}}}, \quad E_2 = -A_2 \boldsymbol{a}_x = \underline{\underline{\frac{-V_o \boldsymbol{a}_x}{a + \varepsilon_2 d / \varepsilon_1 - \varepsilon_2 a / \varepsilon_1}}}$$

## P.E.

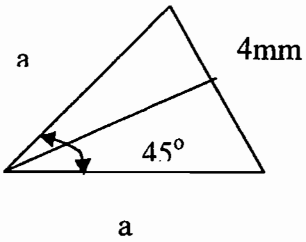
$$\boldsymbol{E} = -\frac{V_o}{\rho\phi_o}\boldsymbol{a}_\phi, \qquad D = \varepsilon_o E$$

$$\rho_s = D_n(\phi = 0) = -\frac{V_o \varepsilon}{\rho \phi_o}$$

The charge on the plate $\phi = 0$ is

$$Q = \int \rho_s dS = -\frac{V_o \varepsilon}{\phi_o}\int_{z=0}^{L}\int_{\rho=a}^{b}\frac{1}{\rho}dz d\rho = -\frac{V_o \varepsilon}{\phi_o}L\ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon L}{\phi_o}\ln\frac{b}{a}$$



$$a\sin\frac{45^o}{2} = 2 \quad \longrightarrow \quad a = \frac{2}{\sin 22.5^o} = 5.226 \text{ mm}$$

$$C = \frac{1.5\times\frac{10^{-9}}{36\pi}}{\frac{\pi}{4}}5\ln\frac{1000}{5.226} = 444 \text{ pF}$$

$$Q = CV_o = 444\times10^{-12}\times 50 \text{ C} = \underline{\underline{22.2 \text{ nC}}}$$

## P.E.

$V_o = 50, \quad \theta_1 = 45°, \quad \theta_2 = 90°, \quad r = \sqrt{3^2+4^2+2^2} = \sqrt{29}, \quad \theta = \tan^{-1}\frac{\rho}{z} =$

$\tan^{-1}\frac{5}{2} \quad \longrightarrow \quad \theta = 68.2°$

$$V = \frac{50\ln(\tan 34.1° / \tan 22.5°)}{\ln(\tan 45° / \tan 22.5°)} = \underline{\underline{27.87 \text{ V}}},$$

$$\boldsymbol{E} = \frac{-50\boldsymbol{a}_\theta}{\sqrt{29}\sin 68.2° \ln(\tan 45° / \tan 22.5°)} = \underline{\underline{-11.35\boldsymbol{a}_\theta \text{ V/m}}}$$

**P.E.**

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{a}_x - \frac{\partial V}{\partial y}\mathbf{a}_y$$

$$= -\frac{4V_o}{b}\sum_{n=\text{odd}}^{\infty}\frac{1}{\sinh n\pi a/b}\left[\cos(n\pi x/b)\sinh(n\pi y/b)\,\mathbf{a}_x + \sin(n\pi x/b)\cosh(n\pi y/b)\mathbf{a}_y\right]$$

(a) At $(x,y) = (a, a/2)$,

$$V = \frac{400}{\pi}(0.3775 - 0.0313 + 0.00394 - 0.000585 + \ldots) = \underline{\underline{44.51\text{ V}}}$$

$$\boldsymbol{E} = 0\mathbf{a}_x + (-115.12 + 19.127 - 3.9411 + 0.8192 - 0.1703 + 0.035 - 0.0074 + \ldots)\mathbf{a}_y$$
$$= \underline{\underline{-99.25\mathbf{a}_y\text{ V/m}}}$$

(b) At $(x,y) = (3a/2, a/4)$,

$$V = \frac{400}{\pi}(0.1238 + 0.006226 - 0.00383 + 0.0000264 + \ldots) = \underline{\underline{16.50\text{ V}}}$$

$$\boldsymbol{E} = (24.757 - 3.7358 - 0.3834 + 0.0369 + 0.00351 - 0.00033 + \ldots)\mathbf{a}_x$$
$$+(-66.25 - 4.518 + 0.3988 + 0.03722 - 0.00352 - 0.000333 + \ldots)\mathbf{a}_y$$
$$= 20.68\mathbf{a}_x - 70.34\mathbf{a}_y\text{ V/m}$$

**P.E.**

$$V(y=a) = V_o\sin(7\pi x/b) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/b)\sinh(n\pi a/b)$$

By equating coefficients, we notice that $c_n = 0$ for $n \neq 7$. For n=7,

$$V_o\sin(7\pi x/b) = c_7\sin(7\pi x/b)\sinh(7\pi a/b) \quad \longrightarrow \quad c_7 = \frac{V_o}{\sinh(7\pi a/b)}$$

Hence

$$\underline{\underline{V(x,y) = \frac{V_o}{\sinh(7\pi a/b)}\sin(7\pi x/b)\sinh(7\pi y/b)}}$$

**P.E.**

Let $V(r,\theta,\phi) = R(r)F(\theta)\Phi(\phi)$.

Substituting this in Laplace's equation gives

$$\frac{\Phi F}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right)+\frac{RF}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0$$

Dividing by $RF\Phi\,/\,r^2\sin^2\theta$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta F')=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\lambda^2$$

$$\underline{\underline{\Phi''+\lambda^2\Phi=0}}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F')=\lambda^2/\sin^2\theta$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right)=\frac{\lambda^2}{\sin^2\theta}-\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F')=\mu^2$$

$$2rR'+r^2R''=\mu^2R$$

or

$$\underline{\underline{R''+\frac{2}{r}R'-\frac{\mu^2}{r^2}R=0}}$$

$$\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta F')-\lambda^2+\mu^2\sin^2\theta=0$$

or

$$F''+\cos\theta F'+(\mu^2\sin\theta-\lambda^2\csc\theta)F=0$$

**P.E.**

(a) This is similar to Example 5.8a except that here $0 < \phi < 2\pi$ instead of $0 < \phi < \pi / 2$. Hence

$$I = \frac{2\pi t V_o \sigma}{\ln(b/a)} \quad \text{and} \quad R = \frac{V_o}{I} = \underline{\underline{\frac{\ln\frac{b}{a}}{2\pi t\sigma}}}$$

(b) This to similar to Example 6.8(b) except that here $0 < \phi < 2\pi$. Hence

$$I = \frac{V_o \sigma}{t}\int_a^b \int_0^{2\pi} \rho\, d\rho\, d\phi = \frac{V_o \sigma \pi (b^2 - a^2)}{t}$$

and $R = \frac{V_o}{I} = \underline{\underline{\frac{t}{\sigma\pi(b^2 - a^2)}}}$

**P.E.**

$$J_1 = \frac{\sigma_1 V_o}{\rho \ln\frac{b}{a}}, \qquad J_2 = \frac{\sigma_2 V_o}{\rho \ln\frac{b}{a}}$$

$$I = \int J \bullet dS = \int_{z=0}^{L}\left[\int_{\phi=0}^{\pi} J_1 \rho\, d\phi + \int_{\phi=\pi}^{2\pi} J_2 \rho\, d\phi\right] dz = \frac{V_o l}{\ln\frac{b}{a}}\left[\pi\sigma_1 + \pi\sigma_2\right]$$

$$R = \frac{V_o}{I} = \frac{\ln\frac{b}{a}}{\pi l\left[\sigma_1 + \sigma_2\right]}$$

**P.E.**

(a) $C = \dfrac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$, $C_1$ and $C_2$ are in series.

$$C_1 = 4\pi x \frac{10^{-9}}{36\pi}\left(\frac{2.5}{\dfrac{10^3}{2} - \dfrac{10^3}{3}}\right) = 5/3 \text{ pF}, \qquad C_2 = 4\pi x \frac{10^{-9}}{36\pi}\left(\frac{3.5}{\dfrac{10^3}{1} - \dfrac{10^3}{2}}\right) = 7/9 \text{ pF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(5/3)(7/9)}{(5/3) + (7/9)} = \underline{\underline{0.53 \text{ pF}}}$$

(b) $C = \dfrac{2\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$, $C_1$ and $C_2$ are in parallel.

$$C_1 = 2\pi x \frac{10^{-9}}{36\pi}\left(\frac{2.5}{\dfrac{10^3}{1} - \dfrac{10^3}{3}}\right) = 5/24 \text{ pF}, \qquad C_2 = 2\pi x \frac{10^{-9}}{36\pi}\left(\frac{3.5}{\dfrac{10^3}{1} - \dfrac{10^3}{3}}\right) = 7/24 \text{ pF}$$

$$C = C_1 + C_2 = \underline{\underline{0.5 \text{ pF}}}$$

**P.E.**

As in Example     assuming $'(\rho = a) = 0$, $V(\rho = b) = V_o$,

$$V = V_o \frac{\ln \rho / a}{\ln b / a}, \qquad E = -\nabla V = -\frac{V_o}{\rho \ln b / a} a_\rho$$

$$Q = \int \varepsilon E \bullet dS = \frac{V_o \varepsilon}{\ln b / a} \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \frac{1}{\rho} dz \rho d\phi = \frac{V_o 2\pi\varepsilon L}{\ln b / a}$$

$$C = \frac{Q}{V_o} = \underline{\underline{\frac{2\pi\varepsilon L}{\ln b / a}}}$$

**P.E.**

(a) Let $C_1$ and $C_2$ be capacitances per unit length of each section and $C_T$ be the total capacitance of 10m length. $C_1$ and $C_2$ are in series.

$$C_1 = \frac{2\pi\varepsilon_{r1}\varepsilon_o}{\ln b/c} = \frac{2\pi x2.5}{\ln 3/2}\frac{10^{-9}}{36\pi} = 342.54 \text{ pF/m,}$$

$$C_2 = \frac{2\pi\varepsilon_{r2}\varepsilon_o}{\ln c/a} = \frac{2\pi x3.5}{\ln 2}\frac{10^{-9}}{36\pi} = 280.52 \text{ pF/m}$$

$$C = \frac{C_1C_2}{C_1+C_2} = \frac{342.54x280.52}{342.54+280.52} = 154.22 \text{ pF}$$

$$C_T = Cl = \underline{\underline{1.54}} \text{ nF}$$

(b) $C_1$ and $C_2$ are in parallel.

$$C = C_1 + C_2 = \frac{\pi\varepsilon_{r1}\varepsilon_o}{\ln b/a} + \frac{\pi\varepsilon_{r2}\varepsilon_o}{\ln b/a} = \frac{\pi(\varepsilon_{r1}+\varepsilon_{r2})\varepsilon_o}{\ln b/a} = \frac{6\pi}{\ln 3}\frac{10^{-9}}{36\pi} = 151.7 \text{ pF/m}$$

$$C_T = Cl = \underline{\underline{1.52}} \text{ nF}$$

**P.E.**

Instead of Eq. ( we now have

$$V = -\int_b^a \frac{Qdr}{4\pi\varepsilon r^2} = -\int_b^a \frac{Qdr}{4\pi\frac{10\varepsilon_o}{r}r^2} = -\frac{Q}{40\pi\varepsilon_o}\ln b/a$$

$$C = \frac{Q}{|V|} = \frac{40\pi}{\ln 4/1.5}\frac{10^{-9}}{36\pi} = \underline{\underline{1.13 \text{ nF}}}$$

## CHAPTER —Problems

**Prob.**

(a)

$$E = -\nabla V = -\left( \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial z} a_y + \frac{\partial V}{\partial z} a_z \right)$$

$$= -(15x^2y^2za_x + 10x^3yza_y + 5x^3y^2a_z)$$

At P, x=-3, y=1, z=2,

$$E = -15(9)(1)(2)a_x + 10(-27)(1)(2)a_y - 5(-27)(1)a_z = \underline{\underline{-270a_x + 540a_y + 135a_z \text{ V/m}}}$$

(b) $\rho_v = \nabla \bullet D$ or $\rho_v = -\varepsilon \nabla^2 V$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial x}(15x^2y^2z) + \frac{\partial}{\partial y}(10x^3yz) + \frac{\partial}{\partial z}(5x^3y^2)$$

$$= 30xy^2z + 10x^3z$$

At P,

$$\rho_v = -\varepsilon \nabla^2 V = -2.25 \times \frac{10^{-9}}{36\pi}\left[30(-3)(1)(2) + 10(-27)(2)\right] = \underline{\underline{14.324 \text{ nC/m}^3}}$$

**Prob.**

If $V'' = f$

$$V' = \int_0^x f(x)\,dx + c_1$$

$$V = \int_0^x \int_0^\lambda f(\mu)\,d\mu\,d\lambda + c_1 x + c_2$$

$$V(x=0) = V_1 = c_2 \longrightarrow c_2 = V_1$$

$$V(x=L) = V_2 = \int_0^L \int_0^\lambda f(\mu)\,d\mu\,d\lambda + c_1 L + c_2$$

$$c_1 = \frac{1}{L}\left[ V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu)\,d\mu\,d\lambda \right]$$

Thus,

$$V = \frac{x}{L}\left[ V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu)\,d\mu\,d\lambda \right] + V_1$$

$$+ \int_0^x \int_0^\lambda f(\mu)\,d\mu\,d\lambda$$

**Prob.**

$$E = -\nabla V = -\frac{\partial V}{\partial x}a_x - \frac{\partial V}{\partial y}a_y - \frac{\partial V}{\partial z}a_z = -6y^2za_x - 12xyza_y - 6xy^2a_z$$

At P(1,2,-5),

$$E = \underline{\underline{120a_x + 120a_y - 12a_z}} \text{ V/m}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 + 12xz + 0$$

$$\rho_v = -\varepsilon_o \nabla^2 V = -12xz\varepsilon_o$$

At P,

$$\rho_v = 60\varepsilon_o = 60x\frac{10^{-9}}{36\pi} = 530.5 \text{ pC/m}^3$$

**Prob.**

(a) $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) = -\frac{\rho_v}{\varepsilon} = \frac{-10\times10^{-12}}{\rho\times\frac{10^{-9}}{36\pi}}$

$\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = -1.131$

$\rho\frac{dV}{d\rho} = -1.131\rho + A \longrightarrow \frac{dV}{d\rho} = -1.131 + \frac{A}{\rho}$

$V = -1.131\rho + A\ln\rho + B$

If $V = 0$ at $\rho = 1$, $0 = -1.131 + B \longrightarrow B = 1.131$

If $V = 100$ at $\rho = 4$, $100 = -1.131\times4 + A\ln 4 + 1.131$

$\longrightarrow A = 74.58$

Hence,

$V = -1.131\rho + 74.58\ln\rho + 1.131$

$V(\rho=3) = \underline{79.67}$ V

(b) $\vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\vec{a}_\rho = \left(1.131 - \frac{A}{\rho}\right)\vec{a}_\rho$

$\vec{E}(\rho=2) = \underline{\underline{-36.14\,\vec{a}_\rho \text{ V/m.}}}$

**Prob.**

**(a)**

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial x}\left(-5e^{-5x}\cos 13y \sinh 12z\right) + \ldots = 25V - 169V + 144V = 0$$

**(b)** $$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(-\frac{z\cos\phi}{\rho}\right) - \frac{z\cos\phi}{\rho} + 0 = \frac{z\cos\phi}{\rho^3} - \frac{z\cos\phi}{\rho^3} = 0$$

(c ) $V = 30r^{-2}\cos\theta,$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(-60r^{-1}\cos\theta) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(-\sin\theta 30r^{-2}\sin\theta) = \frac{60}{r^2}\cos\theta - \frac{30}{r^4\sin\theta}(2\sin\theta\cos\theta) = 0$$

**Prob.**

(a) $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 30yz - 30yz + 0 = 0$$

(b) $$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \cancelto{0}{\frac{\partial^2 V}{\partial z^2}}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(-\frac{\cos\phi}{\rho}\right) + \frac{1}{\rho^2}\left(-\frac{\cos\phi}{\rho}\right)$$

$$= \frac{\cos\phi}{\rho^3} - \frac{\cos\phi}{\rho^3} = 0$$

(c) $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= \frac{10\sin\theta\sin\phi}{r^2}\frac{\partial}{\partial r}\left(-\frac{2}{r}\right) + \frac{10\sin\phi}{r^4\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$-\frac{10}{r^4\sin^2\theta}\sin\theta\sin\phi$$

$$= \frac{20\sin\theta\sin\phi}{r^4} + \frac{10\sin\phi(1-2\sin^2\theta)}{r^4\sin\theta} - \frac{10\sin\phi}{r^4\sin\theta} = 0$$

**Prob.**

$$\frac{\partial V}{\partial x} = (-An\sin nx + Bn\cos nx)(Ce^{ny} + De^{-ny})$$

$$\frac{\partial^2 V}{\partial x^2} = (-An^2\cos nx - n^2 B\sin nx)(Ce^{ny} + De^{-ny}) = -n^2 V$$

$$\frac{\partial V}{\partial y} = (A\cos nx + B\sin nx)(nCe^{ny} - nDe^{-ny})$$

$$\frac{\partial^2 V}{\partial y^2} = n^2 V$$

Thus

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -n^2 V + n^2 V = \underline{\underline{0}}$$

**Prob.**

**(a)**

$$\frac{\partial V}{\partial x} = 4xyz, \qquad \frac{\partial^2 V}{\partial x^2} = 4yz$$

$$\frac{\partial V}{\partial y} = 2x^2 z - 3y^2 z, \qquad \frac{\partial^2 V}{\partial y^2} = -6yz$$

$$\frac{\partial V}{\partial z} = 2x^2 y - y^3, \qquad \frac{\partial^2 V}{\partial z^2} = 0$$

$$\nabla^2 V = 4yz - 6yz + 0 = -2yz$$

$\nabla^2 V \neq 0$, V does not satisfy Laplace's equation.

(b)

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = -2yz \quad \longrightarrow \quad \rho_v = 2yz\varepsilon$$

$$Q = \int \rho_v dv = \int_0^1\int_0^1\int_0^1 (2yz\varepsilon)dxdydz = 2\varepsilon(1)\frac{y^2}{2}\Big|_0^1 \frac{z^2}{2}\Big|_0^1 = \varepsilon/2 = 2\varepsilon_o/2 = \varepsilon_o$$

$$\underline{\underline{Q = 8.854 \text{ pC}}}$$

**Prob.**

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \longrightarrow V = -\frac{A}{r} + B.$$

$$V(r=0.1) = 0 \longrightarrow 0 = -\frac{A}{0.1} + B \rightarrow B = 10A$$

$$V(r=2) = 100 \longrightarrow 100 = -\frac{A}{2} + B = \frac{19}{2}A$$

$$\text{or} \quad A = \frac{200}{19} \qquad B = \frac{2000}{19}$$

$$\underline{\underline{V = -\frac{10.53}{r} + 105.3 \ V.}}$$

$$\vec{E} = -\nabla V = -\frac{dV}{dr}\vec{a}_r = \underline{\underline{-\frac{10.53}{r^2}\vec{a}_r}} \ V/m.$$

$$\vec{D} = \epsilon_o \vec{E} = \underline{\underline{-\frac{9.32\times10^{-11}}{r^2}\vec{a}_r}} \ C/m^2$$

**Prob.**

$$\nabla^2 V = 0 \longrightarrow V = -\frac{A}{r} + B$$

At $r=0.5$, $V=-50 \longrightarrow -50 = -\frac{A}{0.5} + B$

$$\text{or} \quad -50 = -2A + B \quad —(1)$$

At $r=1$, $V=50 \longrightarrow 50 = -A + B \quad -(2)$

from (1) and (2), $A=100$, $B=150$. Hence

$$\underline{\underline{V = -\frac{100}{r} + 150}}$$

$$\vec{E} = -\nabla V = -\frac{A}{r^2}\vec{a}_r = \underline{-\frac{100}{r^2}}\vec{a}_r \ V/m.$$

**Prob.**

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0 \quad \longrightarrow \quad V = A\ln\rho + B$$

Let $\rho$ be in cm.

$$V(\rho = 2) = 60 \quad \longrightarrow \quad 60 = A\ln 2 + B$$

$$V(\rho = 6) = -20 \quad \longrightarrow \quad -20 = A\ln 6 + B$$

Thus, A = -72.82, B = 110.47, and

$$V = 110.47 - 72.82\ln\rho$$

$$E = -\frac{dV}{d\rho}a_\rho = -\frac{A}{\rho}a_\rho = \frac{72.82}{\rho}a_\rho, \qquad D = \varepsilon_o E$$

At $\rho$=4, $\underline{\underline{V = 9.52\ \text{V}}}$, $\underline{\underline{E = 18.21\ a_\rho\ \text{V/m}}}$

$$D = \varepsilon_o E = \frac{10^{-9}}{36\pi}x18.21a_\rho = 0.161a_\rho\ \text{nC/m}^2$$

**Prob.**

**(a)**

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0 \quad \longrightarrow \quad V = A\ln\rho + B$$

$$V(\rho = b) = 0 \quad \longrightarrow \quad 0 = A\ln b + B \quad \longrightarrow \quad B = -A\ln b$$

$$V(\rho = a) = V_o \quad \longrightarrow \quad V_o = A\ln a/b \quad \longrightarrow \quad A = -\frac{V_o}{\ln b/a}$$

$$V = -\frac{V_o}{\ln b/a}\ln\rho/b = \frac{V_o\ln b/\rho}{\ln b/a}$$

$$V(\rho = 15\text{mm}) = 70\frac{\ln 2}{\ln 50} = \underline{\underline{12.4\ \text{V}}}$$

(b) As the electron decelerates, potential energy gained = K.E. loss

$$e[70-12.4]=\frac{1}{2}m[(10^7)^2-u^2] \quad \longrightarrow \quad 10^{14}-u^2=\frac{2e}{m}\times 57.6$$

$$u^2=10^{14}-\frac{2\times 1.6\times 10^{-19}}{9.1\times 10^{-31}}\times 57.6=10^{12}(100-20.25)$$

$$\underline{\underline{u=8.93\times 10^6 \text{ m/s}}}$$

**Prob.**

(a) $$X(x)=A\sin(n\pi x/b)$$

For Y,

$$Y(y)=c_1\cosh(n\pi y/b)+c_2\sinh(n\pi y/b)$$

$$Y(a)=0 \quad \longrightarrow \quad 0=c_1\cosh(n\pi a/b)+c_2\sinh(n\pi a/b) \quad \longrightarrow \quad c_1=-c_2\tanh(n\pi a/b)$$

$$V=\sum_{n=1}^{\infty} a_n \sin(n\pi x/b)\left[\sinh(n\pi y/b)-\tanh(n\pi a/b)\cosh(n\pi y/b)\right]$$

$$V(x,y=0)=V_o=-\sum_{n=1}^{\infty} a_n \tanh(n\pi a/b)\sin(n\pi x/b)$$

$$-a_n\tanh(n\pi a/b)=\frac{2}{b}\int_0^b V_o\sin(n\pi x/b)dx=\begin{cases}\frac{4V_o}{n\pi}, & n=\text{odd}\\ 0, & n=\text{even}\end{cases}$$

Hence,

$$V=-\frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\sin(n\pi x/b)\left[\frac{\sinh(n\pi y/b)}{n\tanh(n\pi a/b)}-\frac{\cosh(n\pi y/b)}{n}\right]$$

$$=-\frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[\sinh(n\pi y/b)\cosh(n\pi a/b)-\cosh(n\pi y/b)\sinh(n\pi a/b)\right]$$

$$\underline{\underline{=\frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}}}$$

Alternatively, for Y

$$Y(y) = c_1 \sinh n\pi(y - c_2)/b$$

$$Y(a) = 0 \longrightarrow 0 = c_1 \sinh[n\pi(a - c_2)/b] \longrightarrow c_2 = a$$

$$V = \sum_{n=1}^{\infty} b_n \sin(n\pi x/b) \sinh[n\pi(y-a)/b]$$

where

$$b_n = \begin{cases} -\dfrac{4V_o}{n\pi \sinh(n\pi a/b)}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

(b) This is the same as **Ex.** except that we exchange y and x. Hence

$$\underline{\underline{V(x,y) = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{\sin(n\pi y/a) \sinh(n\pi x/a)]}{n \sinh(n\pi b/a)}}}$$

(c) This is the same as part (a) except that we must exchange x and y. Hence

$$\underline{\underline{V(x,y) = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{\sin(n\pi y/a) \sinh[n\pi(b-x)/a]}{n \sinh(n\pi b/a)}}}$$

**Prob.**

(a) **X(x) is the same as in Example Hence**

$$V(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x/b)\left[a_n \sinh(n\pi y/b) + b_n \cosh(n\pi y/b)\right]$$

At y=0, $V = V_1$

$$V_1 = \sum_{n=1}^{\infty} b_n \sin(n\pi x/b) \longrightarrow b_n = \begin{cases} \dfrac{4V_1}{n\pi}, & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

At y=a, $V = V_2$

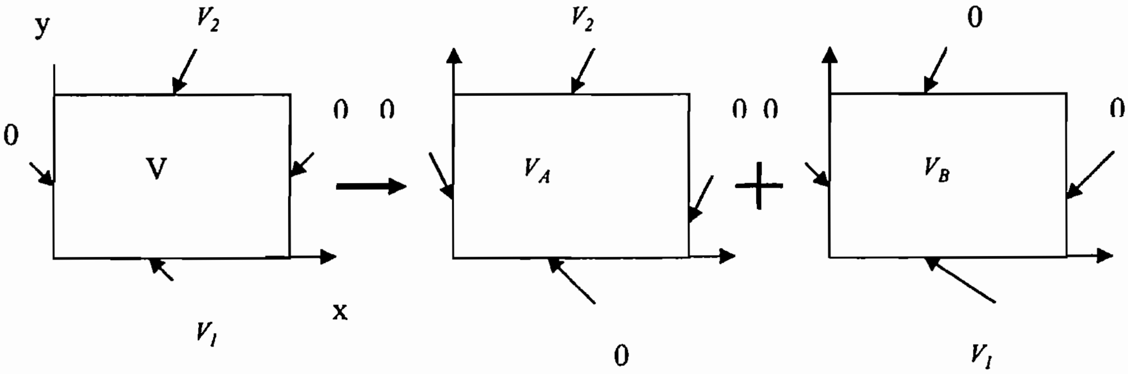
$$V_2 = \sum_{n=1}^{\infty} \sin(n\pi x/b)\left[a_n \sinh(n\pi a/b) + b_n \cosh(n\pi a/b)\right]$$

$$a_n \sinh(n\pi a/b) + b_n \cosh(n\pi a/b) = \begin{cases} \dfrac{4V_2}{n\pi}, & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

or

$$a_n = \begin{cases} \dfrac{4V_2}{n\pi \sinh(n\pi a/b)}\left(V_2 - V_1 \cosh(n\pi a/b)\right), & n = \text{odd} \\ 0, & \text{n = even} \end{cases}$$

Alternatively, we may apply superposition principle.



i.e. $V = V_A + V_B$

$V_A$ is exactly the same as **Example 5.5 with** $V_o = V_2$, while $V_B$ is exactly the same as **Prob. 5.13** (a). Hence

$$V = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[V_1 \sinh[n\pi(a-y)/b] + V_2 \sinh(n\pi y/b)\right]$$

(b)

$$V(x,y) = (a_1 e^{-\alpha x} + a_2 e^{+\alpha x})(a_3 \sin\alpha y + a_4 \cos\alpha y)$$

$$\lim_{x \longrightarrow \infty} V(x,y) = 0 \quad \longrightarrow \quad a_2 = 0$$

$$V(x, y = 0) = 0 \quad \longrightarrow \quad a_4 = 0$$

$$V(x, y = a) = 0 \quad \longrightarrow \quad \alpha = n\pi / a, \quad n = 1,2,3,\ldots$$

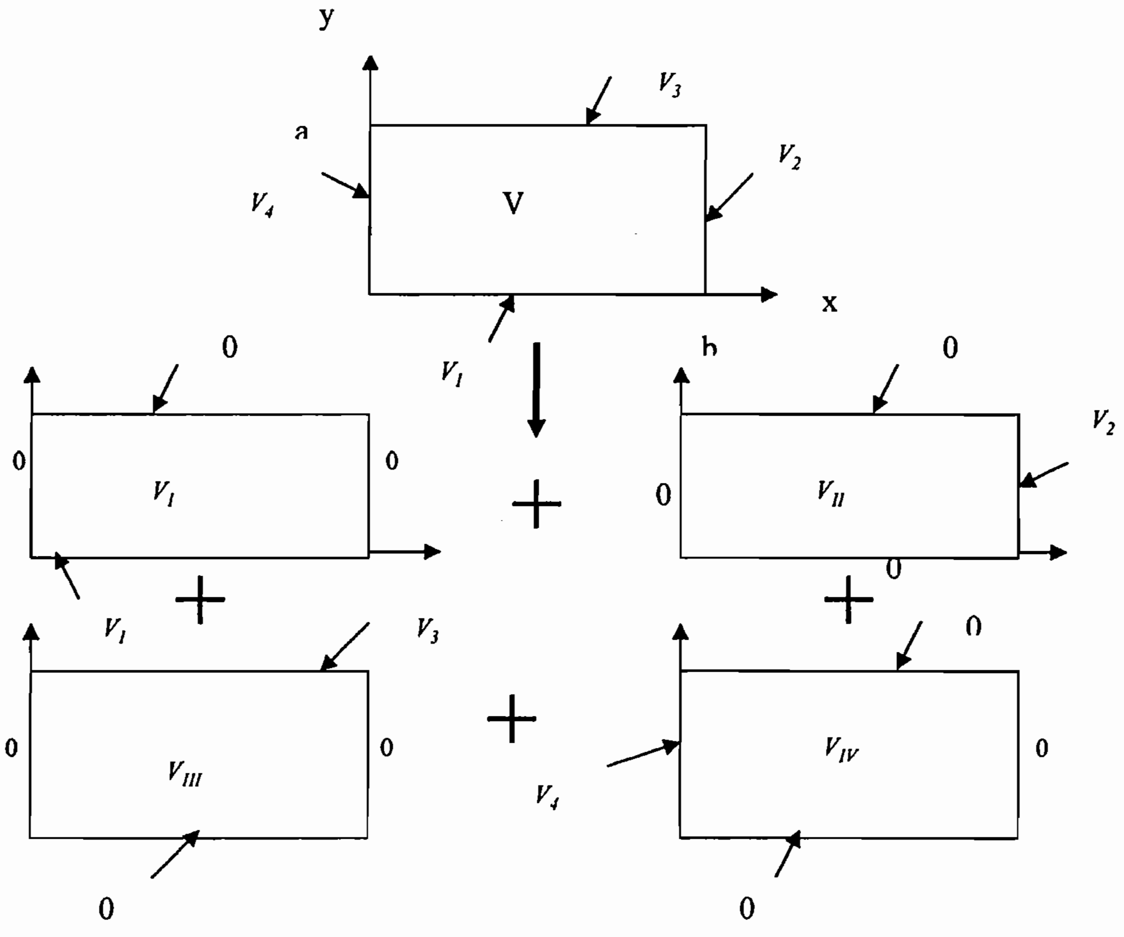
Hence,

$$V(x,y) = \sum_{n=1}^{\infty} a_n e^{-n\pi x/a} \sin(n\pi y / a)$$

$$V(x = 0, y) = V_o = \sum_{n=1}^{\infty} a_n \sin(n\pi y / a) \quad \longrightarrow \quad a_n = \begin{cases} \dfrac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$\underline{V(x,y) = \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y / a)}{n} \exp(-n\pi x / a)}$$

(c ) The problem is easily solved using superposition theorem, as illustrated below.



Therefore,

$$V = V_I + V_{II} + V_{III} + V_{IV}$$

$$= \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\left\{\begin{array}{l}\dfrac{\sin(n\pi x/b)}{\sinh(n\pi a/b)}\left[V_1\sinh(n\pi(a-y)/b)+V_3\sinh(n\pi y/b)\right]\\ +\dfrac{\sin(n\pi x/a)}{\sinh(n\pi b/a)}\left[V_2\sinh(n\pi y/a)+V_4\sinh(n\pi(b-x)/a)\right]\end{array}\right\}$$

where

$$V_I = \frac{4V_1}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}$$

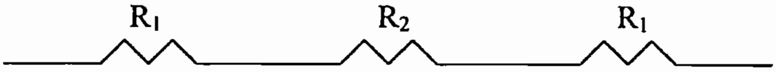
$$V_{II} = \frac{4V_2}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/a)\sinh(n\pi y/a)}{n\sinh(n\pi b/a)}$$

$$V_{III} = \frac{4V_3}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh(n\pi y/b)}{n\sinh(n\pi a/b)}$$

$$V_{IV} = \frac{4V_4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi y/a)\sinh[n\pi(b-x)/a]}{n\sinh(n\pi b/a)}$$

**Prob.**

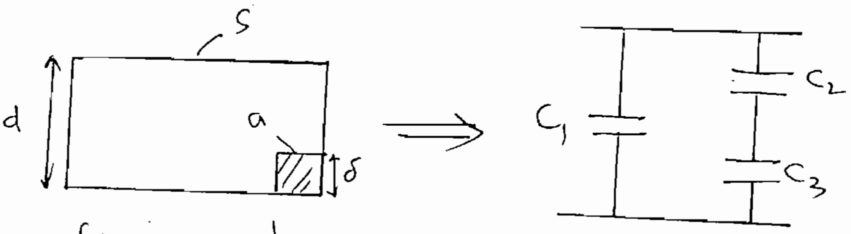
The resistance is calculated horizontally, as shown.



$$R = 2R_1 + R_2 = \frac{1}{\sigma}\left[2\left(\frac{4cm}{0.15cm\times 3cm}\right)+\left(\frac{4cm}{0.15cm\times 1cm}\right)\right]$$

$$= \frac{1}{\sigma}\frac{4}{0.15}\left[\frac{2}{3cm}+\frac{1}{1cm}\right] = \frac{1}{5.8\times10^7}\frac{4}{0.15}100[1+{}^2\!/_3] = \underline{\underline{76.6\ \mu\Omega}}$$

**Prob.**



from the figure above,

$$C = C_1 + \frac{C_2 C_3}{C_2 + C_3}$$

where $C_1 = \frac{\epsilon_o (s-a)}{d}$, $C_2 = \frac{\epsilon_o a}{d-\delta}$, $C_3 = \frac{\epsilon_o \epsilon_r a}{\delta}$

$$C_1 = \frac{10^{-9}}{36\pi} \frac{(2000-9)\times 10^{-6}}{5\times 10^{-3}} = 3.52\,pF.$$

$$C_2 = \frac{10^{-9}}{36\pi} \frac{9\times 10^{-6}}{2\times 10^{-3}} = 0.04\,pF,$$

$$C_3 = \frac{10^{-9}}{36\pi} \frac{4.6\times 9\times 10^{-6}}{3\times 10^{-3}} = 0.122\,pF.$$

Hence, $C = 3.52 + \frac{0.04\times 0.122}{0.162} = \underline{\underline{3.55\,pF}}$

**Prob.**

$$C = \frac{\varepsilon S}{d} \quad \longrightarrow \quad S = \frac{Cd}{\varepsilon_o \varepsilon_r} = \frac{2\times10^{-9}\times10^{-6}}{4\times10^{-9}/36\pi}\ \mathrm{m}^2 = \underline{\underline{0.5655\ \mathrm{cm}^2}}$$

**Prob.**

$$Fdx = dW_E \longrightarrow F = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2}\varepsilon |E|^2\, dv = \frac{1}{2}\varepsilon_o\varepsilon_r E^2 xad + \frac{1}{2}\varepsilon_o E^2 da(l - x)$$

where $E = V_o / d$.

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^2}{d^2}(\varepsilon_r - 1)da \longrightarrow F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^2 a}{2d}$$

Alternatively, $W_E = \frac{1}{2}CV_o^2$, where

$$C = C_1 + C_2 = \frac{\varepsilon_o\varepsilon_r ax}{d} + \frac{\varepsilon_o\varepsilon_r(l - x)}{d}$$

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^2 a}{d}(\varepsilon_r - 1)$$

$$\underline{\underline{F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^2 a}{2d}}}$$

**Prob.**

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

Since $b \longrightarrow \infty$, $C = 4\pi a \varepsilon_o \varepsilon_r$

$$= 4\pi \times 5\times10^{-2} \times 80 \times \frac{10^{-9}}{36\pi}$$

$$= \underline{\underline{444\ \text{pF}}}.$$

**Prob.**

$$C = \frac{2\pi\varepsilon_o\varepsilon_r L}{\ln\frac{b}{a}} = \frac{2\pi \times 3.5 \times 10^{-9} \times 10^{3}}{36\pi \ln\frac{2}{1}} = \frac{3.5\times10^{-6}}{18\ln 2}$$

$$= \underline{\underline{280.5\ \text{nF}/\text{km}}}.$$

**Prob.**

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times \frac{10^{-9}}{36\pi} \times 5.9}{\left(\frac{1}{2} - \frac{1}{5}\right) \cdot 10^{2}} = \frac{5.9}{2.7} \times 10^{-11}$$

$$= \underline{\underline{21.85\ \text{pF}}}$$

**Prob.**

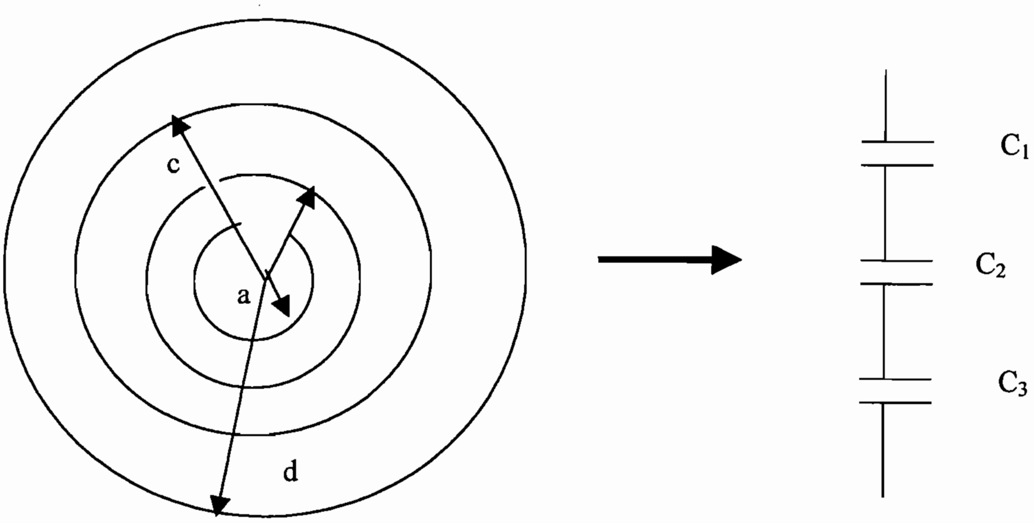
**(a)**

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi x 2.25 x \frac{10^{-9}}{36\pi}}{\frac{1}{5x10^{-2}} - \frac{1}{10x10^{-2}}} = \underline{\underline{25\ \text{pF}}}$$

(b) $Q = C\,V_o = 25x80$ pC

$$\rho_s = \frac{Q}{4\pi r^2} = \frac{25x80}{4\pi x 25x10^{-4}}\ \text{pC/m}^2 = \underline{\underline{63.66\ \text{nC/m}^2}}$$

**Prob.**



$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

where $C_1 = \dfrac{4\pi\varepsilon_1}{\dfrac{1}{c} - \dfrac{1}{d}}$, $\quad C_2 = \dfrac{4\pi\varepsilon_2}{\dfrac{1}{b} - \dfrac{1}{c}}$, $\quad C_3 = \dfrac{4\pi\varepsilon_3}{\dfrac{1}{a} - \dfrac{1}{b}}$,

$$\frac{4\pi}{C} = \frac{1/c - 1/d}{\varepsilon_1} + \frac{1/b - 1/c}{\varepsilon_2} + \frac{1/a - 1/b}{\varepsilon_3}$$

$$C = \frac{4\pi}{\dfrac{\varepsilon_1}{\dfrac{1}{c} - \dfrac{1}{d}} + \dfrac{\varepsilon_2}{\dfrac{1}{b} - \dfrac{1}{c}} + \dfrac{\varepsilon_3}{\dfrac{1}{a} - \dfrac{1}{b}}}$$

**Prob.**

**(a)**

$$C = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}} = \frac{4\pi \times 2.25 \times \dfrac{10^{-9}}{36\pi}}{\dfrac{1}{5x10^{-2}} - \dfrac{1}{10x10^{-2}}} = \underline{\underline{25 \text{ pF}}}$$

(b) $Q = C\,V_o = 25x80$ pC

$$\rho_s = \frac{Q}{4\pi r^2} = \frac{25 \times 80}{4\pi \times 25 \times 10^{-4}} \text{ pC/m}^2 = \underline{\underline{63.66 \text{ nC/m}^2}}$$

**Prob.**

(a)

$$\nabla^2 V = 0 \quad \longrightarrow \quad V = -\frac{A}{r} + B$$

When r=20cm, V=0 $\longrightarrow$ 0 = - A/0.2 + B or B = 5A

When r=30cm, V=50 $\longrightarrow$ 50 = - A/0.3 + 5A or A = 30, B = 150

$$\underline{\underline{V = -\frac{30}{r} + 150 \text{ V}}}$$

$$\boldsymbol{E} = -\nabla V = -\frac{A}{r^2}\boldsymbol{a}_r = \underline{\underline{-\frac{30}{r^2}\boldsymbol{a}_r \text{ V/m}}}$$

$$\boldsymbol{D} = \varepsilon_o \varepsilon_r \boldsymbol{E} = -\frac{30\times 3.1}{r^2}\times\frac{10^{-9}}{36\pi}\boldsymbol{a}_r = \underline{\underline{-\frac{0.8223}{r^2}\boldsymbol{a}_r \text{ nC/m}^2}}$$

(b) $\rho_s = D_n = D \bullet a_n$

On r = 30cm, $\mathbf{a}_n = -\,\mathbf{a}_r$

$$\rho_s = \frac{0.8223}{0.3^2} \text{ nC/m}^2 = \underline{\underline{9.137 \text{ nC/m}^2}}$$

On r = 20cm, $\mathbf{a}_n = +\,\mathbf{a}_r$

$$\rho_s = -\frac{0.8223}{0.2^2} \text{ nC/m}^2 = \underline{\underline{-20.56 \text{ nC/m}^2}}$$

(c)

$$R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma} = \frac{\frac{1}{0.2} - \frac{1}{0.3}}{4\pi\times 10^{-12}} = \underline{\underline{132.6 \text{ G}\Omega}}$$

**Prob.** ________

$$C = 4\pi\epsilon_0 a = 4\pi \times \frac{10^{-9}}{36\pi} \times 6.37 \times 10^{6} = \frac{6.37}{9}\,\text{mF}$$

$$= \underline{\underline{0.7078\ \text{mF}}}$$

**Prob.**

(a)

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{|(6,3,2)|} - \frac{1}{|(6,3,8)|}\right]$$

$$= \frac{10\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{1}{7} - \frac{1}{\sqrt{109}}\right] = \underline{\underline{4.237\,V}}.$$

$$\vec{E} = \frac{10\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{(6,3,2)}{7^3} - \frac{(6,3,8)}{109^{3/2}}\right]$$

$$= \underline{\underline{1.1\vec{a}_x + 0.55\vec{a}_y - 0.108\vec{a}_z\ V/m}}.$$

(b) $$\vec{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2}\vec{a}_r = \frac{-10\times10\times10^{-18}\left[(0,0,-3)-(0,0,3)\right]}{4\pi \times \frac{10^{-9}}{36\pi}\left|(0,0,-3)-(0,0,3)\right|^3}$$

$$= -900\times10^{-9}\,\frac{(0,0,-6)}{6^3} = \underline{\underline{-25\vec{a}_z\ N}}.$$

**Prob.**

$$C = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}} = \frac{4\pi x 5.9x10^{-9}/36\pi}{\left(\frac{1}{2}-\frac{1}{5}\right)x100} = \underline{\underline{21.85\ \text{pF}}}$$

**Prob.**

$$C = \frac{2\pi\varepsilon_o L}{\ln(b/a)} = \frac{2\pi\times\frac{10^{-9}}{36\pi}\times 100\times 10^{-6}}{\ln(600/20)} = 1.633\times 10^{-15}\ \text{F}$$

$$V = Q/C = \frac{50\times 10^{-15}}{1.633\times 10^{-15}} = \underline{\underline{30.62\ \text{V}}}$$

**Prob.**

$V = V_o e^{-t/T_r}$, where $T_r = RC = 10 X 10^{-6} X 100 X 10^{6} = 1000$ s

$$50 = 100e^{-t/T_r} \longrightarrow 2 = e^{t/T_r}$$

$$t = 1000\ln 2 = \underline{\underline{693.1\ \text{s}}}$$

**Prob.**

$$RC = C/G = \varepsilon/\sigma \longrightarrow G = \frac{C\sigma}{\varepsilon}$$

$$\underline{\underline{G = \frac{\pi\sigma}{\cosh^{-1}(d/2a)}}}$$

**Prob.**

(a) $$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho_L}{2\pi\epsilon_0}\left(\frac{\vec{a}_{\rho_1}}{\rho_1} - \frac{\vec{a}_{\rho_2}}{\rho_2}\right)$$

$$= \frac{20\times10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{(1,2,3)-(1,1,5)}{1 \downarrow 1^2} - \frac{(1,2,3)-(1,1,\overset{5}{\cancel{-4}})}{1 \downarrow 1^2}\right]$$

$$= 360\left[\frac{(0,1,-2)}{5} - \frac{(0,1,8)}{65}\right] = \underline{\underline{66.46\,\vec{a}_y - 188.3\,\vec{a}_z \text{ V/m}}}$$

(b) Since $\vec{E}$ does not exist below plane $z=0$,

$$\vec{E} = \underline{\underline{0}}.$$

**Prob.**

$$\vec{E} = \overset{1}{\sum}\frac{\rho_s}{2\epsilon_0}\vec{a}_n = \frac{10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\left[+5\vec{a}_x + 10\vec{a}_x - 10\vec{a}_x + 5\vec{a}_x\right]$$

$$= +180\pi\,\vec{a}_x = \underline{\underline{+565.5\,\vec{a}_x}} \text{ V/m}.$$

**Prob.**

; (a) <u>Method 1</u>: $\boldsymbol{E} = \frac{\rho_s}{\varepsilon}(-\boldsymbol{a}_x)$, where $\rho_s$ is to be determined.

$$V_o = -\int \boldsymbol{E} \bullet d\boldsymbol{l} = -\int \frac{-\rho_s}{\varepsilon} dx = \rho_s \int_0^d \frac{1}{\varepsilon_o} \frac{d}{d+x} dx = \frac{\rho_s}{\varepsilon} d \ln(x+d) \Big|_0^d$$

$$V_o = \rho_s d \ln \frac{2d}{d} \quad \longrightarrow \quad \rho_s = \frac{V_o \varepsilon_o}{d \ln 2}$$

$$\boldsymbol{E} = -\frac{\rho_s}{\varepsilon} \boldsymbol{a}_x = \underline{\underline{-\frac{V_o}{(x+d)\ln 2} \boldsymbol{a}_x}}$$

<u>Method 2</u>: We solve Laplace's equation

$$\nabla \bullet (\varepsilon \nabla V) = \frac{d}{dx}(\varepsilon \frac{dV}{dx}) = 0 \quad \longrightarrow \quad \varepsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\varepsilon} = \frac{Ad}{\varepsilon_o (x+d)} = \frac{c_1}{x+d}$$

$$V = c_1 \ln(x+d) + c_2$$

$$V(x=0) = 0 \quad \longrightarrow \quad 0 = c_1 \ln d + c_2 \quad \longrightarrow \quad c_2 = -c_1 \ln d$$

$$V(x=d) = V_o \quad \longrightarrow \quad V_o = c_1 \ln 2d - c_1 \ln d = c_1 \ln 2$$

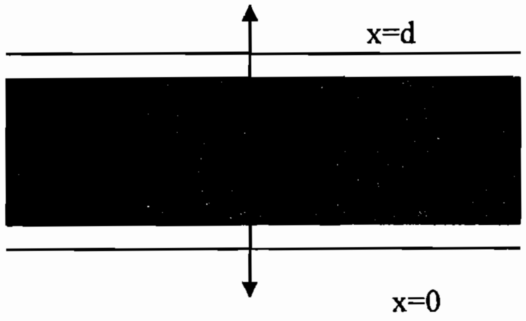
$$c_1 = \frac{V_o}{\ln 2}$$

$$V = c_1 \ln \frac{x+d}{d} = \underline{\underline{\frac{V_o}{\ln 2} \ln \frac{x+d}{d}}}$$

$$\boldsymbol{E} = -\frac{dV}{dx} \boldsymbol{a}_x = \underline{\underline{-\frac{V_o}{(x+d)\ln 2} \boldsymbol{a}_x}}$$

(b) $$\boldsymbol{P} = (\varepsilon_r - 1)\varepsilon_o \boldsymbol{E} = -\left(\frac{x+d}{d} - 1\right) \frac{\varepsilon_o V_o}{(x+d)\ln 2} \boldsymbol{a}_x = \underline{\underline{-\frac{\varepsilon_o x V_o}{d(x+d)\ln 2} \boldsymbol{a}_x}}$$

**(c)**



$$\rho_{ps}|_{x=0} = \mathbf{P}\bullet(-\mathbf{a}_x)|_{x=0} = 0$$

$$\rho_{ps}|_{x=d} = \mathbf{P}\bullet\mathbf{a}_x|_{x=d} = -\frac{\varepsilon_o V_o}{2d\ln 2}$$

**Prob.**

**(a)**

$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{|(6,3,2)|} - \frac{1}{|(6,3,8)|}\right] = \frac{10x10^{-9}}{4\pi x10^{-9}/36\pi}\left[\frac{1}{7} - \frac{1}{\sqrt{109}}\right] = 4.237 \text{ V}$$

$$E = \frac{10x10^{-9}}{4\pi x10^{-9}/36\pi}\left[\frac{(6,3,2)}{7^3} - \frac{(6,3,8)}{109^{3/2}}\right] = \underline{\underline{1.1a_x + 0.55a_y - 0.108a_z \text{ V/m}}}$$

(b)

$$F = \frac{Q_1Q_2}{4\pi\varepsilon_o r^2}a_r = \frac{-10x10x10^{-18}[(0,0,-3)-(0,0,3)]}{4\pi x\frac{10^{-9}}{36\pi}|(0,0,-3)-(0,0,3)|^3} = -900x10^{-9}\frac{(0,0,-6)}{6^3} = \underline{\underline{-25a_z \text{ N}}}$$

**Prob.**

Method 1: Using Gauss's law,

$$Q = \int \mathbf{D} \bullet d\mathbf{S} = 4\pi r^2 D_r \quad \longrightarrow \quad \mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r, \quad \varepsilon = \frac{\varepsilon_o k}{r^2}$$

$$\mathbf{E} = \mathbf{D}/\varepsilon = \frac{Q}{4\pi\varepsilon_o k}\mathbf{a}_r$$

$$V = -\int \mathbf{E} \bullet d\mathbf{l} = -\frac{Q}{4\pi\varepsilon_o k}\int_b^a dr = -\frac{Q}{4\pi\varepsilon_o k}(b-a)$$

$$C = \frac{Q}{|V|} = \underline{\underline{\frac{4\pi\varepsilon_o k}{b-a}}}$$

Method 2: Using the inhomogeneous Laplace's equation,

$$\nabla \bullet (\varepsilon \nabla V) = 0 \quad \longrightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(\frac{\varepsilon_o k}{r^2} r^2 \frac{dV}{dr}\right) = 0$$

$$\varepsilon_o k \frac{dV}{dr} = A' \quad \longrightarrow \quad \frac{dV}{dr} = A \text{ or } V = Ar + B$$

$$V(r=a) = 0 \quad \longrightarrow \quad 0 = Aa + B \quad \longrightarrow \quad B = -Aa$$

$$V(r=b) = V_o \quad \longrightarrow \quad V_o = Ab + B = A(b-a) \quad \longrightarrow \quad A = \frac{V_o}{b-a}$$

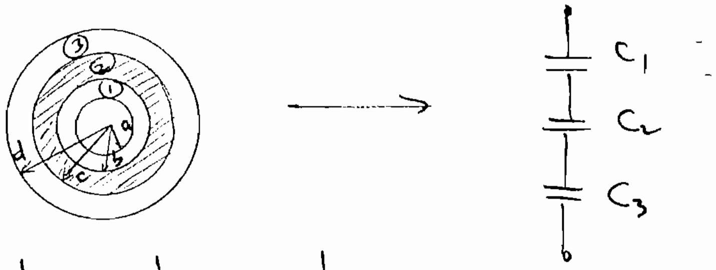
$$\mathbf{E} = -\frac{dV}{dr}\mathbf{a}_r = -A\mathbf{a}_r = -\frac{V_o}{b-a}\mathbf{a}_r$$

$$\rho_s = D_n = -\frac{V_o}{b-a}\frac{\varepsilon_o k}{r^2}\bigg|_{r=a,b}$$

$$Q = \int \rho_s dS = -\frac{V_o \varepsilon_o k}{b-a}\iint \frac{1}{r^2} r^2 \sin\theta \, d\theta \, d\phi = -\frac{V_o \varepsilon_o k}{b-a}4\pi$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{4\pi\varepsilon_o k}{b-a}}}$$

**Prob.**



$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

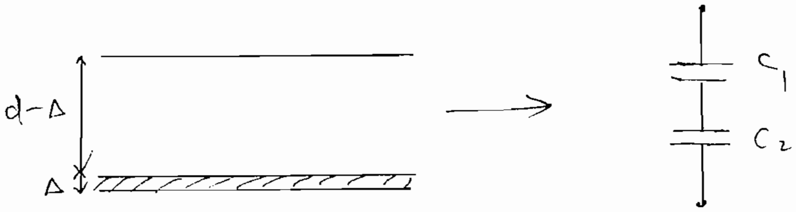
$$C_1 = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}, \quad C_2 = \frac{4\pi\epsilon_0\epsilon_r}{\frac{1}{b} - \frac{1}{c}}, \quad C_3 = \frac{4\pi\epsilon_0}{\frac{1}{c} - \frac{1}{d}}$$

$$\frac{4\pi\epsilon_0}{C} = \frac{1}{a} - \frac{1}{b} + \frac{1}{\epsilon_r}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{c} - \frac{1}{d}$$

$$\frac{4\pi\epsilon_0}{C} = \frac{1}{a} - \frac{1}{d} + \frac{\epsilon_r - 1}{\epsilon_r}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$C = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{d} + \frac{\epsilon_r - 1}{\epsilon_r}\left(\frac{1}{c} - \frac{1}{b}\right)}.$$

**Prob.**



From the figure shown above,

$$C = \frac{C_1 C_2}{C_1 + C_2}, \text{ where } C_1 = \frac{\epsilon_0 S}{d-\Delta}, \quad C_2 = \frac{\epsilon_0 \epsilon_r S}{\Delta}$$

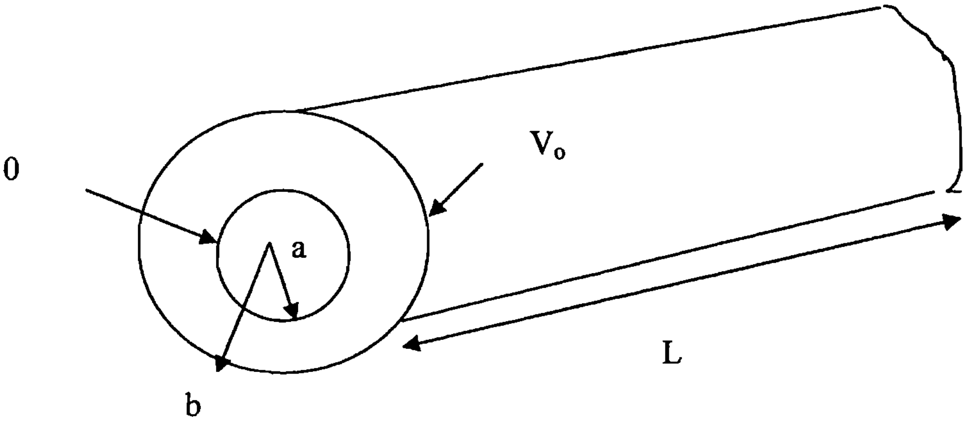
$$C = \frac{\epsilon_0 S \left( \frac{\epsilon_r}{\Delta} \cdot \frac{1}{d-\Delta} \right)}{\frac{\epsilon_r}{\Delta} + \frac{1}{d-\Delta}} = \frac{\epsilon_0 \epsilon_r S}{\epsilon_r (d-\Delta) + \Delta}$$

$$= \frac{\epsilon_r d C_0}{\epsilon_r d + (1-\epsilon_r)\Delta} = \frac{\epsilon_r d C}{\epsilon_r d - \chi_e \Delta}$$

**Prob.**

$$\nabla \bullet \nabla V = 0 = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\varepsilon\frac{dV}{d\rho}\right) = 0$$

$$\frac{d}{d\rho}\left(\rho\frac{\varepsilon_o k}{\rho}\frac{dV}{d\rho}\right) = 0$$



$$\varepsilon_o k \frac{dV}{d\rho} = A' \quad \longrightarrow \quad \frac{dV}{d\rho} = A \ \text{ or } \ \text{V} = \text{A}\rho + \text{B}$$

$$V(\rho = a) = 0 \quad \longrightarrow \quad 0 = Aa + B \quad \longrightarrow \quad B = -Aa$$

$$V(\rho = b) = V_o \quad \longrightarrow \quad V_o = Ab + B = A(b-a) \quad \longrightarrow \quad A = \frac{V_o}{b-a}$$

$$\mathbf{E} = -\frac{dV}{d\rho}\mathbf{a}_\rho = -A\mathbf{a}_\rho = -\frac{V_o}{b-a}\mathbf{a}_\rho$$

$$\rho_s = D_n = \varepsilon E_n$$

On $\rho = b, \quad \mathbf{a}_n = -\mathbf{a}_\rho$

$$\rho_s = \frac{V_o}{b-a}\frac{\varepsilon_o k}{\rho}, \qquad dS = \rho\, d\phi\, dz$$

$$Q = \int \rho_s ds = \iint \frac{V_o}{b-a}\frac{\varepsilon_o k}{\rho} \quad \rho\, d\phi\, dz = 2\pi L \frac{V_o}{b-a}\varepsilon_o k$$

$$C = \frac{Q}{V_o} = \frac{2\pi\varepsilon_o kL}{b-a}$$

$$C' = \frac{C}{L} = \underline{\underline{\frac{2\pi\varepsilon_o k}{b-a}}}$$

Method 2: We use Gauss's law. Assume Q is on the inner conductor and –Q on the outer conductor.

$$\mathbf{D} = \frac{Q}{2\pi\rho L}\mathbf{a}_\rho$$

$$\mathbf{E} = \mathbf{D}/\varepsilon = \frac{Q}{2\pi\varepsilon_o kL}\mathbf{a}_\rho$$

$$V_o = -\int \mathbf{E} \bullet d\mathbf{l} = -\frac{Q}{2\pi\varepsilon_o kL}\int_a^b d\rho = -\frac{Q(b-a)}{2\pi\varepsilon_o kL}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_o kL}{b-a}$$

$$C' = \frac{C}{L} = \underline{\underline{\frac{2\pi\varepsilon_o k}{b-a}}}$$

**Prob.**

Method 1

We can use

$$R = \int \frac{dl}{\sigma S} = \int_a^b \frac{d\rho}{\frac{k}{\rho} 2\pi\rho L} = \frac{1}{2\pi kL} \int_a^b d\rho = \underline{\underline{\frac{b-a}{2\pi kL}}}$$

Method 2: Assume current $I_o$ and find the corresponding V.

$$J = \frac{I_o}{2\pi\rho L} \mathbf{a}_\rho$$

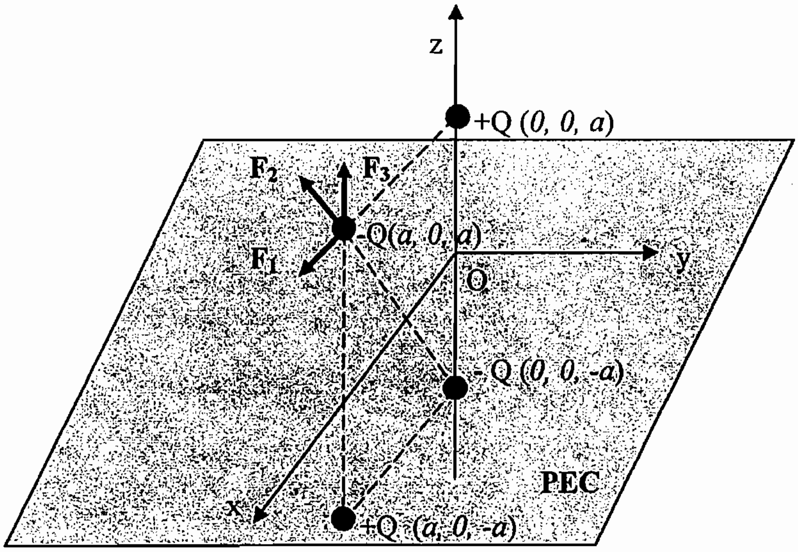
$$E = \frac{J}{\sigma} = \frac{I_o}{2\pi\rho L \frac{k}{\rho}} \mathbf{a}_\rho = \frac{I_o}{2\pi kL} \mathbf{a}_\rho$$

$$V = -\int E \bullet dl = -\int_a^b \frac{I_o}{2\pi kL} d\rho = -\frac{I_o(b-a)}{2\pi kL}$$

$$R = \frac{|V|}{I_o} = \underline{\underline{\frac{b-a}{2\pi kL}}}$$

**Prob.**

The images are shown with proper sign at proper locations. Figure does not show the actual direction of forces but they are expressed a follows:



$$\mathbf{F}_1 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{-\mathbf{a}_x}{a^2}\right]$$

$$\mathbf{F}_2 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{a\mathbf{a}_x + 2a\mathbf{a}_z}{\left(\sqrt{a^2+4a^2}\right)^3}\right]$$

$$\mathbf{F}_3 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{-\mathbf{a}_z}{4a^2}\right]$$

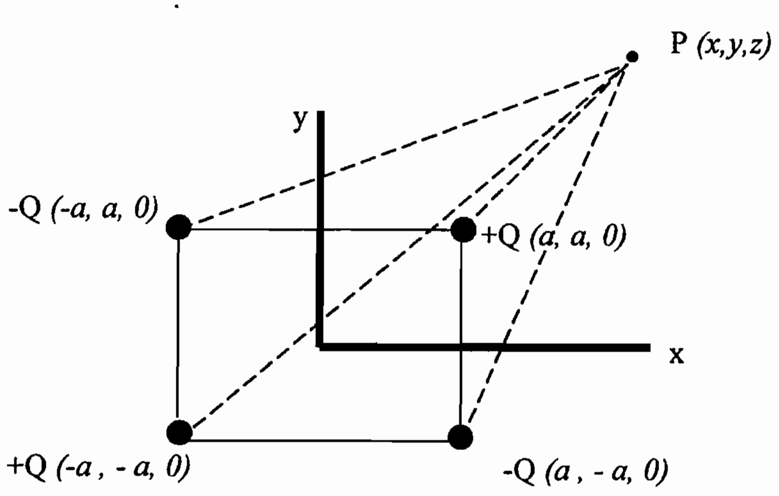
$$\mathbf{F}_{\text{total}} = \frac{Q^2}{4\pi\varepsilon_o a^2}\left[\left(\frac{1}{5\sqrt{5}}-1\right)\mathbf{a}_x + \left(\frac{2}{5\sqrt{5}}-\frac{1}{4}\right)\mathbf{a}_z\right]$$

$$= \frac{Q^2}{4\pi\varepsilon_o a^2}\left[-0.91\mathbf{a}_x \quad -0.071\mathbf{a}_y\right] \text{ N}$$

**Prob.**

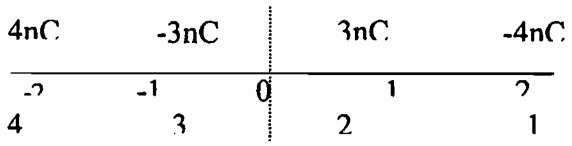
$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right]$$

where $r_1 = \left[(x-a)^2 + (y-a)^2 + z^2\right]^{1/2}$



$$r_2 = \left[(x+a)^2 + (y-a)^2 + z^2\right]^{1/2}$$
$$r_3 = \left[(x+a)^2 + (y+a)^2 + z^2\right]^{1/2}$$
$$r_4 = \left[(x-a)^2 + (y+a)^2 + z^2\right]^{1/2}$$

**Prob.**



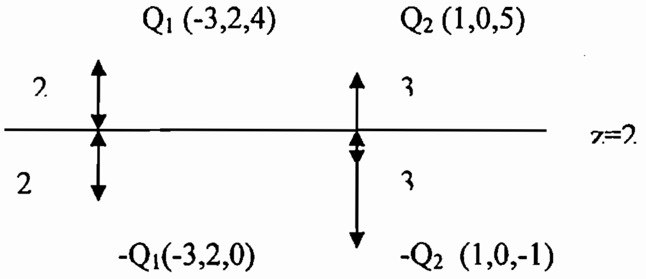
(a) $Q_i$ = -(3nC – 4nC) = <u>1nC</u>

(b) The force of attraction between the charges and the plates is

$$\mathbf{F} = \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{23} + \mathbf{F}_{24}$$

$$|F| = \frac{10^{-18}}{4\pi \times 10^{-9}/36\pi}\left[\frac{9}{2^2} - \frac{2(12)}{3^2} + \frac{16}{4^2}\right] = \underline{\underline{5.25\text{ nN}}}$$

**Prob.**



$Q_1 = 50$ nC, $Q_2 = -20$ nC

For $z \geq 2$,

$$\boldsymbol{D}(x,y,z) = \frac{Q_1}{4\pi}\left[\frac{(x,y,z)-(-3,2,4)}{|(x,y,z)-(-3,2,4)|^3} - \frac{(x,y,z)-(-3,2,0)}{|(x,y,z)-(-3,2,0)|^3}\right]$$

$$+\frac{Q_2}{4\pi}\left[\frac{(x,y,z)-(1,0,5)}{|(x,y,z)-(1,0,5)|^3} - \frac{(x,y,z)-(1,0,-1)}{|(x,y,z)-(1,0,-1)|^3}\right]$$

$$=\frac{50}{4\pi}\left[\frac{(x+3,y-2,z-4)}{|(x+3)^2+(y-2)^2+(z-4)^2|^{3/2}} - \frac{(x+3,y-2,z)}{|(x+3)^2+(y-2)^2+z^2|^{3/2}}\right]$$

$$-\frac{20}{4\pi}\left[\frac{(x-1,y,z-5)}{|(x-1)^2+y^2+(z-5)^2|^{3/2}} - \frac{(x-1,y,z+1)}{|(x-1)^2+y^2+(z+1)^2|^{3/2}}\right] \text{nC/m}^2$$

(a) At (x,y,z) = (7,-2,2),

$$\rho_s = D_z|_{z=2} = \frac{50}{4\pi}\left[\frac{2-4}{(10^2+4^2+2^2)^{3/2}} - \frac{2}{(10^2+4^2+2^2)^{3/2}}\right]$$

$$-\frac{20}{4\pi}\left[\frac{-3}{(6^2+2^2+3^2)^{3/2}} - \frac{3}{(6^2+2^2+3^2)^{3/2}}\right] \text{nC/m}^2$$

$$=\frac{50}{4\pi}\left(\frac{-4}{120^{1.5}}\right) + \frac{120}{4\pi(343)} \text{ nC/m}^2$$

$$\underline{\underline{\rho_s = 15.73 \text{ pC/m}^2}}$$

(b) At (3,4,8)

$$\boldsymbol{D} = \frac{50}{4\pi}\left[\frac{(6,2,4)}{(6^2+2^2+4^2)^{3/2}} - \frac{(6,2,8)}{(6^2+2^2+8^2)^{3/2}}\right]$$

$$-\frac{20}{4\pi}\left[\frac{(2,4,3)}{(2^2+4^2+3^2)^{3/2}}-\frac{(2,4,9)}{(2^2+4^2+9^2)^{3/2}}\right]\text{nC/m}^2$$

$$\underline{\underline{\mathbf{D}=17.21\mathbf{a}_x-23\mathbf{a}_y-8.486\mathbf{a}_z\ \text{ pC/m}^2}}$$

(c) Since (1,1,1) is below the ground plane, $\underline{\underline{\mathbf{D}=0}}$

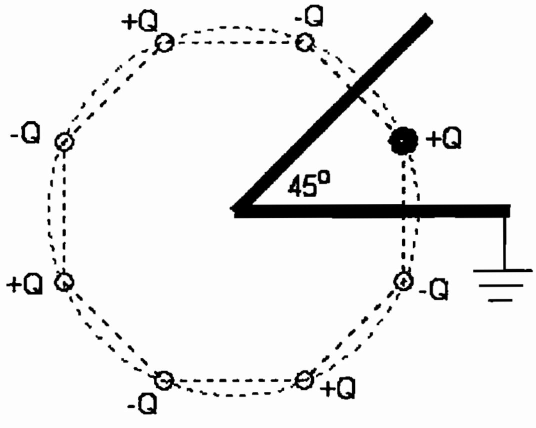
**Prob.**

We have 7 images as follows: -Q at (-1,1,1), -Q at (1,-1,1), -Q at (1,1,-1), -Q at (-1,-1,-1), Q at (1,-1,-1), Q at (-1,-1,1), and Q at (-1,1,-1). Hence,

$$\mathbf{F}=\frac{Q^2}{4\pi\varepsilon_o}\left[\begin{array}{l}-\frac{2}{2^3}\mathbf{a}_x-\frac{2}{2^3}\mathbf{a}_y-\frac{2}{2^3}\mathbf{a}_z-\frac{(2\mathbf{a}_x+2\mathbf{a}_y+2\mathbf{a}_z)}{12^{3/2}}+\frac{(2\mathbf{a}_y+2\mathbf{a}_z)}{8^{3/2}}\\+\frac{(2\mathbf{a}_x+2\mathbf{a}_y)}{8^{3/2}}+\frac{(2\mathbf{a}_x+2\mathbf{a}_z)}{8^{3/2}}\end{array}\right]$$

$$=0.9(\mathbf{a}_x+\mathbf{a}_y+\mathbf{a}_z)\left(-\frac{1}{4}-\frac{1}{12\sqrt{3}}+\frac{1}{4\sqrt{2}}\right)=\underline{\underline{-0.1092(\mathbf{a}_x+\mathbf{a}_y+\mathbf{a}_z)\text{ N}}}$$

**Prob.**



$$N=\left(\frac{360^o}{45^o}-1\right)=\underline{\underline{7}}$$

**Prob.**

**(a)**

$$E = E_+ + E_- = \frac{\rho_L}{2\pi\varepsilon_o}\left(\frac{a_{\rho 1}}{\rho_1} - \frac{a_{\rho 2}}{\rho_2}\right) = \frac{16\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}}\left[\frac{(2,-2,3)-(3,-2,4)}{|(2,-2,3)-(3,-2,4)|^2} - \frac{(2,-2,3)-(3,-2,-4)}{|(2,-2,3)-(3,-2,-4)|^2}\right]$$

$$= 18\text{x}16\left[\frac{(-1,0,-1)}{2} - \frac{(-1,0,7)}{50}\right] = \underline{\underline{-138.2a_x - 184.3a_y \text{ V/m}}}$$

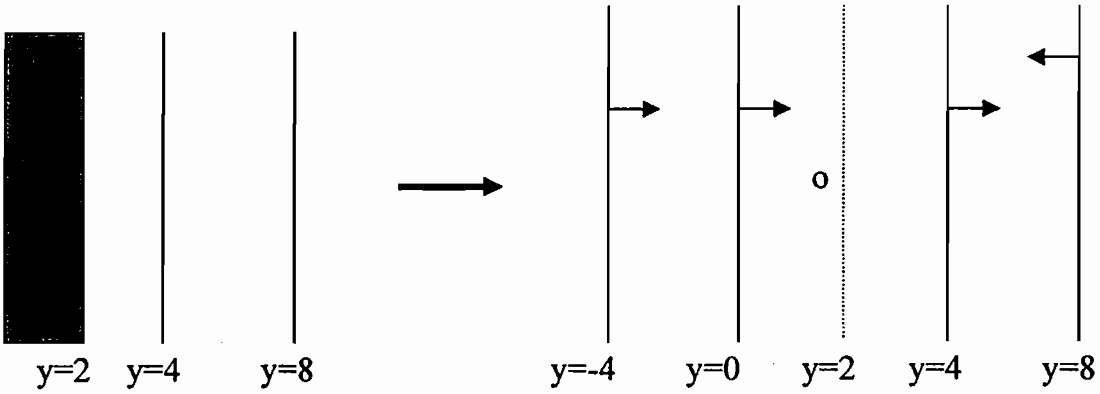
(b) $\rho_s = D_n$

$$D = D_+ + D_- = \frac{\rho_L}{2\pi}\left(\frac{a_{\rho 1}}{\rho_1} - \frac{a_{\rho 2}}{\rho_2}\right) = \frac{16\text{x}10^{-9}}{2\pi}\left[\frac{(5,-6,0)-(3,-6,4)}{|(5,-6,0)-(3,-6,4)|^2} - \frac{(5,-6,0)-(3,-6,-4)}{|(5,-6,0)-(3,-6,-4)|^2}\right]$$

$$= \frac{8}{\pi}\left[\frac{(2,0,-4)}{20} - \frac{(2,0,4)}{20}\right] \text{nC/m}^2 = -1.018a_z \text{ nC/m}^2$$

$$\underline{\underline{\rho_s = -1.018 \text{ nC/m}^2}}$$

**Prob.**



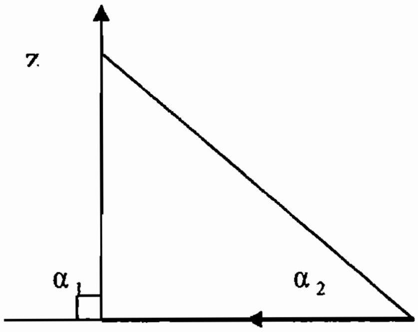
At P(0,0,0), $\underline{E=0}$ since **E** does not exist for y<2.

At Q(-4,6,2), y=6 and

$$E = \sum \frac{\rho_s}{2\varepsilon_o} a_n = \frac{10^{-9}}{2\text{x}10^{-9}/36\pi}(-30a_y + 20a_y - 20a_y - 30a_y) = 18\pi(-60)a_y$$
$$= \underline{\underline{-3.4a_y \text{ kV/m}}}$$

## Chapter —Practice Examples

### P.E.



$$\rho = 5,\ \cos\alpha_1 = 0,\ \ \cos\alpha_2 = \sqrt{\frac{2}{27}}$$

$$a_\phi = a_l \times a_\rho = \left(\frac{-a_x - a_y}{\sqrt{2}}\right) \times a_z = \frac{-a_x + a_y}{\sqrt{2}}$$

$$H_3 = \frac{10}{4\pi(5)}\left(\sqrt{\frac{2}{27}} - 0\right)\left(\frac{-a_x + a_y}{\sqrt{2}}\right) = \underline{\underline{-30.63a_x + 30.63a_y}}\ \ \text{mA/m}$$

### P.E.

(a) $$\boldsymbol{H} = \frac{2}{4\pi(2)}\left(1 + \frac{3}{\sqrt{13}}\right)a_z = \underline{\underline{0.1458\boldsymbol{a}_z}}\ \ \text{A/m}$$

(b) $$\rho = \sqrt{3^2 + 4^2} = 5, \alpha_2 = 0, \cos\alpha_1 = -\frac{12}{13},$$

$$a_\phi = -a_y \times \left(\frac{3a_x - 4a_z}{5}\right) = \frac{4a_x + 3a_z}{5}$$

$$\boldsymbol{H} = \frac{2}{4\pi(5)}\left(1 + \frac{12}{13}\right)\left(\frac{4a_x + 3a_z}{5}\right) = \frac{1}{26\pi}(4a_x + 3a_z)$$

$$= \underline{\underline{48.97a_x + 36.73a_z}}\ \ \text{mA/m}$$

**P.E.**

(a)

$$H = \frac{Ia^2}{2(a^2 + z^2)^{3/2}} a_z$$

At (0,0,-1cm), z = 2cm,

$$H = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 2^2)^{3/2} \times 10^{-6}} a_z \quad \text{A/m}$$

$$= \underline{\underline{400.2a_z \text{ mA/m}}}$$

(b) At (0,0,10cm), z = 9cm,

$$\boldsymbol{H} = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 9^2)^{3/2} \times 10^{-6}} a_z$$

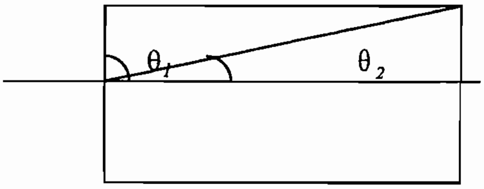
$$= \underline{\underline{57.3a_z \text{ mA/m}}}$$

## P.E.

$$H = \frac{NI}{2L}\left(\cos\theta_2 - \cos\theta_1\right)a_z = \frac{2\times 10^3 \times 50\times 10^{-3}(\cos\theta_2 - \cos\theta_1)a_z}{2\times 0.75}$$

$$= \frac{100}{1.5}\left(\cos\theta_2 - \cos\theta_1\right)\boldsymbol{a}_z$$

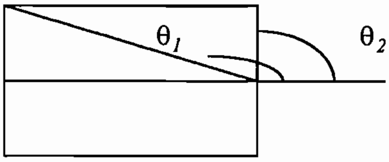
(a) At (0,0,0), $\theta = 90^o$, $\cos\theta_2 = \dfrac{0.75}{\sqrt{0.75^2 + 0.05^2}}$
$= 0.9978$



$$H = \frac{100}{1.5}\left(0.9978 - 0\right)a_z$$

$$= \underline{\underline{66.52\ \mathbf{a}_z\ \text{A/m}}}$$

(b) At (0,0,0.75), $\theta_2 = 90^o, \cos\theta_1 = -0.9978$

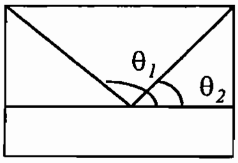


$$\boldsymbol{H} = \frac{100}{1.5}(0 + 0.9978)\boldsymbol{a}_z$$

$$= \underline{\underline{66.52\mathbf{a}_z\ \text{A/m}}}$$

(c) At (0,0,0.5), $\cos\theta_1 = \dfrac{-0.5}{\sqrt{0.5^2 + 0.05^2}} = -0.995$

$$\cos\theta_1 = \frac{0.25}{\sqrt{0.25^2 + 0.05^2}} = 0.9806$$



$$\boldsymbol{H} = \frac{100}{1.5}(0.9806 + 0.995)\boldsymbol{a}_z$$

$$= \underline{\underline{131.7\mathbf{a}_z\ \text{A/m}}}$$

## P.E.

$$\boldsymbol{H} = \frac{1}{2}\boldsymbol{K}\times\boldsymbol{a}_n$$

(a) $H(0,0,0) = \frac{1}{2}50a_z \times (-a_y) = \underline{\underline{25a_x}}$ mA/m

(b) $H(1,5,-3) = \frac{1}{2}50a_z \times a_y = \underline{\underline{-25a_x}}$ mA/m

**P.E.**

$$|H| = \begin{cases} \dfrac{NI}{2\pi\rho}, \rho - a < \rho < \rho + a, \quad 9 < \rho < 11 \\ \quad 0, \qquad\qquad \text{otherwise} \end{cases}$$

(a) At (3,-4,0), $\rho = \sqrt{3^2 + 4^2}$ =5cm ‹ 9cm

$$|H| = \underline{\underline{0}}$$

(b) At (6,9,0), $\rho = \sqrt{6^2 + 9^2} = \sqrt{117}$ ‹ 11

$$|H| = \frac{10^3 \times 100 \times 10^{-3}}{2\pi\sqrt{117} \times 10^2} = \underline{\underline{147.1}} \text{ A/m}$$

**P.E.**

(a) $\boldsymbol{B} = \nabla \times \boldsymbol{A} = (-4xz - 0)\boldsymbol{a}_x + (0 + 4yz)\boldsymbol{a}_y + (y^2 - x^2)\boldsymbol{a}_z$

$$B(-1,2,5) = \underline{\underline{20a_x + 40a_y + 3a_z}} \text{ Wb/m}^2$$

(b) $$\psi = \int B.\partial s = \int_{y=-1}^{4} \int_{x=0}^{1} (y^2 - x^2)\partial x \partial y = \int_{-1}^{4} y^2 \partial y - 5\int_{0}^{1} x^2 \partial x$$

$$= \frac{1}{3}(64 + 1) - \frac{5}{3} = \underline{\underline{20}} \text{ Wb}$$

Alternatively,

$$\psi = \int \mathbf{A}.\partial\mathbf{l} = \int_{0}^{1} \mathbf{x}^2(-1)\partial\mathbf{x} + \int_{-1}^{4} \mathbf{y}^2(1)\partial\mathbf{y} + \int_{1}^{0} \mathbf{x}^2(4)\partial\mathbf{x} + 0$$

$$= -\frac{5}{3} + \frac{65}{3} = \underline{\underline{20}} \text{ Wb}$$

**CHAPTER —Problems**

**Prob.**

Since the element is so small, from Biot-Savart law,

$$\vec{H} = \frac{I dl \times \vec{R}}{4\pi R^3},$$

where $\vec{R} = (3,-6,2) - (0,0,0) = (3,-6,2)$,

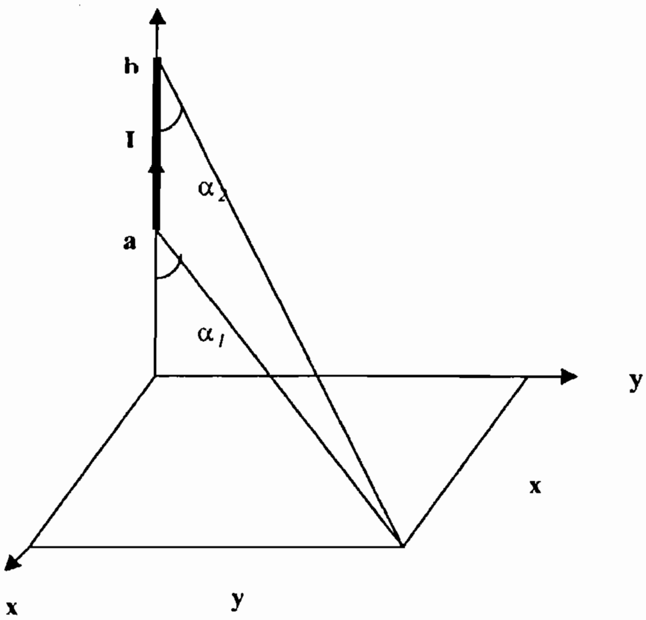
$$R = |\vec{R}| = 7, \quad \vec{dl} = dy\,\vec{a}_y$$

$$\vec{dl} \times \vec{R} = \begin{vmatrix} 0 & dy & 0 \\ 3 & -6 & 2 \end{vmatrix} = 2\,dy\,\vec{a}_x - 3\,dy\,\vec{a}_z$$

$$\vec{H} = \frac{16\cdot10^{-3}(2\vec{a}_x - 3\vec{a}_z)\ 2\times10^{-3}}{4\pi\times7^3}\ A/m$$

$$= \underline{\underline{14.85\,\vec{a}_x - 22.27\,\vec{a}_z\ nA/m}}.$$

**Prob.**



$$H = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)a_\phi$$

$$\rho = \sqrt{x^2 + y^2}, \cos\alpha_1 = \frac{a}{\sqrt{a^2 + \rho^2}}, \cos\alpha_2 = \frac{b}{\sqrt{b^2 + \rho^2}}$$

$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$. Hence,

$$\boldsymbol{H} = \frac{I}{4\pi\sqrt{x^2 + y^2}}\underline{\underline{\left[\frac{b}{\sqrt{x^2 + y^2 + b^2}} - \frac{a}{\sqrt{x^2 + y^2 + a^2}}\right]}}\boldsymbol{a}_\phi$$

**Prob.**

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi$$

$$\rho = 5, \quad \boldsymbol{a}_\phi = \boldsymbol{a}_\ell\, x\boldsymbol{a}_\rho = \boldsymbol{a}_x x\left(\frac{3\boldsymbol{a}_y + 4\boldsymbol{a}_z}{5}\right) = \frac{3}{5}\boldsymbol{a}_z - \frac{4}{5}\boldsymbol{a}_y$$

$$\boldsymbol{H} = \frac{10\times10^{-3}}{2\pi\times5}\left(-\frac{4}{5}\boldsymbol{a}_y + \frac{3}{5}\boldsymbol{a}_z\right) = \underline{\underline{-0.2546\boldsymbol{a}_y + 0.19099\boldsymbol{a}_z \text{ mA/m}}}$$

**Prob.**

Let $\vec{H} = \vec{H}_z + \vec{H}_x$

For $\vec{H}_z = \frac{I}{2\pi\rho}\vec{a}_\phi$, $\rho = \sqrt{6^2 + 8^2} = 10$,

$$\vec{a}_\phi = \vec{a}_z \times \left(\frac{6\vec{a}_y + 8\vec{a}_y}{10}\right)$$

$$H_z = \frac{20}{2\pi(100)}\left(-8\vec{a}_x + 6\vec{a}_y\right) = \frac{1}{\pi}\left(-0.8\vec{a}_x + 0.6\vec{a}_y\right)$$

For $H_x = \frac{I}{2\pi\rho}\vec{a}_\phi$, $\rho = \sqrt{8^2 + 6^2} = 10$

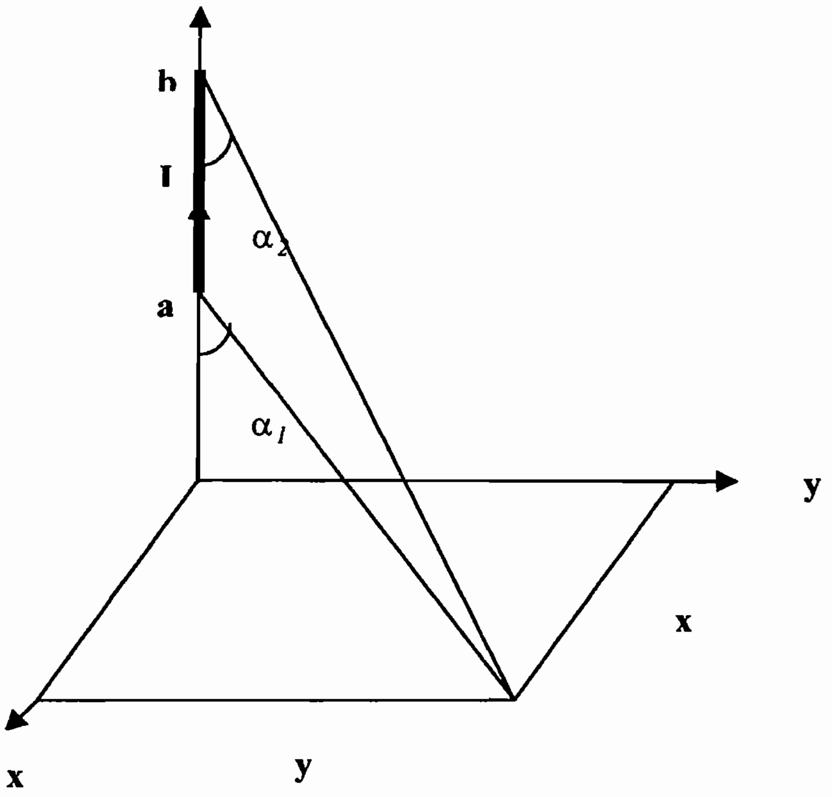
$$\vec{a}_\phi = \vec{a}_x \times \left(\frac{8\vec{a}_y - 6\vec{a}_z}{10}\right)$$

$$\vec{H}_x = \frac{30}{2\pi(100)}\left(6\vec{a}_y + 8\vec{a}_z\right) = \frac{1}{\pi}\left(0.9\vec{a}_y + 1.2\vec{a}_z\right)$$

$$\vec{H} = \vec{H}_x + \vec{H}_z = \frac{1}{\pi}\left(-0.8\vec{a}_x + 1.5\vec{a}_y + 1.2\vec{a}_z\right)$$

$$= \underline{\underline{-0.255\vec{a}_x + 0.4775\vec{a}_y + 0.382\vec{a}_z \ \ A/m.}}$$

**Prob.**



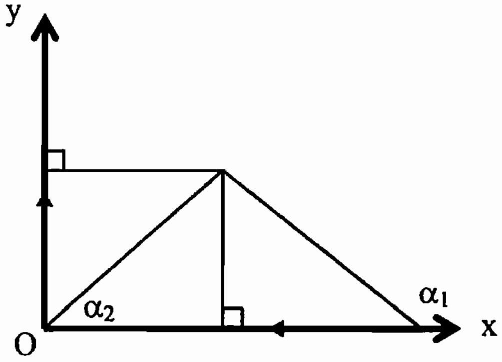
$$H = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)a_\phi$$

$$\rho = \sqrt{x^2 + y^2}, \cos\alpha_1 = \frac{a}{\sqrt{a^2+\rho^2}}, \cos\alpha_2 = \frac{b}{\sqrt{b^2+\rho^2}}$$

$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$. Hence,

$$H = \frac{I}{4\pi\sqrt{x^2+y^2}} \underline{\underline{\left[\frac{b}{\sqrt{x^2+y^2+b^2}} - \frac{a}{\sqrt{x^2+y^2+a^2}}\right]}} a_\phi$$

**Prob.**



(a) Let $\overline{H} = \overline{H}_x + \overline{H}_y = 2\overline{H}_x$

$$\underline{\underline{\overline{H}_x = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\bar{a}_\phi}}$$

where $\bar{a}_\phi = -\bar{a}_x \times \bar{a}_y = -\bar{a}_z$, $\alpha_1 = 180°$, $\alpha_2 = 45°$

$$\bar{H}_x = \frac{5}{4\pi(2)}(\cos 45° - \cos 180°)\,(-\bar{a}_z)$$

$$= \underline{\underline{-0.6792\,\bar{a}_z \text{ A/m}}}$$

(b) $\bar{H} = \bar{H}_x + \bar{H}_y$

where $\bar{H}_x = \frac{5}{4\pi(2)}(1-0)\,\bar{a}_\phi$, $\bar{a}_\phi = -\bar{a}_x \times -\bar{a}_y = \bar{a}_z$

$$= 198.9\,\bar{a}_z \text{ mA/m}$$

$\bar{H}_y = 0$ since $\alpha_1 = \alpha_2 = 0$

$\bar{H} = \underline{\underline{0.1989\,\bar{a}_z \text{ A/m}}}$

(c) $\bar{H} = \bar{H}_x + \bar{H}_y$

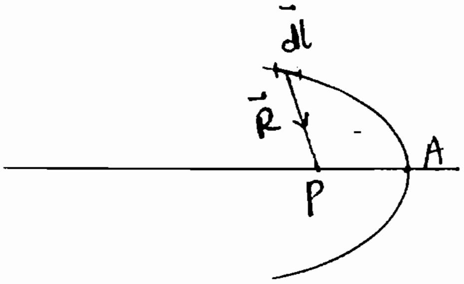
where $\bar{H}_x = \dfrac{5}{4\pi(2)}(1-0)\ (-\bar{a}_x \times \bar{a}_z) = 198.9\,\bar{a}_y$ mA/m

$\bar{H}_y = \dfrac{5}{4\pi(2)}(1-0)\ (\bar{a}_y \times \bar{a}_z) = 198.9\,\bar{a}_x$ mA/m

$\bar{H} = \underline{\underline{0.1989\,\bar{a}_x + 0.1989\,\bar{a}_y \text{ A/m.}}}$

**Prob.**

$$\vec{H} = \int \frac{I\vec{dl} \times \vec{R}}{4\pi R^3}$$



$$\left| \frac{\hat{R} \times \vec{dl}}{R^3} \right| = \frac{1}{r^2}\, r\, d\theta = \frac{d\theta}{r}$$

$$H = \frac{I}{4\pi} \int \frac{d\theta}{r}$$

If P is the focus of the parabola and the apex A is the point $r = d,\ \theta = 0$, then $r(1+\cos\theta) = 2d$. Hence at P,

$$H = \frac{I}{4\pi} \int_{-\pi}^{\pi} \frac{(1+\cos\theta)}{2d}\, d\theta = \frac{i}{8\pi d}(\theta - \sin\theta)\Big|_{-\pi}^{\pi}$$

$$= \frac{I}{4d} = \frac{3}{0{\cdot}1} = \underline{\underline{30 \text{ A/m}}}.$$

**Prob.**

$$\boldsymbol{H} = \frac{\sqrt{2}I}{\pi a}\boldsymbol{a}_z = \frac{\sqrt{2}\times 10\boldsymbol{a}_z}{\pi(1.5x10^{-2})} = \underline{\underline{300.1\boldsymbol{a}_z \text{ A/m}}}$$

**Prob.**

Let $\vec{H} = \vec{H}_L + \vec{H}_s$

where $\vec{H}_L = \frac{I}{2\pi\rho}\vec{a}_\phi$, $I = 50\pi$ mA,

$$\rho = |(3,4,5)-(3,1,4)| = \sqrt{10}$$

$$\vec{a}_\phi = \vec{a}_l \times \vec{a}_\rho = \vec{a}_x \times \left(\frac{3\vec{a}_y + \vec{a}_z}{\sqrt{10}}\right) = \frac{-\vec{a}_y + 3\vec{a}_z}{\sqrt{10}}$$

$$\vec{H}_L = \frac{50\pi}{2\pi \times 10}(-\vec{a}_y + 3\vec{a}_z) = -2.5\vec{a}_y + 7.5\vec{a}_z \text{ mA/m}$$

$$\vec{H}_s = \frac{1}{2}(20\vec{a}_x \times \vec{a}_z) = -10\vec{a}_y \text{ mA/m.}$$

Thus $\vec{H} = \vec{H}_L + \vec{H}_s = \underline{\underline{-12.5\vec{a}_y + 7.5\vec{a}_z \text{ mA/m.}}}$

**Prob.**

(a) See text.

(b) for $0 < \rho < a$, $\oint \vec{H}\cdot d\vec{l} = H_\phi \cdot 2\pi\rho = I \cdot \frac{\pi\rho^2}{\pi \cdot a^2}$

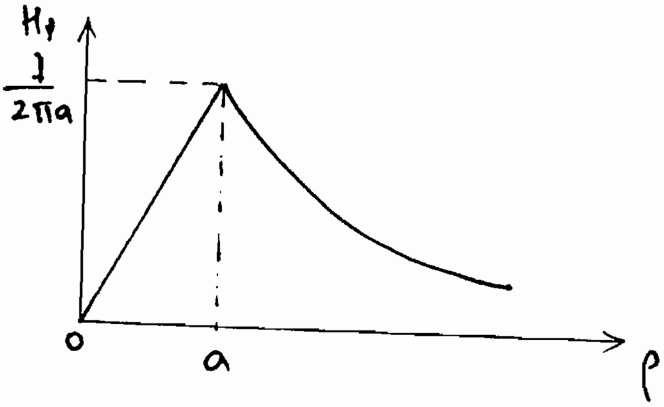
$$H_\phi = \frac{I\rho}{2\pi a^2}.$$

for $\rho > a$, $\oint \vec{H}\cdot d\vec{l} = H_\phi \cdot 2\pi\rho = I$

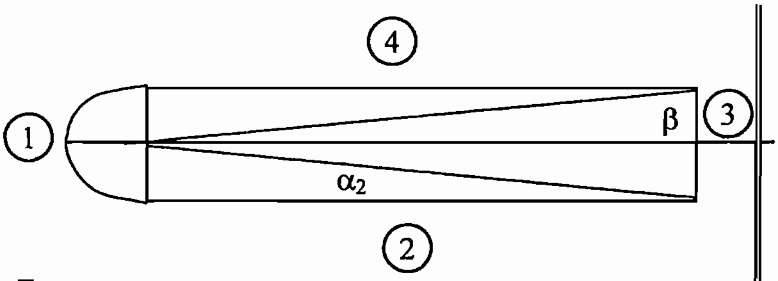
or $H_\phi = \frac{I}{2\pi\rho}$

Thus $$\vec{H} = \begin{cases} \frac{I\rho}{2\pi a^2}\,\bar{a}_\phi, & 0 < \rho < a \\ \frac{I}{2\pi\rho}\,\bar{a}_\phi, & \rho > a \end{cases}$$

$|\vec{H}|$ is sketched below.



**Prob.**



$$Let \quad \bar{H} = \bar{H}_1 + \bar{H}_2 + \bar{H}_3 + \bar{H}_4$$

$$\bar{H}_1 = \frac{I}{4a}\,\bar{a}_z = \frac{10}{4\times 4\times 10^{-2}}\,\bar{a}_z = 62.5\,\bar{a}_z$$

$$\bar{H}_2 = \bar{H}_4 = \frac{I}{4\pi\times 4\times 10^{-2}}\left(\cos\alpha_2 - \cos 90^o\right)\bar{a}_z\,, \quad \alpha_2 = \tan^{-1}\frac{4}{100} = 2.29^\circ$$

$$= 19.88\,\bar{a}_z$$

$$\bar{H}_3 = \frac{I}{4\pi(1)}\,2\cos\beta\,\bar{a}_z\,, \quad \beta = \tan^{-1}\frac{100}{4} = 87.7^\circ$$

$$= \frac{10}{4\pi}\,2\cos 87.7^\circ\,\bar{a}_z = 0.06361\,\bar{a}_z$$

$$\bar{H} = \left(62.5 + 2\times 19.88 + 0.06361\right)\bar{a}_z$$

$$= 102.32\,\bar{a}_z \text{ A/m.}$$

**Prob.**

$\bar{H}$ due to circular loop is

$$\bar{H}_1 = \frac{I\rho^2}{2\left(\rho^2+z^2\right)}\bar{a}_z$$

(a) $$\bar{H}(0,0,0) = \frac{5\times 2^2}{2\left(2^2+0^2\right)^{3/2}}\bar{a}_z + \frac{5\times 2^2}{2\left(2^2+4^2\right)^{3/2}}\bar{a}_z$$

$$= \underline{\underline{1.36\,\bar{a}_z \text{ A/m}}}$$

(b) $$\bar{H}(0,0,2) = 2\frac{5\times 2^2}{2\left(2^2+2^2\right)^{3/2}}\bar{a}_z$$

$$= \underline{\underline{0.884\,\bar{a}_z \text{ A/m}}}$$

**Prob.**

$$\bar{B} = \mu_o\bar{H} = \frac{\mu_o NI}{L}$$

$$N = \frac{Bl}{\mu\cdot I} = \frac{5\times10^{-3}\times3\times10^{-2}}{4\pi\times10^{-7}\times400\times10^{-3}} = 298.4$$

$N \approx$ 300 turns.

**Prob.**

(a) $$\vec{J} = \nabla\times\vec{H} = -2y(x+1)\vec{a}_x + \left(y^2+z^2\right)\vec{a}_y$$

$$\vec{J}(1,0,-3) = \underline{\underline{9\vec{a}_y \text{ A/m}^2}}.$$

(b) $$I = \int\vec{J}\cdot d\vec{S} = \int_{x=0}^{1}\int_{z=0}^{1}\left(1+z^2\right)dx\,dz = (1)\left(z+\frac{z^3}{3}\right)\Big|_0^1 = \frac{4}{3}$$

$I = \underline{\underline{1.333 \text{ A}}}$

**Prob.**

(*a*) $$\bar{J} = \nabla\times\overline{\mathrm{H}} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz(x^2+y^2) & -y^2\times z & -4x^2y^2 \end{vmatrix}$$

$$= (8x^2y+xy^2)\bar{a}_x + \left[y(x^2+y^2)-4xy^2\right]\bar{a}_y$$

$$+ \left[-y^2z - z(x^2+y^2)\right]\bar{a}_z$$

At $(5,2,-3)$, $x=5,\ y=2,\ z=-3$

$$\bar{J} \underline{\underline{=420\,\bar{a}_z - 22\,\bar{a}_y + 99\,\bar{a}_z\ \mathrm{A/m^2}}}$$

(*b*) $$\mathrm{I} = \int J\cdot dS = \iint (8x^2y+xy^2)dydz\Big|_{x=-1}$$

$$= \int_0^2 dz\int_0^2 (8y-y^2)dz = 2\left(4y^2-\frac{y^3}{3}\right)\Bigg|_0^2$$

$$= 4\left(16-\frac{8}{3}\right) = 53.33\ \mathrm{A}$$

(*c*) $$\overline{\mathrm{B}} = \mu\overline{\mathrm{H}},\quad \nabla\cdot\overline{\mathrm{B}} = 0 \ \rightarrow\ \bar{\nabla}\cdot\overline{\mathrm{H}} = 0$$

$$\nabla\cdot\overline{\mathrm{H}} = \frac{\partial}{\partial x}\mathrm{H}_x + \frac{\partial}{\partial y}\mathrm{H}_y + \frac{\partial}{\partial z}\mathrm{H}_z = 2xy - 2yxz = 0$$

Hence $\underline{\underline{\nabla\cdot\overline{\mathrm{B}} = 0}}$

**Prob.**

$$\Psi = \int \vec{B}\cdot d\vec{S} = \mu_0\int_{\phi=0}^{\pi/2}\int_{z=0}^{2}\frac{10^6}{\rho}\cos\phi\,\rho\,d\phi\,dz$$

$$= 4\pi\times10^{-7}\times10^{6}\int_0^2 dz\int_0^{\pi/2}\cos\phi\,d\phi = 0.4\pi(2)(1)$$

$$\Psi = \underline{\underline{2.513\ \mathrm{Wb}}}.$$

**Prob.**

(*a*) $\overline{B} = \frac{\mu_o I}{2\pi\ell}\bar{a}_\phi$. At $(-3,4,5)$, $\rho=5$

$$\overline{B} = \frac{4\pi\times10^{-7}\times2}{2\pi(5)}\bar{a}_\phi = 80\,\bar{a}_\phi \text{ nWb/m}^2$$

(*b*) $\phi = \int \overline{B}.dS = \frac{\mu_o I}{2\pi}\iint\frac{d\rho\, dz}{\rho}$

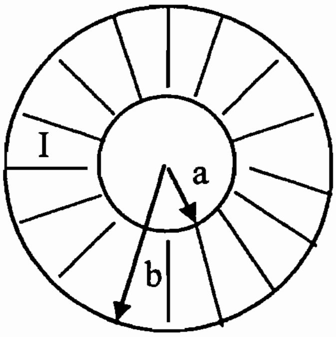
$$= \frac{4\pi\times10^{-7}\times2}{2\pi}\,\text{In}\,\rho\Big|_2^6 z\Big|_0^4 = 16\times10^{-7}\,\text{In}\,3$$

$$= 1.756\,\mu\text{Wb.}$$

**Prob.**

(a) See text.

(b)



For $\rho < a$, $\oint \overline{H}\cdot dl = I_{enc} = 0 \rightarrow \overline{H} = 0$

For $0 < \rho < b$, $H_\phi \cdot 2\pi\rho = \frac{I\pi(\rho^2 - a^2)}{\pi(b^2 - a^2)}$

$$H_\phi = \frac{I}{2\pi\rho}\left(\frac{\rho^2 - a^2}{b^2 - a^2}\right)$$

For $\rho < b$, $H_\phi \cdot 2\pi\rho = I \rightarrow \overline{H}_\varphi = \frac{I}{2\pi\rho}$

Thus,

$$H_\phi = \begin{cases} 0, & \rho < a \\ \dfrac{I}{2\pi\rho}\left(\dfrac{\rho^2 - a^2}{b^2 - a^2}\right), & a < \rho < 1 \\ \dfrac{I}{2\pi\rho}, & \rho > b \end{cases}$$

**Prob.**

The stoke of lightning may be regarded as a semi-infinite filamentary current

$$B = \frac{\mu_o I}{4\pi\rho} = \frac{4\pi \times 10^{-7} \times 50 \times 10^3}{4\pi(100)}$$

$$= \underline{\underline{50\ \mu Wb/m^2}}$$

**Prob.**

(*a*) Applying Ampere's law,

$$H_\phi \cdot 2\pi\rho = I \cdot \frac{\pi\rho^2}{\pi a^2} \quad \rightarrow \quad H_\phi = I \cdot \frac{\rho^2}{2\pi a^2}$$

$$\text{i.e} \qquad \bar{H} = \frac{I\rho}{2\pi a^2}\,\bar{a}_\phi$$

$$\bar{J} = \nabla x \bar{H} = -\frac{\partial H_\phi}{\partial z}\bar{a}_\rho + \frac{I}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\,\bar{a}_z$$

$$= \frac{I}{\rho}\frac{1}{2\pi a^2}\cdot 2\rho\bar{a}_z = \underline{\underline{\frac{I}{\pi a^2}\bar{a}_z}}$$

$(b)$ **From Eq. (**

$$H_\phi = \left| \begin{array}{ll} \dfrac{I\rho}{2\pi a^2}, & \rho < a \\ \dfrac{I}{2\pi\rho}, & \rho > a \end{array} \right.$$

At $(0, 1\text{ cm}, 0)$,

$$H_\phi = \frac{3 \times 1 \times 10^{-2}}{2\pi \times 4 \times 10^{-4}} = \frac{300}{8\pi}$$

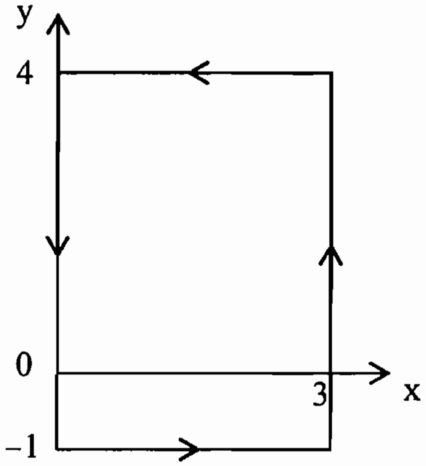
$$\bar{H} = \underline{\underline{11.94\,\bar{a}_\phi \text{ A/m}}}$$

At $(0, 4\text{ cm}, 0)$,

$$H_\phi = \frac{3}{2\pi \times 4 \times 10^{-2}} = \frac{300}{8\pi}$$

$$\bar{H} = \underline{\underline{11.94\,\bar{a}_\phi \text{ A/m}}}$$

**Prob.**



$(a)$ $$\bar{J} = \nabla \cdot \bar{H} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$

$$\bar{J} = \underline{\underline{-2\,\bar{a}_z \text{ A/m}^2}}$$

(*b*) $\oint \overline{\text{H}}.\text{d}\overline{\text{l}} = \text{I}_{\text{enc}}$

$$\text{I}_{\text{enc}} = \int \overline{\text{J}}.\text{d}\overline{\text{s}} = \int_{x=0}^{3}\int_{y=-1}^{4}(-2)dxdy = (-2)(3)(5) = -30\,\text{A}$$

$$\oint \overline{\text{H}}.\text{d}\overline{\text{l}} = \int_{0}^{3} y\,dy\Big|_{y=-1} + \int_{y=-1}^{4}(-x)\,dy\Big|_{x=3} + \int_{3}^{0} y\,dx\Big|_{y=4}$$

$$+ \int_{4}^{-1}(-x)\,dy\Big|_{x=0} = (-1)(3) + (-3)(5) + (4)(-3)$$

$$= -30\,\text{A}$$

Thus, $\oint \overline{\text{H}}.\text{d}\overline{\text{l}} = \text{I}_{\text{enc}} = \underline{\underline{-30\,\text{A}}}$

**Prob.**

(a) $\nabla\cdot\vec{A} = 0, \quad \nabla\times\vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = 0$

i.e. $\vec{A}$ is $\underline{\underline{\bar{E}\text{-field in a charge-free region.}}}$

(b) $\nabla\cdot\vec{B} = \frac{1}{(x^2+y^2)^2}(-2xy + 2xy) = 0, \quad \nabla\times\vec{B} \neq 0$

i.e. $\vec{B}$ is $\underline{\underline{\vec{H}\text{-field.}}}$

(c) $\nabla\cdot\vec{C} = e^{-y}(-\sin x + \sin x) = 0$

$\nabla\times\vec{C} = e^{-y}(-\cos x + \cos x) = 0$

i.e. $\vec{C}$ is $\underline{\underline{\bar{E}\text{-field in a charge-free region.}}}$

(d) $\nabla\cdot\vec{D} = 0, \quad \nabla\times\vec{D} = -10\rho e^{-2z} \neq 0$

i.e. $\vec{D}$ is $\underline{\underline{\vec{H}\text{-field.}}}$

(e) $\nabla\cdot\vec{E} = \frac{4\cos\theta}{r^2} \neq 0, \quad \nabla\times\vec{E} = \frac{2\sin\theta}{r^2}\vec{a}_\phi \neq 0$

i.e. $\bar{E}$ is $\underline{\underline{\text{neither electric nor magnetic.}}}$

**Prob.**

(a) $\boldsymbol{B} = \dfrac{\mu_o I}{2\pi\rho}\boldsymbol{a}_\phi$

At (-3,4,5), ρ=5.

$$B = \frac{4\pi\times10^{-7}\times2}{2\pi(5)}a_\phi = \underline{\underline{80a_\phi \text{ nW/m}^2}}$$

(b) $$\Psi = \int \boldsymbol{B}\bullet dS = \frac{\mu_o I}{2\pi}\iint\frac{d\rho dz}{\rho} = \frac{4\pi\times10^{-7}\times2}{2\pi}\ln\rho\Big|_2^6 z\Big|_0^4$$

$$= 16\times10^{-7}\ln3 = \underline{\underline{1.756\ \mu\text{Wb}}}$$

**Prob.**

from Eq. ), $\bar{B} = \frac{\mu_o}{4\pi}\int_L \frac{I\bar{dl}'\times\bar{R}}{R^3}$

Using this leads to Eq. , namely

$$\vec{B} = -\frac{\mu_o}{4\pi}\int I d\vec{l}' \times \nabla\left(\frac{1}{R}\right)$$

$$\nabla\cdot\vec{B} = -\frac{\mu_o I}{4\pi}\nabla\cdot\left[\int d\vec{l}' \times \nabla\left(\frac{1}{R}\right)\right]$$

Since $\int dl'$ operates on $(x', y', z')$ and $\nabla$ operates on $(x, y, z)$, the two operators can be reversed in order, i.e.

$$\nabla\cdot\vec{B} = -\frac{\mu_o I}{4\pi}\int \nabla\cdot\left[d\vec{l}' \times \nabla\left(\frac{1}{R}\right)\right]$$

But $\nabla\cdot(\vec{A}\times\vec{C}) = \vec{C}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{C})$.

Taking $\vec{A} = d\vec{l}'$ and $\vec{C} = \nabla\left(\frac{1}{R}\right)$,

$$\nabla\cdot\vec{B} = -\frac{\mu_o I}{4\pi}\int\left[\nabla\left(\frac{1}{R}\right)\cdot\underbrace{(\nabla\times d\vec{l}')}_{0} - d\vec{l}'\cdot\underbrace{\left(\nabla\times\nabla\left(\frac{1}{R}\right)\right)}_{0}\right] = 0$$

Since $\nabla$ operates on $(x, y, z)$ and $d\vec{l}'$ depends on $(x', y', z')$, $\nabla\times d\vec{l}' = 0$. Also, since $\nabla\times\nabla V = 0$, $\nabla\times\nabla\left(\frac{1}{R}\right) = 0$. Thus

$$\nabla\cdot\vec{B} = 0$$

**Prob.**

(a) $$\overline{B} = \overline{\nabla}\times\overline{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2y + yz & xy^2 - xz^2 & -6xy + 2z^2y^2 \end{vmatrix}$$

$$\underline{\underline{\overline{B} = (-6xz + 4z^2y + 2xz^2)\overline{a}_x + (y + 4yz)\overline{a}_y + (y^2 - z^2 - 2x^2 - z)\overline{a}_z \text{ Wb/m}^2}}$$

$$(b) \quad \psi = \int \overline{B} \cdot dS, \qquad dS = dy\,dz\,dx$$

$$\psi = \int_{z=0}^{2} \int_{y=0}^{2} \left(-6xz + 4zy - 2xy\right) dy\,dz \bigg|_{x=1}$$

$$= \int \int_{0} \left(-6z\right) dy\,dz + 4 \int \int_{0} z^2 y\,dy\,dz + 2 \int \int_{0} y\,dy\,dz$$

$$= -8 \int_{0}^{2} z\,dz \int_{0}^{2} dy + 4 \int_{0}^{2} z^2\,dz \int_{0}^{2} y\,dy$$

$$= -8 \frac{z^2}{2} \bigg|_0^2 (2) + 4 \frac{z^3}{3} \bigg|_0^2 \left( \frac{y^2}{2} \bigg|_0^2 \right) = -32 + \frac{64}{3}$$

$$\psi = -10.67 \text{ Wb}$$

$\overline{E}$ can be a magnetostatic field.

$$(c) \quad \overline{V} \cdot \overline{A} = \partial A_x + \frac{\partial A_y}{} + \frac{\partial A_z}{xz} = 4xy + 2xy - 6xy = 0$$

$$\overline{V} \cdot \overline{B} = -6z + 3z^3 + 1 + 6z - 3z^3 - 1 = 0$$

**Prob.**

$$\psi = \int \overline{B} . d\overline{s} = \mu_o \int_{z=0}^{0.2} \int_{\phi=0}^{50^\circ} \frac{10^6}{\rho} \sin 2\phi \; \rho \, d\phi \, dz$$

$$\psi = 4\pi \times 10^{-7} \times 10^6 (0.2) \left( -\frac{\cos 2\phi}{2} \right) \bigg|_0^{50^\circ}$$

$$= 0.04\pi \left(1 - \cos 100^o\right)$$

$$= \underline{\underline{0.1475 \text{ Wb}}}$$

**Prob.**

$$\text{(a)} \quad \boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi) \boldsymbol{a}_z = \frac{1}{\rho} \frac{d}{d\rho} (k_o \frac{\rho^2}{a}) \boldsymbol{a}_z = \underline{\underline{\frac{2k_o}{a} \boldsymbol{a}_z}}$$

(b)For ρ>a,

$$\oint \boldsymbol{H}\cdot d\boldsymbol{l} = I_{enc} = \int \boldsymbol{J}\cdot d\boldsymbol{S} = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{2k_o}{a}\rho d\rho d\phi = \frac{2k_o}{a}(2\pi)\frac{\rho^2}{2}\Big|_0^a$$

$$H_\phi 2\pi\rho = 2\pi k_o a \qquad \longrightarrow \qquad H_\phi = \frac{k_o a}{\rho}$$

$$\underline{\underline{\boldsymbol{H} = k_o\left(\frac{a}{\rho}\right)\boldsymbol{a}_\phi, \qquad \rho > a}}$$

**Prob.**

$$\overline{B} = \overline{V}\times\overline{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\overline{a}_\rho - \frac{\partial A_z}{\partial\rho}\overline{a}_\phi \qquad =$$

$$=$$

$$= \frac{15}{\rho}e^{-\rho}\cos\phi\,\overline{a}_\rho + 15\,e^{-\rho}\sin\phi\,\overline{a}_\phi$$

$$\overline{B}\left(3,\frac{\pi}{4},-10\right) = 5\,e^{-3}\frac{1}{\sqrt{2}}\overline{a}_\rho + 15\,e^{-3}\frac{1}{\sqrt{2}}\overline{a}_\phi$$

$$\overline{H} = \frac{\overline{B}}{\mu_o} = \frac{10^7}{4\pi}\frac{15}{\sqrt{2}}e^{-3}\left(\frac{1}{3}\overline{a}_\rho + \overline{a}_\phi\right)$$

$$\overline{H} = \left(14\,\overline{a}_\rho + 42\,\overline{a}_\phi\right)\cdot 10^4 \text{ A/m}$$

$$\psi = \int\overline{B}\cdot d\overline{s} = \iint\frac{15}{\rho}e^{-\rho}\cos\phi\,\rho\,d\phi\,dz$$

$$= 15\,z\Big|_0^{10}\left(-\sin\phi\right)\Big|_0^{\pi/2}e^{-5} = -150\,e^{-5} \qquad \Rightarrow \qquad \psi = -1.011\,\text{Wb}$$

**Prob.**

Let $\bar{H} = \bar{H}_1 + \bar{H}_2$

where $\bar{H}_1$ and $\bar{H}_2$ are due to the wires centered at x = 0 and x = 10cm respectively.

(*a*) For $\bar{H}_1$, $\rho = 50$ cm, $\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho = \bar{a}_z \times \bar{a}_x = \bar{a}_y$

$$\bar{H}_1 = \frac{5}{2\pi\left(5\times10^{-2}\right)}\bar{a}_y = \frac{50}{\pi}\bar{a}_y$$

For $\bar{H}_2$, $\rho = 5$ cm, $\bar{a}_\phi = -\bar{a}_z \times -\bar{a}_x = \dot{a}_y$, $\bar{H}_2 = \bar{H}_1$

$$\bar{H} = 2\bar{H}_1 = \frac{100}{\pi}\bar{a}_y$$

$$= \underline{\underline{31.83\,\bar{a}_y \text{ A/m}}}$$

(*b*) For $\bar{H}_1$, $\bar{a}_\phi = \bar{a}_z \times \left(\frac{2\bar{a}_x + \bar{a}_y}{\sqrt{5}}\right) = \frac{2\bar{a}_y - \bar{a}_x}{\sqrt{5}}$

$$\bar{H}_1 = \frac{5}{2\pi 5\sqrt{5}\times10^{-2}}\left(\frac{-\bar{a}_x + 2\bar{a}_y}{\sqrt{5}}\right) = -3.183\,\bar{a}_x + 6.366\,\bar{a}_y$$

For $\bar{H}_2$, $\bar{a}_\rho = -\bar{a}_z \times \bar{a}_y = \dot{a}_x$

$$\bar{H}_2 = \frac{5}{2\pi(5)}\bar{a}_x = 15.915\,\bar{a}_x$$

$$\bar{H} = \bar{H}_1 + \bar{H}_2$$

$$= \underline{\underline{12.3\,\bar{a}_x + 6.366\,\bar{a}_y \text{ A/m}}}$$

**Prob.**

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho}\bar{a}_\varphi$$

$$\psi = \int \boldsymbol{B}\cdot dS = \int_{\rho=d}^{d+a}\int_{z=0}^{b}\frac{\mu_0 I}{2\pi\rho}\,d\rho dz$$

$$= \underline{\underline{\frac{\mu_0 Ib}{2\pi}\ln\frac{d+a}{d}}}$$

**Prob.**

$$\Psi = \int \boldsymbol{B} \bullet d\boldsymbol{S} = \iint 4\cos(\pi y/4)e^{-3z}dydz = 4\int_0^\infty e^{-3z}dz\int_0^1 \cos(\pi y/4)dy$$

$$= 4\frac{e^{-3z}}{-3}\Bigg|_0^\infty \frac{\sin(\pi y/4)}{\pi/4}\Bigg|_0^1 = \frac{4}{3}(0-1)\frac{4}{\pi}(\sin(\pi/4)-0) = \frac{16}{3\pi}\sin(\pi/4) = \underline{\underline{1.2004 \text{ Wb}}}$$

**Prob.**

(a) $\nabla \bullet \boldsymbol{B} = \dfrac{\partial B_x}{dx} + \dfrac{\partial B_y}{dy} + \dfrac{\partial B_z}{dz} = 0$

showing that **B** satisfies Maxwell's equation.

(b) $dS = dydz\boldsymbol{a}_x$

$$\Psi = \int \boldsymbol{B} \bullet dS = \int_{z=1}^{4}\int_{y=0}^{1} y^2 dydz = \frac{y^3}{3}\Bigg|_0^1 (z)\Bigg|_1^4 = \underline{\underline{1 \text{ Wb}}}$$

(c ) $\nabla x\mathbf{H} = \mathbf{J} \longrightarrow \boldsymbol{J} = \nabla x \dfrac{\boldsymbol{B}}{\mu_o}$

$$\nabla x\boldsymbol{B} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\boldsymbol{a}_x - 2x\boldsymbol{a}_y - 2y\boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{J} = -\frac{2}{\mu_o}(z\boldsymbol{a}_x + x\boldsymbol{a}_y + y\boldsymbol{a}_z) \text{ A/m}}}$$

**Prob.**

$$\vec{A} = \frac{\mu_o I}{4\pi} \int \frac{d\vec{l}}{R},$$

$\vec{R} = (x, y, z) - (0, 0, z')$, $R = \sqrt{x^2 + y^2 + (z - z')^2}$. At a distant point, $z >> z'$ so that

$$R \simeq \sqrt{x^2 + y^2 + z^2} = r .$$

Hence, $\vec{A} = \dfrac{\mu_o I}{4\pi} \displaystyle\int_{-L}^{L} dz' \, \vec{a}_z = \dfrac{\mu_o I L}{2\pi r} \, \vec{a}_z$

$$\vec{A} = \frac{\mu_o I L \, a_z}{2\pi (x^2 + y^2 + z^2)^{1/2}}$$

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_o I L}{2\pi} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{r} \end{vmatrix}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{r^3}, \quad \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3}$$

$$\vec{B} = \frac{\mu_o I L}{2\pi r^3} (-y \, \vec{a}_x + x \, \vec{a}_y) = \underline{\underline{\frac{\mu_o I L (-y \, \vec{a}_x + x \, \vec{a}_y)}{2\pi (x^2 + y^2 + z^2)^{3/2}}}}$$

**Prob.**

On the slant side of the ring, $z = \frac{h}{6}(\rho - a)$

where $\bar{H}_1$ and $\bar{H}_2$ are due to the wires centered at x = 0 and x = 10cm respectively

$$\psi = \int \bar{B}.d\bar{s} = \int \frac{\mu_o I}{2\pi\rho}\, d\rho\, dz$$

$$= \frac{\mu_o I}{2\pi} \int_{\rho=a}^{a+b} \int_{z=0}^{\frac{h}{b}(\rho-a)} \frac{dz\, d\rho}{\rho} = \frac{\mu_o I h}{2\pi b} \int_{\rho=a}^{a+b} \left(1 - \frac{a}{\rho}\right) d\rho$$

$$= \underline{\underline{\frac{\mu_o I h}{2\pi b}\left(b - a \ln \frac{a+b}{a}\right)}} \text{ as required.}$$

If a = 30 cm, b = 10 cm, h = 5 cm, I = 10 A,

$$\psi = \frac{2\pi \times 10^{-7} \times 10 \times 0.05}{2\pi\left(5 \times 10^{-2}\right)} \left(0.1 - 0.3 \text{ In } \frac{4}{3}\right)$$

$$= \underline{\underline{1.37 \times 10^{-8} \text{ Wb}}}$$

**Prob.**

$$\vec{H} = \frac{\nabla \times \vec{A}}{\mu_o} = \frac{I}{2\pi\rho}\vec{a}_\rho$$

$$\nabla \times \vec{A} = \frac{\mu_o I}{2\pi\rho}\vec{a}_\phi = \left[\underset{\swarrow 0}{\frac{\partial \cancel{A_\rho}}{\partial z}} - \frac{\partial A_z}{\partial \rho}\right]\vec{a}_\phi$$

$$\frac{dA_z}{d\rho} = -\frac{\mu_o I}{2\pi\rho} \longrightarrow A_z = -\frac{\mu_o I}{2\pi}\ln\rho$$

$$\underline{\underline{\vec{A} = -\frac{\mu_o I}{2\pi}\ln\rho\, \vec{a}_z}} \, .$$

**Prob.**

(*a*) $\nabla\cdot\bar{A} = -ya \sin ax \neq 0$

$$\nabla\times\bar{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos ax & 0 & y+e^{-x} \end{vmatrix}$$

$$= \bar{a}_x + e^{-x}\bar{a}_y - \cos ax\bar{a}_z \neq 0$$

$\bar{A}$ is neither electrostatic nor magnetostatic field

(*b*) $\nabla\cdot\bar{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho B_\rho\right) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(20\right) = 0$

$\nabla\times\bar{B} = 0$

$\bar{B}$ can be $\bar{E}$-field in a charge-free region.

(*c*) $\nabla\cdot\bar{C} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\sin\theta) = 0$

$\nabla\times\bar{C} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r^2 \sin^2\theta\right) - \frac{1}{r}\frac{\partial}{\partial r}(r^3\sin\theta) \neq 0$

$\bar{C}$ is possibly $\bar{H}$ field.

**Prob.**

(*a*) $$\bar{B} = \nabla\times\bar{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2y+yz & xy^2-xz^3 & -6xy+2z^2y^2 \end{vmatrix}$$

$$\bar{B} = (-6xz+4x^2y+3xz^2)\bar{a}_x + \left(y+6yz\text{-}4xy^2\right)\bar{a}_y + \left(y^2-z^3-2x^2-z\right)\bar{a}_z \text{ Wb/m}^2$$

(*b*) $\psi = \int\bar{B}\cdot dS, \quad dS = dy\,dz\,dx$

$$\psi = \int_{z=0}^{2}\int_{y=0}^{2}\left(-6xz+4x^2y+3xz^2\right) dy\,dz \,\Big|_{x=1}$$

$$= \int\int_0\left(-6xz\right) dy\,dz + 4\int\int_0 x^2y\,dy\,dz + 3\int\int_0 xz^2\,dy\,dz$$

$$= -6\int_0^2 dz \int_0^2 dy + 4\int_0^2 dz \int_0^2 y\, dy + 3\int_0^2 dy \int_0^2 z^2 dz$$

$$= -6(2)(2) + 4(2)\left(\left.\frac{y^2}{2}\right|_0^2\right) + 3(2)\left(\left.\frac{z^3}{3}\right|_0^2\right) = -24+16+16$$

$$\psi = \underline{\underline{8 \text{ Wb}}}$$

$$(c) \quad \nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 4xy + 2xy - 6xy = 0$$

$$\nabla\cdot\bar{B} = -6z + 3z^3 + 1 + 6z - 3z^3 - 1 = 0$$

As a matter of mathematical necessity,

$$\nabla \bullet \boldsymbol{B} = \nabla \bullet (\nabla x A) = 0$$

**Prob.**

$$\vec{A} = -\frac{I_o \mu_o}{4\pi a^2}(x^2+y^2)\vec{a}_z = -\frac{I_o \mu_o \rho^2}{4\pi a^2}\vec{a}_z$$

$$\vec{B} = \nabla \times \vec{A} = \frac{I_o \mu_o \rho}{2\pi a^2}\vec{a}_\phi = \mu_o \vec{H}$$

$$\text{i.e.} \quad \vec{H} = \frac{I_o \rho}{2\pi a^2}\vec{a}_\phi = \frac{I_o\sqrt{x^2+y^2}}{2\pi a^2}\vec{a}_\phi .$$

By Ampere's law, $\oint \vec{H}\cdot d\vec{l} = I_{enc}$

$$H_\phi \cdot 2\pi\rho = I_o \cdot \frac{\rho^2}{a^2}$$

$$\text{or} \quad \vec{H} = \underline{\underline{\frac{I_o \rho}{2\pi a^2}\vec{a}_\phi}} .$$

**Prob.**

$(a)\quad \nabla\cdot\bar{\mathrm{D}} \;=\; 0$

$$\nabla\times\bar{\mathrm{H}} \;=\; \begin{vmatrix} \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ \mathrm{u^2z} & 2(\mathrm{x}+1)\mathrm{yz} & -(\mathrm{x}+1)\mathrm{z}^2 \end{vmatrix}$$

$$= \; 2(\mathrm{x}+1)\mathrm{y}\bar{a}_x \; + \; . \; . \; . \; \neq \; 0$$

$\bar{\mathrm{D}}$ is possibly a magnetostatic field.

$$(b)\quad \nabla\cdot\bar{\mathrm{E}} \;=\; \frac{1}{\rho}\frac{\partial}{\partial\rho}\big((z+1)\cos\phi\big)+\frac{\partial}{\partial z}\left(\frac{\sin\phi}{\rho}\right)=0$$

$$\nabla\times\bar{\mathrm{E}} \;=\; \frac{1}{\rho^2}\cos\theta\,\bar{a}_\rho \; + \; . \; . \; . \; \neq \; 0$$

$\bar{\mathrm{E}}$ could be a magnetostatic field.

$$(c)\quad \nabla\cdot\bar{\mathrm{F}} \;=\frac{1}{r^2}\frac{\partial}{\partial \mathrm{r}}\big(2\cos\theta\big) + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\mathrm{r}^2}\right) \;\neq\; 0$$

$$\nabla\times\bar{\mathrm{F}} \;=\; \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\mathrm{r}^{-1}\sin\theta\right) \;+\; \frac{2\sin\theta}{r^2}\right]\bar{a}_\theta \;\neq\; 0$$

$\bar{\mathrm{F}}$ can be neither electrostatic nor magnetostatic field.

**Prob.**

$$\text{(a)}\quad \boldsymbol{B}=\nabla\times\boldsymbol{A}=\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-x}\sin y & 1+\cos y & 0 \end{vmatrix}=\underline{\underline{-e^{-x}\cos y\boldsymbol{a}_z\,T}}$$

$$\text{(b)}\quad \boldsymbol{B}=\nabla\times\boldsymbol{A}=-\frac{\partial A_z}{\partial\rho}\boldsymbol{a}_\phi=-\frac{(\rho^2+1)(4)-4\rho(2\rho)}{(\rho^2+1)^2}\boldsymbol{a}_\phi=\underline{\underline{\frac{4(\rho^2-1)}{(\rho^2+1)^2}\boldsymbol{a}_\phi}}$$

(c )

$$\boldsymbol{B}=\nabla\times\boldsymbol{A}=\frac{1}{r}\left[\frac{\partial}{\partial r}(\sin\theta)+\frac{\sin\theta}{r^2}\right]\boldsymbol{a}_\phi=\underline{\underline{\frac{\sin\theta}{r^3}\boldsymbol{a}_\phi}}$$

**Prob.**

For an infinite line current, $\vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi$.

But $\vec{H} = -\nabla V_m \quad (\vec{J} = 0)$

$$\frac{I}{2\pi\rho}\vec{a}_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}\vec{a}_\phi \longrightarrow V_m = -\frac{I}{2\pi}\phi + c$$

(a) If $V_m = 0$ at $\phi = 0 \rightarrow c = 0$

$$V(6, \pi/4, 0) = -\frac{20}{2\pi}\cdot\frac{\pi}{4} = \underline{\underline{-2.5\text{ A}}}.$$

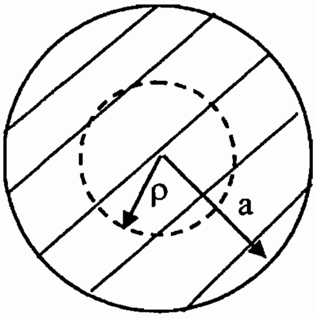
(b) If $V_m = 1$ at $\phi = \pi/2 \longrightarrow c = 6$.

$$V_m = \frac{-20\phi}{2\pi} + 6$$

At $(-3, 4, 2)$, $\phi = \tan^{-1}\frac{4}{-3} = 126.87^\circ = 2.214\text{ rad}$

$$V_m = -\frac{20}{2\pi}(2.214) + 6 = \underline{\underline{-1.05\text{ A}}}.$$

**Prob.**



Applying Ampere's law gives

$$H_\varphi \cdot 2\pi\rho = \mathrm{J}_o \cdot \pi\rho^2$$

$$H_\varphi = \frac{J_o}{2}\rho$$

$$B_\varphi = \mu_o H_\varphi = \mu_o \frac{J_o\rho}{2}$$

But $B_\varphi = \nabla \times \bar{\mathrm{A}} = -\frac{\partial A_Z}{\partial\rho}\bar{\mathrm{a}}_\varphi + \ldots$

$$-\frac{\partial A_Z}{\partial\rho} = \frac{1}{2}\mu J_o\rho$$

$$\mathrm{A_Z} = -\mu_o \frac{J_o\rho^2}{4}$$

or $\bar{\mathrm{A}} = -\frac{1}{4}\mu_o J_o\rho^2\,\bar{\mathrm{a}}_z$

**Prob.**

A has the same direction as **K**, i.e. **A**= A **a**$_x$. **From eq. (6.23).**

$$\boldsymbol{B} = \mu_o\boldsymbol{H} = \frac{1}{2}\mu_o\boldsymbol{K}\times\boldsymbol{a}_n = \begin{cases} \frac{1}{2}\mu_o k_o \boldsymbol{a}_z, & y>0 \\ -\frac{1}{2}\mu_o k_o \boldsymbol{a}_z, & y<0 \end{cases}$$

But $\boldsymbol{B} = \nabla\times\boldsymbol{A} = -\frac{\partial A_x}{\partial y}\boldsymbol{a}_z$

Integrating with respect to y and assuming A(0) =0,

$$\boldsymbol{A} = \begin{cases} -\frac{1}{2}\mu_o k_o y\boldsymbol{a}_z, & y>0 \\ \frac{1}{2}\mu_o k_o y\boldsymbol{a}_z, & y<0 \end{cases}$$

**Prob.**

(a) $$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2 & x^2 z & -y^2 z \end{vmatrix} = (-2yz - x^2)\boldsymbol{a}_x + (2xz - 2xy)\boldsymbol{a}_z$$

At (2,-1,3), x=2, y=-1, z=3.

$$\underline{\underline{\boldsymbol{J} = \boldsymbol{2a}_x + 16\boldsymbol{a}_z \ \text{A/m}^2}}$$

(b) $$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet \boldsymbol{J} = 0 - 2x + 2x = 0$$

At (2,-1,3),

$$\frac{\partial \rho_v}{\partial t} = \underline{\underline{0 \ \text{C/m}^3\text{s}}}$$

**Prob.**

(a) $$\nabla \times \psi \bar{F} = \left( \frac{\partial}{\partial y}(\psi F_z) - \frac{\partial}{\partial z}(\psi F_y) \right) \vec{a}_x + \left( \frac{\partial}{\partial z}(\psi F_x) - \frac{\partial}{\partial x}(\psi F_z) \right) \vec{a}_y + \left( \frac{\partial}{\partial x}(\psi F_y) - \frac{\partial}{\partial y}(\psi F_x) \right) \vec{a}_z$$

$$= \psi \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \vec{a}_x + \psi \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \vec{a}_y + \psi \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \vec{a}_z + \left( F_z \frac{\partial \psi}{\partial y} - F_y \frac{\partial \psi}{\partial z} \right) \vec{a}_x$$

$$+ \left( F_x \frac{\partial \psi}{\partial z} - F_z \frac{\partial \psi}{\partial x} \right) \vec{a}_y + \left( F_y \frac{\partial \psi}{\partial x} - F_x \frac{\partial \psi}{\partial y} \right) \vec{a}_z$$

$$\underline{\underline{= \psi \nabla \times \bar{F} + \nabla \psi \times \vec{F}}}$$

(b) $$\bar{\nabla} \times \nabla \times \bar{A} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left( \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \right) & \left( \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} \right) & \left( \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \right) \end{vmatrix}$$

$$= \left( \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_x}{\partial^2 y^2} \right) \vec{a}_x$$

$$- \left( \frac{\partial^2 A_y}{\partial z^2} + \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_x}{\partial x \partial y} \right) \vec{a}_y$$

$$- \left( \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_y}{\partial z \partial y} - \frac{\partial^2 A_z}{\partial x \partial z} \right) \vec{a}_z$$

$$\nabla(\nabla \cdot \vec{A}) = \left( \frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z \right) \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

$$= \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} \right) \vec{a}_x$$

$$+ \left( \frac{\partial^2 A_x}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z} \right) \vec{a}_y$$

$$+ \left( \frac{\partial^2 A_x}{\partial z \partial x} + \frac{\partial^2 A_y}{\partial z \partial y} + \frac{\partial^2 A_z}{\partial z^2} \right) \vec{a}_z$$

$$\nabla^2 \vec{A} = \nabla^2 A_x \vec{a}_x + \nabla^2 A_y \vec{a}_y + \nabla^2 A_z \vec{a}_z$$

$$= \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \right) \vec{a}_x +$$

$$\left( \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \right) \vec{a}_y + \left( \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right) \vec{a}_z$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \left( \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_z}{\partial z^2} \right) \vec{a}_x$$

$$+ \left( \frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial y^2} - \frac{\partial^2 A_y}{\partial z^2} \right) \vec{a}_z$$

$$= \nabla \times \nabla \times \vec{A} .$$

Alternatively, we may use the bac-cab rule

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B})$$

and let $\vec{A} \to \nabla$, $\vec{B} \to \nabla$, $\vec{C} \to \vec{A}$, we obtain

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \vec{A}\,(\nabla \cdot \nabla)$$

$$= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} .$$

**Prob.**

$$\bar{H} = -\bar{\nabla} V_m \rightarrow V_m = -\int \bar{H} \cdot d\bar{l} = -mmf$$

$$\bar{H} = \frac{Ia^2}{2\left(z^2+a^2\right)^{3/2}}\,\bar{a}_z$$

$$V_m = -\frac{Ia^2}{2}\int \left(z^2+a^2\right)^{-3/2} dz = \frac{-Iz}{2\left(z^2+a^2\right)^{1/2}} + c$$

As $z \to \infty$, $V_m = 0$, i.e.

$$0 = -\frac{I}{2} + c \rightarrow c = \frac{I}{2}$$

Hence,

$$V_m = \frac{I}{2}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]$$

**Prob.**

For an infinite current sheet,

$$\bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_n = \frac{1}{2} 50\bar{a}_y \times \bar{a}_n = 25\bar{a}_x$$

But $\bar{H} = -\nabla V_m \; \left(\bar{J} = 0\right)$

$$25\,\bar{a}_x = -\frac{\partial V_m}{\partial x}\,\bar{a}_n \rightarrow V_m = -25x + c$$

At the origin, $x = 0,\ V_m = 0,\ c = 0,$ i.e.

$$V_m = -25x$$

(a) At $(-2, 0, 5)$, $\underline{\underline{V_m = 50A.}}$

(b) At $(10, 3, 1)$, $\underline{\underline{V_m = -250A.}}$

**Prob.**

---

$$R = |\bar{r} - \bar{r}| = \left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}$$

$$\nabla\frac{1}{R} = \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= -\frac{1}{2}\,2\,(x-x')\,\bar{a}_x\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} + a_y \text{ and } a_z \text{ terms}$$

$$= -\left[(x-x')\bar{a}_x+(y-y')\bar{a}_y+(z-z')\bar{a}_z\right]\Big/ R^3 = -\frac{\bar{R}}{R^3}$$

$$\nabla'\frac{1}{R} = \left(\frac{\partial}{\partial x'}\bar{a}_x + \frac{\partial}{\partial y'}\bar{a}_y + \frac{\partial}{\partial z'}\bar{a}_z\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= \left(-\frac{1}{2}\right)(-2)\,(x-x')\bar{a}_x\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} + a_y \text{and } a_y \text{ terms}$$

$$= \frac{\bar{R}}{R^3}$$

**Chapter —Practice Examples**

## P.E.

(a) $F = m\frac{\partial \bar{u}}{\partial t} = Q\bar{E} = \underline{\underline{6\bar{a}_z N}}$

(b) $\frac{\partial \bar{u}}{\partial t} = 6\bar{a}_z = \frac{\partial}{\partial t}(u_x, u_y, u_z) \Rightarrow$

$$\frac{\partial u_x}{\partial t} = 0 \rightarrow u_x = A$$

$$\frac{\partial u_y}{\partial t} = 0 \rightarrow u_y = B$$

$$\frac{\partial u_z}{\partial t} = 6 \rightarrow u_z = 6t + C$$

Since $\bar{u}(t=0) = 0$, A = B = C = 0

$u_x = 0 = u_y, \quad u_z = 6t$

$$u_x = \frac{\partial x}{\partial t} = 0 \rightarrow x = A$$

$$u_y = \frac{\partial y}{\partial t} = 0 \rightarrow y = B$$

$$u_z = \frac{\partial z}{\partial t} = 6t \rightarrow z = 3t^2 + C_1$$

At t = 0, (x,y,z) = (0,0,0) → $A_1 = 0 = B_1 = C_1$

Hence , (x,y,z) = $(0,0,3t^2)$,

$u = 6ta_z$ at any time. At P(0,0,12), z = 12 =$3t^2$ → t =2s

$\underline{\underline{t = 2s}}$

(c) $u = 6t\bar{a}_z = 12a_z m/s$ .

$$a = \frac{\partial \bar{U}}{\partial t} = \underline{\underline{6a_z\, {}^{m}\!/_{s^2}}}$$

(d) $K.E = \frac{1}{2}m|\bar{U}|^2 = \frac{1}{2}(1)(144) = \underline{\underline{72J}}$

## P.E.

(a) $m\vec{a} = e\vec{u}XB$ = ($eB_o$uy, -$eB_o$ux, 0)

$$\frac{d^2x}{dt^2} = \frac{eBo}{m}\frac{dy}{dt} = \omega\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = -\frac{eBo}{m}\frac{dx}{dt} = -\omega\frac{dx}{dt} \qquad (2)$$

$$\frac{d^2z}{dt^2} = 0; \Rightarrow \frac{dz}{dt} = C_1 \qquad (3)$$

From (1) and (2),

$$\frac{d^3x}{dt^3} = \omega\frac{d^2y}{dt^2} = -\omega^2\frac{dx}{dt}$$

$(D^2 + w^2 D)x = 0 \rightarrow Dx = (0, \pm j\omega)x$

$x = c_2 + c_3\cos\omega t + c_4\sin\omega t$

$$\frac{dy}{dt} = \frac{1}{\omega}\frac{d^2x}{dt^2} = -c_3\omega\cos\omega t - c_4\omega\sin\omega t$$

At $t = 0$, $\vec{u} = (\alpha, 0, \beta)$. Hence,

$$c_1 = \beta, c_3 = 0, c_4 = \frac{\alpha}{\omega}$$

$$\underline{\underline{\frac{dx}{dt} = \alpha\cos\omega t, \frac{dy}{dt} = -\alpha\sin\omega t, \frac{dz}{dt} = \beta}}$$

(b) Solving these yields

$$x = \frac{a}{\omega}\sin\omega t, y = \frac{\alpha}{\omega}\cos\omega t, z = \beta t$$

The starting point of the particle is $(0, \frac{\alpha}{\omega}, 0)$

(c) $x^2 + y^2 = \frac{\alpha^2}{\omega^2}$, $z = \beta t$

showing that the particles move along a helix of radius $\alpha/\omega$ placed along the z-axis.

**P.E.**

(a) From Example , QuB = QE regardless of the sign of the charge.

$E = uB = 8 \times 10^6 \times 0.5 \times 10^{-3} = \underline{4\text{ kV/m}}$

(b) Yes, since QuB = QE holds for any Q and m.

## P.E.

By Newton's 3rd law, $\vec{F}_{12} = \vec{F}_{21}$, the force on the infinitely long wire is:

$$\vec{F}_l = -\vec{F} = \frac{\mu_o I_1 I_2 b}{2\pi}\left(\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right)\vec{a}_\rho$$

$$= \frac{4\pi\times10^{-7}\times50\times3}{2\pi}\left(\frac{1}{2}-\frac{1}{3}\right)\vec{a}_\rho = \underline{\underline{5\vec{a}_\rho \;\; \mu N}}$$

## P. E.

$$\vec{m} = IS\vec{a}_n = 10\times10^{-4} x50 \frac{(2,6,-3)}{7}$$

$$= 7.143 \text{ x } 10^{-3} (2, 6, -3)$$

$$= \underline{\underline{(1.429\,\vec{a}_x + 4.286\,\vec{a}_y - 2.143\,\vec{a}_z) \text{ x } 10^{-2} \text{ A-m}^2}}$$

## P.E.

(a) $$\vec{T} = \vec{m}\times B = \frac{10\times10^{-4}\times50}{7\times10}\begin{vmatrix} 2 & 6 & -3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$= \underline{\underline{0.03\,\vec{a}_x - 0.02\,\vec{a}_y - 0.02\,\vec{a}_z \text{ N-m}}}$$

(b) $$|\vec{T}| = ISB\sin\theta \rightarrow |\vec{T}|_{\max} = ISB$$

$$|\boldsymbol{T}|_{max} = \frac{50x10^{-3}}{10}|6a_x + 4a_y + 5a_z| = \underline{\underline{0.04387 \text{ Nm}}}$$

## P.E.

$$\vec{a}_n = \frac{3a_x + 4\vec{a}_y}{5} = \frac{6\vec{a}_x + 8\vec{a}_y}{10}$$

$$\vec{B}_{1n} = (\vec{B}_1 \bullet \vec{a}_n)\vec{a}_n = \frac{(6+32)(6\vec{a}_x + 8\vec{a}_y)}{1000}$$

$$= 0.228\,\vec{a}_x + 0.304\,\vec{a}_y = B_{2n}$$

$$\vec{B}_{1t} = \vec{B}_1 - \vec{B}_{1n} = -0.128\vec{a}_x + 0.096\vec{a}_y + 0.2\vec{a}_z$$

$$\vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t} = 10\vec{B}_{1t} = -1.28\vec{a}_x + 0.96\vec{a}_y + 2\vec{a}_z$$

$$\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} = \underline{\underline{-1.052\vec{a}_x + 1.264\vec{a}_y + 2\vec{a}_z}} \;\; \text{Wb/m}^2$$

**P.E.**

(a) $B_{1n} = \vec{B}_{2n} \rightarrow \mu_1\vec{H}_{1n} =_z \mu_2\vec{H}_{2n}$

or $\mu_1\vec{H}_1 \bullet \vec{a}_{n21} = \mu_2\vec{H}_2 \bullet \vec{a}_{n21}$

$$\mu_o \frac{(60+2-36)}{7} = 2\mu_o \frac{(6H_{2x}-10-12)}{7}$$

$$35 = 6H_{2x}$$

$$\underline{\underline{H_{2x} = 5.833 \text{ A/m}}}$$

(b) $\vec{K} = (\vec{H}_1 - \vec{H}_2) \times \vec{a}_{n12} = \vec{a}_{n21} \times (\vec{H}_1 - \vec{H}_2)$

$$= \vec{a}_{n21} \times \left[(10,1,12) - ({}^{35}\!/_{6}, -5, 4)\right]$$

$$= \frac{1}{7}\begin{vmatrix} 6 & 2 & -3 \\ {}^{25}\!/_{6} & 6 & 8 \end{vmatrix}$$

$$\vec{K} = \underline{\underline{4.86\vec{a}_x - 8.64\vec{a}_y + 3.95\vec{a}_z}} \text{ A/m}$$

(c) Since $\vec{B} = \mu\vec{H}$, $\vec{B}_1$ and $\vec{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos\theta_1 = \frac{\vec{H}_1 \bullet \vec{a}_{n21}}{|\vec{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$$\underline{\underline{\theta_1 = 76.27^o}}$$

$$\cos\theta_2 = \frac{\vec{H}_2 \bullet \vec{a}_{n21}}{|\vec{H}_2|} = \frac{13}{7\sqrt{(5.833)^2 + 25 + 16}} = 0.2144$$

$$\underline{\underline{\theta_2 = 77.62^o}}$$

**P.E.**

(a) $$L' = \mu_o \mu_r n^2 S = 4\pi \times 10^{-7} \times 1000 \times 16 \times 10^6 \times 4 \times 10^{-4}$$

$$= \underline{\underline{8.042 \text{ H/m}}}$$

(b) $$W_m' = \frac{1}{2} L' I^2 = \frac{1}{2}(8.042)(0.5^2) = \underline{\underline{1.005}} \text{ J/m}$$

**P.E.**

$$\text{L}_{\text{in}} = \frac{\mu_o l}{8\pi}$$

$$\text{L}_{\text{ext}} = \frac{2w_m}{I^2} = \frac{1}{I^2} \iiint \frac{\mu I^2}{4\pi^2 \rho^2} \rho d\rho d\phi dz$$

$$= \frac{1}{4\pi^2} \int_0^l dz \int_0^{2\pi} d\phi \int_a^b \frac{2\mu_o}{(1+\rho)\rho} d\rho$$

$$= \frac{2\mu_o}{4\pi^2} \bullet 2\pi l \int_a^b \left[ \frac{1}{\rho} - \frac{1}{(1+\rho)} \right] d\rho$$

$$= \frac{\mu_o l}{\pi} \left[ \ln\frac{b}{a} - \ln\frac{1+b}{1+a} \right]$$

$$\text{L} = \text{L}_{\text{in}} + \text{L}_{\text{ext}} = \underline{\underline{\frac{\mu_o l}{8\pi} + \frac{\mu_o l}{\pi} \left[ \ln\frac{b}{a} - \ln\frac{1+b}{1+a} \right]}}$$

## P.E.

(a) $$L'_{in} = \frac{\mu_o}{8\pi} = \frac{4\pi\times10^{-7}}{8\pi} = \underline{\underline{0.05\ \mu H/m}}$$

$$L'_{ext} = L' - L'_{in} = 1.2 - 0.05 = \underline{\underline{1.15\ \mu H/m}}$$

(b) $$L' = \frac{\mu_o}{2\pi}\left[\frac{1}{4} + \ln\frac{d-a}{a}\right]$$

$$\ln\frac{d-a}{a} = \frac{2\pi L'}{\mu_o} - 0.25 = \frac{2\pi\times1.2\times10^{-6}}{4\pi x 10^{-7}} - 0.25$$

$$= 6 - 0.25 = 5.75$$

$$\frac{d-a}{a} = e^{5.75} = 314.19$$

$$d - a = 314.19a = 314.19\times\frac{2.588\times10^{-3}}{2} = 406.6mm$$

$$d = 407.9mm = \underline{\underline{40.79cm}}$$

## P.E.

In this case. however. h-0 so that

$$\vec{A}_1 = \frac{\mu_o I_1 a^2 b}{4b^3}\vec{a}_\phi$$

$$\phi_{12} = \frac{\mu_o I_1 a^2}{4b^2}\bullet 2\pi b = \frac{\mu_o \pi I_1 a^2}{2b}$$

$$m_{12} = \frac{\phi_{12}}{I_1} = \frac{\mu_o \pi a^2}{2b} = \frac{4\pi\times10^{-7}\times\pi\times4}{2\times3}$$

$$= \underline{\underline{2.632\ \mu H}}$$

## CHAPTER —Problems

**Prob.**

$$\bar{F} = q\left(\bar{E} + u\times\bar{B}\right)$$

$$\text{If } \bar{F} = 0, \quad \bar{E} = -u\times\bar{B} = \bar{B}\times u$$

$$= \begin{vmatrix} 10 & 20 & 30 \\ 3 & 12 & -4 \end{vmatrix} \times 10^5 \times 10^{-3}$$

$$\bar{E} = \underline{\underline{-44\,\boldsymbol{a}_x + 13\,\boldsymbol{a}_y + 6\,\boldsymbol{a}_z \text{ kV/m}}}$$

**Prob.**

(a) $\vec{F}_e = Q\vec{E} = \underline{\underline{10\bar{a}_x - 30\vec{a}_y + 80\vec{a}_z \text{ N}}}.$

(b) $\vec{F}_m = Q\vec{u}\times\vec{B} = 10\begin{vmatrix} 2 & 0 & -4 \\ 0.3 & 0.1 & 0 \end{vmatrix} = \underline{\underline{4\bar{a}_x - 12\bar{a}_y + 2\bar{a}_z \text{ N}}}.$

(c) $\vec{F} = \vec{F}_e + \vec{F}_m = \underline{\underline{14\vec{a}_x - 42\bar{a}_y + 82\bar{a}_z}} \text{ N}.$

**Prob.**

From Lorentz force equation,

$$\vec{F} = Q\vec{u}\times\vec{B}$$

$$dW = \vec{F}\cdot d\vec{l} = Q\left(\frac{d\vec{l}}{dt}\times\vec{B}\right)\cdot d\vec{l} = 0$$

since $d\vec{l}\times\vec{B}$ is perpendicular to $d\vec{l}$. Thus a magnetic field does not contribute to the work done on a charged particle undergoing displacement $d\vec{l}$.

**Prob.**

(a) $F = m\vec{a} = Q(\vec{E} + \vec{u} \times \vec{B})$

$$\frac{d}{dt}(u_x, u_y, u_z) = 2\left[-4\vec{a}_y + \begin{vmatrix} u_x & u_y & u_z \\ 5 & 0 & 0 \end{vmatrix}\right] = -8\vec{a}_y + 10u_z\vec{a}_y - 10u_y\vec{a}_z$$

i.e. $\frac{du_x}{dt} = 0 \to u_x = A_1$ (1)

$$\frac{du_y}{dt} = -8 + 10u_z \quad (2)$$

$$\frac{du_z}{dt} = -10u_y \quad (3)$$

$$\frac{d^2u_y}{dt^2} = 0 + 10\frac{du_z}{dt} = -100u_y$$

$$\ddot{u}_y + 100u_y = 0 \to u_y = B_1\cos 10t + B_2\sin 10t$$

From (2),

$$10u_z = 8 + \dot{u}_y = 8 - 10B_1\sin 10t + 10B_2\cos 10t$$

$u_z$=0.8-$B_1$sin10t +$B_2$cos10t

At t=0, $\bar{u} = 0 \to A_1 = 0, B_1 = 0, B_2 = -0.8$

Hence,

$$\bar{u} = (0,\ -0.8\sin 10t,\ 0.8 - 0.8\cos 10t) \quad (4)$$

$$u_x = \frac{dx}{dt} = 0 \to x = c_1$$

$$u_y = \frac{dy}{dt} = -0.8\sin 10t \to y = 0.08\cos 10t + c_2$$

$$u_z = \frac{dz}{dt} = 0.8 - 0.8\cos 10t \to z = 0.8t + c_3 - 0.08\sin 10t$$

At t=0, (x, y, z) = (2, 3, -4) ⇒ $c_1$=2, $c_2$=2.92, $c_3$=-4

Hence (x, y, z) = (2, 2.92 + 0.08cos10t, 0.8t – 0.08sin10t – 4)

At t=1,

$$(x, y, z) = \underline{(2, 2.853, -3.156)}$$

(b) From (4), at t=1, $\vec{u} = \underline{\underline{(0,0.435,1.471)}}$ m/s

$$\text{K.E.} = \frac{1}{2}m|\vec{u}|^2 = \frac{1}{2}(1)(0.435^2 + 1.471^2) \ = \underline{1.177\text{J}}$$

**Prob.**

(a) $\vec{a} = \frac{Q\vec{E}}{m} = \frac{3}{2}(12\vec{a}_x + 10\vec{a}_y) = \underline{\underline{18\vec{a}_x + 15\vec{a}_y \ m/s^2}}$

(b) $\vec{a} = \left(\frac{du_x}{dt}, \frac{du_y}{dt}, \frac{du_z}{dt}\right) = 18\vec{a}_x + 15\vec{a}_y$

$$\frac{du_x}{dt} = 18 \longrightarrow u_x = 18t + A$$

$$\frac{du_y}{dt} = 15 \longrightarrow u_y = 15t + B$$

$$\frac{du_z}{dt} = 0 \longrightarrow u_z = C.$$

At $t=0$, $\vec{u} = (4, 0, 3)$, hence $A = 4, B = 0, C = 3$

and $\vec{u} = (18t + 4, 15t, 3)$

$$= 22\vec{a}_x + 15\vec{a}_y + 3\vec{a}_z \ m/s.$$

$$\vec{u}(t=1s) = \underline{\underline{22\vec{a}_x + 15\vec{a}_y}}$$

(c) $K.E. = \frac{1}{2}m|\vec{u}|^2 = \frac{1}{2}(2)(22^2 + 15^2 + 3^2) = \underline{\underline{718\ J.}}$

(d) $\frac{dx}{dt} = u_x = 18t + 4 \longrightarrow x = 9t^2 + 4t + A_1$

$$\frac{dy}{dt} = u_y = 15t \longrightarrow y = 7.5t^2 + B_1$$

$$\frac{dz}{dt} = u_z = 3 \longrightarrow z = 3t + C_1$$

At $t=0$, $(x, y, z) = (1, -2, 0)$. Hence

$$A_1 = 1, \quad B_1 = -2, \quad C_1 = 0.$$

$$(x, y, z) = (9t^2 + 4t + 1, 7.5t^2 - 2, 3t)$$

At $t=1$, $(x, y, z) = \underline{\underline{(14, 5.5, 3)}}.$

**Prob.**

(a) $m\vec{a} = -e(\vec{u} \times \vec{B})$

$$-\frac{m}{e}\frac{d}{dt}(u_x, u_y, u_z) = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & B_o \end{vmatrix} = u_y B_o \vec{a}_x - B_o u_x \vec{a}_y$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = c = 0$$

$$\frac{du_x}{dt} = -u_y \frac{B_o e}{m} = -u_y w, \text{ where } w = \frac{B_o e}{m}$$

$$\frac{du_y}{dt} = u_x w$$

Hence,

$$\ddot{u}_x = -w\ddot{u}_y = -w^2 u_x$$

$$\text{or } \ddot{u}_x + w^2 u_x = 0 \rightarrow u_x = A\cos wt + B\sin wt$$

$$u_y = -\frac{\dot{u}_x}{w} = A\sin wt - B\cos wt$$

$$u_y = \frac{\dot{u}_x}{w} = -A\sin wt + B\cos wt$$

At t=0, $u_x = u_o$, $u_y = 0 \rightarrow A = u_o$, B=0

Hence,

$$u_x = u_o \cos wt = \frac{dx}{dt} \rightarrow x = \frac{u_o}{w}\sin wt + c_1$$

$$u_y = u_o \sin wt = \frac{dy}{dt} \rightarrow y = -\frac{u_o}{w}\cos wt + c_2$$

At t=0, x = 0 = y $\rightarrow$ $c_1$=0, $c_2 = \frac{u_o}{w}$. Hence,

$$x = \frac{u_o}{w}\sin wt, y = \frac{u_o}{w}(1 - \cos wt)$$

$$\frac{u^2{}_o}{w^2}(\cos^2 wt + \sin^2 wt) = \left(\frac{u_o}{w}\right)^2 = x^2 + (y - \frac{u_o}{w})^2$$

showing that the electron would move in a circle centered at (0, $\frac{u_o}{w}$). But since the field does not exist throughout the circular region, the electron passes through a semi-circle and leaves the field horizontally.

(b) d = twice the radius of the semi-circle

$$= \frac{2u_o}{w} = \frac{2u_o m}{B_o e}$$

**Prob.**

$$\bar{\mathrm{B}}_1 = \frac{\mu \mathrm{I}_1}{2\pi\rho}\,\bar{\mathrm{a}}_\phi, \quad \rho = 5$$

$$\bar{\mathrm{a}}_\phi = \bar{\mathrm{a}}_z \times \left(\frac{-3\bar{\mathrm{a}}_x - 4\bar{\mathrm{a}}_y}{5}\right) = \frac{-3\bar{\mathrm{a}}_y + 4\bar{\mathrm{a}}_x}{5}$$

$$\bar{\mathrm{B}}_1 = \frac{4\pi\times10^{-7}\times15}{2\pi(5)}\left(\frac{4}{5}\bar{\mathrm{a}}_x - \frac{3}{5}\bar{\mathrm{a}}_y\right) = \frac{6\times10^{-7}}{5}\left(4\mathrm{a}_x - 3\bar{\mathrm{a}}_y\right)$$

$$\bar{\mathrm{F}}_2 = d\bar{\mathrm{F}} = \mathrm{I}d\bar{\mathrm{l}}\times\bar{\mathrm{B}} = 2\times10^{-2}\times12\times10^{-3}\,\bar{\mathrm{a}}_x\times\frac{6\times10^{-7}}{5}\left(4\mathrm{a}_x - 3\bar{\mathrm{a}}_y\right)$$

$$= \underline{\underline{-86.4\,\bar{\mathrm{a}}_z\ \mathrm{pN}}}$$

**Prob.**

$$m\vec{a} = Q\vec{u}\times\vec{B}$$

$$10^{-3}\vec{a} = -2\times10^{-3}\begin{vmatrix} u_x & u_y & u_z \\ 0 & 6 & 0 \end{vmatrix}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (12u_z, 0, -12u_x)$$

i.e. $\dfrac{du_x}{dt} = 12u_z$ (1)

$\dfrac{du_y}{dt} = 0 \rightarrow u_y = A_1$ (2)

$\dfrac{du_z}{dt} = -12u_x$ (3)

From (1) and (3),

$$\ddot{u}_x = 12\dot{u}_z = -144u_x$$

or

$$\ddot{u}_x + 144u_x = 0 \rightarrow u_x = c_1\cos 12t + c_2\sin 12t$$

From (1), $u_z = -c_1\sin 12t + c_2\cos 12t$

At t=0,

$u_x=5$, $u_y=0$, $u_z=0 \rightarrow A_1=0=c_2$, $c_1=5$

Hence,

$$\vec{u} = (5\cos 12t, 0, -5\sin 12t)$$

$$\vec{u}(t=10s) = (5\cos 120, 0, -5\sin 120) = \underline{\underline{4.071\vec{a}_x - 2.903\vec{a}_z}} \text{ m/s}$$

$$u_x = \frac{dx}{dt} = 5\cos 12t \to x = {}^5\!/_{12}\sin 12t + B_1$$

$$u_y = \frac{dy}{dt} = 0 \to y = B_2$$

$$u_z = \frac{dz}{dt} = -5\sin 12t \to z = {}^5\!/_{12}\cos 12t + B_3$$

At t=0, (x, y, z) = (0, 1, 2) → $B_1$=0, $B_2$=1, $B_3$=$\frac{19}{12}$

$$(x,y,z) = \left(\frac{5}{12}\sin 12t, 1, \frac{5}{12}\cos 12t + \frac{19}{12}\right) \qquad (4)$$

At t=10s,

$$(x,y,z) = \left(\frac{5}{12}\sin 120, 1, \frac{5}{12}\cos 120 + \frac{19}{12}\right) = \underline{(0.2419, 1, 1.923)}$$

By eliminating t from (4),

$x^2 + (z - {}^{19}\!/_{12})^2 = ({}^5\!/_{12})^2$, $y = 1$ which is a circle in the y=1 plane with center at (0,1,19/12). The particle gyrates.

**Prob.**

$$\vec{\Im} = I\vec{L}\times\vec{B} \to \vec{\Im} = \frac{\vec{F}}{L} = I_1\vec{a}_l \times \vec{B}_2 = \frac{\mu_o I_1 I_2 a_l \times \vec{a}_\phi}{2\pi\rho}$$

(a) $$F_{21} = \frac{a_z \times (-a_y) 4\pi \times 10^{-7}(-100)(200)}{2\pi} = \underline{\underline{a_x \text{ mN/m}}} \text{ (repulsive)}$$

(b) $$F_{12} = -F_{21} = -a_x \text{ mN/m (repulsive)}$$

(c) $$\vec{a}_l \times \vec{a}_\phi = \vec{a}_z \times (-\frac{4}{5}\vec{a}_x + \frac{3}{5}\vec{a}_y) = -\frac{3}{5}\vec{a}_x - \frac{4}{5}\vec{a}_y, \rho = 5$$

$$\vec{F}_{31} = \frac{4\pi \times 10^{-7}(-3\times 10^4)}{2\pi(5)}\left(-\frac{3}{5}\vec{a}_x - \frac{4}{5}\vec{a}_y\right)$$

$$= \underline{\underline{0.72\vec{a}_x + 0.96\vec{a}_y}} \text{ mN/m (attractive)}$$

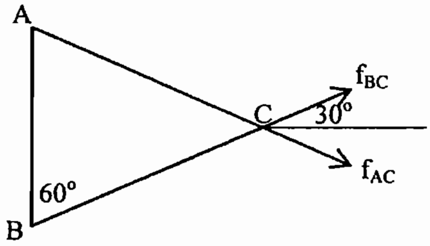
(d) $\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32}$

$$\vec{F}_{32} = \frac{4\pi\times10^{-7}\times6\times10^{4})}{2\pi(3)}\left(\vec{a}_z\times\vec{a}_y\right) = -4\vec{a}_x \text{ mN/m(attractive)}$$

$$\underline{\underline{\vec{F}_3 = -3.28\vec{a}_x + 0.96\vec{a}_y}} \text{ mN/m}$$

(attractive due to L2 and repulsive due to L1)

**Prob.**



**From Prob.**

$$f = \frac{\mu_o I_1 I_2}{2\pi\rho}\vec{a}_\rho$$

$$\vec{f} = \vec{f}_{AC} + \vec{f}_{BC}$$

$$|\vec{f}_{AC}| = |\vec{f}_{BC}| = \frac{4\pi\times10^{-7}\times75\times150}{2\pi\times2} = 1.125\times10^{-3}$$

$$\vec{f} = 2\times1.125\cos30^o\,\vec{a}_x \text{ mN/m}$$

$$= \underline{\underline{1.949\vec{a}_x}} \text{ mN/m}$$

**Prob.**

The field due to the current sheet is

$$\boldsymbol{B} = \frac{\mu}{2}\boldsymbol{K}\times\boldsymbol{a}_n = \frac{\mu_o}{2}10\boldsymbol{a}_x\times(-\boldsymbol{a}_z) = 5\mu_o\boldsymbol{a}_y$$

$$\boldsymbol{F} = I_2\int dl_2\times\boldsymbol{B} = 2.5\int_0^L dx\boldsymbol{a}_x\times(5\mu_o\boldsymbol{a}_y) = 2.5L\times5\mu_o(\boldsymbol{a}_z)$$

$$\frac{\boldsymbol{F}}{L} = 12.5x4\pi x10^{-7}(\boldsymbol{a}_z) = \underline{\underline{15.71\boldsymbol{a}_z \quad \mu\text{N/m}}}$$

**Prob.**

$$T = mB\sin\theta \longrightarrow T_{max} = mB$$

$$\text{or} \quad B = \frac{T_{max}}{ISN} = \frac{50\times10^{-6}}{10\times10^{-3}\times300\times0.6\times10^{-4}}$$

$$= \underline{\underline{104.2 \text{ mWb/m}^2}}.$$

**Prob.**

$$\vec{F} = \int L d\vec{l} \times \vec{B} = \int \vec{J} dv \times \vec{B}$$

$$\vec{J} = \frac{I}{\pi(b^2 - a^2)}\vec{a}_z, \vec{B} = B_o \vec{a}_\rho$$

$$F = \frac{I}{\pi(b^2 - a^2)} \int a_z dv x B_o a_\rho = \frac{IBa_\phi}{\pi(b^2 - a^2)} \int dv$$

$$= \frac{-IB_o}{\pi(b^2 - a^2)} \pi(a^2 - b^2)l$$

$$\vec{f} = \frac{\vec{F}}{l} = \underline{\underline{IB_o \vec{a}_\phi}}$$

**Prob.**

(a) $\vec{m} = IS\vec{a}_n = 1(\pi)(0.1)^2\vec{a}_z = \underline{\underline{0.03142\vec{a}_z}}$ A·m

(b) $\vec{H} = \dfrac{m}{4\pi r^3}(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta)$

where $r = \sqrt{4+4+4} = 2\sqrt{3}$, $\tan\theta = \frac{\rho}{z} = \frac{2\sqrt{2}}{2} = \sqrt{2}$,

i.e. $\sin\theta = \sqrt{\frac{2}{3}}$, $\cos\theta = \frac{1}{\sqrt{3}}$. Hence

$$\vec{H} = \frac{0.01\pi}{4\pi(24\sqrt{3})}\left(\frac{2}{\sqrt{3}}\vec{a}_r + \sqrt{\frac{2}{3}}\vec{a}_\theta\right)$$

$$= \underline{\underline{69.44\vec{a}_r + 49.1\vec{a}_\theta \ \mu A/m.}}$$

(c) $\vec{B} = \dfrac{\mu_o m}{4\pi r^3}(2\cos\theta\,\vec{a}_r + \sin\theta\,\hat{a}_\theta)$

where $r = \sqrt{36+64+100} = 10\sqrt{2}$, $\tan\theta = \frac{10}{10} = 1$. Th

$$\vec{B} = \frac{4\pi \times 10^{-7} \times 0.01\pi}{4\pi(2000\sqrt{2})}\left(\frac{2}{\sqrt{2}}\vec{a}_r + \frac{1}{\sqrt{2}}\hat{a}_\theta\right)$$

$$= \underline{\underline{1.571\vec{a}_r + 0.7853\vec{a}_\theta \ pWb/m^2.}}$$

**Prob.**

$$\vec{B} = \frac{4\pi \times 10^{-7} \times 10^{-2}}{288}(2\vec{a}_r + \sqrt{2}\,\vec{a}_\theta)$$

$$\vec{m} = 4\vec{a}_z = 4(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta) = 4\left(\frac{\vec{a}_r}{\sqrt{3}} - \sqrt{\frac{2}{3}}\,\vec{a}_\theta\right)$$

$$\vec{T} = \vec{m} \times \vec{B} = \frac{4 \times 10^{-2} \times 4\pi \times 10^{-7}}{288}\begin{vmatrix} \vec{a}_r & \vec{a}_\theta & \vec{a}_\phi \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ 2 & \sqrt{2} & 0 \end{vmatrix}$$

$$= -\sqrt{\frac{2}{3}} \cdot \frac{\pi}{8} \cdot 10^{-9}\,\vec{a}_\phi = \underline{\underline{-1.425 \times 10^{-10}\,\vec{a}_\phi\ \text{N}\cdot\text{m}}}$$

**Prob.**

$T = mB = NISB = 1000 \times 2 \times 10^{-3} \times 300 \times 10^{-6} \times 0.4$

$= \underline{\underline{240\mu\text{Nm}}}$

**Prob.**

$$f(x,y,z) = x + 2y - 5z - 12 = 0 \quad \longrightarrow \quad \nabla f = \boldsymbol{a}_x + 2\boldsymbol{a}_y - 5\boldsymbol{a}_z$$

$$\boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\boldsymbol{a}_x + 2\boldsymbol{a}_y - 5\boldsymbol{a}_z}{\sqrt{30}}$$

$$\boldsymbol{m} = NIS\boldsymbol{a}_n = 2x60x8x10^{-4}\frac{(\boldsymbol{a}_x + 2\boldsymbol{a}_y - 5\boldsymbol{a}_z)}{\sqrt{30}} = \underline{\underline{17.53\boldsymbol{a}_x + 35.05\boldsymbol{a}_y - 87.64\boldsymbol{a}_z\ \text{mAm}}}$$

**Prob.**

$$\vec{B} = \frac{k}{r^3}(2\cos\theta\vec{a}_r + \sin\theta\vec{a}_\theta)$$

At (10, 0, 0), r = 10; $\theta = \pi/2, \vec{a}_r = \vec{a}_x, \vec{a}_\theta = -\vec{a}_z$

$$-0.5 \times 10^{-3}\vec{a}_z = \frac{k}{10^3}(0 - \vec{a}_z) \rightarrow k = 0.5$$

Thus,

$$\vec{B} = \frac{0.5}{r^3}(2\cos\theta\vec{a}_r + \sin\theta\vec{a}_\theta)$$

(a) At (0, 3, 0), r=3, $\theta = \pi/2, \vec{a}_r = \vec{a}_y, \vec{a}_\theta = -\vec{a}_z$

$$\vec{B} = \frac{0.5}{27}(0 - \vec{a}_z) = \underline{\underline{-18.52\vec{a}_z}} \text{ mWb/m}^2$$

(b) At (3, 4, 0), r=5, $\theta = \pi/2, \vec{a}_\theta = -\vec{a}_z$

$$\vec{B} = \frac{0.5}{125}(0 - \vec{a}_z) = \underline{\underline{-4\vec{a}_z}} \text{ mWb/m}^2$$

(c) At (1, 1, -1), r=$\sqrt{3}$, $\tan\theta = \rho/z = \sqrt{2}/-1$, i.e.

$$\sin\theta = \sqrt{2}/\sqrt{3}, \cos\theta = -1/\sqrt{3}$$

$$\vec{B} = \frac{0.5}{3\sqrt{3}}(-2/\sqrt{3}\vec{a}_r + \sqrt{2}/\sqrt{3}\vec{a}_\theta) = \underline{\underline{-111\vec{a}_r + 78.6\vec{a}_\theta}} \text{ mWb/m}^2$$

**Prob.**

(a) $\mu_r = \chi_m + 1 = \underline{\underline{5.2}}$

(b) $\mu = \mu_o\mu_r = 5.2 \times 4\pi \times 10^{-7} = \underline{\underline{6.534 \times 10^{-6} \text{ H/m}}}$

(c) $\boldsymbol{M} = \chi_m \boldsymbol{H} = 4.2(0.2 \times \boldsymbol{a}_y) = \underline{\underline{0.84 \times \boldsymbol{a}_y \text{ A/m}}}$

(d) $\boldsymbol{B} = \mu_o\mu_r\boldsymbol{H} = 4\pi x 10^{-7}(5.2)(0.2x\boldsymbol{a}_y) = \underline{\underline{1.307x\boldsymbol{a}_y \ \mu\text{Wb/m}^2}}$

(e) $\boldsymbol{J} = \nabla x \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0.2x & 0 \end{vmatrix} = \underline{\underline{0.2\boldsymbol{a}_z \text{ A/m}^2}}$

(f) $\boldsymbol{J}_b = \chi_m \boldsymbol{J} = 4.2x0.2\boldsymbol{a}_z = \underline{\underline{0.84\boldsymbol{a}_z \text{ A/m}^2}}$

**Prob.**

(a) $$\bar{M} = x_m H = x_m \frac{B}{\mu_o \mu}$$

$$= \frac{4999}{5000} \times \frac{1.5}{4\pi \times 10^{-7}} = \underline{\underline{1.193 \times 10^6 \text{ A/m}}}$$

(b) $$\bar{M} = \frac{\sum_{k=1}^{N} m_k}{\Delta v}$$

If we assume that all $\bar{m}_k$ align with the applied $\bar{B}$ field,

$$M = \frac{Nm_k}{\Delta v} \rightarrow m_k = \frac{Nm_k}{N/\Delta v} = \frac{1.193 \times 10^6}{8.5 \times 10^{28}}$$

$$m_k = \underline{\underline{1.404 \times 10^{-23} \text{ A} \cdot \text{m}^2}}$$

**Prob.**

For case 1,

$$\mu = \frac{B_1}{H_1} = \frac{2}{1200}$$

$$\mu_r = \frac{\mu}{\mu_o} = \frac{1}{600} \times \frac{1}{4\pi \times 10^{-7}} = 1326.3$$

$$\psi_m = \mu_r - 1 = 1325.3$$

$$\overline{M}_1 = \psi_m H_1 = 1{,}590{,}366$$

For case 2,

$$\mu = \frac{B_2}{H_2} = \frac{1.4}{400}$$

$$\mu_r = \frac{\mu}{\mu_o} = \frac{1.4}{400} \times \frac{1}{4\pi \times 10^{-7}} = 2785.2$$

$$\psi_m = \mu_r - 1 = 2784.2$$

$$M = \psi_m H = 1{,}113{,}630$$

$$\Delta M = M_1 - M_2 = 476{,}680$$

$$= \underline{\underline{476.7 \text{ kA/m}}}$$

**Prob.**

Let $\boldsymbol{H}_2 = (H_x, H_y, H_z)$

$$(\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \boldsymbol{a}_{n12} = \boldsymbol{K} \longrightarrow \boldsymbol{H}_2 \times \boldsymbol{a}_{n12} = \boldsymbol{H}_1 \times \boldsymbol{a}_{n12} - \boldsymbol{K}$$

But $\boldsymbol{a}_{n12} = \boldsymbol{a}_z$

$$(H_x, H_y, H_z) \times \boldsymbol{a}_z = (24, -30, 40) \times \boldsymbol{a}_z - 6\boldsymbol{a}_x \longrightarrow (H_y, -H_x, 0) = (-30, -24, 0) - 6\boldsymbol{a}_x$$

Equating components,

$$H_y = -30 - 6 = -36$$

$$-H_x = -24 \longrightarrow H_x = 24$$

$$\boldsymbol{B}_{1n} = \boldsymbol{B}_{2n} \longrightarrow \mu_1 \boldsymbol{H}_{1n} = \mu_2 \boldsymbol{H}_{2n}$$

$$\boldsymbol{H}_{2n} = \frac{\mu_1}{\mu_2} \boldsymbol{H}_{1n} = \frac{50\mu_o}{100\mu_o}(40\boldsymbol{a}_z) = 20\boldsymbol{a}_z$$

$$H_z = H_{2n} = 20$$

Hence,

$$\boldsymbol{H}_2 = (24, -36, 20) \text{ kA/m}$$

$$\boldsymbol{B}_2 = 100\mu_o(24, -36, 20) \times 10^3 = 10^5 \times 4\pi x 10^{-7}(24, -36, 20)$$

$$= \underline{\underline{3.016\boldsymbol{a}_x - 4.524\boldsymbol{a}_y + 2.513\boldsymbol{a}_z \text{ Wb/m}^2}}$$

**Prob.**

(a)

$$\bar{B}_{1n} = \bar{B}_{2n} = 15\,\bar{a}_\phi$$

$$\bar{H}_{1t} = \bar{H}_{2t} \rightarrow \frac{\bar{B}_{1t}}{\mu_1} = \frac{\bar{B}_{2t}}{\mu_2}$$

$$\bar{B}_{1t} = \frac{\mu_1}{\mu_2}\bar{B}_{2t} = \frac{2}{5}\left(10\,\bar{a}_\rho - 20\,\bar{a}_z\right) = 4\,\bar{a}_\rho - 8\,\bar{a}_z$$

Hence,

$$\bar{B}_1 = \underline{\underline{4\,\bar{a}_\rho + 15\,\bar{a}_\phi - 8\,\bar{a}_z \text{ mWb/m}^2}}$$

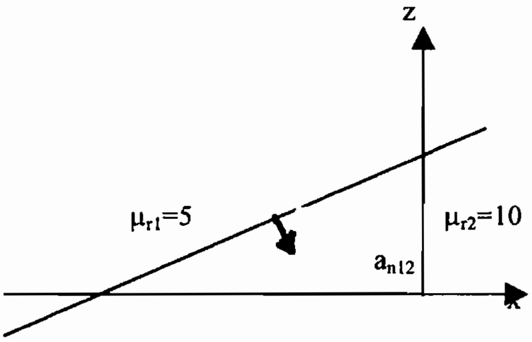
(b)

$$w_{m1} = \frac{1}{2}\bar{B}_1 \cdot \bar{H}_1 = \frac{B_1^2}{2\mu_1} = \frac{\left(4^2 + 15^2 + 8^2\right) \times 10^{-6}}{2 \times 2 \times 4\pi \times 10^{-7}}$$

$$w_{m1} = \underline{\underline{60.68 \;\; J/m^3}}$$

$$w_{m2} = \frac{B_2^2}{2\mu_2} = \frac{\left(10^2 + 15^2 + 20^2\right) \times 10^{-6}}{2 \times 5 \times 4\pi \times 10^{-7}} = \underline{\underline{57.7 \;\; J/m^3}}$$

**Prob.**



Let $\vec{H}_2 = (H_x, H_y, H_z)$

$$(\vec{H}_1 - \vec{H}_2) \times \vec{a}_{n12} = \vec{k}$$

where f(x, z)= 5z – 4x = 0 and

$$a_{n12} = -\frac{\nabla f}{|\nabla f|} = \frac{4\vec{a}_x - 5\vec{a}_z}{\sqrt{41}}$$

$$(\vec{H}_1 - \vec{H}_2) \times \vec{a}_{n12} = \frac{1}{\sqrt{41}} \begin{vmatrix} 25 - H_x & -30 - H_y & 45 - H_z \\ 4 & 0 & -5 \end{vmatrix}$$

$$= \frac{1}{\sqrt{41}} \left[ 150 + 5H_y, 180 - 4H_z, 120 + 4H_y \right] = \vec{k} = 35\vec{a}_y$$

Equating components,

$\vec{a}_x :$ $150 + 5H_y = 0 \rightarrow H_y = -30$

$\vec{a}_y :$ $300 - 4H_z - 5H_x = 35 \rightarrow 4H_z + 5H_x = 270$

$\vec{a}_z :$ $120 + 4H_y = 0 \rightarrow H_y = -30$

Also, $\vec{B}_{1n} = \vec{B}_{2n} \rightarrow \mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$

$$5\mu_o(2, -30, 45)\frac{(4,0,5)}{\sqrt{41}} = 10\mu_o(H_x, H_y, H_z)\frac{(4,0,5)}{\sqrt{41}}$$

$100 - 225 = 68H_x - 10H_z$

or $125 = 10H_z - 8H_x$
$= 10H_z - 8(54\text{-}0.8H_z) \longrightarrow H_z = 33.96$

and $H_x = 54 - 0.8\ H_z = 26.83$

Thus,

$$\vec{H}_z = \underline{\underline{26.83\vec{a}_x - 30\vec{a}_y + 33.96\vec{a}_z}}\ \text{A/m}$$

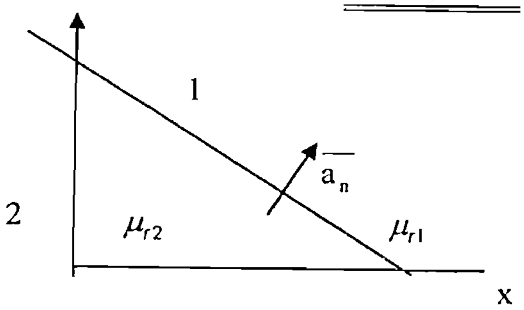
**Prob.**

(a) $w_{m1} = \frac{1}{2}\bar{B}_1 \cdot \bar{H}_1 = \frac{1}{2}\mu_0\mu_{r1}\bar{H}_1 \cdot \bar{H}_1\ ,\ \mu_r = 1$

$w_{m1} = \frac{1}{2} \times 4\pi \times 10^{-7} \times 1\ (16+9+1)$

$= \underline{\underline{16.34\ \mu J/m^3}}$

(b)



$f(x,y) = 2x + y - 8 = 0$

$\nabla f = 2\bar{a}_x + \bar{a}_y\ ,\quad \bar{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{2\bar{a}_x + \bar{a}_y}{\sqrt{5}}$

$\bar{H}_{1n} = (\bar{H}_1 \cdot \bar{a}_n)\bar{a}_n = \left(\frac{-8+3}{5}\right)(2\bar{a}_x + \bar{a}_y) = -2\bar{a}_x - \bar{a}_y$

$\bar{H}_{1t} = \bar{H}_1 - \bar{H}_{1n} = -2\bar{a}_x + 4\bar{a}_y - \bar{a}_z = \bar{H}_{2t}$

$\bar{B}_{2n} = \bar{B}_{1n} \rightarrow \mu_2\bar{H}_{2n} = \mu_1\bar{H}_{1n}$

$$\bar{H}_{2n} = \frac{\mu_1}{\mu_2}\bar{H}_{1n} = \frac{1}{10}\left(-2\bar{a}_x - \bar{a}_y\right)$$

$$= -0.2\bar{a}_x - 0.1\bar{a}_y$$

$$H_2 = H_{2t} + H_{2n} = -2.2a_x + 3.9a_y - a_z$$

$$M_2 = \chi_{m2}H_2 = 9H_2 = \underline{\underline{-19.8a_x + 35.1a_y - 9a_z \text{ A/m}}}$$

$$B_2 = \mu_2 H_2 = 10\mu_o H_2 = 4\pi x(-2.2, 3.9, -1) \ \mu\text{Wb/m}^2$$

$$= \underline{\underline{-27.75a_x + 49a_y - 12.56a_z \ \mu\text{Wb/m}^2}}$$

(c ) $H_l \bullet a_n = H_l \cos\theta_l$

$$\cos\theta_1 = \frac{H_1 \bullet a_n}{H_1} = \frac{(-8+3)/\sqrt{5}}{\sqrt{16+9+1}} = -0.4385 \quad \longrightarrow \quad \underline{\underline{\theta_1 = 116^\circ}}$$

$$\cos\theta_2 = \frac{H_2 \bullet a_n}{H_2} = \frac{(-4.4+3.9)/\sqrt{5}}{4.588} = -0.0487 \quad \longrightarrow \quad \underline{\underline{\theta_1 = 92.8^o}}$$

**Prob.**

Let f(x,y,z) = x+y+z-2, $\nabla f = \boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z$

$$\boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z}{\sqrt{3}}$$

$$\boldsymbol{B}_{1n} = (\boldsymbol{B}_1 \bullet \boldsymbol{a}_n)\boldsymbol{a}_n = \frac{(50+30-20)}{3}(1,1,1) = (20,20,20)$$

$$\boldsymbol{B}_{1t} = \boldsymbol{B}_1 - \boldsymbol{B}_{1n} = (30,10,-40)$$

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = (20,20,20)$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \quad \longrightarrow \quad \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$

$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = \frac{400}{1}(30,10,-40) = (12000, 4000, -16000)$$

$$\boldsymbol{B}_2 = \boldsymbol{B}_{2n} + \boldsymbol{B}_{2t} = \underline{\underline{12020a_x + 4020a_y - 15980a_z \ \text{Wb/m}^2}}$$

$$\boldsymbol{H}_2 = \frac{\boldsymbol{B}_2}{\mu_2} = \frac{(12020, 4020, -15980)}{400x4\pi x10^{-7}} = \underline{\underline{23.913\boldsymbol{a}_x + 7.998\boldsymbol{a}_y - 31.79\boldsymbol{a}_z \ \text{MA/m}}}$$

**Prob.**

(a) $\mathrm{B} = 70 + (210)^2 = 44.17\mathrm{Wb/m^2}$

$$\mu_r = \frac{B}{\mu_o H} = \frac{44.17\times10^3}{4\pi\times10^{-7}\times210} = \underline{\underline{167.4}}$$

(b) $$W_m = \int_0^{H_o} HdB = \int_0^{H_o} H(1/3 + 2H)dH$$

$$= \frac{H_o^{\ 2}}{6} + \frac{2}{3}H_o^{\ 3} = 7350 + 6174000$$

$$= \underline{\underline{6181.35}} \quad \mathrm{kJ/m^3}$$

**Prob.**

$$\boldsymbol{a}_n = \boldsymbol{a}_\rho$$

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = 22\mu_o\boldsymbol{a}_\rho$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \qquad \longrightarrow \qquad \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$

$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = \frac{\mu_o}{800\mu_o}(45\mu_o\boldsymbol{a}_\phi) = 0.05625\mu_o\boldsymbol{a}_\phi$$

$$\underline{\underline{B_2 = \mu_o(22a_\rho + 0.05625a_\phi)\ \mathrm{Wb/m^2}}}$$

**Prob.**

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$$L' = \frac{L}{L} = \mu_o n^2 S = \frac{\mu_o N^2 S}{l^2} = \frac{\mu_o N^2 \pi d^2}{4 l^2}$$

$$L = \frac{4\pi\times10^{-7}\times10^6\pi\times100\times10^{-4}}{4(0.4)} = \underline{\underline{24.674\ \mathrm{mH}}}$$

**Prob.**

$$\vec{H}_{1n} = -3\vec{a}_z, \qquad \vec{H}_{1t} = 10\vec{a}_x + 15\vec{a}_y$$

$$\vec{H}_{2t} = \vec{H}_{1t} = 10\vec{a}_x + 15\vec{a}_y$$

$$\vec{H}_{2n} = \frac{\mu_1}{\mu_2}\vec{H}_{1n} = \frac{1}{200}(-3\vec{a}_z) = -0.015\vec{a}_z$$

$$\vec{H}_2 = 10\vec{a}_x + 15\vec{a}_y - 0.015\vec{a}_z$$

$$\vec{B}_2 = \mu_2\vec{H}_2 = 200\times 4\pi\times 10^{-7}(10,15,-0.015)$$

$$\vec{B}_2 = \underline{\underline{2.51\vec{a}_x + 3.77\vec{a}_y - 0.0037\vec{a}_z}} \text{ mWb/m}^2$$

$$\tan\alpha = \frac{B_{2n}}{B_{2t}}$$

or $\alpha = \tan^{-1}\dfrac{0.0037}{\sqrt{2.51^2 + 3.77^2}} = \underline{\underline{0.047^\circ}}$

**Prob.**

(a) $\psi_m = \mu_r - 1 = \underline{\underline{3.5}}$

(b) $\overline{H} = \dfrac{\overline{B}}{\mu} = \dfrac{4y\,\bar{a}_z \times 10^{-3}}{4\pi\times 10^{-7}\times 4.5} = \underline{\underline{707.3y\,\bar{a}_z\ A/m}}$

(c) $\overline{M} = \psi_m\overline{H} = \underline{\underline{2.476y\,\bar{a}_z \text{ kA/m}}}$

(d) $$\bar{\tau}_b = \nabla\times\overline{M} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & M_z(y) \end{vmatrix} = \frac{dM_z}{dy}\bar{a}_x$$

$$= \underline{\underline{2.476\,\bar{a}_x \text{ kA/m}^2}}$$

**Prob.**

(a) $\psi_m = \mu_r - 1 = \underline{\underline{5.5}}$

(b) $\overline{B} = \mu_0 \mu_r \overline{H} = 4\pi \times 10^{-7} \times 6.5(10, 25, -40)$

$= \underline{\underline{81.68\,\bar{a}_x + 204.2\,\bar{a}_y - 326.7\,\bar{a}_z\ \mu\text{Wb/m}^2}}$

(c) $\overline{M} = \psi_m \overline{H} = \underline{\underline{55\,\bar{a}_x + 137.5\,\bar{a}_y - 220\,\bar{a}_z\ \text{A/m}}}$

(d) $W_m = \frac{1}{2}\mu\overline{H}\cdot\overline{H} = \frac{1}{2}(6.5)4\pi \times 10^{-7} \times 6.5(100 + 625 + 1600)$

$= \underline{\underline{9.5\ \text{mJ/m}^2}}$

**Prob.**

$$\mu_r = \psi_m + 1 = 20$$

$$W_m = \frac{1}{2}\overline{B}_1 \cdot \overline{H}_1 = \frac{1}{2}\mu\overline{H}\cdot\overline{H}$$

$$= \frac{1}{2}\mu\left(25x^4y^2z^2 + 100x^2y^4z^2 + 225x^2y^2z^4\right)$$

$$W_m = \int W_m dv$$

$$= \frac{1}{2}\mu\left[25\int_0^1 x^4dx \int_0^2 y^2dy \int_{-1}^2 z^2dz + 100\int_0^1 x^2dx \int_0^2 y^4dy \int_{-1}^2 z^2dz\right.$$

$$= \left. + 225\int_0^1 x^2dx \int_0^2 y^2dy \int_{-1}^2 zdz\right]$$

$$= \frac{25\mu}{2}\left[\frac{x^5}{5}\Big|_0^1 \frac{y^3}{3}\Big|_0^2 \frac{z^3}{3}\Big|_{-1}^2 + 4\frac{x^3}{3}\Big|_0^1 \frac{y^5}{5}\Big|_0^2 \frac{z^3}{3}\Big|_{-1}^2\right.$$

$$= \left. + 9\frac{x^3}{3}\Big|_0^1 \frac{y^3}{3}\Big|_0^2 \frac{z^5}{5}\Big|_{-1}^2\right]$$

$$= \frac{25\mu}{2}\left(\frac{1}{5}\cdot\frac{8}{3}\cdot\frac{9}{3} + \frac{4}{3}\cdot\frac{32}{3}\cdot\frac{9}{3} + \frac{9}{3}\cdot\frac{8}{3}\cdot\frac{33}{5}\right)$$

$$= \frac{25}{2}\times 4\pi\times 10^{-7}\times 20\times\frac{3600}{45}$$

$$W_m = \underline{\underline{25.13\,\text{mJ}}}$$

**Prob.**

$$L = \frac{\mu N^2 S}{l} \rightarrow N^2 = \frac{Ll}{\mu S} = \frac{L2\pi\rho_o}{\mu_o\mu_r S}$$

$$= \frac{2.5\times 2\pi\times 0.5}{4\pi\times 10^{-7}\times 200\times 12\times 10^{-4}} = \frac{25}{96}\times 10^{8}$$

$$N = \underline{\underline{5103\text{ turns}}}$$

**Prob.**

(a) $$\vec{H} = \tfrac{1}{2}\vec{k}\times\vec{a}_n = \tfrac{1}{2}(30-40)\vec{a}_x\times(-\vec{a}_r) = \underline{\underline{-5\vec{a}_y}}\text{ A/m}$$

$$\vec{B} = \mu_o\vec{H} = 4\pi\times 10^{-7}(-5\vec{a}_y) = \underline{\underline{-6.28\vec{a}_y}}\mu\text{ Wb/m}^2$$

(b) $$\vec{H} = \tfrac{1}{2}(-30-40)\vec{a}_y = \underline{\underline{-35\vec{a}_y}}\text{ A/m}$$

$$\boldsymbol{B} = \mu_o\mu_r\boldsymbol{H} = 4\pi\times 10^{-7}(2.5)(-35\boldsymbol{a}_y) = \underline{\underline{-110\boldsymbol{a}_y\ \mu\text{Wb/m}^2}}$$

(c) $$\vec{H} = \tfrac{1}{2}(-30+40)\vec{a}_y = \underline{\underline{5\vec{a}_y}}$$

$$\vec{B} = \mu_o\vec{H} = \underline{\underline{6.283\vec{a}_y}}\mu\text{ Wb/m}^2$$

**Prob.**

For d >> a,

$$L' = L/l = \frac{\mu_o}{\pi}\ln\frac{d}{a} = \frac{4\pi\times10^{-7}}{\pi}\ln\frac{d}{a} = 2.5\times10^{-6}$$

$$\text{or } \ln\frac{d}{a} = 6.25 \rightarrow \frac{d}{a} = e^{6.25} = 518.01$$

$$a = \frac{3}{518.01} = 5.78mm$$

$$D = 2a = \underline{\underline{11.58 \text{ mm}}}$$

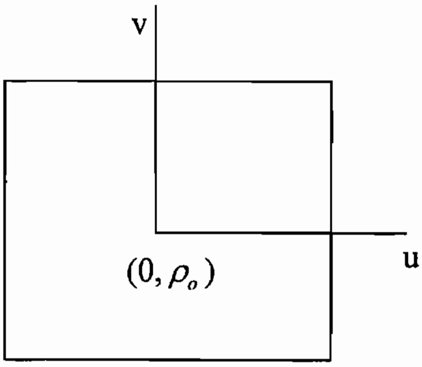
**Prob.**

**From Problem**

$$L = \frac{\mu_o N^2 S}{\ell} = \frac{4\pi x10^{-7}x(450)^2 x\pi(10^{-2})^2}{0.1} = \underline{\underline{800\mu H}}$$

**Prob.**

(a) The square cross-section of the toroid is shown below. Let (u,v) be the local coordinates and $\rho_o$ =mean radius. Using Ampere's law around a circle passing through P, we get



$$H(2\pi)(\rho_o + v) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + v)}$$

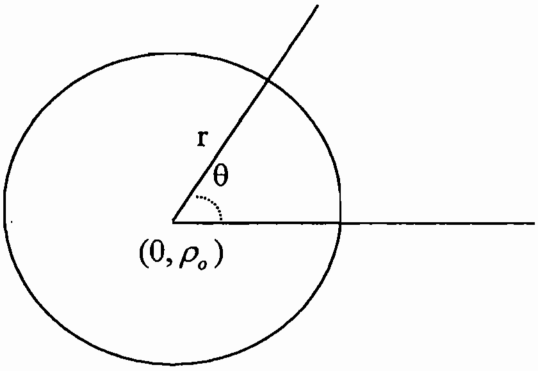
The flux per turn is

$$\Psi = \int_{u=-a/2}^{a/2} \int_{v=-a/2}^{a/2} B du dv = \frac{\mu_o N I a}{2\pi} \ln\left(\frac{\rho_o + a/2}{\rho_o - a/2}\right)$$

$$L = \frac{N\Psi}{I} = \frac{\mu_o N^2 I a}{2\pi} \ln\left(\frac{2\rho_o + a}{2\rho_o - a}\right)$$

(b) The circular cross-section of the toroid is shown below. Let (r,θ) be the local coordinates. Consider a point P($r\cos\theta, \rho_o + r\sin\theta$) and apply Ampere's law around a circle that passes through P.

$$H(2\pi)(\rho_o + r\sin\theta) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + r\sin\theta)} \approx \frac{NI}{2\pi\rho_o}\left(1 - \frac{r\sin\theta}{\rho_o}\right)$$



Flux per turn $\Psi = \int_{r=0}^{a} \int_{\theta}^{2\pi} \frac{\mu N I}{2\pi\rho_o}\left(1 - \frac{r\sin\theta}{\rho_o}\right) r dr d\theta = \frac{\mu N I}{2\pi\rho_o} \frac{a^2}{2}(2\pi)$

$$L = \frac{N\Psi}{I} = \frac{\mu N^2 a^2}{2\rho_o}$$

**Prob.**

**From Problem**

$$L = \frac{\mu N^2 a}{2\pi} \ln\left[\frac{2\rho_o + a}{2\rho_o - a}\right] \longrightarrow N^2 = \frac{2\pi L}{\mu a} \frac{1}{\ln\left[\frac{2\rho_o + a}{2\rho_o - a}\right]}$$

$$a^2 = 40 \text{ cm}^2 \longrightarrow a = 6.324\text{cm}$$

$$\rho_o = 50/2 = 25 \text{ cm}$$

$$N^2 = \frac{2\pi x2}{4\pi x10^{-7} x2000x6.324x10^{-2}} \frac{1}{\ln\left[\frac{50+6.324}{50-6.324}\right]} = \frac{10^6}{3.2172}$$

$$N = \underline{\underline{558}}$$

**Prob.**

We may approximate the longer solenoid as infinite so that $B_1 = \frac{\mu_o N_1 I_1}{l_1}$. The flux linking the second solenoid is:

$$\psi_2 = N_2 B_1 S_1 = \frac{\mu_o N_1 I_1}{l_1} \bullet \pi r_1^2 \bullet N_2$$

$$M = \frac{\psi_2}{I_1} = \frac{\mu_o N_1 N_2}{l_1} \bullet \pi r_1^2$$

**Prob.**

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a}_\rho$$

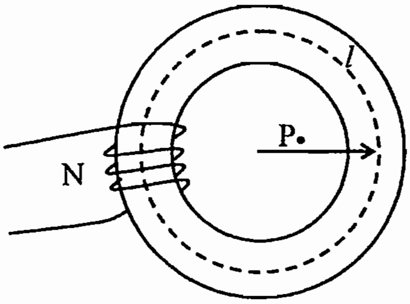
$$w_m = \frac{1}{2}\mu|\boldsymbol{H}|^2 = \frac{1}{2}\mu\frac{I^2}{4\pi^2\rho^2}$$

$$W = \int w_m dv = \iiint \frac{1}{2}\mu\frac{I^2}{4\pi^2\rho^2}\rho d\phi d\rho dz = \frac{1}{4\pi}\mu I^2 L\ln(b/a)$$

$$= \frac{1}{4\pi}x4x4\pi x10^{-7}(625x10^{-6})3\ln(18/12) = \underline{\underline{304.1 \text{ pJ}}}$$

**Prob.**

$$NI = Hl = \frac{Bl}{\mu}$$



$$N = \frac{Bl}{\mu_o\mu_r I} = \frac{1.5\times0.6\pi}{4\pi\times10^{-7}\times600\times12}$$

$$\underline{\underline{= 313 \text{ turns}}}$$

**Prob.**

Total F = NI = 2000 x 10 = 20,000 A.t

$$R_c = \frac{l_c}{\mu_o\mu_r S} = \frac{(24+20-0.6)\times10^{-2}}{4\pi\times10^{-7}\times1500\times2\times10^{-4}} = \underline{\underline{0.115 \text{ x } 10^7 \text{ A.t/m}}}$$

$$R_a = \frac{l_a}{\mu_o\mu_r S} = \frac{0.6\times10^{-2}}{4\pi\times10^{-7}(1)\times2\times10^{-4}} = \underline{\underline{2.387 \text{ x } 10^7 \text{ A.t/m}}}$$

$R = R_a + R_c = 2.502 \text{ x } 10^7$ A.t/m

$$\psi = \frac{\Im}{R} = \psi_a = \psi_c = \frac{20,000}{2.502\times10^7} = \underline{\underline{8 \text{ x } 10^{-4} \text{ Wb/m}^2}}$$

$$\Im_a = \frac{R_a}{R_a+R_c}\Im = \frac{2.387\times20,000}{2.502} = \underline{\underline{19,081}} \text{ A.t}$$

$$\Im_c = \frac{R_c}{R_a+R_c}\Im = \frac{0.115\times20,000}{2.502} = \underline{\underline{919}} \text{ A.t}$$

**Prob.**

$$NI = \Psi R = \Psi \frac{l}{\mu S} = \frac{2.56x10^{-3}x2.5x10^{-3}}{4\pi x10^{-7}x40x60x10^{-6}} = \underline{\underline{2122.1 \ \text{A.t}}}$$

## Chapter —Practice Examples

### P.E.

(a) $V_{emf} = \int (\bar{u} \times \bar{B}) \cdot \partial \bar{l} = uBl = 8(0.5)(0.1) = \underline{\underline{0.4}}$ V

(b) $I = \frac{V_{emf}}{R} = \frac{0.4}{20} = \underline{\underline{20}}$ mA

(c) $\overline{F_m} = I\bar{l} \times \bar{B} = 0.02\left(-0.1\overline{a_y} \times 0.5\overline{a_z}\right) = \underline{\underline{-\overline{a_x}}}$ mN

(d) $P = FU = I^2 R = 8$ mW

or $\quad P = \frac{V_{emf}}{R} = \frac{(0.4)^2}{20} = \underline{\underline{8}}$ mW

### P.E.

(a) $V_{emf} = \int (\bar{u} \times \bar{B}) \cdot d\bar{l}$

where $\bar{B} = B_o\overline{a_y} = B_o\left(\sin\phi\overline{a_\rho} + \cos\phi\overline{a_\phi}\right)$ , $B_o = 0.05$ Wb/m$^2$

$$\left(\bar{u} \times \bar{B}\right) \cdot \overline{dl} = -\rho\omega B_o \sin\phi dz = -0.2\pi \sin\left(\omega t + \pi/2\right) dz$$

$$V_{emf} = \int_0^{0.03} (\bar{u} \times \bar{B}) \cdot d\bar{l} = -6\pi\cos(100\pi t) \text{ mV}$$

At t = 1ms,

$$V_{emf} = -6\pi\cos 0.1\pi = \underline{\underline{-17.93}} \text{ mV}$$

$$i = \frac{V_{emf}}{R} = -60\pi\cos(100\pi.t) \text{ mA}$$

At t = 3ms, $i = -60\pi\cos 0.3\pi = \underline{\underline{-110.8 \text{ mA}}}$ A

(b) Method 1:

$$\Psi = \int \bar{B} \cdot d\bar{S} = \int B_o t\left(\cos\phi\overline{a_\rho} - \sin\phi\overline{a_\phi}\right) \cdot d\rho dz\overline{a_\phi} = -\int_0^{\rho_o}\int_0^{z_o} B_o t \sin\phi d\rho dz = -B_o\rho_o z_o t\sin\phi$$

where $B_o = 0.02$, $\rho_o = 0.04$, $z_o = 0.03$

$$\phi = \omega t + \pi/2$$

$$\Psi = -B_o\rho_o z_o t\cos\omega t$$

$$V_{emf} = -\frac{\partial\Psi}{\partial t} = B_o\rho_o z_o \cos\omega t - B_o\rho_o z_o t\omega\sin\omega t$$

$$= (0.02)(0.04)(0.03)\left[\cos\omega t - \omega t\sin\omega t\right]$$

$$= 24\left[\cos\omega t - \omega t\sin\omega t\right]\mu V$$

Method 2:

$$V_{emf} = -\int \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s} + \int (\vec{u} \times \vec{B}).d\vec{l}$$

$$\vec{B} = B_o t\vec{a}_x = B_o t(\cos\phi\vec{a}_\rho - \sin\phi\vec{a}_\phi), \phi = \omega t + \pi/2$$

$$\frac{\partial \vec{B}}{\partial t} = B_o(\cos\phi\vec{a}_\rho - \sin\phi\vec{a}_\phi)$$

Note that only explicit dependence of $\vec{B}$ on time is accounted for, i.e. we make ϕ = constant because it is transformer (stationary) emf. Thus,

$$V_{emf} = -B_o \int_0^{\rho_o}\int_0^{z_o} (\cos\phi\vec{a}_\rho - \sin\phi\vec{a}_\phi) dpdz\vec{a}_\phi + \int_{z_o}^{0} -\rho_o \omega B_o t\cos\phi dz$$

$$= B_o\rho_o z_o(\sin\phi - \omega t\cos\phi), \phi = \omega t + \pi/2$$

$$= B_o\rho_o z_o(\cos\omega t - \omega t\sin\omega t)$$ as obtained earlier.

At t = 1ms,

$$V_{emf} = 24[\cos 18^o - 100\pi \times 10^{-3}\sin 18^o]\mu V$$

$$= \underline{\underline{20.5\mu V}}$$

At t = 3ms,

$$i = 240[\cos 54^o - .03\pi\sin 54^o]mA$$

$$= \underline{\underline{-41.93\ mA}}$$

**P.E.**

$$V_1 = -N_1\frac{d\psi}{dt}, V_2 = -N_2\frac{d\psi}{dt}$$

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \rightarrow V_2 = \frac{N_2}{N_1}V_1 = \frac{300 \times 120}{500} = \underline{\underline{72V}}$$

**P.E.**

(a) $\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \underline{\underline{-20\omega\varepsilon_o \sin(\omega t - 50x)\vec{a}_y A / m^2}}$

(b) $\nabla \times \vec{H} = \vec{J}_d \rightarrow -\frac{\partial \vec{H}_z}{\partial x}\vec{a}_y = -20\omega\varepsilon_o \sin(\omega t - 50x)\vec{a}_y$

or $\vec{H} = \frac{20\omega\varepsilon_o}{50}\cos(\omega t - 50x)\vec{a}_z$

$= \underline{\underline{0.4\omega\varepsilon_o \cos(\omega t - 50x)\vec{a}_z}}$ A/m

(c) $\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} \rightarrow \frac{\partial \vec{E}_y}{\partial x}\vec{a}_z = 0.4\mu_o\omega^2\varepsilon_o \sin(\omega t - 50x)\vec{a}_z$

$1000 = 0.4\mu_o\varepsilon_o\omega^2 = 0.4\frac{\omega^2}{c^2}$

or ω = $\underline{\underline{1.5 \text{ x } 10^{10} \text{ rad/s}}}$

**P.E.**

(a) $j^3\left(\frac{1+j}{2-j}\right)^2 = -j\left[\frac{\sqrt{2}\angle 45^o}{\sqrt{5}\angle -26.56^o}\right]^2 = -j\left(\frac{2}{5}\angle 143.13^o\right)$

$= \underline{0.24 + j0.32}$

(b) $6\angle 30^o + j5 - 3 + e^{j45^o} = 5.196 + j3 + j5 - 3 + 0.7071(1 + j)$

$= \underline{\underline{2.903 + j8.707}}$

**P.E.**

$$-\mu\frac{\partial\vec{H}}{\partial t}=\nabla\times\vec{E}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\phi\sin\theta)\vec{a}_r-\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\vec{a}_\theta$$

$$=\frac{2\cos\theta}{r^2}\cos(\omega t-\beta r)\vec{a}_r-\frac{\beta}{r}\sin\theta\sin(\omega t-\beta r)\vec{a}_\theta$$

$$\vec{H}=-\frac{2\cos\theta}{\mu\omega r^2}\sin(\omega t-\beta r)\vec{a}_r-\frac{\beta}{\mu\omega r}\sin\theta\cos(\omega t-\beta r)\vec{a}_\theta$$

$$\beta=\frac{\omega}{c}=\frac{6\times10^7}{3\times10^8}=\underline{\underline{0.2}}\ \text{rad/m}$$

$$\vec{H}=\underline{\underline{-\frac{1}{12\pi r^2}\cos\theta\sin(6\times10^7-0.2r)\vec{a}_r-\frac{1}{120\pi r}\sin\theta\cos(6\times10^7-0.2r)\vec{a}_\theta}}$$

**P.E.**

$$\omega=\frac{3}{\sqrt{\mu\varepsilon}}=\frac{3c}{\sqrt{\mu_r\varepsilon_r}}=\frac{9\times10^8}{\sqrt{10}}=\underline{\underline{2.846\times10^8}}\ \text{rad/s}$$

$$\vec{E}=\frac{1}{\varepsilon}\int\nabla\times\vec{H}dt=-\frac{6}{\omega\varepsilon}\cos(\omega t-3y)\vec{a}_x$$

$$=\frac{-6}{\dfrac{9\times10^8}{\sqrt{10}}\bullet\dfrac{10^{-9}}{36}(5)}\cos(\omega t-3y)\vec{a}_x$$

$$\vec{E}=\underline{\underline{-151.8\cos(2.846\times10^8t-3y)\vec{a}_x}}\ \text{V/m}$$

## CHAPTER —Problems

**Prob.**

$$V_{emf} = -\frac{\partial \lambda}{\partial t} = -N\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S} = -NBS\frac{d\phi}{dt}$$

$$= -NBSw = -50 \times 0{\cdot}06 \times 0{\cdot}3 \times 0{\cdot}4$$

$$= \underline{\underline{-54\ V}}\cdot$$

**Prob.**

$$V = -\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial t}\int \vec{B} \bullet dS = -\frac{\partial \vec{B}}{\partial t} \bullet S$$

$= 3770 \sin 377t \text{ x } \pi(0.2)^2 \text{ x } 10^{-3}$

$= \underline{\underline{0.4738 \sin 377t \text{ V}}}$

**Prob.**

$$\Psi = \int \boldsymbol{B} \cdot d\boldsymbol{S} = BS$$

$$V_{emf} = -\frac{d\Psi}{dt} = -\frac{dB}{dt}S = 0.6x10x10^{-4} = 0.6 \text{ mV}$$

The emf is divided in the ratio of the resistances

$$v_1 = \frac{10}{15}x0.6 = \underline{\underline{0.4 \text{ mV}}}$$

$$v_2 = \frac{5}{15}x0.6 = \underline{\underline{0.2 \text{ mV}}}$$

**Prob.**

$$V_{emf} = \int\left(\bar{u}\times\bar{\mathrm{B}}\right)\cdot d\bar{\mathrm{l}},\quad d\bar{\mathrm{l}} = d\rho\bar{\mathrm{a}}_\rho,\quad \bar{\mathrm{u}} = \rho\frac{d\phi}{dt}\bar{\mathrm{a}}_\phi = \rho\omega\bar{\mathrm{a}}_\phi$$

$$\bar{\mathrm{u}}\times\bar{\mathrm{B}} = \rho\omega\,\bar{\mathrm{a}}_\phi\times\mathrm{B}_o\bar{\mathrm{a}}_z = \mathrm{B}_o\rho\omega\,\bar{\mathrm{a}}_\rho$$

$$V_{emf} = \int_{\rho=0}^{1}\mathrm{B}_o\rho\,\bar{\mathrm{a}}_\rho\cdot d\rho\bar{\mathrm{a}}_\rho = \mathrm{B}_o\omega\left.\frac{\rho^2}{2}\right|_0^1 = \frac{1}{2}\mathrm{B}_o\omega$$

$$V_{emf} = \underline{\underline{\frac{1}{2}\mathrm{B}_o\omega}}$$

**Prob.**

$$V_{emf} = \int_{\rho}^{\rho+a} 3\boldsymbol{a}_z\times\frac{\mu_o I}{2\pi\rho}\boldsymbol{a}_\phi\bullet d\rho\boldsymbol{a}_\rho = -\frac{3\mu_o I}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$= -\frac{4\pi\times10^{-7}}{2\pi}\times15\times3\ln\frac{60}{20} = -9.888\mu V$$

Thus the induced emf = <u>9.888μV, point A at higher potential.</u>

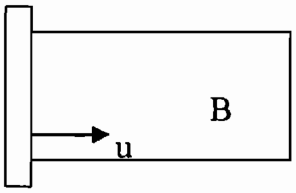
**Prob.**

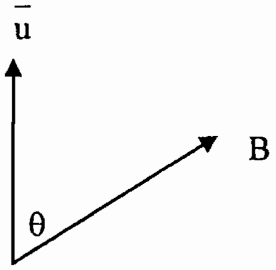
$$V_{emf} = \int\left(\vec{u}\times\vec{B}\right)\cdot d\vec{l} = uBl$$

$$= \frac{80\times10^{3}\times70\times10^{-6}\times2.5}{3600}$$

$$= \underline{\underline{3.89\ mV}}.$$

**Prob.**





$$V_{emf} = \int (\bar{u}\times\bar{B})\cdot d\bar{l} = uBl\cos\theta$$

$$= \left(\frac{120\times10^{3}}{3600}m/s\right)\left(4.3\times10^{-5}\right)\left(1.6\right)cos65^{o}$$

$= 2.293cos65^{o} = \underline{\underline{0.97}}$ mV

**Prob.**

$$V_{emf} = -\int\frac{\partial\vec{B}}{\partial t}\bullet dS + \int(\vec{u}\times\vec{B})\bullet d\vec{l}$$

where $\vec{B} = B_o\cos\omega t\vec{a}_x, \vec{u} = u_o\cos\omega t\vec{a}_y, d\vec{l} = dz\vec{a}_z$

$$V_{emf} = \int_{z=0}^{l}\int_{y=-a}^{y} B_o\omega\sin\omega t dydz - \int_{0}^{l} B_o u_o\cos^2\omega t dz$$

$= \mathrm{B_o\omega l(y+a)sin\omega t - B_ou_olcos^2\omega t}$

Alternatively,

$$\psi = \int\vec{B}\bullet d\vec{s} = \int_{z=0}^{l}\int_{y=-a}^{y} B_o\cos\omega t\vec{a}_x\bullet dydz\vec{a}_x = \mathrm{B_o(y+a)lcoswt}$$

$$V_{emf} = -\frac{\partial\psi}{\partial t} = B_o(y+a)l\omega\sin\omega t - B_o\frac{dy}{dt}l\cos\omega t$$

But $\frac{dy}{dt} = u = u_o\cos\omega t \to y = \frac{u_o}{\omega}\sin\omega t$

$\mathrm{V_{emf} = B_o\omega l(y+a)sin\omega t - B_ou_olcos^2\omega t}$

$= B_o u_o l\sin^2\omega t + B_o\omega a l\sin\omega t - B_o u_o l\cos^2\omega t$

$= -B_o u_o l\cos 2\omega t + B_o\omega a l\sin\omega t$

$= 6 \times 10^{-3} \times 5[10 \times 10\sin 10t - 2\cos 20t]$

$V_{emf} = \underline{\underline{3\sin 10t - 0.06\cos 20t \text{ V}}}$

**Prob.**

$V = \int(\vec{u} \times \vec{B}) \bullet d\vec{l}$, where $\vec{u} = \rho\omega\vec{a}_\phi$, $\vec{B} = B_o\vec{a}_z$

$$V = \int_{\rho_1}^{\rho_2} \rho\omega B_o d\rho = \frac{\omega B_o}{2}(\rho^2{}_2 - \rho^2{}_1)$$

$$V = \frac{60 \times 15}{2} \bullet 10^{-3}(100-4) \bullet 10^{-4} = \underline{\underline{4.32 \text{ mV}}}$$

**Prob.**

$$J = \sigma E = \underline{\underline{4\cos 10^7 t \ \text{nA}/\text{m}^2}}$$

$$J_d = \frac{\partial D}{\partial t} = \epsilon_o\epsilon_o \frac{\partial E}{\partial t} = -20 \times 10^7 \times 10^{-6} \times \frac{10^{-9}}{36\pi} \times 5\sin 10^7 t$$

$$= \underline{\underline{-8.842 \sin 10^7 t \ \text{nA}/\text{m}^2}}.$$

**Prob.**

$$\frac{J_c}{J_d} = \frac{\sigma E}{\omega\varepsilon E} = \frac{\sigma}{\omega\varepsilon}$$

(a) $$\frac{\sigma}{\omega\varepsilon} = \frac{2x10^{-3}}{2\pi x10^9 x81x\dfrac{10^{-9}}{36\pi}} = \underline{\underline{0.444x10^{-3}}}$$

(b) $\frac{\sigma}{\omega\varepsilon} = \frac{25}{2\pi x10^9 x81x\frac{10^{-9}}{36\pi}} = \underline{\underline{5.555}}$

(c) $\frac{\sigma}{\omega\varepsilon} = \frac{2x10^{-4}}{2\pi x10^9 x5x\frac{10^{-9}}{36\pi}} = \underline{\underline{7.2x10^{-4}}}$

**Prob.**

$$\overline{B} = \frac{\mu_o I}{2\pi y}(-\overline{a}_x)$$

$$\psi = \int \vec{B} \bullet d\vec{S} = \frac{\mu_o I}{2\pi}\int_{z=0}^{a}\int_{y=\rho}^{\rho+a}\frac{dzdy}{y} = \frac{\mu_o Ia}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$V_{emf} = -\frac{\partial\psi}{\partial t} = -\frac{\partial\psi}{\partial\rho}\bullet\frac{\partial\rho}{\partial t} = -\frac{\mu_o Ia}{2\pi}u_o\frac{d}{d\rho}[\ln(\rho+a)-\ln\rho]$$

$$= -\frac{\mu_o Ia}{2\pi}u_o\left[\frac{1}{\rho+a}-\frac{1}{\rho}\right] = \underline{\underline{\frac{\mu_o a^2 Iu_o}{2\pi\rho(\rho+a)}}}$$

where $\rho = \rho_o + u_o t$

**Prob.**

$d\psi = 0.64 - 0.45 = 0.19$, $dt = 0.02$

$$V_{emf} = N\frac{d\psi}{dt} = 10\left(\frac{0.19}{0.02}\right) = 95V$$

$$I = \frac{V_{emf}}{R} = \left(\frac{95}{15}\right) = \underline{\underline{6.33\ \text{A}}}$$

Using Lenz's law, the direction of the induced current is <u>counterclockwise.</u>

**Prob.**

$$\frac{J}{J_d} = \frac{\sigma E}{\omega\varepsilon E} = \frac{\sigma}{\omega\varepsilon} = 10$$

$$\omega = \frac{\sigma}{10\varepsilon} = 2\pi f \quad \longrightarrow \quad f = \frac{\sigma}{20\pi\varepsilon} = \frac{20}{20\pi x \frac{10^{-9}}{36\pi}}$$

$$f = \underline{\underline{36 \text{ GHz}}}$$

**Prob.**

$$J_c = \frac{I_c}{S} = \sigma E \to E = \frac{I_c}{\sigma S}$$

$$J_d = j\omega\varepsilon E \to |J_d| = \omega\varepsilon E = \frac{\omega\varepsilon I_c}{\sigma S}$$

$$|J_d| = \frac{10^9 \times 6 \times 10^{-9}/_{36\pi} \times 0.2 \times 10^{-3}}{2.5 \times 10^6 \times 10 \times 10^{-4}} A/m = \underline{\underline{4.2}} \text{ nA/m}^2$$

**Prob.**

(a) $\nabla \bullet \vec{E}_s = {}^{\rho_s}/_{\varepsilon}, \nabla \bullet \vec{H}_s = 0$

$$\underline{\underline{\nabla \times \vec{E}_s = j\omega\mu\vec{H}_s, \nabla \times \vec{H}_s = (\sigma - j\omega\varepsilon)\vec{E}_s}}$$

(b) $\nabla \bullet \vec{D} = \rho_v \to \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v$ (1)

$\nabla \bullet \vec{B} = 0 \to \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$ (2)

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \to \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$ (3)

$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$ (4)

$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}$ (5)

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \tag{6}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t} \tag{7}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t} \tag{8}$$

**Prob.**

If $\rho_v = 0, \quad \nabla \cdot \vec{E} = 0 \quad —(1)$

$$\vec{H} = -\frac{1}{\mu_o} \int \nabla \times \vec{E}\, dt$$

But $\nabla \times \vec{E} = -30k \sin 2x \cos(kz - wt)\,\vec{a}_x + 60 \cos 2x \sin(kz - wt)\,\vec{a}_z$

$$\vec{H} = -\frac{30k}{\mu_o w} \sin 2x \sin(kz - wt)\,\vec{a}_x - \frac{60}{\mu w} \cos 2x \cos(kz - wt)\,\vec{a}_z \text{ A/m}$$

$\nabla \cdot \vec{H} = 0 \quad — (2)$

$$\vec{J}_d = \epsilon_o \frac{\partial \vec{E}}{\partial t} = -30 w \epsilon_o \sin 2x \cos(kz - wt)\,\vec{a}_y \text{ A/m}^2$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{a}_y$$

$$= -\frac{30\epsilon_o}{w} \sin 2x \cos(kz - wt)\left[\frac{k^2}{\mu_o \epsilon_o} + \frac{4}{\mu_o \epsilon_o}\right]\vec{a}_y$$

$$= -30\epsilon_o w \sin 2x \cos(kz - wt)\,\vec{a}_y = \vec{J}_d \quad —(3)$$

since $\frac{k^2}{\mu_o \epsilon_o} + \frac{4}{\mu_o \epsilon_o} = w^2$. Thus the four haxwell's equations are satisfied by $\vec{E}$ howing that $\vec{E}$ is a genuine EM field.

**Prob.**

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times \nabla \times E = -\frac{\partial}{\partial t}\nabla \times B = -\mu\frac{\partial}{\partial t}\nabla \times H = -\mu\frac{\partial J}{\partial t} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2}$$

But

$$\nabla \times \nabla \times E = \nabla(\nabla \bullet E) - \nabla^2 E$$

$$\nabla(\nabla \bullet E) - \nabla^2 E = -\mu\frac{\partial J}{\partial t} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2}, \qquad J = \sigma E$$

In a source-free region, $\nabla \bullet E = \rho_v / \varepsilon = 0$. Thus,

$$\underline{\underline{\nabla^2 E = \mu\sigma\frac{\partial E}{\partial t} + \mu\varepsilon\frac{\partial^2 E}{\partial t^2}}}$$

**Prob.**

$$\nabla \bullet J = (0 + 0 + 3z^2)\sin 10^4 t = -\frac{\partial \rho_v}{\partial t}$$

$$\rho_v = -\int \nabla \bullet J dt = -\int 3z^2 \sin 10^4 t dt = \frac{3z^2}{10^4}\cos 10^4 t + C_o$$

If $\rho_v |_{z=0} = 0$, then $C_o = 0$ and

$$\underline{\underline{\rho_v = 0.3z^2 \cos 10^4 t \ \text{ mC/m}^3}}$$

**Prob.**

$$\nabla \times H = J_d$$

$$\boldsymbol{J}_d = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z(x,t) \end{vmatrix} = -\frac{\partial H_z}{\partial x}\boldsymbol{a}_y$$

$$\underline{\underline{\boldsymbol{J}_d = 20\cos(10^8 t - 2x)\boldsymbol{a}_y \text{ A/m}^2}}$$

But $\boldsymbol{J}_d = \varepsilon \dfrac{\partial \boldsymbol{E}}{\partial t} \longrightarrow \boldsymbol{E} = \dfrac{1}{\varepsilon}\displaystyle\int \boldsymbol{J}_d dt = \dfrac{1}{\varepsilon}\dfrac{20}{10^8}\sin(10^8 t - 2x)\boldsymbol{a}_y$

Also,

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y(x) & 0 \end{vmatrix} = \frac{\partial E_y}{\partial x}\boldsymbol{a}_z = -\frac{40}{10^8 \varepsilon}\cos(10^8 t - 2x)\boldsymbol{a}_z$$

$$-\mu \frac{\partial \boldsymbol{H}}{\partial t} = -10\mu_o x 10^8 \cos(10^8 t - 2x)\boldsymbol{a}_z$$

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \longrightarrow -\frac{40}{10^8 \varepsilon} = -10\mu x 10^8$$

$$\frac{4}{\mu_o \varepsilon_r \varepsilon_o} = 10^{16} \longrightarrow \varepsilon_r = \frac{4}{4\pi x 10^{-7} x \frac{10^{-9}}{36\pi} x 10^{16}} = \underline{\underline{36}}$$

$$\boldsymbol{E} = \frac{20}{36 x \frac{10^{-9}}{36\pi} x 10^8}\sin(10^8 t - 2x)\boldsymbol{a}_y = \underline{\underline{628.32\sin(10^8 t - 2x)\boldsymbol{a}_y \text{ V/m}}}$$

**Prob.**

$$\varepsilon\frac{\partial\vec{E}}{\partial t} = \nabla\times\vec{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z \end{vmatrix} = -\frac{\partial H_z}{\partial x}\vec{a}_y$$

$$= 0.6\beta\sin\beta x\cos wt\vec{a}_y$$

$$E = \frac{1}{\varepsilon}\int\nabla\times\vec{H}\partial t = \frac{0.6\beta}{w\varepsilon}\sin\beta x\sin wt\vec{a}_y$$

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial x}\vec{a}_z$$

$$= \frac{0.6\beta^2}{w\varepsilon}\cos\beta x\sin wt\vec{a}_z$$

$$\vec{H} = -\frac{1}{\mu}\int\nabla\times\vec{E}\partial t = \frac{0.6\beta^2}{w^2\mu\varepsilon}\cos\beta x\cos wt\vec{a}_z$$

Thus $\beta = w\sqrt{\mu\varepsilon} = \frac{w}{c}\sqrt{\mu_r\varepsilon_r} = \frac{10^8(2.25)}{3\times10^8}$

$= \underline{\underline{0.8333 \text{ rad/m}}}$

$$E_o = \frac{0.6\beta}{w\varepsilon} = \frac{0.6w\sqrt{\mu\varepsilon}}{w\varepsilon} = 0.6\sqrt{\mu/\varepsilon} = 0.6(377)\sqrt{\mu_r/\varepsilon_r}$$

$$= \frac{0.6\times337}{2.25} = 100.5$$

$$\underline{\underline{\vec{E} = 100.5\sin\beta x\sin wt\vec{a}_y}} \text{ V/m}$$

**Prob.**

$$\nabla\times \boldsymbol{E} = -\mu\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla\times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(y,t) & 0 & 0 \end{vmatrix} = -\frac{\partial E_x}{\partial y}\boldsymbol{a}_z = 10\pi\sin(\omega t+\pi y)\boldsymbol{a}_z$$

$$-\mu\frac{\partial \boldsymbol{H}}{\partial t} = \frac{10\mu\omega}{\eta}\sin(\omega t+\pi y)\boldsymbol{a}_z$$

$$\nabla\times \boldsymbol{E} = -\mu\frac{\partial \boldsymbol{H}}{\partial t} \longrightarrow 10\pi = \frac{10\mu\omega}{\eta} \longrightarrow \eta\pi = \mu\omega \qquad (1)$$

$$\nabla\times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla\times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z(y,t) \end{vmatrix} = \frac{\partial H_z}{\partial y}\boldsymbol{a}_x = -\frac{10\pi}{\eta}\sin(\omega t+\pi y)\boldsymbol{a}_x$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = -10\varepsilon\omega\sin(\omega t+\pi y)\boldsymbol{a}_x$$

$$\nabla\times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \longrightarrow \frac{10\pi}{\eta} = 10\varepsilon\omega \longrightarrow \frac{\pi}{\eta} = \varepsilon\omega \qquad (2)$$

Multiplying (1) and (2) gives

$$\pi^2 = \mu\varepsilon\omega^2 \longrightarrow \omega^2 = \frac{\pi^2}{\mu\varepsilon} = \frac{\pi^2}{4\pi\times10^{-7}\times9\times\frac{10^{-9}}{36\pi}} = \pi^2 10^{16}$$

$$\omega = \underline{\underline{3.142\times10^8 \text{ rad/s}}}$$

From (1),

$$\eta = \frac{\mu\omega}{\pi} = \frac{4\pi\times10^{-7}\times10^8\pi}{\pi} = 40\pi = \underline{\underline{125.66\ \Omega}}$$

**Prob.**

(a) $$\frac{2+j3}{j(-10+j2)} = \frac{3.606\angle 56.31^\circ}{10.198\angle 258.69^\circ} = \underline{\underline{0.354\angle -202.38^\circ}}$$

(b) $$\frac{8\angle 30^\circ - 5\angle 60^\circ}{1+j} = \frac{4.4282 - j0.33}{1+j} = \frac{4.44\angle -4.26^\circ}{1.414\angle 45^\circ}$$

$$= \underline{\underline{3.14\angle -49.26^\circ}}$$

(c) $$\frac{(1+j3)(4-j2)}{(1+j3)+(4-j2)} = \frac{10+j10}{5+j} = \underline{\underline{2.77\angle 33.7^\circ}}$$

(d) $$\left[\frac{(15-j7)(3+j2)^*}{(4+j6)^*(3\angle 70^\circ)}\right]^* = \left(\frac{16.555\angle -25.02^\circ}{7.211\angle -56.31^\circ}\cdot\frac{3.606\angle -33.69^\circ}{3\angle 70^\circ}\right)^*$$

$$= (2.7596\angle -92.4)^* = \underline{\underline{2.76\angle 72.4^\circ}}.$$

(e) $$\left[(-10\angle 30^\circ)(4e^{j\pi/4})\right]^{1/2} = (40\angle 210^\circ + 45^\circ)^{1/2}$$

$$= \underline{\underline{6.325\angle 127.5^\circ}}.$$

**Prob.**

(a) $\nabla \bullet A = 0$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ & & \\ 0 & 0 & E_z(x,t) \end{vmatrix} = -\frac{\partial E_z(x,t)}{\partial x}\boldsymbol{a}_y \neq 0$$

<u>Yes</u>, $A$ is a possible EM field.

(b) $\nabla \bullet B = 0$

$$\nabla \times \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[10\cos(\omega t - 2\rho)\right]\boldsymbol{a}_z \neq 0$$

Yes, $B$ is a possible EM field.

(c) $$\nabla \bullet \boldsymbol{C} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(3\rho^3 \cot\phi \sin\omega t\right) - \frac{\sin\phi \sin\omega t}{\rho^2} \neq 0$$

$$\nabla \times \boldsymbol{C} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\cos\phi \sin\omega t\right)\boldsymbol{a}_z - 3\rho^2 \frac{\partial}{\partial \phi}(\cot\phi \sin\omega t)\boldsymbol{a}_z \neq 0$$

No, $C$ cannot be an EM field.

(d) $$\nabla \bullet D = \frac{1}{r^2 \sin\theta} \sin(\omega t - 5r)\frac{\partial}{\partial \theta}(\sin^2\theta) \neq 0$$

$$\nabla \times \boldsymbol{D} = -\frac{\partial D_\theta}{\partial \phi}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial}{\partial r}(rD_\theta)\boldsymbol{a}_\phi = \frac{1}{r}\sin\theta(-5)\sin(\omega t - 5r)\boldsymbol{a}_\phi \neq 0$$

No, $D$ cannot be an EM field.

**Prob.**

$$\nabla x H = J + J_d$$

$J = \sigma E = 0$ in free space.

$$J_d = \nabla x H = \left[\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right]a_\rho + \left[\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right]a_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\phi) - \frac{\partial H_\rho}{\partial \phi}\right]a_z$$

$$= 0 + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(2\rho^2\cos\phi) - \rho\cos\phi\right]\cos 4x10^6 t a_z = \frac{a_z}{\rho}(4\cos\phi - \rho\cos\phi)\cos 4x10^6 t$$

$$J_d = \underline{\underline{3\cos\phi \cos 4x10^6 t a_z}}$$

$$J_d = \frac{\partial D}{\partial t} = \varepsilon_o \frac{\partial E}{\partial t} \quad \longrightarrow \quad E = \frac{1}{\varepsilon_o}\int J_d dt$$

$$E = \frac{3}{\varepsilon_o}\frac{\cos\phi}{4x10^6}\sin 4x10^6 t a_z = \frac{3}{4x10^6 x \frac{10^{-9}}{36\pi}}\cos\phi \sin 4x10^6 t a_z$$

$$E = \underline{\underline{84.82\cos\phi \sin 4x10^6 t a_z}} \text{ kV/m}$$

**Prob.**

Using Maxwell's equations,

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \qquad (\sigma = 0) \qquad \longrightarrow \qquad \boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

But

$$\nabla \times \boldsymbol{H} = -\frac{1}{r\sin\theta}\frac{\partial H_\theta}{\partial \phi}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)\boldsymbol{a}_\phi = \frac{12\sin\theta}{r}\beta \sin(2\pi x 10^8 t - \beta r)\boldsymbol{a}_\phi$$

$$\boldsymbol{E} = \frac{12\sin\theta}{\varepsilon_o}\beta \int \sin(2\pi x 10^8 t - \beta r) dt \boldsymbol{a}_\phi$$

$$= -\frac{12\sin\theta}{\omega \varepsilon_o r}\beta \cos(\omega t - \beta r)\boldsymbol{a}_\phi, \qquad \omega = 2\pi x 10^8$$

**Prob.**

$$\vec{E} = \mathrm{Re}[\vec{E}_s e^{jwt}) = 20 \sin(k_x x) \sin(k_y y) \cos wt\, \vec{a}_z \ \mathrm{V/m}$$

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \frac{\partial E_z}{\partial y}\vec{a}_x - \frac{\partial E_z}{\partial x}\vec{a}_y$$

$$= 20 k_y \sin(k_x x) \cos(k_y y) \cos wt\, \vec{a}_x$$

$$-20 k_x \cos(k_x x) \sin(k_y y) \cos wt\, \vec{a}_y$$

$$\vec{B} = -\int \nabla \times \vec{E}\, dt$$

$$= -\frac{20}{w}\Big[ k_y \sin(k_x x) \cos(k_y y) \sin wt\, \vec{a}_x$$

$$- k_x \cos(k_x x) \sin(k_y y) \sin wt\, \vec{a}_y \Big] \ \mathrm{Wb/m^2}.$$

**Prob.**

$$\nabla \times \vec{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi)\vec{a}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 t e^{-\rho-t})\vec{a}_z$$

$$= (2-\rho)te^{-\rho-t}\vec{a}_z$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} \rightarrow \vec{B} = -\int \nabla \times \vec{E} dt = \int \frac{(\rho-2)t}{V}\frac{e^{-\rho-t}dt}{du}\vec{a}_z$$

Integrating by parts yields

$$\vec{B} = [-(\rho-2)te^{-\rho-t} + \int(\rho-2)e^{-\rho-t}dt]\vec{a}_z$$

$$= \underline{\underline{(2-\rho)(1+t)e^{-\rho-t}\vec{a}_z}} \text{ Wb/m}^2$$

$$\vec{J} = \nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_o} = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial \rho}\vec{a}_\phi$$

$$= -\frac{1}{\mu_o}(1+t)(-1-2+\rho)e^{-\rho-t}\vec{a}_\phi$$

$$\vec{J} = \underline{\underline{\frac{(1+t)(3-\rho)e^{-\rho-t}}{4\pi x 10^{-7}}}}\vec{a}_\phi \text{ A/m}^2$$

**Prob.**

(a) $\sin\theta = \cos(\theta - 90^o)$

$$E = 4\cos(\omega t - 3x - 10^o)a_y - 5\cos(\omega t + 3x - 70^o)a_z$$

$$= \text{Re}\left[4e^{j(-3x-10^o)}e^{j\omega t}a_y - 5e^{j(3x-70^o)}e^{j\omega t}a_z\right] = \text{Re}\left[E_s e^{j\omega t}\right]$$

$$E_s = \underline{\underline{4e^{-j(3x+10^o)}a_y - 5e^{j(3x-70^o)}a_z}}$$

(b) $$H = \text{Re}\left[\frac{\sin\theta}{r}e^{j\omega t}e^{-j5r}a_\theta\right] = \text{Re}\left[\text{H}_s e^{j\omega t}\right]$$

$$H_s = \underline{\underline{\frac{\sin\theta}{r}e^{-j5r}a_\theta}}$$

(c) $$J = \text{Re}\left[6e^{-3x}e^{-j2x}e^{-j90^o}e^{j\omega t}a_y + \ldots\right] = \text{Re}\left[\text{J}_s e^{j\omega t}\right]$$

$$J_s = \underline{\underline{-j6e^{-(3+j2)x}a_y + 10e^{-(1+j5)x}a_z}}$$

**Prob.**

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

But $\boldsymbol{B} = \nabla x A$

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}(\nabla \times A) = \nabla \times \left(-\frac{\partial A}{\partial t}\right)$$

Hence,

$$\boldsymbol{E} = -\frac{\partial A}{\partial t}$$

**Prob.**

(a) $\vec{A} = Re\left(\vec{A}_s e^{j\omega t}\right) = Re\left[5e^{j90^\circ} e^{-j20^\circ} e^{j\omega t} \vec{a}_x - 5x e^{j\alpha} e^{j\omega t} \vec{a}_y\right]$

where $\alpha = \tan^{-1}\frac{4}{3} = 53.13^\circ$

$$\vec{A} = \underline{\underline{5\cos(\omega t + 70^\circ)\vec{a}_x - 5x\cos(\omega t + 53.13^\circ)\vec{a}_y}}.$$

(b) $\vec{B} = Re\left[10e^{-jkz} e^{j\omega t} \vec{a}_x + 5e^{j\pi/2} e^{jkz + \pi/4} e^{j\omega t} \vec{a}_y\right]$

$$= 10\cos(\omega t - kz)\vec{a}_x + 5\cos(\omega t + kz + \pi/2 + \pi/4)\vec{a}_y$$

$$= 10\cos(\omega t - kz)\vec{a}_x \underline{\underline{- 5\sin(\omega t + kz + \pi/4)\vec{a}_y}}.$$

**Prob.**

$$\nabla xE = -\frac{\partial B}{\partial t} = -\mu_o \frac{\partial H}{\partial t}$$

$$\nabla xE = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y(x) & E_z(x) \end{vmatrix} = -\frac{\partial E_z}{\partial x}a_y + \frac{\partial E_y}{\partial x}a_z \qquad = 40x8\cos(10^9 t - 8x)a_y + 50x8\sin(10^9 t - 8x)a_z$$

$$H = -\frac{1}{\mu_o}\int \nabla xEdt = -\frac{10^{-9}}{\mu_o}\left[40x8\sin(10^9 t - 8x)a_y - 50x8\cos(10^9 t - 8x)a_z\right]$$

$$= -\frac{10^{-2}}{4\pi}\left[320\sin(10^9 t - 8x)a_y - 400\cos(10^9 t - 8x)a_z\right]$$

$$\underline{\underline{H = -0.2546\sin(10^9 t - 8x)a_y + 0.3184\cos(10^9 t - 8x)a_z \text{ A / m}}}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}, \quad (\mu_r = 1) \quad \longrightarrow \quad \sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{8x3x10^8}{10^9} = 2.4$$

$$\underline{\underline{\varepsilon_r = 5.76}}$$

**Prob.**

From Maxwell's equations,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (1)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (2)$$

Dotting both sides of (2) with $\vec{E}$ gives:

$$\vec{E} \bullet (\nabla \times \vec{H}) = \vec{E} \bullet \vec{J} + \vec{E} \bullet \frac{\partial \vec{D}}{\partial t} \qquad (3)$$

But for any arbitrary vectors $\vec{A}$ and $\vec{B}$,

$$\nabla \bullet (\vec{A} \times \vec{B}) = \vec{B} \bullet (\nabla \times \vec{A}) - \vec{A} \bullet (\nabla \times \vec{B})$$

Applying this on the left-hand side of (3) by letting $\vec{A} \equiv \vec{H}$ and $\vec{B} \equiv \vec{E}$, we get

$$\vec{H} \bullet (\nabla \times \vec{E}) + \nabla \bullet (\vec{H} \times \vec{E}) = \vec{E} \bullet \vec{J} + \tfrac{1}{2}\frac{\partial}{\partial t}(\vec{D} \bullet \vec{E}) \qquad (4)$$

From (1),

$$\vec{H} \bullet (\nabla \times \vec{E}) = \vec{H} \bullet \left(-\frac{\partial \vec{B}}{\partial t}\right) = \tfrac{1}{2}\frac{\partial}{\partial t}(\vec{B} \bullet \vec{H})$$

Substituting this in (4) gives:

$$-\tfrac{1}{2}\frac{\partial}{\partial t}(\vec{B} \bullet \vec{H}) - \nabla \bullet (\vec{E} \times \vec{H}) = \vec{J} \bullet \vec{E} + \tfrac{1}{2}\frac{\partial}{\partial t}(\vec{D} \bullet \vec{E})$$

Rearranging terms and then taking the volume integral of both sides

$$\int_v \nabla \bullet (\vec{E} \times \vec{H})dv = -\frac{\partial}{\partial t}\tfrac{1}{2}\int_v (\vec{E} \bullet \vec{D} + \vec{H} \bullet \vec{B})dv - \int_v \vec{J} \bullet \vec{E}dv$$

Using the divergence theorem, we get

$$\oint_s (\vec{E} \times \vec{H}) \bullet d\vec{S} = -\frac{\partial W}{\partial t} - \int_v \vec{J} \bullet \vec{E}dv$$

or $\frac{\partial W}{\partial t} = -\oint_s (\vec{E} \times \vec{H}) \bullet d\vec{S} - \int_v \vec{E} \bullet \vec{J}dv$ as required.

**Prob.**

(a)

$$z = 4\angle 30^o - 10\angle 50^o = 3.464 + 2j - 6.427 - j7.66 = -2.963 - j5.66$$

$$= 6.389\angle -117.64^o$$

$$z^{1/2} = \underline{\underline{2.5277\angle -58.82^o}}$$

(b)

$$\frac{1+j2}{6-j8-7\angle 15^o} = \frac{2.236\angle 63.43^o}{6-j8-7.761-j1.812} = \frac{2.236\angle 63.43^o}{9.841\angle 265.57^o}$$

$$= \underline{\underline{0.2272\angle -202.1^o}}$$

(c) $$z = \frac{(5\angle 53.13^o)^2}{12-j7-6-j10} = \frac{25\angle 106.26^o}{18.028\angle -70.56^o}$$

$$= \underline{\underline{1.387\angle 176.8^o}}$$

(d)

$$\frac{1.897\angle -100^o}{(5.76\angle 90^o)(9.434\angle -122^o)} = \underline{\underline{0.0349\angle -68^o}}$$

**Prob.**

(a) $(4-j3) = 5e^{-j36.87^o}$

$$\boldsymbol{A}_s = 5e^{-j(\beta x + 36.87^o)}\boldsymbol{a}_y$$

$$\boldsymbol{A} = \mathrm{Re}\left[\boldsymbol{A}_s e^{j\omega t}\right] = \underline{\underline{5\cos(\omega t - \beta x - 36.87^o)\boldsymbol{a}_y}}$$

**(b)**

$$\boldsymbol{B} = \mathrm{Re}\left[\boldsymbol{B}_s e^{j\omega t}\right] = \mathrm{Re}\left[\frac{20}{\rho} e^{j(\omega t - 2z)} \boldsymbol{a}_\rho\right]$$

$$= \underline{\underline{\frac{20}{\rho}\cos(\omega t - 2z)\boldsymbol{a}_\rho}}$$

(c ) $1 + j2 = 2.23 e^{j63.43^o}$

$$\boldsymbol{C}_s = \frac{10}{r^2}(2.236) e^{j63.43^o} e^{-j\phi} \sin\theta \boldsymbol{a}_\phi$$

$$\boldsymbol{C} = \mathrm{Re}\left[C_s e^{j\omega t}\right] = \mathrm{Re}\left[\frac{22.36}{r^2} e^{j(\omega t - \phi + 63.43^o)} \sin\theta \boldsymbol{a}_\phi\right]$$

$$= \underline{\underline{\frac{22.36}{r^2}\cos(\omega t - \phi + 63.43^o)\sin\theta \boldsymbol{a}_\phi}}$$

**Prob.**

(a) $\nabla \cdot \boldsymbol{E}_s = 0$

$$\nabla \times \boldsymbol{E}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{sx}(y) & 0 & 0 \end{vmatrix} = -\frac{\partial E_{sx}}{\partial y}\boldsymbol{a}_z = j40 e^{-j4y}\boldsymbol{a}_z$$

But $\nabla \times \boldsymbol{E}_s = -j\mu_o \omega \boldsymbol{H}_s \longrightarrow \boldsymbol{H}_s = -\frac{40}{\mu_o \omega} e^{-j4y}\boldsymbol{a}_z$

$\nabla \cdot \boldsymbol{H}_s = 0$

$$\nabla \times \boldsymbol{H}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_{sz}(y) \end{vmatrix} = \frac{\partial H_{sz}}{\partial y}\boldsymbol{a}_x = \frac{j160}{\mu_o \omega} e^{-j4y}\boldsymbol{a}_x$$

But $\nabla \times \boldsymbol{H}_s = j\omega\varepsilon_o \boldsymbol{E}_s \longrightarrow \boldsymbol{E}_s = \frac{160}{\mu_o \varepsilon_o \omega^2} e^{-j4y}\boldsymbol{a}_x = 10 e^{-j4y}\boldsymbol{a}_x$

$$\omega^2 = \frac{16}{\mu_o \varepsilon_o} = \frac{16}{4\pi \times 10^{-7} \times \frac{10^{-9}}{36\pi}} = 16 \times 9 \times 10^{16}$$

$\omega = \underline{\underline{12 \times 10^8 \text{ rad/s}}}$

(b) $\boldsymbol{H}_s = -\frac{40}{4\pi x 10^{-7} x 12 x 10^8} e^{-j4y}\boldsymbol{a}_z = \underline{\underline{-26.53 e^{-j4y}\boldsymbol{a}_z \text{ mA/m}}}$

**Prob.**

$$A = 4\cos(\omega t - 90^o)\boldsymbol{a}_x + 3\cos\omega t\boldsymbol{a}_y = \text{Re}\left[4\text{e}^{j(\omega t-90^o)}\boldsymbol{a}_x + 3e^{j\omega t}\boldsymbol{a}_y\right] = \text{Re}\left[A_s e^{j\omega t}\right]$$

$$A_s = 4e^{-j90^o}\boldsymbol{a}_x + 3\boldsymbol{a}_y = \underline{\underline{-j4\boldsymbol{a}_x + 3\boldsymbol{a}_y}}$$

$$\boldsymbol{B}_s = 10ze^{j90^o}e^{-jz}\boldsymbol{a}_x$$

$$\boldsymbol{B} = \text{Re}\left[\boldsymbol{B}_s e^{j\omega t}\right] = 10z\cos(\omega t - z + 90^o)\boldsymbol{a}_x = \underline{\underline{-10z\sin(\omega t - z)\boldsymbol{a}_x}}$$

**Prob.**

We begin with Maxwell's equations:

$$\nabla \bullet \boldsymbol{D} = \rho_v / \varepsilon = 0, \qquad \nabla \bullet \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

We write these in phasor form and in terms of $E_s$ and $H_s$ only.

$$\nabla \bullet \boldsymbol{E}_s = 0 \qquad (1)$$

$$\nabla \bullet \boldsymbol{H}_s = 0 \qquad (2)$$

$$\nabla \times \boldsymbol{E}_s = -j\omega\mu \boldsymbol{H}_s \qquad (3)$$

$$\nabla \times \boldsymbol{H}_s = (\sigma + j\omega\varepsilon)\boldsymbol{E}_s \qquad (4)$$

Taking the curl of (3),

$$\nabla \times \nabla \times \boldsymbol{E}_s = -j\omega\mu \nabla \times \boldsymbol{H}_s$$

$$\nabla(\nabla \bullet \boldsymbol{E}_s) - \nabla^2 \boldsymbol{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\boldsymbol{E}_s$$

$$\nabla^2 \boldsymbol{E}_s + (\omega^2 \mu\varepsilon - j\omega\mu\sigma)\boldsymbol{E}_s = 0 \quad \longrightarrow \quad \underline{\underline{\nabla^2 \boldsymbol{E}_s + \gamma^2 \boldsymbol{E}_s = 0}}$$

Similarly, by taking the curl of (4),

$$\nabla \times \nabla \times \boldsymbol{H}_s = (\sigma + j\omega\varepsilon)\nabla \times \boldsymbol{E}_s$$

$$\nabla(\nabla \bullet \boldsymbol{H}_s) - \nabla^2 \boldsymbol{H}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\boldsymbol{H}_s$$

$$\nabla^2 \boldsymbol{H}_s + (\omega^2 \mu\varepsilon - j\omega\mu\sigma)\boldsymbol{H}_s = 0 \quad \longrightarrow \quad \underline{\underline{\nabla^2 \boldsymbol{H}_s + \gamma^2 \boldsymbol{H}_s = 0}}$$

## Chapter Practice Examples

### P.E.

Let $x_o = \sqrt{1+(\sigma/\omega\varepsilon)^2}$ , then

$$\alpha = \omega\sqrt{\frac{\mu_o\varepsilon_o}{2}\mu_r\varepsilon_r(x_o-1)} = \frac{\omega}{c}\sqrt{\frac{16}{2}}\sqrt{x_o-1}$$

or $$\sqrt{x_o-1} = \frac{\alpha c}{\omega\sqrt{8}} = \frac{1/3\times3\times10^8}{10^8\sqrt{8}} = \frac{1}{\sqrt{8}} \longrightarrow x_o = 9/8$$

$$x_o^2 = \frac{81}{64} = 1+(\sigma/\omega\varepsilon)^2 \longrightarrow \frac{\sigma}{\omega\varepsilon} = 0.5154$$

$$\tan 2\theta_\eta = 0.5154 \longrightarrow \theta_\eta = 13.63^o$$

$$\frac{\beta}{\alpha} = \sqrt{\frac{x_o+1}{x_o-1}} = \sqrt{17}$$

(a) $\beta = \alpha\sqrt{17} = \dfrac{\sqrt{17}}{3} = \underline{\underline{1.374 \text{ rad/m}}}$

(b) $\dfrac{\sigma}{\omega\varepsilon} = \underline{\underline{0.5154}}$

(c) $|\eta| = \dfrac{\sqrt{\mu/\varepsilon}}{\sqrt{x_o}} = \dfrac{120\pi\sqrt{2/8}}{\sqrt{9/8}} = 177.72$

$\eta = \underline{\underline{177.72\angle 13.63^o\,\Omega}}$

(d) $u = \dfrac{\omega}{\beta} = \dfrac{10^8}{1.374} = \underline{\underline{7.278\times10^7}}$ m/s

(e) $\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E \longrightarrow \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_H \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_y$

$$\boldsymbol{H} = \frac{0.5}{177.5}e^{-z/3}\sin(10^8t-\beta z-13.63^o)\boldsymbol{a}_y = \underline{2.817e^{-z/3}\sin(10^8t-\beta z-13.63^o)\boldsymbol{a}_y \text{ mA/m}}$$

### P.E.

(a) Along $\underline{\underline{\text{-z direction}}}$

$$\text{(b)}\ \lambda = \frac{2\pi}{\beta} = 2\pi / 2 = \underline{\underline{3.142\text{ m}}}$$

$$f = \frac{\omega}{2\pi} = \frac{10^8}{2\pi} = \underline{\underline{15.92\text{ MHz}}}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{(1)\varepsilon_r}$$

$$\text{or}\sqrt{\varepsilon_r} = \beta c/\omega = \frac{3x10^8x2}{10^8} = 6 \quad \longrightarrow \quad \underline{\underline{\varepsilon_r = 36}}$$

$$\text{(c)}\ \theta_\eta = 0, |\eta| = \sqrt{\mu/\varepsilon} = \sqrt{\mu_o/\varepsilon_o}\sqrt{1/\varepsilon_r} = \frac{120\pi}{6} = 20\pi$$

$$\boldsymbol{a}_k = \boldsymbol{a}_E \times \boldsymbol{a}_H \quad \longrightarrow \quad -\boldsymbol{a}_z = \boldsymbol{a}_y \times \boldsymbol{a}_H \quad \longrightarrow \quad \boldsymbol{a}_H = \boldsymbol{a}_x$$

$$\boldsymbol{H} = \frac{50}{20\pi}\sin(\omega t + \beta z)\boldsymbol{a}_x = \underline{\underline{795.8\sin(10^8 t + 2z)\boldsymbol{a}_x}} \text{ mA/m}$$

**P.E.**

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{10^9\pi \times 4 \times \frac{10^{-9}}{36\pi}} = 0.09$$

$$\alpha \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \frac{\omega}{2c}\sqrt{\mu_r\varepsilon_r}\frac{\sigma}{\omega\varepsilon} = \frac{10^9\pi}{2\times3\times10^8}(2)(0.09) = 0.9425$$

$$\beta \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2+1\right]} = \frac{10^9\pi}{3\times10^8}\sqrt{2[2+0.5(0.09)^2]} = 20.965$$

$$\boldsymbol{E} = 30e^{-0.9425y}\cos(10^9\pi t - 20.96y + \pi/4)\boldsymbol{a}_z$$

At t = 2ns, y = 1m,

$$\boldsymbol{E} = 30e^{-0.9425}\cos(2\pi - 20.96 + \pi/4)\boldsymbol{a}_z = \underline{\underline{2.844\boldsymbol{a}_z}} \text{ V/m}$$

(b) $\beta y = 10^o = \dfrac{10\pi}{180}$ rad

or

$$y = \frac{\pi}{18}\frac{1}{\beta} = \frac{\pi}{18\times 20.965} = \underline{\underline{8.325 \text{ mm}}}$$

(c ) $30(0.6) \;=\; 30\; e^{-\alpha y}$

$$y = \frac{1}{\alpha}\ln(1/0.6) = \frac{1}{0.9425}\ln\frac{1}{0.6} = \underline{\underline{542 \text{ mm}}}$$

(d)

$$|\eta| \cong \frac{\sqrt{\mu/\varepsilon}}{[1+\frac{1}{4}(0.09)^2]} = \frac{60\pi}{1.002} = 188.11$$

$$2\theta_\eta = \tan^{-1} 0.09 \quad \xrightarrow{\qquad\qquad} \quad \theta_\eta = 2.571^o$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$$

$$\boldsymbol{H} = \frac{30}{188.11} e^{-0.9425y}\cos(10^9\pi t - 20.96y + \pi/4 - 2.571^o)\boldsymbol{a}_x$$

At y = 2m, t = 2ns,

$$\boldsymbol{H} = (0.1595)(0.1518)\;\cos(-34.8963 rad) a_x = \underline{\underline{-22.83 a_x}} \text{ mA/m}$$

## P.E.

$$I_s = \int_0^w\int_0^\infty J_{xs}dydz = J_{xs}(0)\int_0^w dy\int_0^\infty e^{-z(1+j)/\delta}dz = \frac{J_{xs}(0)w\delta}{1+j}$$

$$\underline{\underline{|I_s| = \frac{J_{xs}(0)w\delta}{\sqrt{2}}}}$$

## P.E.

(a)

$$\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} = \frac{a}{2}\sqrt{\pi f \mu \sigma} = \frac{1.3\times10^{-3}}{2}\sqrt{\pi\times10^{7}\times4\pi\times10^{-7}\times3.5\times10^{7}} = \underline{\underline{24.16}}$$

(b)

$$\frac{R_{ac}}{R_{dc}} = \frac{1.3x10^{-3}}{2}\sqrt{\pi\times2\times10^{9}\times4\pi\times10^{-7}\times3.5\times10^{7}} = \underline{\underline{341.7}}$$

## P.E.

$$\mathcal{P}_{\text{ave}} = \frac{1}{2}\eta H_o^2 \boldsymbol{a}_x$$

(a) Let $f(x,z) = x + y - 1 = 0$

$$\boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}, \qquad d\boldsymbol{S} = \text{dS}\boldsymbol{a}_\text{n}$$

$$P_t = \int \mathbf{P}.\text{dS} = \mathbf{P}.\mathbf{S}\mathbf{a}_\text{n} = \frac{1}{2}\eta H_o^2 \boldsymbol{a}_x . \frac{\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}(120\pi)(0.2)^2(0.1)^2 = \underline{\underline{53.31\text{ mW}}}$$

(b) $d\boldsymbol{S} = \text{dydz}\boldsymbol{a}_x$, $P_t = \int \mathcal{P}.\text{dS} = \frac{1}{2}\eta H_o^2 S$

$$P_t = \frac{1}{2}(120\pi)(0.2)^2\pi(0.05)^2 = \underline{\underline{59.22\text{ mW}}}$$

## P.E.

$$\eta_1 = \eta_o = 120\pi, \eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{2}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = 2/3, \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1/3$$

$$E_{ro} = \Gamma E_{io} = -\frac{10}{3}$$

$$\underline{\underline{\mathbf{E}_{rs} = -\frac{10}{3} e^{j\beta_1 z} \mathbf{a}_x \text{ V/m}}}$$

where $\beta_1 = \omega / c = 100\pi / 3$.

$$E_{to} = \tau E_{io} = \frac{20}{3}$$

$$\underline{\underline{\mathbf{E}_{ts} = \frac{20}{3} e^{-j\beta_2 z} \mathbf{a}_x \text{ V/m}}}$$

where $\beta_2 = \omega \sqrt{\varepsilon_r} / c = 2\beta_1 = 200\pi / 3$.

## P.E.

$$\alpha_1 = 0, \quad \beta_1 = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{2\omega}{c} = 5 \longrightarrow \omega = 5c/2 = 7.5\times 10^8$$

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{0.1}{7.5\times 10^8 \times 4 \times \dfrac{10^{-9}}{36\pi}} = 1.2\pi$$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1 + 1.44\pi^2} - 1\right]} = 6.021$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1 + 1.44\pi^2} + 1\right]} = 7.826$$

$$|\eta_2| = \frac{60\pi}{\sqrt[4]{1+1.44\pi^2}} = 95.445, \eta_1 = 120\pi\sqrt{\varepsilon_{r1}} = 754$$

$$\tan 2\theta_{\eta_2} = 1.2\pi \longrightarrow \theta_{\eta_2} = 37.57^o$$

$$\eta_2 = 95.445\angle 37.57^o$$

(a)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{95.445\angle 37.57^o - 754}{95.445\angle 37.57^o + 754} = \underline{\underline{0.8186\angle 171.08^o}}$$

$$\tau = 1 + \Gamma = \underline{\underline{0.2295\angle 33.56^o}}$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.8186}{1-0.8186} = \underline{\underline{10.025}}$$

(b) $\boldsymbol{E}_i = 50\sin(\omega t - 5x)\boldsymbol{a}_y = \mathrm{Im}(\boldsymbol{E}_{is}e^{j\omega t})$, where $E_{is} = 50e^{-j5x}a_y$.

$$E_{ro} = \Gamma E_{io} = 0.8186e^{j171.08^o}(50) = 40.93e^{j171.08^o}$$

$$\boldsymbol{E}_{rs} = 40.93e^{j5x+j171.08^o}\boldsymbol{a}_y$$

$$\boldsymbol{E}_r = \mathrm{Im}(\boldsymbol{E}_{rs}e^{j\omega t}) = \underline{\underline{40.93\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_y}} \text{ V/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = -\boldsymbol{a}_x \times \boldsymbol{a}_y = -\boldsymbol{a}_z$$

$$\boldsymbol{H}_r = -\frac{40.93}{754}\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_z = \underline{\underline{-0.0543\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_z}} \text{ A/m}$$

(c)

$$E_{to} = \tau E_{io} = 0.229e^{j33.56^o}(50) = 11.475e^{j33.56^o}$$

$$\boldsymbol{E}_{ts} = 11.475e^{-j\beta_2 x + j33.56^o}e^{-\alpha_2 x}\boldsymbol{a}_y$$

$$\boldsymbol{E}_t = \mathrm{Im}(\boldsymbol{E}_{ts}e^{j\omega t}) = \underline{\underline{11.475e^{-6.021x}\sin(\omega t - 7.826x + 33.56^o)\boldsymbol{a}_y}} \text{ V/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_x \times \boldsymbol{a}_y = \boldsymbol{a}_z$$

$$\mathbf{H}_t = \frac{11.495}{95.445} e^{-6.021x} \sin(\omega t - 7.826x + 33.56^o - 37.57^o)\mathbf{a}_z$$

$$= \underline{\underline{0.1202 e^{-6.021x} \sin(\omega t - 7.826x - 4.01^o)\mathbf{a}_z}} \quad \text{A/m}$$

(d)

$$\mathcal{P}_{1\text{ave}} = \frac{E_{io}^{\ 2}}{2\eta_1}\mathbf{a}_x + \frac{E_{ro}^{\ 2}}{2\eta_1}(-\mathbf{a}_x) = \frac{1}{2(240\pi)}[50^2\mathbf{a}_x - 40.93^2\mathbf{a}_x] = \underline{\underline{0.5469\mathbf{a}_x}} \quad \text{W/m}^2$$

$$\mathbf{P}_{2\text{ave}} = \frac{E_{to}^{\ 2}}{2|\eta_2|} e^{-2\alpha_2 x} \cos\theta_{\eta_2} \mathbf{a}_x = \frac{(11.475)^2}{2(95.445)} \cos 37.57^o e^{-2(6.021)x}\mathbf{a}_x = \underline{\underline{0.5469 e^{-12.04x}\mathbf{a}_x}} \quad \text{W/m}^2$$

**P.E.**

$$\mathbf{k} = -2\mathbf{a}_y + 4\mathbf{a}_z \longrightarrow k = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$\omega = kc = 3x10^8\sqrt{20} = \underline{\underline{1.342x10^9 \ \text{rad/s}}},$$

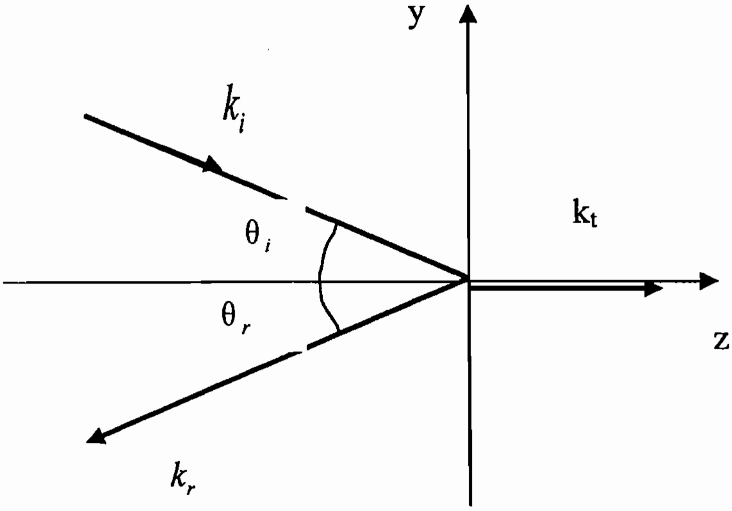
$$\lambda = 2\pi / k = \underline{\underline{1.405\text{m}}}$$

$$\text{(b)}\ \mathbf{H} = \frac{\mathbf{a}_k \times \mathbf{E}}{\eta_o} = \frac{(-2\mathbf{a}_y + 4\mathbf{a}_z)}{\sqrt{20}(120\pi)} \times (10\mathbf{a}_y + 5\mathbf{a}_z)\cos(\omega t - \mathbf{k}.\mathbf{r})$$

$$= \underline{\underline{-29.66\cos(1.342x10^9 t + 2y - 4z)\mathbf{a}_x}} \quad \text{mA/m}$$

$$\text{(c)}\ \mathcal{P}_{\text{ave}} = \frac{|E_o|^2}{2\eta_o}\mathbf{a}_k = \frac{125}{2(120\pi)} \frac{(-2\mathbf{a}_y + 4\mathbf{a}_z)}{\sqrt{20}} = \underline{\underline{-74.15\mathbf{a}_y + 148.9\mathbf{a}_z}} \quad \text{mW/m}^2$$

**P.E.**



$$\tan\theta_i = \frac{k_{iy}}{k_{iz}} = \frac{2}{4} \longrightarrow \underline{\underline{\theta_i = 26.56 = \theta_r}}$$

$$\sin\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sin\theta_i = \frac{1}{2}\sin 26.56^o \longrightarrow \underline{\underline{\theta_t = 12.92^o}}$$

(b) $\eta_1 = \eta_o, \eta_2 = \eta_o/2$ **E** is parallel to the plane of incidence. Since $\mu_1 = \mu_2 = \mu_o$,

we may use the result of Prob. 10.42, i.e.

$$\Gamma_{\backslash\backslash} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\tan(-13.64^o)}{\tan(39.48^o)} = \underline{\underline{-0.2946}}$$

$$\tau_{\backslash\backslash} = \frac{2\cos 26.56^o \sin 12.92^o}{\sin 39.48^o \cos(-13.64^o)} = \underline{\underline{0.6474}}$$

(c) $\boldsymbol{k}_r = -\beta_1 \sin\theta_r \boldsymbol{a}_y - \beta_1 \cos\theta_r \boldsymbol{a}_z$. Once $k_r$ is known, $E_r$ is chosen such that

$\boldsymbol{k}_r . \boldsymbol{E}_r = 0$ or $\nabla . \boldsymbol{E}_r = 0$. Let

$$\boldsymbol{E}_r = \pm E_{or}(-\cos\theta_r \boldsymbol{a}_y + \sin\theta_r \boldsymbol{a}_z)\cos(\omega t + \beta_1 \sin\theta_r y + \beta_1 \cos\theta_r z)$$

Only the positive sign will satisfy the boundary conditions. It is evident that

$$\boldsymbol{E}_i = E_{oi}(\cos\theta_i \boldsymbol{a}_y + \sin\theta_i \boldsymbol{a}_z)\cos(\omega t + 2y - 4z)$$

Since $\theta_r = \theta_i$,

$$E_{or}\cos\theta_r = \Gamma_{//}E_{oi}\cos\theta_i = 10\Gamma_{//} = -2.946$$

$$E_{or}\sin\theta_r = \Gamma_{//}E_{oi}\sin\theta_i = 5\Gamma_{//} = -1.473$$

$$\beta_1\sin\theta_r = 2, \beta_1\cos\theta_r = 4$$

i.e.

$$E_r = -(2.946a_y - 1.473a_z)\cos(\omega t + 2y + 4z)$$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{(10\boldsymbol{a}_y + 5\boldsymbol{a}_z)\cos(\omega t + 2y - 4z) + (-2.946\boldsymbol{a}_y + 1.473\boldsymbol{a}_z)\cos(\omega t + 2y + 4z)}}$$

V/m

(d) $\boldsymbol{k}_t = -\beta_2\sin\theta_t\boldsymbol{a}_y + \beta_2\cos\theta_t\boldsymbol{a}_z$. Since $k_r \bullet E_r = 0$, let

$$\boldsymbol{E}_t = E_{ot}(\cos\theta_t\boldsymbol{a}_y + \sin\theta_t\boldsymbol{a}_z)\cos(\omega t + \beta_2 y\sin\theta_t - \beta_2 z\cos\theta_t)$$

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \beta_1\sqrt{\varepsilon_{r2}} = 2\sqrt{20}$$

$$\sin\theta_t = \frac{1}{2}\sin\theta_i = \frac{1}{2\sqrt{5}}, \qquad \cos\theta_t = \frac{\sqrt{9}}{\sqrt{20}}$$

$$\beta_2\cos\theta_t = 2\sqrt{20}\sqrt{\frac{19}{20}} = 8.718$$

$$E_{ot}\cos\theta_t = \tau_{//}E_{oi}\cos\theta_t = 0.6474\sqrt{125}\sqrt{\frac{19}{20}} = 7.055$$

$$E_{ot}\sin\theta_t = \tau_{//}E_{oi}\sin\theta_t = 0.6474\sqrt{125}\sqrt{\frac{1}{20}} = 1.6185$$

Hence

$$\boldsymbol{E}_2 = \boldsymbol{E}_t = \underline{\underline{(7.055\boldsymbol{a}_y + 1.6185\boldsymbol{a}_z)\cos(\omega t + 2y - 8.718z)}} \quad \text{V/m}$$

(d) $\tan\theta_{B//} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 2 \longrightarrow \underline{\underline{\theta_{B//} = 63.43^o}}$

**P.E.**

$$S_i = \frac{1+0.4}{1-0.4} = \frac{1.4}{0.6} = \underline{\underline{2.333}}$$

$$S_o = \frac{1+0.2}{1-0.2} = \frac{1.2}{0.8} = \underline{\underline{1.5}}$$

## CHAPTER —Problems

**Prob.**

(a) Let $\vec{E} = E_o \sin(\omega t - \beta x + \phi)\vec{a}_z$

$\frac{\lambda}{2} = 170 - 20 = 150\,m \longrightarrow \lambda = 300\,m$

$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{300}, \quad \omega = \beta c = \frac{2\pi}{300} \times 3 \times 10^8 = 2\pi \times 10^6$

If $\vec{E}(x=0, t=0) = 0$, then $\phi = 0$. Hence

$$\underline{\underline{\vec{E} = 0.01 \sin\left(2\pi \times 10^6 t - \frac{\pi x}{150}\right)\vec{a}_z \ V/m.}}$$

(b) At $x = 100\,m$, $t = 2\mu s$,

$$\vec{E} = 0.01 \sin\left(4\pi - \frac{100\pi}{150}\right)\vec{a}_z = \underline{\underline{-8.66\,\vec{a}_z \ mV/m}}.$$

**Prob.**

If $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = -\omega^2\mu\varepsilon + j\omega\mu\sigma$ and $\gamma = \alpha + j\beta$, then

$$|\gamma^2| = \sqrt{(\alpha^2 - \beta^2) + 4\alpha^2\beta^2} = \sqrt{(\alpha^2 + \beta^2)^2} = \alpha^2 + \beta^2$$

i.e.

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\sigma^2 + \omega^2\varepsilon^2)} \qquad (1)$$

$$\mathrm{Re}(\gamma^2) = \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

$$\beta^2 - \alpha^2 = \omega^2\mu\varepsilon \qquad (2)$$

Subtracting and adding (1) and (2) lead respectively to

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

(b) From **Eq. (9.25),** $\boldsymbol{E}_s(z) = E_o e^{-\gamma z} \boldsymbol{a}_x$.

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}_s \quad \longrightarrow \quad \boldsymbol{H}_s = \frac{j}{\omega\mu} \nabla \times \boldsymbol{E}_s = \frac{j}{\omega\mu}(-\gamma E_o e^{-\gamma z} \boldsymbol{a}_y)$$

But $\boldsymbol{H}_s(z) = H_o e^{-\gamma z} \boldsymbol{a}_y$, hence $H_o = \dfrac{E_o}{\eta} = -\dfrac{j\gamma}{\omega\mu} E_o$

$$\eta = \frac{j\omega\mu}{\gamma}$$

(c) From (b),

$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - j\dfrac{\sigma}{\omega\varepsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}}, \tan 2\theta_\eta = \left(\frac{\omega\varepsilon}{\sigma}\right)^{-1} = \frac{\sigma}{\omega\varepsilon}$$

**Prob.**

(a) Along the positive x direction.

(b) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6} = \underline{\underline{1.0472 \text{ m}}}$.

(c) $w = 10^8$, $\beta = 6$, $u = \dfrac{w}{\beta} = \dfrac{10^8}{6} = \underline{\underline{1.667 \times 10^7 \text{ m/s}}}$.

**Prob.**

$$\frac{\sigma}{\omega\varepsilon} = \frac{8x10^{-2}}{2\pi x 50 x 10^6 x 3.6 x \dfrac{10^{-9}}{36\pi}} = 8$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = \frac{2\pi \times 50 \times 10^6}{3 \times 10^8}\sqrt{\frac{2.1 \times 3.6}{2}[\sqrt{65} - 1]} = 5.41$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} = 6.129$$

$\gamma = \alpha + j\beta = \underline{\underline{5.41 + j6.129}}$ /m

(b) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6.129} = \underline{\underline{1.025}}$ m

(c) $u = \dfrac{\omega}{\beta} = \dfrac{2\pi \times 50 \times 10^6}{6.129} = \underline{\underline{5.125 \times 10^7}}$ m/s

(d) $$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{120\pi\sqrt{\frac{2.1}{3.6}}}{\sqrt[4]{65}} = 101.4$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 8 \longrightarrow \theta_\eta = 41.44^o$$

$\eta = \underline{\underline{101.41\angle 41.44^o\,\Omega}}$

(e) $$\boldsymbol{H}_s = \boldsymbol{a}_k \times \frac{\boldsymbol{E}_s}{\eta} = \boldsymbol{a}_x \times \frac{6}{\eta} e^{-\gamma z}\boldsymbol{a}_z = -\frac{6}{\eta} e^{-\gamma z}\boldsymbol{a}_y = \underline{\underline{-59.16 e^{-j41.44^o} e^{-\gamma z}\boldsymbol{a}_y}} \text{ mA/m}$$

**Prob.**

(a) Let $u = \dfrac{\sigma}{\omega\varepsilon}$ = loss tangent

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+u^2}+1\right]}$$

$$10 = \frac{\omega}{c}\sqrt{\frac{5\times 2}{2}\left[\sqrt{1+u^2}+1\right]} = \frac{2\pi\times 5\times 10^6\sqrt{5}}{3\times 10^8}\sqrt{\left[\sqrt{1+u^2}+1\right]}$$

which leads to

$$u = \frac{\sigma}{\omega\varepsilon} = \underline{\underline{1823}}$$

(b) $\sigma = \omega\varepsilon u = 2\pi\times 5\times 10^6 \times 2\times 1823\times 2\times\frac{10^{-9}}{36\pi} = \underline{\underline{1.013}}$ S/m

(c ) $\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\frac{\sigma}{\omega} = 2\times\frac{10^{-9}}{36\pi} - j\frac{1.023}{2\pi\times 5\times 10^6} = \underline{\underline{1.768\times 10^{-11} - j3.224\times 10^{-8}}}$ F/m

d) $\frac{\alpha}{\beta} = \frac{\sqrt{\sqrt{1+u^2}-1}}{\sqrt{\sqrt{1+u^2}+1}} = \sqrt{\frac{1822}{1824}}$

$\alpha = \underline{\underline{0.9995}}$ Np/m

(e) $|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+u^2}} = \frac{120\pi\sqrt{\frac{5}{2}}}{\sqrt[4]{1+1823^2}} = 13.96$

$\tan 2\theta_\eta = u = 1823 \longrightarrow \theta_\eta = 44.98^o$

$\eta = \underline{\underline{13.96\angle 44.98^o\,\Omega}}$

**Prob.**

(a) $|E| = E_o e^{-\alpha z}$

$E_o e^{-\alpha(1)} = (1-0.18)E_o \longrightarrow e^{-\alpha} = 0.82$

$\alpha = \ln\frac{1}{0.82} = 0.1984$

$\theta_\eta = 24^o \longrightarrow \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 1.111$

$$\frac{\alpha}{\beta} = \frac{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1}}{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1}} = \sqrt{\frac{\sqrt{2.233} - 1}{\sqrt{2.233} + 1}} = 2.247\,, \quad \beta = 0.4458$$

$$\gamma = \alpha + j\beta = \underline{\underline{0.1984 + j0.4458}} \ /\text{m}$$

(b) $\lambda = \frac{2\pi}{\beta} = 2\pi / 0.4458 = \underline{\underline{14.09}}$ m

(c ) $\delta = 1/\alpha = \underline{\underline{5.04}}$ m

(d) Since

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = \frac{\omega}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}}\sqrt{0.494}, \qquad \mu_r = 1$$

$$\sqrt{\frac{\varepsilon_r}{2}} = \frac{\alpha c}{\omega\sqrt{0.494}} = \frac{0.1984x3x10^8}{2\pi x10^7\sqrt{0.494}} = 1.348 \longrightarrow \varepsilon_r = 3.633$$

Since $\frac{\sigma}{\omega\varepsilon} = 1.111$

$$\sigma = \omega\varepsilon_o\varepsilon_r x1.111 = 2\pi x10^7 x\frac{10^{-9}}{36\pi}x3.633x1.111 = \underline{\underline{2.24x10^{-3}}} \ \text{S/m}$$

**Prob.**

(a) $\frac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60^o = \underline{\underline{1.732}}$

(b) $|\eta| = 240 = \frac{\frac{120\pi}{\sqrt{\varepsilon_r}}}{\sqrt[4]{1+3}} = \frac{120\pi}{\sqrt{2\varepsilon_r}} \longrightarrow \varepsilon_r = \frac{\pi^2}{8} = \underline{\underline{1.234}}$

(c) $\varepsilon_c = \varepsilon(1 - j\frac{\sigma}{\omega\varepsilon}) = 1.234\times\frac{10^{-9}}{36\pi}(1 - j1.732) = \underline{\underline{(1.091 - j1.89)\times 10^{-11}}}$ F/m

(d)

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \frac{2\pi\times10^6}{3\times10^8}\sqrt{\frac{1}{2}\frac{\pi^2}{8}\left[\sqrt{1+3}-1\right]} = \underline{\underline{0.0164}}\ \text{Np/m}$$

**Prob.**

$\beta = 0.5$ rad/m

$$0.8E_o = E_o e^{-\alpha 1} \quad \longrightarrow \quad e^{\alpha} = \frac{1}{0.8} = 1.25$$

$$\alpha = \underline{\underline{0.2231\ \text{Np/m}}}$$

$$\delta = \frac{1}{\alpha} = \underline{\underline{4.4823\ \text{m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times20\times10^6}{0.5} = \underline{\underline{2.513\times10^8\ \text{m/s}}}$$

**Prob.**

$$\tan 2\theta_\eta = \tan 2\times35.26^\circ = 2.827 = \frac{\sigma}{\omega\varepsilon}$$

loss tangent $= \underline{\underline{2.827}}$ .

loss angle $= \theta = 2\theta_\eta = \underline{\underline{70.52^\circ}}$ .

$$\frac{\alpha}{\beta} = \sqrt{\frac{\sqrt{1+(\sigma/\omega\varepsilon)^2}-1}{\sqrt{1+(\sigma/\omega\varepsilon)^2}+1}} = \sqrt{\frac{\sqrt{9}-1}{\sqrt{9}+1}} = \frac{1}{\sqrt{2}}$$

$$\beta = \alpha\sqrt{2} = 0.1\sqrt{2}, \quad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.1\sqrt{2}} = \underline{\underline{44.43\ \text{m}}}$$

**Prob.**

(a)

Let $x=\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$, $\alpha=1,\beta=5$

$$\frac{\alpha}{\beta}=\left(\frac{x-1}{x+1}\right)^{1/2}=\frac{1}{5}$$

Solving for x gives

$$x=\frac{13}{12}=\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$$

$$\alpha=1=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \longrightarrow 1=\frac{\omega}{c}\sqrt{\frac{\varepsilon_r}{2}\left[\frac{15}{12}-1\right]}$$

$$\sqrt{\frac{\varepsilon_r}{24}}=\frac{c}{\omega}=\frac{3\times10^8}{2\pi\times10^8} \longrightarrow \varepsilon_r=\frac{24\times9}{4\pi^2}=\underline{\underline{5.471}}$$

$$\frac{13}{12}=\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \longrightarrow \frac{\sigma}{\omega\varepsilon}=\frac{5}{12}$$

$$\sigma=\frac{5}{12}\omega\varepsilon=\frac{5}{12}\times2\pi\times10^8\times5.471\times\frac{10^{-9}}{36\pi}=\underline{\underline{0.0127\ \text{S/m}}}$$

(b) Let $E=E_oe^{-y}\cos(\omega t-2y+\theta_\eta)a_E$

$$|\eta|=\frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}=\frac{\frac{120\pi}{\sqrt{5.471}}}{\sqrt{\frac{13}{12}}}=154.85$$

$$E_o=H_o|\eta|=30.97$$

$$\tan2\theta_\eta=\frac{\sigma}{\omega\varepsilon}=5/12 \longrightarrow \theta_\eta=11.31^o$$

$$\boldsymbol{a}_E\times\boldsymbol{a}_H=\boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E\times\boldsymbol{a}_z=\boldsymbol{a}_y \longrightarrow \boldsymbol{a}_E=-\boldsymbol{a}_x$$

$$\boldsymbol{E}=\underline{\underline{-30.97e^{-y}\cos(2\pi\times10^8t-5y+11.31^o)\boldsymbol{a}_x\text{V/m}}}$$

**Prob.**

(a) $\gamma = \alpha + j\beta = \underline{\underline{0.05 + j2}}$ /m

(b) $\lambda = 2\pi / \beta = \pi = \underline{\underline{3.142}}$ m

(c) $u = \omega / \beta = \dfrac{2 \times 10^8}{2} = \underline{\underline{10^8}}$ m/s

(d) $\delta = 1 / \alpha = \dfrac{1}{0.05} = \underline{\underline{20}}$ m

**Prob.**

(a) Along $\underline{\underline{\vec{a}_z}}$.

(b) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{1} = \underline{\underline{6.283\text{ m}}}$.

$T = \dfrac{2\pi}{w} = \dfrac{2\pi}{3 \times 10^8} = \underline{\underline{20.94\text{ ms}}}$

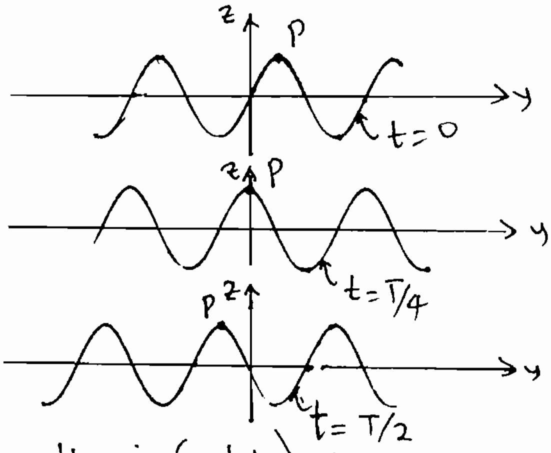
$u = c = \underline{\underline{3 \times 10^8\text{ m/s}}}$.

(c) At $t = 0$, $\vec{E} = 5 \sin y\, \vec{a}_z$ V/m

At $t = T/4$, $\vec{E} = 5 \sin\left(w \cdot \dfrac{2\pi}{4w} + y\right)\vec{a}_z = 5 \cos y\, \vec{a}_z$

At $t = T/2$, $\vec{E} = 5 \sin\left(y + w \cdot \dfrac{2\pi}{2w}\right)\vec{a}_z = -5 \sin y\, \vec{a}_z$

The wave is sketched below.



d) Let $\vec{H} = H_o \sin(wt + y)\, \vec{a}_H$,

$H_o = \dfrac{E_o}{|\eta|} = \dfrac{5}{377} = 0.01326$, $a_z \times a_H = -\vec{a}_y \rightarrow \vec{a}_H = -\vec{a}_x$

$$\vec{H} = \underline{\underline{-13.26 \sin\left(3\times10^{8}t + y\right)\hat{a}_x \quad mA/m.}}$$

**Prob.**

(a) $\beta = \omega / c = \dfrac{2\pi x 10^{6}}{3x10^{8}} = \underline{\underline{0.02094}}$ rad/m,

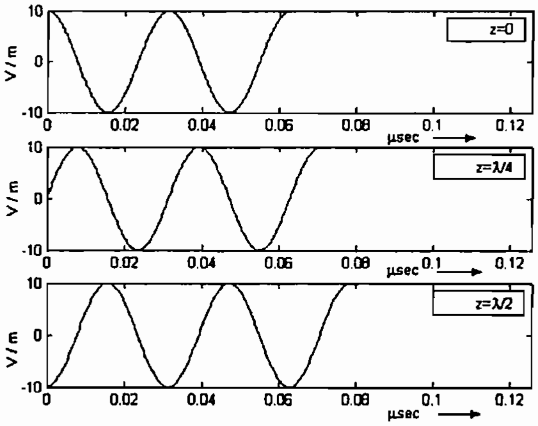
$\lambda = 2\pi / \beta = \underline{\underline{300}}$ m

(b) When $z = 0, \quad E_y = 10\cos\omega t$

$$z = \lambda / 4, \quad E_y = 10\cos(\omega t - \frac{2\pi}{\lambda}\frac{\lambda}{4}) = 10\sin\omega t$$

$$z = \lambda / 2, \quad E_y = 10\cos(\omega t - \pi) = -10\cos\omega t$$

Thus E is sketched below.



(c )

$$\boldsymbol{H} = \frac{10}{120\pi}\cos(2\pi\times10^{6}t - 2\pi z/300)\boldsymbol{a}_x = \underline{\underline{26.53\cos(2\pi\times10^{6}t - 0.02094z)\boldsymbol{a}_x}} \text{ mA/m}$$

**Prob.**

$$u = \frac{c}{\sqrt{\varepsilon_r\mu_r}} = \frac{3\times10^{8}}{\sqrt{3\times4}} = \underline{\underline{8.66x10^{7} \text{ m/s}}}$$

$$\lambda = \frac{u}{f} = \frac{8.66\times10^{7}}{60\times10^{6}} = \underline{\underline{1.443 \text{ m}}}$$

$$\eta = \eta_o\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{4}{3}} = \underline{\underline{435.32\ \Omega}}$$

**Prob.**

$$\frac{\sigma}{w\epsilon} = \frac{5\times10^{-9}}{2\pi\times10^{9}\times9\times\frac{10^{-9}}{36\pi}} = 10^{-8} \simeq 0$$

$$\alpha \simeq 0, \quad \beta \simeq w\sqrt{\mu\epsilon} = \frac{2\pi f}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi\times10^{9}\sqrt{9}}{3\times10^{8}} = 20\pi$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{20\pi} = \underline{\underline{0.1\text{ m}}}$$

It is a $\underline{\underline{\text{lossless medium}}}$ since $\frac{\sigma}{w\epsilon} \simeq 0$.

**Prob.**

$\beta = 4 \longrightarrow \lambda = 2\pi/\beta = \underline{\underline{1.571}}$ m

Also, $\beta = \omega/u = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}$

$$\omega = \frac{\beta c}{\sqrt{\mu_r\varepsilon_r}} = \frac{4x3x10^8}{\sqrt{4}} = \underline{\underline{6x10^8}}\text{ rad/s}$$

$$J_d = \nabla xH = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(z) & 0 & 0 \end{vmatrix} = \frac{\partial H_x}{\partial z}a_y$$

$$J_d = -40\cos(\omega t - 4z)x10^{-3}a_y = \underline{\underline{-40\cos(\omega t - 4z)a_y}}\ \text{mA/m}^2$$

**Prob.**

(a) Along <u>-x direction.</u>

(b) $\beta = 6, \quad \omega = 2\times10^8$,

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \beta c/\omega = \frac{6\times3\times10^8}{2\times10^8} = 9 \longrightarrow \varepsilon_r = 81$$

$$\varepsilon = \varepsilon_o \varepsilon_r = \frac{10^{-9}}{36\pi} \times 81 = \underline{\underline{7.162 \times 10^{-10}}} \text{ F/m}$$

$$\text{(c) } \eta = \sqrt{\mu / \varepsilon} = \sqrt{\mu_o / \varepsilon_o} \sqrt{\mu_r / \varepsilon_r} = \frac{120\pi}{9} = 41.89\ \Omega$$

$$E_o = H_o \eta = 25 \times 10^{-3} \times 41.88 = 1.047$$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E \times \boldsymbol{a}_y = -\boldsymbol{a}_x \longrightarrow \boldsymbol{a}_E = \boldsymbol{a}_z$$

$$\boldsymbol{E} = \underline{\underline{1.047 \sin(2 \times 10^8 t + 6x) \boldsymbol{a}_z}} \text{ V/m}$$

**Prob.**

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \frac{4\pi \times 10^6}{3 \times 10^8} = \frac{4\pi}{300} = \underline{\underline{0.04189 \text{ rad/m}}}.$$

$$\vec{a}_k = \underline{\underline{\vec{a}_z}}$$

$$E_o = \eta_o H_o = 120\pi(0.3) = 36\pi$$

$$-\vec{a}_E = \vec{a}_k \times \vec{a}_H = \vec{a}_z \times (-\vec{a}_x) \longrightarrow \vec{a}_E = \vec{a}_y$$

$$\vec{E} = (36\pi \vec{a}_y) \sin(4\pi \times 10^8 t - kz) \longrightarrow \vec{A} = \underline{\underline{36\pi \vec{a}_y}}.$$

**Prob.**

$$\text{(a) } \frac{\sigma}{\omega \varepsilon} = \frac{10^{-6}}{2\pi x 10^7 x 5 x \frac{10^{-9}}{36\pi}} = 3.6x10^{-4} << 1$$

Thus, the material is $\underline{\underline{\text{lossless}}}$ at this frequency.

$$\text{(b) } \beta = \omega \sqrt{\mu\varepsilon} = \frac{2\pi x 10^7}{3x10^8} \sqrt{5x750} = \underline{\underline{12.83}} \text{ rad/m}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{12.83} = \underline{\underline{0.49}} \text{ m}$$

(c ) Phase difference $= \beta l = \underline{\underline{25.66 \text{ rad}}}$

(d) $\eta = \sqrt{\mu/\varepsilon} = 120\pi\sqrt{\dfrac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\dfrac{750}{5}} = \underline{\underline{4617\Omega}}$

**Prob.**

If $\vec{A}$ is a uniform vector and $\phi(r)$ is a scalar,

$$\nabla \times (\phi\vec{A}) = \nabla\phi \times \vec{A} + \phi\,\cancel{(\nabla \times \vec{A})}^{\,0} = \nabla\phi \times \vec{A}$$

$$\nabla \times \vec{E} = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right) \times \vec{E}_o\, e^{j(k_x x + k_y y + k_z z - wt)}$$

$$= j(k_x\vec{a}_x + k_y\vec{a}_y + k_z\vec{a}_z)\, e^{j(\quad)} \times \vec{E}_o$$

$$= j\vec{k} \times \vec{E}_o\, e^{j(\,)} = jk \times \vec{E}$$

Also $-\dfrac{\partial\vec{B}}{\partial t} = jw\mu\vec{H}$. Hence $\nabla \times \vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ become

$$\vec{k} \times \vec{E} = w\mu\vec{H}.$$

from this, $\vec{a}_k \times \vec{a}_E = \vec{a}_H$.

**Prob.**

**(a)**

$$\beta = \frac{\omega}{c}\sqrt{\varepsilon_r} \qquad \longrightarrow \qquad \sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{5\times 3\times 10^8}{2\pi\times 10^8} = \frac{15}{2\pi}$$

$$\underline{\underline{\varepsilon_r = 5.6993}}$$

(b) $\lambda = 2\pi / \beta = 2\pi / 5 = \underline{\underline{1.2566}}$ m

$$u = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{3\times 10^8}{\frac{15}{2\pi}} = \underline{\underline{1.257\times 10^8}} \text{ m/s}$$

$$(c)\ \eta = \eta_o\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{120\pi}{\frac{15}{2\pi}} = \underline{\underline{157.91\Omega}}$$

$$(d) \quad \boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E \times \boldsymbol{a}_z = \boldsymbol{a}_x \longrightarrow \boldsymbol{a}_E = \underline{\underline{\boldsymbol{a}_y}}$$

$$(e) \quad \boldsymbol{E} = 30x10^{-3}(157.91)\sin(\omega t - \beta x)\boldsymbol{a}_E = \underline{\underline{4.737\sin(2\pi x 10^8 t - 5x)\boldsymbol{a}_y}} \text{ V/m}$$

$$(f)\ \boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times \boldsymbol{H} = \underline{\underline{0.15\cos(2\pi\times 10^8 t - 5x)\boldsymbol{a}_y}} \text{ A/m}$$

**Prob.**

$$\nabla \bullet E = \left(\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right) \bullet E_o e^{j(k_x x + k_y y + k_z z - \omega t)} = j(k_x a_x + k_y a_y + k_z a_z)e^{j(\quad)} \bullet E_o$$

$$= jk \bullet E_o e^{j(\quad)} = jk \bullet E = 0 \qquad \longrightarrow \qquad k \bullet E = 0$$

Similarly,

$$\nabla \bullet H = jk \bullet H = 0 \qquad \longrightarrow \qquad k \bullet H = 0$$

$$\nabla x E = -\frac{\partial B}{\partial t} \qquad \longrightarrow \qquad k x E = \omega \mu H$$

Similarly,

$$\nabla x H = \frac{\partial D}{\partial t} \xrightarrow{\qquad\qquad} kxH = -\varepsilon\omega E$$

From $kxE = \omega\mu H$, $\quad a_k x a_E = a_H$ and

From $kxH = -\varepsilon\omega E$, $\quad a_k x a_H = -a_E$

**Prob.**

(a) $\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{40,000}{3\times10^8} = \underline{\underline{1.333\times10^{-4}\text{ rad/m}}}$

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\frac{4}{3}\times10^{-4}} = \underline{\underline{4.712\times10^4\text{ m}}}$

(c) $u = c = \underline{\underline{3\times10^8\text{ m/s}}}$

(d) Let $\quad \boldsymbol{E} = E_o\cos(\beta z + 40,000t)\boldsymbol{a}_E$

$$E_o = 25\times120\pi = 9.425\times10^3$$

$$\boldsymbol{a}_E x \boldsymbol{a}_H = \boldsymbol{a}_k \quad\longrightarrow\quad \boldsymbol{a}_E x \boldsymbol{a}_y = -\boldsymbol{a}_z \quad\longrightarrow\quad \boldsymbol{a}_E = -\boldsymbol{a}_x$$

$$\boldsymbol{E} = \underline{\underline{-9.425\cos(\beta z + 40,000t)\boldsymbol{a}_x\text{ kV/m}}}$$

**Prob.**

(a) $\frac{\sigma}{w\epsilon} = \frac{4}{2\pi\times10^6 \times \frac{10^{-9}}{36\pi} \times 80} = \underline{\underline{900}}$.

(b) $\frac{\sigma}{w\epsilon} = 18000 \times \frac{10^{-3}}{80} = \underline{\underline{0.225}}$.

(c) $\frac{\sigma}{w\epsilon} = 18000 \times \frac{10^{-3}}{10} = \underline{\underline{1.8}}$.

(d) $\frac{\sigma}{w\epsilon} = 18000 \times \frac{10^{-5}}{3} = \underline{\underline{0.06}}$.

**Prob.**

(a) along $+\mathbf{a}_x$

(b) $u = \frac{\omega}{\beta} = \frac{1}{3} \times 10^8$ m/s

(c) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{3} = 2.0944$ m

**Prob.**

$$\delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

or $$f = \frac{1}{\delta^2 \pi \mu_o \sigma} = \frac{1}{4 \times 10^{-6} \times \pi \times 4\pi \times 10^{-7} \times 6.1 \times 10^{7}}$$

$$= 1.038 \text{ kHz}$$

$$\lambda = \frac{2\pi}{\beta} = 2\pi\delta = 4\pi = 12.566 \text{ mm}$$

$$u = \frac{w}{\beta} = \frac{2}{\delta \mu_o \sigma} = \frac{2}{2 \times 10^{-3} \times 4\pi \times 10^{-7} \times 6.1 \times 10^{7}}$$

$$= 13.04 \text{ m/s}.$$

**Prob.**

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{2\pi \times 10^7}{3 \times 10^8}(10) = 2.0943 \text{ rad/m}$$

$$\boldsymbol{H} = -\frac{1}{\mu}\int \nabla \times \boldsymbol{E} dt$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y(x) & E_z(x) \end{vmatrix} = -\frac{\partial E_z}{\partial x}\boldsymbol{a}_y + \frac{\partial E_y}{\partial x}\boldsymbol{a}_z = -10\beta \sin(\omega t - \beta x)(a_y - a_z)$$

$$\boldsymbol{H} = -\frac{10\beta}{\omega\mu}\cos(\omega t - \beta x)(\boldsymbol{a}_y - \boldsymbol{a}_z) = -\frac{10 \times 2\pi/3}{2\pi \times 10^7 \times 50 \times 4\pi \times 10^{-7}}\cos(\omega t - \beta x)(\boldsymbol{a}_y - \boldsymbol{a}_z)$$

$$\underline{\underline{\boldsymbol{H} = 5.305\cos(2\pi \times 10^7 t - 2.0943x)(-\boldsymbol{a}_y + \boldsymbol{a}_z)}} \quad \text{mA/m}$$

**Prob.**

For a good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1$, say $\frac{\sigma}{\omega\varepsilon} > 100$

(a) $\frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{2\pi x 8x10^6 x15x\frac{10^{-9}}{36\pi}} = 1.5 \longrightarrow$ lossy

No, not conducting.

(b) $\frac{\sigma}{\omega\varepsilon} = \frac{0.025}{2\pi x 8x10^6 x16x\frac{10^{-9}}{36\pi}} = 3.515 \longrightarrow$ lossy

No, not conducting.

(c ) $\frac{\sigma}{\omega\varepsilon} = \frac{25}{2\pi x 8x10^6 x81x\frac{10^{-9}}{36\pi}} = 694.4 \longrightarrow$ conducting

Yes, conducting.

**Prob.**

$$P_{ave} = \frac{E_o^2}{2\eta_o} = \frac{1}{2}\eta_o H_o^2 = \frac{1}{2}(120\pi)(0.1)^2 = \underline{\underline{1.885 \text{ W}/\text{m}^2}}.$$

$\vec{P}$ is along $\underline{\underline{-\vec{a}_y}}$.

**Prob.**

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \frac{2\pi f}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[\sqrt{1.0049}-1\right]} = \frac{2\pi\times 6\times 10^6}{3\times 10^8}\sqrt{\frac{4}{2}\times 2.447\times 10^{-3}}$$

$\alpha = 8.791x10^{-3}$

$\delta = 1/\alpha = \underline{\underline{113.75 \text{ m}}}$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} = \frac{4\pi}{100}\sqrt{\frac{4}{2}\left[\sqrt{1.0049}+1\right]} = 0.2515$$

$$u = \omega/\beta = \frac{2\pi\times 6\times 10^6}{0.2515} = \underline{\underline{1.5\times 10^8}} \text{ m/s}$$

**Prob.**

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\varepsilon_r}\sqrt{\mu_r}, \qquad \mu_r = 1$$

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{8x3x10^8}{10^9} = 2.4 \longrightarrow \underline{\underline{\varepsilon_r = 5.76}}$$

Let $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2,$

$$E_1 = 50\cos(10^9 t - 8x)a_y, \qquad E_2 = 40\sin(10^9 t - 8x)a_z$$

$$H_1 = H_{o1}\cos(10^9 t - 8x)a_{H1}, \qquad H_{o1} = \frac{50x2.4}{120\pi} = \frac{1}{\pi}$$

$$a_{E1}xa_{H1} = a_{k1} \longrightarrow a_y xa_{H1} = a_x \longrightarrow a_{H1} = a_z$$

$$H_1 = \frac{1}{\pi}\cos(10^9 t - 8x)a_z,$$

$$H_2 = H_{o2}\sin(10^9 t - 8x)a_{H2}, \qquad H_{o2} = \frac{40x2.4}{120\pi} = \frac{0.8}{\pi}$$

$$a_{E2}xa_{H2} = a_{k2} \longrightarrow a_z xa_{H2} = a_x \longrightarrow a_{H2} = -a_y$$

$$H_2 = -\frac{0.8}{\pi}\sin(10^9 t - 8x)a_y,$$

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = \underline{\underline{-0.2546\sin(10^9 t - 8x)a_y + 0.3183\cos(10^9 t - 8x)a_z}} \quad \text{A/m}$$

**Prob.**

**(a)**

$$R_{dc} = \frac{l}{\sigma S} = \frac{l}{\sigma \pi a^2} = \frac{600}{5.8\times10^7\times\pi\times(1.2)^2\times10^{-6}} = \underline{\underline{2.287\ \Omega}}$$

(b) $R_{ac} = \dfrac{l}{\sigma\, 2\pi a\delta}$. At 100 MHz, $\delta = 6.6x10^{-3}$ mm for copper **(see Table 9.2).**

$$R_{ac} = \frac{600}{5.8\times10^7\times2\pi\times(1.2\times10^{-3})\times6.6\times10^{-3}} = \underline{\underline{0.2079\ \Omega}}$$

(c) $\dfrac{R_{ac}}{R_{dc}} = \dfrac{a}{2\delta} = 1 \longrightarrow \delta = a/2 = \dfrac{66.1\times10^{-3}}{\sqrt{f}}$

$$\sqrt{f} = \frac{66.1\times2\times10^{-3}}{a} = \frac{66.1\times2}{1.2} \longrightarrow f = \underline{\underline{12.137}}\ \text{kHz}$$

**Prob.**

$$\omega = 10^6\pi = 2\pi f \quad \longrightarrow \quad f = 0.5x10^6$$

$$\delta = \frac{1}{\sqrt{\pi f \sigma \mu}} = \frac{1}{\sqrt{\pi x 0.5x10^6 x 3.5x10^7 x 4\pi x 10^{-7}}} = \underline{\underline{0.1203}} \text{ mm}$$

$$R_{ac} = \frac{l}{\sigma \delta w}$$

since $\delta$ is very small, $w = 2\pi\rho_{outer}$

$$R_{ac} = \frac{l}{\sigma 2\pi\rho_{outer}\delta} = \frac{40}{3.5x10^7 x 0.1203 x 2\pi x 12 x 10^{-6}} = \underline{\underline{0.126\Omega}}$$

**Prob.**

(a) $\vec{E} = Re\left(\vec{E}_s e^{j\omega t}\right) = \underline{\underline{40\cos(\omega t - 8y + 20^\circ)\vec{a}_z}}$.

(b) $\eta = 377\sqrt{\frac{\mu_r}{\epsilon_r}} = 377\sqrt{2.25} = 565.5\ \Omega$.

$\vec{a}_E \times \vec{a}_H = \vec{a}_k \longrightarrow \vec{a}_z \times \vec{a}_H = \vec{a}_y \longrightarrow \vec{a}_H = \vec{a}_x$.

$$\vec{H} = \frac{40}{565.5}\cos(\omega t - 8y + 20^\circ)$$

$$= \underline{\underline{0.0707\cos(\omega t - 8y + 20^\circ)\vec{a}_x \text{ A/m}}}.$$

(c) $\lambda = \frac{2\pi}{\beta} = \pi/4 = \underline{\underline{0.7854\text{ m}}}$, $\eta = \underline{\underline{565.5\ \Omega}}$.

(d) $\vec{P}_{ave} = \frac{E_o^2}{2\eta}\vec{a}_y = \frac{(40)^2}{2(565.5)}\vec{a}_y = \underline{\underline{1.415\vec{a}_y \text{ W/m}^2}}$.

**Prob.**

(a) $$\tan\theta = \frac{\sigma}{\omega\varepsilon} = \frac{3.5\times10^7}{2\pi\times150\times10^6\times\dfrac{10^{-9}}{36\pi}} = \frac{3.5\times18\times10^9}{15} >> 1$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} = \sqrt{150\pi\times10^6\times4\pi\times10^{-7}\times3.5\times10^7} = 143{,}965.86$$

$$\gamma = \alpha + j\beta = \underline{\underline{1.44(1+j)x10^5\ /\text{m}}}$$

(b) $$\delta = 1/\alpha = \underline{\underline{6.946\times10^{-6}\ \text{m}}}$$

(c) $$u = \frac{\omega}{\beta} = \frac{150\times2\pi\times10^6}{1.44\times10^5} = \underline{\underline{6547\ \text{m/s}}}$$

**Prob.**

$\alpha = \beta = 1/\delta$

$\lambda = 2\pi/\beta = 2\pi\delta = 6.283\delta \quad \longrightarrow \quad \delta = 0.1591\lambda$

showing that $\delta$ is shorter than $\lambda$.

**Prob.**

$$\omega = 10^6\pi = 2\pi f \quad \longrightarrow \quad f = 0.5\times10^6$$

$$\delta = \frac{1}{\sqrt{\pi f \sigma\mu}} = \frac{1}{\sqrt{\pi\times0.5\times10^6\times3.5\times10^7\times4\pi\times10^{-7}}} = \underline{\underline{0.1203}} \text{ mm}$$

$$R_{ac} = \frac{l}{\sigma\delta w}$$

since $\delta$ is very small, $w = 2\pi\rho_{outer}$

$$R_{ac} = \frac{l}{\sigma 2\pi\rho_{outer}\delta} = \frac{40}{3.5\times10^7\times2\pi\times12\times10^{-6}\times0.1203} = \underline{\underline{0.126\Omega}}$$

**Prob.**

**(a)**

$$E = \text{Re}[E_s e^{j\omega t}] = (5a_x + 12a_y)e^{-0.2z}\cos(\omega t - 3.4z)$$

At z = 4m, t = T/8, $\omega t = \frac{2\pi}{T}\frac{T}{8} = \frac{\pi}{4}$

$$E = (5a_x + 12a_y)e^{-0.8}\cos(\pi/4 - 13.6)$$

$$|E| = 13e^{-0.8}|\cos(\pi/4 - 13.6)| = \underline{\underline{5.662}}$$

(b) loss $= \alpha\Delta z = 0.2(3) = 0.6$ Np. Since 1 Np = 8.686 dB,

loss = 0.6 x 8.686 = $\underline{\underline{5.212 \text{ dB}}}$

(c ) Let $x = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2} = 0.2/3.4 = \frac{1}{17}$$

$$\frac{x-1}{x+1} = 1/289 \quad \longrightarrow \quad x = 1.00694$$

$$\alpha = \omega\sqrt{\mu\varepsilon/2}\sqrt{x-1} = \frac{\omega}{c}\sqrt{\varepsilon_r/2}\sqrt{x-1}$$

$$\sqrt{\frac{\varepsilon_r}{2}} = \frac{\alpha c}{\omega\sqrt{x-1}} = \frac{0.2x3x10^3}{10^8\sqrt{0.00694}} = 2.4 \quad \longrightarrow \quad \varepsilon_r = 11.52$$

$$|\eta| = \frac{\sqrt{\frac{\mu_o}{\varepsilon_o}}.\frac{1}{\sqrt{\varepsilon_r}}}{\sqrt{x}} = \frac{120\pi}{\sqrt{11.52x1.00694}} = 32.5$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = \sqrt{x^2-1} = 0.118 \quad \longrightarrow \quad \theta_\eta = 3.365^o$$

$$\eta = 32.5\angle 3.365^o$$

$$H_s = a_k x\frac{E_s}{\eta} = \frac{a_z}{\eta}x(5a_x + 12a_y)e^{-\gamma z} = \frac{(5a_x + 12a_y)}{|\eta|}e^{-j3.365^o}e^{-\gamma z}$$

$$H = (-369.2a_x + 153.8a_y)e^{-0.2z}\cos(\omega t - 3.4z - 3.365^o) \text{ mA}$$

$$P = ExH = \begin{vmatrix} 5 & 12 & 0 \\ -369.2 & 153.8 & 0 \end{vmatrix} x10^{-3}e^{-0.4z}\cos(\omega t - 3.4z)\cos(\omega t - 3.4z - 3.365^o)$$

$$P = 5.2e^{-0.4z}\cos(\omega t - 3.4z)\cos(\omega t - 3.4z - 3.365^o)a_z$$

At z = 4, t = T/4,

$$P = 5.2e^{-1.6}\cos(\pi/4 - 13.6)\cos(\pi/4 - 13.6 - 0.0587)a_z = \underline{\underline{0.9702a_z \text{W}/\text{m}^2}}$$

**Prob.**

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \longrightarrow \omega = \frac{2}{\mu\sigma\delta^2}$$

$$\omega = \frac{2}{4\pi\times10^{-7}\times5.8\times10^7\times(10^{-4})^2} = 2.744\times10^6$$

where the value of $\sigma$ for copper is obtained from **Table C.1 in Appendix C.**

$$f = \frac{\omega}{2\pi} = \underline{\underline{436.7 \text{ kHz}}}$$

**Prob.**

$$t = 5\delta = \frac{5}{\sqrt{\pi f\mu\sigma}} = \frac{5}{\sqrt{\pi\times12\times10^9\times4\pi\times10^{-7}\times6.1\times10^7}} = \underline{\underline{2.94\times10^{-6} \text{ m}}}$$

**Prob.**

(a) This is a lossless medium,

$$\beta = \omega\sqrt{\mu\varepsilon}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\eta = \frac{\omega\mu_o}{\beta} = \frac{2\pi x10^8 x4\pi x10^{-7}}{6} = \underline{\underline{131.6\Omega}}$$

(b) $E_o = \eta H_o = 131.6x30x10^{-3} = 3.948$

$$a_E x a_H = a_k \longrightarrow a_E x a_y = a_x \longrightarrow a_E = -a_z$$

$$P = ExH = \eta H_o^2\cos^2(2\pi x10^8 t - 6x)a_x = \underline{\underline{0.1184\cos^2(2\pi x10^8 t - 6x)a_x \text{ W}/\text{m}^2}}$$

(c) $\mathcal{P}_{ave} = \frac{1}{2}\eta H_o^2 = 0.0592a_x \text{ W}/\text{m}^2$

$$P_{ave} = \int \mathcal{P}_{ave}\bullet dS = \mathcal{P}_{ave}\bullet S = 0.0592x3x2 = \underline{\underline{0.3535 \text{ W}}}$$

**Prob.**

(a) $\beta = \frac{w}{c} = \frac{10^7}{3\times10^8} = \underline{\underline{0.0333 \text{ rad/m.}}}$

$E_o = \eta_o H_o = 377, \quad -\vec{a}_E = \vec{a}_k \times \vec{a}_H = \vec{a}_r \times \vec{a}_\theta \rightarrow \vec{a}_E = -\vec{a}_\phi$

$$\underline{\underline{\vec{E} = -\frac{377\sin\theta}{r}\cos(10^7 t - 0.0333r)\,\vec{a}_\phi \text{ V/m.}}}$$

(c) $P_t = \int \vec{\mathcal{P}}_{ave}\cdot d\vec{s} = \int_{\theta=0}^{\pi/3}\int_{\phi=0}^{2\pi} \frac{60\pi\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi$

$= 60\pi(2\pi)\int_0^{\pi/3}(1-\cos^2\theta)\,d(-\cos\theta)$

$= 120\pi^2\left(\frac{5}{24}\right) = \underline{\underline{246.74 \text{ W.}}}$

(b) $\vec{\mathcal{P}} = \vec{E}\times\vec{H} = 377\frac{\sin^2\theta}{r^2}\cos^2(10^7 t - \beta r)\,\vec{a}_r$

$$\underline{\underline{\mathcal{P}_{ave} = \frac{188.5\sin^2\theta}{r^2}\,\vec{a}_r \text{ W/m}^2}}$$

**Prob.**

(a) $$\underline{\underline{\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{H} = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)}\cos^2(\omega t - \beta z)\boldsymbol{a}_z}}$$

(b) $$\boldsymbol{P}_{ave} = \frac{1}{T}\int_0^T \boldsymbol{E}\times\boldsymbol{H}dt = \underline{\underline{\frac{V_o I_o}{4\pi\rho^2 \ln(b/a)}\boldsymbol{a}_z}}$$

**Prob.**

Let $\boldsymbol{E}_s = \boldsymbol{E}_r + j\boldsymbol{E}_i$ and $\boldsymbol{H}_s = \boldsymbol{H}_r + j\boldsymbol{H}_i$

$$\boldsymbol{E} = \mathrm{Re}(\boldsymbol{E}_s e^{j\omega t}) = \boldsymbol{E}_r \cos\omega t - \boldsymbol{E}_i \sin\omega t$$

Similarly,

$$\boldsymbol{H} = \boldsymbol{H}_r \cos\omega t - \boldsymbol{H}_i \sin\omega t$$

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} = \boldsymbol{E}_r\times\boldsymbol{H}_r \cos^2\omega t + \boldsymbol{E}_i\times\boldsymbol{H}_i \sin^2\omega t - \frac{1}{2}(\boldsymbol{E}_r\times\boldsymbol{H}_i + \boldsymbol{E}_i\times\boldsymbol{H}_r)\sin\ 2\omega t$$

$$\mathscr{P}_{\mathrm{ave}} = \frac{1}{T}\int_0^T \mathscr{P}dt = \frac{1}{T}\int_0^T \cos^2\omega dt(\boldsymbol{E}_r\times\boldsymbol{H}_r) + \frac{1}{T}\int_0^T \sin^2\omega dt(\boldsymbol{E}_i\times\boldsymbol{H}_i) - \frac{1}{2T}\int_0^T \sin 2\omega dt(\boldsymbol{E}_i\times\boldsymbol{H}_i + \boldsymbol{E}_i\times\boldsymbol{H}_r)$$

$$= \frac{1}{2}(\boldsymbol{E}_r\times\boldsymbol{H}_r + \boldsymbol{E}_i\times\boldsymbol{H}_i) = \frac{1}{2}\mathrm{Re}[(\boldsymbol{E}_r + j\boldsymbol{E}_i)\times(\boldsymbol{H}_r - j\boldsymbol{H}_i)]$$

$$\mathscr{P}_{\mathrm{ave}} = \frac{1}{2}\ \mathrm{Re}(\boldsymbol{E}_s\times\boldsymbol{H}_s^*)$$

as required.

**Prob.**

$$P_{i\,ave} = \frac{E_{io}^2}{2\eta_1}, \quad P_{r\,ave} = \frac{E_{ro}^2}{2\eta_1}, \quad P_{t\,ave} = \frac{E_{to}^2}{2\eta_2}$$

$$R = \frac{P_{r\,ave}}{P_{i\,ave}} = \frac{E_{ro}^2}{E_{ro}^2} = \Gamma^2 = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2$$

$$T = \frac{P_{t\,ave}}{P_{i\,ave}} = \frac{\eta_1}{\eta_2}\frac{E_{to}^2}{E_{io}^2} = \frac{\eta_1}{\eta_2}\tau^2 = \frac{4\eta_1\eta_2}{(\eta_1+\eta_2)^2}$$

$$R + T = \frac{\eta_2^2 - 2\eta_1\eta_2 + \eta_1^2 + 4\eta_1\eta_2}{(\eta_1 + \eta_2)^2} = 1$$

Alternatively,

$$P_{i\,ave} = P_{r\,ave} + P_{t\,ave}$$

$$1 = \frac{P_{r\,ave}}{P_{i\,ave}} + \frac{P_{t\,ave}}{P_{i\,ave}} = R + T.$$

**Prob.**

(a) $$H_s = \frac{j30\beta I_o dl}{120\pi r}\sin\theta e^{-j\beta r} a_H$$

where $a_E \times a_H = a_k \longrightarrow a_\theta \times a_H = a_r \longrightarrow a_H = a_\phi$

$$\underline{\underline{H_s = \frac{j\beta I_o dl}{4\pi r}\sin\theta e^{-j\beta r} a_\phi}}$$

(b) $$P_{ave} = \frac{1}{2}\mathrm{Re}[E_s \times H_s^*] = \frac{1}{2}\mathrm{Re}[\frac{30\beta^2 I_o^2 dl^2 \sin^2\theta}{4\pi r^2} a_r] = \underline{\underline{\frac{15\beta^2 I_o^2 dl^2 \sin^2\theta}{4\pi r^2} a_r}}$$

**Prob.**

**(a)**

$a_E x a_H = a_k \longrightarrow a_E x a_z = a_x \longrightarrow a_E = -a_y$

i.e. polarization is along the y-axis.

(b) $$\beta = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c}\sqrt{\mu_r\varepsilon_r} = \frac{2\pi x 30x10^6}{3x10^8}\sqrt{4x9} = \underline{\underline{3.77}}\ \text{rad/m}$$

(c) $$J_d = \nabla x H = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ 0 & 0 & H_z(x,t) \end{vmatrix} = -\frac{\partial H_z}{\partial x} a_y$$

$$= -10\beta\cos(\omega t + \beta x) a_y = \underline{\underline{-37.6\cos(\omega t + \beta x) a_y \text{ mA / m}}}$$

(d) $\eta_2 = \eta_o, \qquad \eta_1 = \eta_o\sqrt{\frac{4}{9}} = \frac{2}{3}\eta_o$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 1/5, \qquad \tau = 1 + \Gamma = 6/5$$

$$E_i = 10\eta_1 \sin(\omega t + \beta x)a_E \text{ mV/m}, \;\; a_E = -a_y$$

$$E_r = \Gamma 10\eta_1 \sin(\omega t - \beta x)(-a_y) \text{ mV/m}$$

$$a_E x a_H = a_k \longrightarrow -a_y x a_H = a_x \longrightarrow a_H = -a_z$$

$$H_r = \Gamma 10 \sin(\omega t - \beta x)(-a_z) \text{ mA/m} = \underline{\underline{-2\sin(\omega t - \beta x)a_z \text{ mA/m}}}$$

$$E_t = \tau 10\eta_1 \sin(\omega t + \beta x)(-a_y) \text{ mV/m}$$

$$a_E x a_H = a_k \longrightarrow -a_y x a_H = -a_x \longrightarrow a_H = a_z$$

$$H_t = 10(6/5)(\eta_1/\eta_2)\sin(\omega t + \beta x)a_z \text{ mA/m} = \underline{\underline{8\sin(\omega t + \beta x)a_z \text{ mA/m}}}$$

(e) $\mathcal{P}_{\text{ave1}} = \frac{E_{io}^2}{2\eta_1}(-a_x) + \frac{E_{ro}^2}{2\eta_1}(+a_x) = \frac{-E_{io}^2}{2\eta_1}(1 - \Gamma^2)a_x$

$$= -\frac{\eta_1^2 H_{io}^2}{2\eta_1}(1 - \Gamma^2)a_x = -\frac{1}{3}\eta_o 100(1 - \frac{1}{25})a_x = \underline{\underline{-0.012064a_x \text{ W/m}^2}}$$

$$E_{ot} = \tau E_{oi} = \tau\eta_1 H_{io}$$

$$\mathcal{P}_{\text{ave2}} = \frac{E_{to}^2}{2\eta_2}(-a_x) = \frac{\tau^2\eta_1^2 H_{io}^2}{2\eta_2}(-a_x) = 32\eta_o(-a_x)\ \mu\text{W/m}^2 = \underline{\underline{-0.012064a_x \text{ W/m}^2}}$$

**Prob.**

(a)

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$

$$\tan\theta = \frac{\sigma}{\omega\varepsilon} = 10^{-2}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+10^{-4}}-1\right]} = \frac{2\pi\times2\times10^9}{3\times10^8}\sqrt{\frac{2.5}{2}}\sqrt{0.00005} = 0.3311$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+10^{-4}}+1\right]} = \frac{2\pi\times2\times10^9}{3\times10^8}\sqrt{\frac{2.5}{2}}\sqrt{2.00005} = 66.23$$

(In this case, $\beta \simeq \omega\sqrt{\mu\varepsilon}$ .)

$$(1-0.2)E_o = E_o e^{-\alpha z} \quad \longrightarrow \quad 0.8E_o = E_o e^{-\alpha z}$$

$$e^{\alpha z} = 1.25 \quad \longrightarrow \quad z = \frac{1}{\alpha}\ln 1.25 = \underline{\underline{0.674\text{ m}}}$$

(b) $$\beta z = 180^o = \pi \quad \longrightarrow \quad z = \frac{\pi}{\beta} = \underline{\underline{0.04743\text{ m}}}$$

(c) $$P = P_o e^{-2\alpha z} \quad \longrightarrow \quad 0.9P_o = P_o e^{-2\alpha z}$$

$$e^{2\alpha z} = 1/0.9 = 1.111 \quad \longrightarrow \quad z = \frac{1}{2\alpha}\ln 1.111 = \underline{\underline{0.159\text{ m}}}$$

**Prob.**

Both media are lossless, hence $\Gamma$ is real.

$$\Gamma = \frac{s-1}{s+1} = \frac{1.2}{3.2} = \frac{3}{8}, \quad \tau = 1 + \Gamma = \frac{11}{8}$$

from Prob. 10.29, $P_t = T P_i = \frac{\eta_1}{\eta_2} \tau^2 P_i$.

But $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1-x}{1+x} = \frac{3}{8}$, where $x = \frac{\eta_1}{\eta_2}$

$$3x + 3 = 8 - 8x \longrightarrow x = \frac{5}{11} = \frac{\eta_1}{\eta_2}$$

$$P_t = \frac{5}{11}\left(\frac{11}{8}\right)^2 (3) = \underline{\underline{2.578 \text{ W/m}^2}}.$$

**Prob.**

(a) $P_{i,ave} = \dfrac{E_{io}^2}{2\eta_1}$, $\quad P_{r,ave} = \dfrac{E_{ro}^2}{2\eta_1}$, $\quad P_{t,ave} = \dfrac{E_{to}^2}{2\eta_2}$

$$R = \frac{P_{r,ave}}{P_{i,ave}} = \frac{E_{ro}^2}{E_{io}^2} = \Gamma^2 = \underline{\underline{\left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2}}$$

$$R = \left(\frac{\sqrt{\frac{\mu_o}{\varepsilon_2}} - \sqrt{\frac{\mu_o}{\varepsilon_1}}}{\sqrt{\frac{\mu_o}{\varepsilon_2}} + \sqrt{\frac{\mu_o}{\varepsilon_1}}}\right)^2 = \left(\frac{\sqrt{\mu_o\varepsilon_1} - \sqrt{\mu_o\varepsilon_2}}{\sqrt{\mu_o\varepsilon_1} + \sqrt{\mu_o\varepsilon_2}}\right)^2$$

Since $n_1 = c\sqrt{\mu_1\varepsilon_1} = c\sqrt{\mu_o\varepsilon_1}$, $\quad n_2 = c\sqrt{\mu_o\varepsilon_2}$,

**Prob.**

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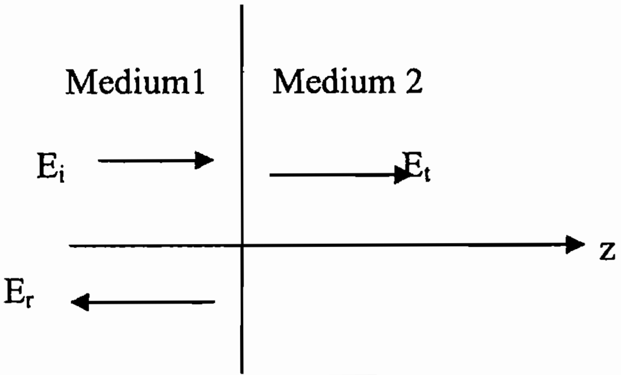
$$\eta_1 = \eta_o, \quad \eta_2 = \sqrt{\frac{\mu}{\epsilon}} = 3\eta_o$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1}{2}, \quad \tau = 1 + \Gamma = \frac{3}{2}, \quad E_{to} = \tau E_{io} = \frac{3}{2}(2) = 2$$

$$P_{t\,ave} = \frac{E_{to}^2}{2\eta_2} = \frac{\frac{1}{2}(3)^2}{3(120\pi)} = \frac{1}{80\pi} = \underline{\underline{3.98 \text{ mW/m}^2}}.$$

**Prob.**



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$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \frac{120\pi}{\sqrt{4}} = 60\pi, \qquad \eta_2 = \eta_o = 120\pi$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{120\pi - 60\pi}{180\pi} = \frac{1}{3}, \qquad \tau = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{2x120\pi}{180\pi} = \frac{4}{3}$$

$$E_{io} = 10, \qquad E_{ro} = \Gamma E_{io} = 10/3, \qquad E_{to} = \tau E_{io} = 40/3$$

$$\boldsymbol{P}_1 = \boldsymbol{P}_i + \boldsymbol{P}_r = \frac{E_{io}^2}{2\eta_1}\boldsymbol{a}_z + \frac{E_{ro}^2}{2\eta_1}(-\boldsymbol{a}_z) = \frac{100}{2\times 60\pi}\boldsymbol{a}_z - \frac{100}{92\times 60\pi}\boldsymbol{a}_z = \underline{\underline{0.2358\boldsymbol{a}_z \text{ W/m}^2}}$$

$$\boldsymbol{P}_2 = \boldsymbol{P}_t = \frac{E_{to}^2}{2\eta_2}\boldsymbol{a}_z = \frac{1600}{9\times 2\times 120\pi}\boldsymbol{a}_z = \underline{\underline{0.2358\boldsymbol{a}_z \text{ W/m}^2}}$$

**Prob.**

(a) In air, $\beta_l = 1, \lambda_l = 2\pi/\beta_l = 2\pi = \underline{\underline{6.283 \text{ m}}}$

$\omega = \beta_l c = \underline{\underline{3x10^8 \text{ rad/s}}}$

In the dielectric medium, ω is the same .

$$\omega = \underline{\underline{3x10^{8}\ \text{rad/s}}}$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\varepsilon_{r2}} = \beta_1\sqrt{\varepsilon_{r2}} = \sqrt{3}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = \underline{\underline{3.6276\ \text{m}}}$$

(b) $H_o = \frac{E_o}{\eta_o} = \frac{10}{120\pi} = 0.0265$

$$a_H = a_k x a_E = a_z x a_y = a_x$$

$$H_i = \underline{\underline{-26.5\cos(\omega t - z)a_x\ \text{mA/m}}}$$

(c) $\eta_1 = \eta_o, \quad \eta_1 = \eta_o/\sqrt{3}$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(1/\sqrt{3}) - 1}{(1/\sqrt{3}) + 1} = \underline{\underline{-0.268}}, \qquad \tau = 1 + \Gamma = \underline{\underline{0.732}}$$

(d) $E_{to} = \tau E_{io} = 7.32, \quad E_{ro} = \Gamma E_{io} = -2.68$

$$E_1 = E_i + E_r = \underline{\underline{10\cos(\omega t - z)a_y - 2.68\cos(\omega t + z)a_y\ \text{V/m}}}$$

$$E_2 = E_t = \underline{\underline{7.32\cos(\omega t - z)a_y\ \text{V/m}}}$$

$$\mathcal{P}_{\text{ave1}} = \frac{1}{2\eta_1}(a_z)[E_{io}^{\ 2} - E_{ro}^{\ 2}] = \frac{1}{2(120\pi)}(a_z)(10^2 - 2.68^2) = \underline{\underline{0.1231a_z\ \text{W/m}}}$$

$$\mathcal{P}_{\text{ave2}} = \frac{E_{to}^{\ 2}}{2\eta_2}(a_z) = \frac{\sqrt{3}}{2x120\pi}(7.32)^2(a_z) = \underline{\underline{0.1231a_z\ \text{W/m}^2}}$$

**Prob.**

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \eta_o / 2, \qquad \eta_2 = \eta_o$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 1/3, \qquad \tau = 1 + \Gamma = 4/3$$

$$E_{or} = \Gamma E_{io} = (1/3)(5) = 5/3, \qquad E_{ot} = \tau E_{io} = 20/3$$

$$\beta = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{10^8}{3x10^8}\sqrt{4} = 2/3$$

(a) $\boldsymbol{E}_r = \frac{5}{3}\cos(10^8 t - 2y/3)\boldsymbol{a}_z$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{5\cos(10^8 t + \frac{2}{3}y)\boldsymbol{a}_z + \frac{5}{3}\cos(10^8 t - \frac{2}{3}y)\boldsymbol{a}_z}} \text{ V/m}$$

(b) $\mathcal{P}_{\text{ave1}} = \frac{E_{io}^{\ 2}}{2\eta_1}(-\boldsymbol{a}_y) + \frac{E_{ro}^{\ 2}}{2\eta_1}(+\boldsymbol{a}_y) = \frac{25}{2(60\pi)}(1-\frac{1}{9})(-\boldsymbol{a}_y) = \underline{\underline{-0.0589\boldsymbol{a}_y \text{ W/m}^2}}$

(c ) $\mathcal{P}_{\text{ave2}} = \frac{E_{to}^{\ 2}}{2\eta_2}(-\boldsymbol{a}_y) = \frac{400}{9(2)(120\pi)}(-\boldsymbol{a}_y) = \underline{\underline{-0.0589\boldsymbol{a}_y \text{ W/m}^2}}$

**Prob.**

(a) $\omega = \beta c = 3\times 3\times 10^8 = \underline{\underline{9\times 10^8 \text{ rad/s}}}$

(b) $\lambda = 2\pi/\beta = 2\pi/3 = \underline{\underline{2.094 \text{ m}}}$

(c) $\frac{\sigma}{\omega\varepsilon} = \frac{4}{9\times 10^8 \times 80\times 10^{-9}/36\pi} = 2\pi = \underline{\underline{6.288}}$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 6.288 \quad \longrightarrow \quad \theta_\eta = 40.47^o$$

$$|\eta_2| = \frac{\sqrt{\mu_2/\varepsilon_2}}{\sqrt[4]{1+\left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2}} = \frac{377/\sqrt{80}}{\sqrt[4]{1+4\pi^2}} = 16.71$$

$$\eta_2 = \underline{\underline{16.71\angle 40.47^o\ \Omega}}$$

(d) $\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{16.71\angle 40.47^o - 377}{16.71\angle 40.47^o + 377} = 0.935\angle 179.7^o$

$$E_{or} = \Gamma E_{oi} = 9.35\angle 179.7^o$$

$$\mathbf{E}_r = \underline{\underline{9.35\sin(\omega t - 3z + 179.7)\mathbf{a}_x \text{ V/m}}}$$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{\mu_{r2}\varepsilon_{r2}}{2}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2} - 1\right]} = \frac{9\times 10^8}{3\times 10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2} - 1\right]} = 43.94 \text{ Np/m}$$

$$\beta_2 = \frac{9\times 10^8}{3\times 10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2} + 1\right]} = 51.48 \text{ rad/m}$$

$$\tau = \frac{2\eta_2}{\eta_2+\eta_1} = \frac{2\times 16.71\angle 40.47^o}{16.71\angle 40.47^o + 377} = 0.0857\angle 38.89^o$$

**Prob.**

$$\vec{H} = H_o \vec{a}_H = \frac{E_o}{\eta} \vec{a}_k \times \vec{a}_E = \frac{50}{120\pi} \vec{a}_k \times \vec{a}_E$$

$$\text{where } \vec{a}_k = \sin 45^\circ \vec{a}_x + \cos 45^\circ \vec{a}_z$$

$$\vec{a}_k \times \vec{a}_E = -\cos 45^\circ \vec{a}_x + \sin 45^\circ \vec{a}_z$$

$$\vec{H} = \frac{50}{120\pi} (-\cos 45^\circ \vec{a}_x + \sin 45^\circ a_z) \cos(wt - \dots)$$

$$\underline{\underline{\vec{H} = 93.78 (-\vec{a}_x + \vec{a}_z) \cos(wt - \beta_1 x \sin 45^\circ - \beta_1 z \cos 45^\circ) \text{ mA/m}}}$$

$$\tau_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{2}{1 + \frac{\eta_1}{\eta_2} \frac{\cos\theta_t}{\cos\theta_i}},$$

$$\cos\theta_i = \vec{a}_n \cdot \vec{a}_k = \cos 45^\circ \longrightarrow \theta_i = 45^\circ$$

$$k_i \sin\theta_i = k_t \sin\theta_t \longrightarrow w\sqrt{\mu_o \varepsilon_o} \sin 45^\circ = w\sqrt{\mu_o \varepsilon_o (2.25)} \sin\theta_t$$

$$\sin\theta_t = \frac{\sin 45^\circ}{1.5} \longrightarrow \theta_t = 28.125^\circ, \frac{\eta_1}{\eta_2} = 1.5$$

$$\tau_\perp = \frac{2}{1 + 1.5 \frac{\cos 28.125^\circ}{\cos 45^\circ}} = \underline{\underline{0.6967}}.$$

**Prob.**

Since $\mu_o = \mu_1 = \mu_2$,

$$\sin\theta_{t1} = \sin\theta_i\sqrt{\frac{\varepsilon_o}{\varepsilon_1}} = \frac{\sin 45^o}{\sqrt{4.5}} = 0.3333 \quad \longrightarrow \quad \underline{\underline{\theta_{t1} = 19.47^o}}$$

$$\sin\theta_{t2} = \sin\theta_{t1}\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{1}{3}\sqrt{\frac{4.5}{2.25}} = 0.4714 \quad \longrightarrow \quad \underline{\underline{\theta_{t2} = 28.13^o}}$$

**Prob.**

$$\boldsymbol{E}_s = \frac{20(e^{jk_xx} - e^{-jk_xx})}{j2}\frac{(e^{jk_yy} + e^{-jk_yy})}{2}\boldsymbol{a}_z$$

$$= -j5\left[e^{j(k_xx+k_yy)} + e^{j(k_xx-k_yy)} - e^{-j(k_xx-k_yy)} - e^{-j(k_xx+k_yy)}\right]\boldsymbol{a}_z$$

which consists of four plane waves.

$$\nabla\times\boldsymbol{E}_s = -j\omega\mu_o\boldsymbol{H}_s \quad \longrightarrow \quad \boldsymbol{H}_s = \frac{j}{\omega\mu_o}\nabla\times\boldsymbol{E}_s = \frac{j}{\omega\mu_o}\left(\frac{\partial E_z}{\partial y}\boldsymbol{a}_x - \frac{\partial E_z}{\partial x}\boldsymbol{a}_y\right)$$

$$\underline{\underline{\boldsymbol{H}_s = -\frac{j20}{\omega\mu_o}\left[k_y\sin(k_xx)\sin(k_yy)\boldsymbol{a}_x + k_x\cos(k_xx)\cos(k_yy)\boldsymbol{a}_y\right]}}$$

**Prob.**

Total transmission occurs at Brewster angle
Since $\mu_1 = \mu_2 = \mu_o$, this can only occur for
<u>parallel polarization.</u>

$\eta_1 = 1, \quad \eta_2 = \sqrt{\epsilon_{r2}} = 2$

$\theta_{B\parallel} = \theta_i = \tan^{-1}\frac{n_2}{n_1} = \tan^{-1}\frac{2}{1} \longrightarrow \theta_i = 63.43°$

from Snell's law, $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{\sin 63.43°}{2}$

$\longrightarrow \quad \theta_t = \underline{\underline{26.56°}}$

**Prob.**

If $\mu_o = \mu_1 = \mu_2,\quad \eta_1 = \dfrac{\eta_o}{\sqrt{\varepsilon_{r1}}}, \eta_2 = \dfrac{\eta_o}{\sqrt{\varepsilon_{r2}}}$

$$\Gamma_{\parallel} = \frac{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t - \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i}{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t + \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i}$$

$$\sqrt{\varepsilon_{r1}}\sin\theta_i = \sqrt{\varepsilon_{r2}}\sin\theta_t \qquad \longrightarrow \qquad \frac{\sqrt{\varepsilon_{r2}}}{\sqrt{\varepsilon_{r1}}} = \frac{\sin\theta_i}{\sin\theta_t}$$

$$\Gamma_{\parallel} = \frac{\cos\theta_t - \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}{\cos\theta_t + \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i}$$

$$= \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} = \frac{\sin(\theta_t - \theta_i)\cos(\theta_t + \theta_i)}{\cos(\theta_t - \theta_i)\sin(\theta_t + \theta_i)} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}$$

Similarly,

$$\tau_{\parallel} = \frac{\dfrac{2}{\sqrt{\varepsilon_{r2}}}\cos\theta_i}{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t + \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i} = \frac{2\cos\theta_i}{\cos\theta_t + \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin\theta_t\cos\theta_t(\sin^2\theta_i + \cos^2\theta_i) + \sin\theta_i\cos\theta_i(\sin^2\theta_t + \cos^2\theta_t)}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{(\sin\theta_i\cos\theta_t + \sin\ \theta_t\cos\ \theta_i)(\cos\theta_i\cos\theta_t + \sin\ \theta_i\sin\ \theta_t)}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

$$\Gamma_{\perp} = \frac{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i - \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t}{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i + \dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t} = \frac{\cos\theta_i - \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_t}{\cos\theta_i + \dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

$$\tau_{\perp} = \frac{\frac{2}{\sqrt{\varepsilon_{r2}}}\cos\theta_i}{\frac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i + \frac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} = \frac{2\cos\theta_i \sin\theta_i}{\sin(\theta_t + \theta_i)}$$

**Prob.**

$\beta_1 = \sqrt{3^2+4^2} = 5 = \frac{w}{c} \rightarrow w = \beta_1 c = \underline{\underline{15 \times 10^8 \text{ rad/s.}}}$

Let $\vec{E}_r = (E_{ox}, E_{oy}, E_{oz}) \sin(wt + 3x + 4y)$. In order for $\nabla \cdot \vec{E}_r = 0$, $3E_{ox} + 4E_{oy} = 0$ —(1)

Also at $y = 0$, $\vec{E}_{1tan} = E_{2tan} = 0$

$\vec{E}_{1tan} = 0 \longrightarrow 8\bar{a}_x + 5\bar{a}_z + E_{ox}\bar{a}_x + E_{oz}\bar{a}_z = 0$

Equating components, $E_{ox} = -8$, $E_{oz} = -5$.

from (1), $4E_{oy} = -3E_{ox} = 24 \longrightarrow E_{oy} = 6$

Hence $\vec{E}_r = \underline{\underline{(-8\bar{a}_x + 6\bar{a}_y - 5\bar{a}_z)\sin(15\times10^8 t + 3x + 4y) \text{ V/m}}}$

**Prob.**

(a) $\boldsymbol{k}_i = 4\boldsymbol{a}_y + 3\boldsymbol{a}_z$

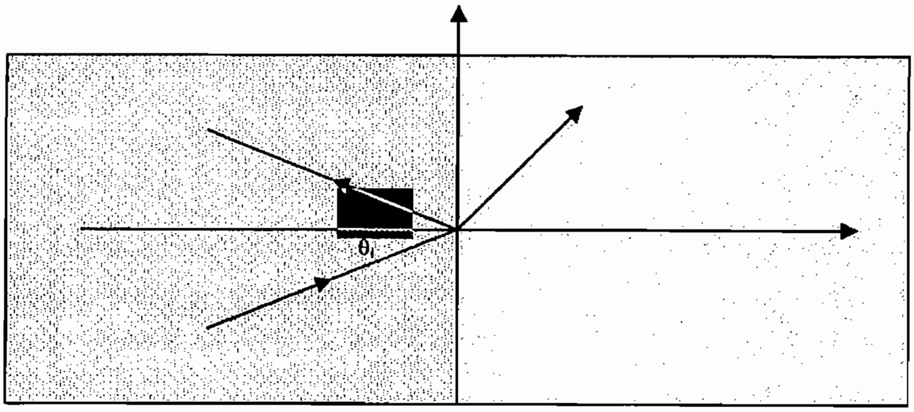
$\boldsymbol{k}_i \bullet \boldsymbol{a}_n = k_i \cos\theta_i \longrightarrow \cos\theta_i = 4/5 \longrightarrow \underline{\underline{\theta_i = 36.87^o}}$

(b)

$$\boldsymbol{P}_{ave} = \frac{1}{2}\text{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s^*) = \frac{E_o^2}{2\eta}\boldsymbol{a}_k = \frac{(\sqrt{8^2+6^2})^2}{2\times120\pi}\frac{(4\boldsymbol{a}_y + 3\boldsymbol{a}_z)}{5} = \underline{\underline{106.1\boldsymbol{a}_y + 79.58\boldsymbol{a}_z \text{ mW/m}^2}}$$

(c) $\theta_r = \theta_i = 36.87^o$. Let

$$\boldsymbol{E}_r = (E_{ry}\boldsymbol{a}_x + E_{rz}\boldsymbol{a}_z)\sin(\omega t - \boldsymbol{k}_r \bullet \boldsymbol{r})$$



From the figure, $\boldsymbol{k}_r = k_{rz}\boldsymbol{a}_z - k_{ry}\boldsymbol{a}_y$. But $k_r = k_i = 5$

$$k_{rz} = k_r \sin\theta_r = 5(3/5) = 3, \qquad k_{ry} = k_r \cos\theta_r = 5(4/5) = 4,$$

Hence, $\boldsymbol{k}_r = -4\boldsymbol{a}_y + 3\boldsymbol{a}_z$

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{c\sqrt{\mu_1\varepsilon_1}}{c\sqrt{\mu_2\varepsilon_2}}\sin\theta_i = \frac{3/5}{\sqrt{4}} = 0.3$$

$$\theta_t = 17.46, \quad \cos\theta_t = 0.9539, \qquad \eta_1 = \eta_o = 120\pi, \eta_2 = \eta_o/2 = 60\pi$$

$$\Gamma_{//} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{\eta_o}{2}(0.9539) - \eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = -0.253$$

$$E_{ro} = \Gamma_{//}E_{io} = -0.253(10) = -2.53$$

But $(E_{ry}\boldsymbol{a}_y + E_{rz}\boldsymbol{a}_z) = E_{ro}(\sin\theta_r\boldsymbol{a}_y + \cos\theta_r\boldsymbol{a}_z) = -2.53(\frac{3}{5}\boldsymbol{a}_y + \frac{4}{5}\boldsymbol{a}_z)$

$$\underline{\underline{\boldsymbol{E}_r = -(1.518\boldsymbol{a}_y + 2.024\boldsymbol{a}_z)\sin(\omega t + 4y - 3z) \text{ V/m}}}$$

Similarly, let

$$\boldsymbol{E}_t = (E_{ty}\boldsymbol{a}_y + E_{tz}\boldsymbol{a}_z)\sin(\omega t - \boldsymbol{k}_t \bullet \boldsymbol{r})$$

$$k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \omega\sqrt{4\mu_o\varepsilon_o}$$

But $k_i = \beta_1 = \omega\sqrt{\mu_o\varepsilon_o}$

$$\frac{k_t}{k_i} = 2 \quad \longrightarrow \quad k_t = 2k_i = 10$$

$$k_{ty} = k_t\cos\theta_t = 9.539, \qquad k_{tz} = k_t\sin\theta_t = 3,$$

$$k_t = 9.539a_y + 3a_z$$

Note that $k_{iz} = k_{rz} = k_{tz} = 3$

$$\tau_{\backslash\backslash} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = 0.6265$$

$$E_{to} = \tau_{\backslash\backslash}E_{io} = 6.265$$

But

$$(E_{ty}\boldsymbol{a}_y + E_{tz}\boldsymbol{a}_z) = E_{to}(\sin\theta_t\boldsymbol{a}_y - \cos\theta_t\boldsymbol{a}_z) = 6.256(0.3\boldsymbol{a}_y - 0.9539\boldsymbol{a}_z)$$

Hence,

$$\underline{\underline{\boldsymbol{E}_t = (1.879\boldsymbol{a}_y - 5.968\boldsymbol{a}_z)\sin(\omega t - 9.539y - 3z)}} \ \text{V/m}$$

**Prob.**

$$\beta_1 = \sqrt{3^2 + 4^2} = 5 = \omega / c \quad \longrightarrow \quad \underline{\underline{\omega = \beta_1 c = 15\times10^8 \text{ rad/s}}}$$

Let $\boldsymbol{E}_r = (E_{ox}, E_{oy}, E_{oz})\sin(\omega t + 3x + 4y)$. In order for

$$\nabla \bullet \boldsymbol{E}_r = 0, \qquad 3E_{ox} + 4E_{oy} = 0 \qquad (1)$$

Also, at y=0, $E_{1\tan} = E_{2\tan} = 0$

$E_{I\tan} = 0, \quad 8a_x + 5a_z + E_{ox}a_x + E_{oz}a_z = 0$

Equating components, $E_{ox} = -8, \quad E_{oz} = -5$

From (1), $4E_{oy} = -3E_{ox} = 24 \quad E_{oy} = 6$

Hence,

$$\underline{\underline{\boldsymbol{E}_r = (-8\boldsymbol{a}_x + 6\boldsymbol{a}_y - 5\boldsymbol{a}_z)\sin(15\times10^8 t + 3x + 4y) \text{ V/m}}}$$

**Prob.**

(a) In terms of the S-parameters, the T-parameters are given by

$T_{11} = 1/S_{21}$, $T_{12} = -S_{22}/S_{21}$, $T_{21} = S_{11}/S_{21}$, $T_{22} = S_{12} - S_{11} S_{22}/S_{21}$

(b) $T_{11} = 1/0.4 = 2.5$, $T_{12} = -0.2/0.4$,

$T_{21} = 0.2/0.4$, $T_{22} = 0.4 - 0.2 \times 0.2/0.4 = 0.3$

Hence,

$$T = \underline{\underline{\begin{bmatrix} 2.5 & -0.5 \\ 0.5 & 0.3 \end{bmatrix}}}$$

**Prob.**

The microwave wavelengths are of the same magnitude as the circuit components. The wavelength in air at a microwave frequency of 300 GHz, for example, is 1 mm. The physical dimension of the lumped element must be in this range to avoid interference. Also, the leads connecting the lumped element probably have much more inductance and capacitance than is needed.

**Prob.**

$$\lambda = c/f = \frac{3\times10^8}{8.4\times10^9} = \underline{\underline{35.71 \text{ mm}}}$$

## Chapter —Practice Examples

### P. E.

Since $Z_o$ is real and α ≠ 0, this is a distortionless line.

$$Z_o = \sqrt{\frac{R}{G}} \qquad (1)$$

or $$\frac{L}{R} = \frac{C}{G} \qquad (2)$$

$$\alpha = \sqrt{RG} \qquad (3)$$

$$\beta = \omega L\sqrt{\frac{G}{R}} = \frac{\omega L}{Z_o} \qquad (4)$$

$$(1) \times (3) \rightarrow R = \alpha Z_o = 0.04 \times 80 = \underline{\underline{3.2\Omega/\mathrm{m}}},$$

$$(3) \div (1) \rightarrow G = \frac{\alpha}{Z_o} = \frac{0.04}{80} = \underline{\underline{5 \times 10^{-4} S/\mathrm{m}}}$$

$$L = \frac{\beta Z_o}{\omega} = \frac{1.5 \times 80}{2\pi \times 5 \times 10^8} = \underline{\underline{38.2\ \mathrm{nH/m}}}$$

$$C = \frac{LG}{R} = \frac{12}{\pi}.10^{-8} \times \frac{0.04}{80} \times \frac{1}{0.04 \times 80} = \underline{\underline{5.97\ \mathrm{pF/m}}}$$

### P.E.

(a) $$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{0.03 + j2\pi \times 0.1 \times 10^{-3}}{0 + j2\pi \times 0.02 \times 10^{-6}}}$$

$$= 70.73 - j1.688 = \underline{\underline{70.75\angle - 1.367^\circ\,\Omega}}$$

(b) $$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(0.03 + j0.2\pi)(j0.4 \times 10^{-4}\pi)}$$

$$\underline{\underline{= 2.121 \times 10^{-4} + j8.888 \times 10^{-3}\ /\mathrm{m}}}$$

(c) $$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^3}{8.888 \times 10^{-3}} = \underline{\underline{7.069 \times 10^5\ \mathrm{m/s}}}$$

## P.E.

(a) $Z_o = Z_l \rightarrow Z_{in} = Z_o = \underline{\underline{30 + j60\Omega}}$

(b) $V_{in} = V_o = \dfrac{Z_{in}}{Z_{in} + Z_o} V_g = \dfrac{V_g}{2} = \underline{\underline{7.5\angle 0^o \text{ V}_{\text{rms}}}}$

$$I_{in} = I_o = \frac{V_g}{Z_g + Z_{in}} = \frac{V_g}{2Z_o} = \frac{15\angle 0^o}{2(30 + j60)}$$

$$\underline{\underline{= 0.118\angle -63.43^o A}}$$

(c ) Since $\text{Z}_0 = \text{Z}_\text{r}$, $\Gamma = 0 \rightarrow V_o^- = 0, V_o^+ = V_o$

The load voltage is $V_L = V_s(z = l) = V_o^+ e^{-\gamma l}$

$$e^{-\gamma l} = \frac{V_o^+}{V_L} = \frac{7.5\angle 0^o}{5\angle - 48^o} 1.5\angle 48^o$$

$$e^{\alpha l} e^{j\beta l} = 1.5\angle 48^o$$

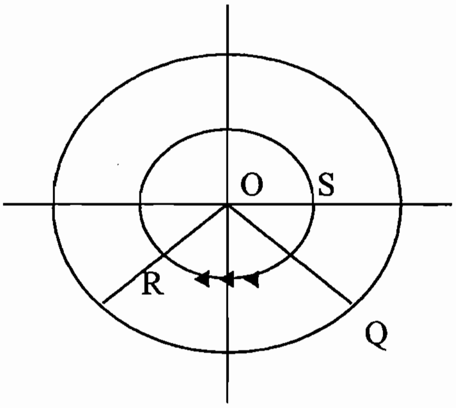
$$e^{\alpha l} = 1.5 \rightarrow \alpha = \frac{1}{l}\ln(1.5) = \frac{1}{40}\ln(1.5) = 0.0101$$

$$e^{j\beta l} = e^{j48^o} \rightarrow \beta = \frac{1}{l}\frac{48^o}{180^o}\pi rad = 0.02094$$

$$\underline{\underline{\gamma = 0.0101 + j0.02094 \text{ /m}}}$$

## P. E.

(a) Using the Smith chart, locate S at s = 1.6. Draw a circle of radius OS. Locate P where $\theta_\Gamma = 300^\circ$. At P,



$$|\Gamma| = \frac{OP}{OQ} = \frac{2.1cm}{9.2cm} = 0.228$$

$$\underline{\underline{\Gamma = 0.228\angle 300^o}}$$

Also at P, $\underline{\underline{z_L = 1.15 - j0.48}}$,

$Z_L = Z_o z_L = 70(1.15 - j0.48) = \underline{\underline{80.5 - j33.6\Omega}}$

$\ell = 0.6\lambda \rightarrow 0.6 \times 720^\circ = 432^\circ = \underline{\underline{360^\circ + 73^\circ}}$

From P, move $432^\circ$ to R. At R, $z_{in} = 0.68 - j025$

$$Z_{in} = Z_o Z_{in} = 70(0.68 - j0.25) = \underline{\underline{47.6 - j17.5\Omega}}$$

(b) The minimum voltage (the only one) occurs at $\theta_\Gamma = 180^o$; its distance from the load is $\frac{180 - 60}{720}\lambda = \frac{\lambda}{6} = \underline{\underline{0.1667\lambda}}$

Values obtained using formulas are as follows:

$$\Gamma = \frac{s-1}{s+1} < 300^o = 0.2308 < 300^o$$

$Z_L = 80.5755 - j34.018\ \Omega$

$Z_{in} = 48.655 - j17.63\ \Omega$

These are pretty close.

## P.E.

(a) $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j60 - 60}{60 + j60 + 60} = \frac{j}{2 + j} = \underline{\underline{0.4472\angle 63.43^o}}$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.4472}{1 - 0.4472} = \underline{\underline{2.618}}$$

Let $x = \tan(\beta l) = \tan\frac{2\pi l}{\lambda}$

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o\tan(\beta l)}{Z_o + jZ_L\tan(\beta l)}\right]$$

$$120 - j60 = 60\left[\frac{60 + j60 + j60x}{60 + j(60 + j60)x}\right]$$

Or $2 - j = \frac{1 + j(1 + x)}{1 - x + jx} \rightarrow 1 - x + j(2x - 2) = 0$

Or $x = 1 = \tan(\beta l)$

$$\frac{\pi}{4} + n\pi = \frac{2\pi l}{\lambda}$$

i.e $\underline{\underline{l = \frac{\lambda}{8}(1 + 4n), n = 0,1,2,3...}}$

(b) $z_L = \frac{Z_L}{Z_o}\frac{60 + j60}{60} = 1 + j$

Locate the load point P on the Smith chart.

$$|\Gamma| = \frac{OP}{OQ} = \frac{4.1cm}{9.2cm} = 0.4457, \theta_\Gamma = 62^o$$

$\underline{\underline{\Gamma = 0.4457\angle 62^o}}$

Locate the point S on the Smith chart. At S, r = $\underline{s = 2.6}$

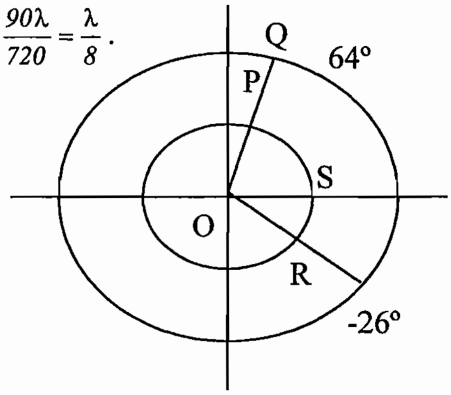
$z_{in} = \frac{Z_{in}}{Z} = \frac{120 + j60}{60} = 2 - j$, which is located at R on the chart. The angle between OP and OR is 64°-(-25°) = 90° which is equivalent to $\frac{90\lambda}{720} = \frac{\lambda}{8}$.

Hence $l = \frac{\lambda}{8} + n\frac{\lambda}{2} = \frac{\lambda}{8}(1+4n), n = 0,1,2........$

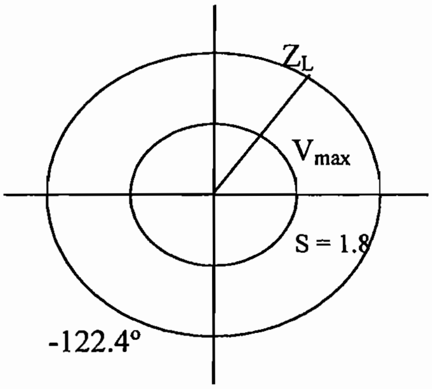
$$(Z_{in})_{\max} = sZ_o = 2.618(60) = 157.08\Omega$$

$$(Z_{in})_{\min} = Z_o / s = 60 / 2.618 = \underline{\underline{22.92\ \Omega}}$$

$$l = \frac{62^o}{720^o}\lambda = 0.0851\lambda$$



**P.E.**



$$\frac{\lambda}{2} = 37.5 - 25 = 12.5cm \text{ or } \lambda = 25cm$$

$$l = 37.5 - 35.5 = 2cm = \frac{2\lambda}{25}$$

$$l = 0.08\lambda \rightarrow 57.6^o$$

$$z_L = 1.184 + j0.622$$

$$Z_L = Z_o z_L = 50(1.184 - j0.622)$$

$$\underline{\underline{= 59.22 + j31.11\Omega}}$$

**P.E.** See the Smith chart

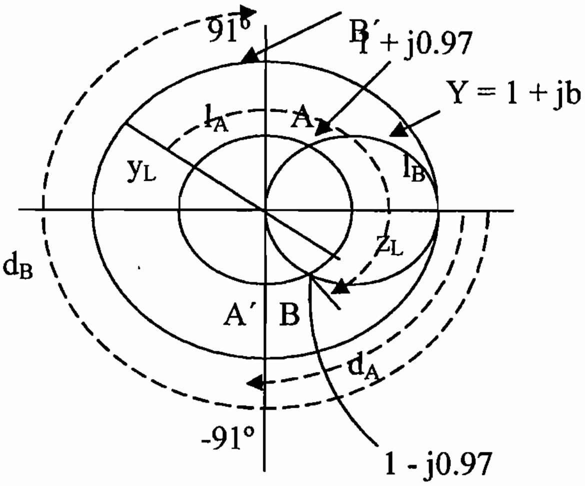
$$z_L = \frac{100 - j80}{75} = 1.33 - j1.067$$

$$l_A = \frac{132^o - 65}{72}\lambda = \underline{\underline{0.093\lambda}}$$

$$l_B = \frac{132^o + 64^o}{720^o} = \underline{\underline{0.272\lambda}}$$

$$d_A = \frac{91}{720}\lambda = 0.126\lambda$$

$$d_B = 0.5\lambda - d_A \quad \underline{\underline{= 0.374\lambda}}$$



$$Y_s = \pm\frac{j0.95}{75} = \underline{\underline{\pm j12.67 \text{ mS}}}$$

**P.E.**

(a) For $w/h = 0.8$, $\varepsilon_{eff} = \dfrac{4.8}{2} + \dfrac{2.8}{2}\left[1 + \dfrac{12}{0.8}\right]^{-1/2} = \underline{\underline{2.75}}$

(b) $Z_o = \dfrac{60}{\sqrt{2.75}}\ln\left(\dfrac{8}{0.8} + \dfrac{0.8}{4}\right) = 36.18\ln 10.2 = \underline{\underline{84.03\Omega}}$

(c) $\lambda = \dfrac{3 \times 10^8}{10^{10}\sqrt{2.75}} = \underline{\underline{18.09 \text{ mm}}}$

**P.E.**

$$R_s = \sqrt{\frac{\pi f \mu_o}{\sigma_c}} = \sqrt{\frac{\pi \times 20 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$

$$= 3.69 \times 10^{-2}$$

$$\alpha_c = 8.685\frac{R_s}{wZ_o} = \frac{8.686 \times 3.69 \times 10^{-2}}{2.5 \times 10^{-3} \times 50}$$

$$= \underline{\underline{2.564 \text{ dB/m}}}$$

## CHAPTER —Problems

**Prob.**

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = \frac{1}{\sqrt{\pi \times 500 \times 10^6 \times 4\pi \times 10^{-7} \times 7 \times 10^7}}$$

$$\delta = 2.6902 \times 10^{-6}$$

$$R = \frac{2}{w\delta\sigma_c} = \frac{2}{0.3 \times 2.6902 \times 10^{-6} \times 7 \times 10^7} = \underline{\underline{0.0354 \Omega / \text{m}}}$$

$$L = \frac{\mu_o d}{w} = \frac{4\pi \times 10^{-7} \times 1.2 \times 10^{-2}}{0.3} = \underline{\underline{50.26 \text{ nH/m}}}$$

$$C = \frac{\varepsilon_o w}{d} = \frac{10^{-9}}{36\pi} \times \frac{0.3}{1.2 \times 10^{-2}} = \underline{\underline{221 \text{ pF/m}}}$$

Since $\sigma = 0$ for air,

$$G = \frac{\sigma w}{d} = 0$$

**Prob.**

(a) For a lossless line, $R = 0 = G$.

$$\gamma = jw\sqrt{LC} \longrightarrow \beta = w\sqrt{LC} = w\sqrt{\mu_o \varepsilon_o} = \frac{w}{c}$$

$$u = \frac{w}{\beta} = c = \frac{1}{\sqrt{LC}}$$

(b) $Z_o = \sqrt{\frac{L}{C}}$, $\beta = w\sqrt{LC}$

$$\frac{Z_o}{\beta} = \frac{1}{wC} \longrightarrow Z_o = \frac{\beta}{wC} = \frac{1}{cC}$$

These proofs do not require the actual values of L and C. Hence the formulas are generally valid for any lossless line.

**Prob.**

$$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}} = \frac{1}{\sqrt{\pi x 80x10^6 x 4\pi x10^{-7} x5.28x10^7}} = 7.744x10^{-6}$$

$$R = \frac{1}{2\pi\delta\sigma_c}\left[\frac{1}{a}+\frac{1}{b}\right] = \frac{\left[\frac{1}{0.8x10^{-3}}+\frac{1}{2.6x10^{-3}}\right]}{2\pi x7.744x10^{-6}x5.28x10^7} = \frac{10^3(1.25+0.3836)}{2569.09} = \underline{\underline{0.6359\ \Omega/\text{m}}}$$

$$L = \frac{\mu}{2\pi}\ln\frac{b}{a} = \frac{4\pi x10^{-7}}{2\pi}\ln\frac{2.6}{0.8} = \underline{\underline{2.357x10^{-7}\ \text{H/m}}}$$

$$G = \frac{2\pi\sigma}{\ln\frac{b}{a}} = \frac{2\pi x10^{-5}}{\ln\frac{2.6}{0.8}} = \underline{\underline{5.33x10^{-5}\ \text{S/m}}}$$

$$C = \frac{2\pi\varepsilon}{\ln\frac{b}{a}} = \frac{2\pi x3.5x\frac{10^{-9}}{36\pi}}{\ln\frac{2.6}{0.8}} = \underline{\underline{1.65x10^{-10}\ \text{F/m}}}$$

**Prob.**

Since $R = 0 = G$,

$$-\frac{\partial V}{\partial z} = L\frac{\partial I}{\partial t} \quad — (1)$$

$$-\frac{\partial I}{\partial z} = C\frac{\partial V}{\partial t} \quad — (2)$$

If $V = V_0 \sin(wt - \beta z)$, from (1)

$$L\frac{\partial I}{\partial t} = V_0\beta\cos(wt - \beta z).$$

$$I = \frac{V_0\beta}{L}\sin(wt - \beta z)$$

Using (2),

$$\frac{V_0\beta^2}{wL}\cos(wt - \beta z) = wCV_0\cos(wt - \beta z)$$

---

i.e. $\frac{\beta^2}{wL} = wC \longrightarrow \beta = w\sqrt{LC}$

but $Z_o = \sqrt{\frac{L}{C}}$, hence $Z_o = \frac{wL}{\beta}$ and

$I = \frac{V_o}{Z_o} \sin(wt - \beta z)$.

**Prob.**

$$C = \frac{\pi\varepsilon l}{\cosh^{-1}(d/2a)} \cong \frac{\pi\varepsilon l}{\ln(d/a)}$$

since $(d/2a)^2 = 11.11 >> 1$.

$$C = \frac{\pi x \frac{10^{-9}}{36\pi} x16x10^{-3}}{\ln(2/0.3)} = \underline{\underline{0.2342 \text{ pF}}}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi x10^7 x4\pi x10^{-7} x5.8x10^7}} = 2.09x10^{-5} \text{ m} << a$$

$$R_{ac} = \frac{l}{\pi a \delta \sigma_c} = \frac{16x10^{-3}}{\pi x0.3x10^{-3} x2.09x10^{-5} x5.8x10^7} = \underline{\underline{1.4x10^{-2}\ \Omega}}$$

**Prob.**

$$Z_o = \sqrt{\frac{L}{c}} = \sqrt{\frac{\mu d}{w} \cdot \frac{d}{\varepsilon w}} = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_o = \eta_o \frac{d}{w} = 78$$

$$Z_o^{'} = \eta_o \frac{d}{w^{'}} = 75$$

$$\frac{78}{75} = \frac{w^{'}}{w} \rightarrow w^{'} 1.04w$$

i.e. the width must be increased by $\underline{\underline{4\%}}$.

**Prob.**

(a)

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = j\omega\sqrt{LC}\sqrt{(1+\frac{R}{j\omega L})(1+\frac{G}{j\omega C})}$$

$$= j\omega\sqrt{LC}\sqrt{1-\frac{RG}{\omega^2 LC}+\frac{R}{j\omega L}+\frac{G}{j\omega C}}$$

As R<< ωL and G<<ωC, dropping the $\omega^2$ term gives

$$\gamma \cong j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}+\frac{G}{j\omega C}} \cong j\omega\sqrt{LC}\left[1+\frac{R}{2j\omega L}+\frac{G}{2j\omega C}\right]$$

$$\underline{\underline{=\left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)+j\omega\sqrt{LC}}}$$

(b)

$$Z_o = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}\sqrt{\frac{1+\frac{R}{j\omega L}}{1+\frac{G}{j\omega C}}} = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left(1+\frac{R}{2j\omega L}+...\right)\left(1-\frac{G}{j2\omega C}+...\right) = \sqrt{\frac{L}{C}}\left(1-j\frac{R}{2\omega L}+j\frac{G}{2\omega C}+...\right)$$

$$\underline{\underline{\cong \sqrt{\frac{L}{C}}\left[1+j\left(\frac{G}{2\omega C}-\frac{R}{2\omega L}\right)\right]}}$$

**Prob.**

(a) $R + j\omega L = \gamma Z_o = (0.25 + j4.2)(100 - j5) = 46 + j418.5$

$\underline{\underline{R = 46\ \Omega/m.}}$

$L = \frac{418.5}{2\pi \times 60 \times 10^6} = \underline{\underline{1.111\ \mu H/m.}}$

$$G + jwC = \frac{\gamma}{Z_0} = \frac{(0.25 + j4.2)}{100 - j5} = 4.007 \times 10^{-4} + j0.042$$

$$G = \underline{\underline{4.007 \times 10^{-4}\ \mho/m}}$$

$$C = \frac{0.042}{2\pi \times 60 \times 10^{6}} = \underline{\underline{111.5\ pF/m.}}$$

(b) $R = 0 = G$

$$C = \frac{\beta}{R_0 w} = \frac{3}{50 \times 2\pi \times 10 \times 10^{6}} = \underline{\underline{955\ pF/m.}}$$

$$L = \frac{R_0 \beta}{w} = \frac{50 \times 3}{2\pi \times 10 \times 10^{6}} = \underline{\underline{2.387\ \mu H/m.}}$$

(c) $Z_0 = 80 = \sqrt{\frac{R + jwL}{G + jwC}}$

or $80\sqrt{G + jwC} = \sqrt{R + jwL}$ — (1)

$\gamma = 0.04 + j1.5 = \sqrt{(R + jwL)(G + jwC)}$ — (2)

From (1) and (2),

$$G + jwC = \frac{\gamma}{Z_0} = \frac{0.04 + j1.5}{80} = 0.0005 + j0.0188$$

$$G = \underline{\underline{0.5\ m\mho/m.}}$$

$$C = \frac{0.0188}{w} = \underline{\underline{5.98\ pF/m}}$$

$$R + jwL = 6400(G + jwC) = 3.2 + j120.32$$

$$R = \underline{\underline{3.2\ \Omega/m.}}$$

$$L = \frac{120.32}{w} = \underline{\underline{0.038\ \mu H/m.}}$$

**Prob.**

(a) $R + j\omega L = 40 + j2\pi \times 10^7 \times 0.2 \times 10^{-6} = 41.93\angle 17.44^o$

$$G + j\omega C = 400 \times 10^{-6} + j2\pi \times 10^7 \times 0.5 \times 10^{-9} = 3.142 \times 10^{-2}\angle 89.27^o$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \underline{\underline{29.59 - j21.43\ \Omega}}$$

$$\gamma = \sqrt{\left(R + j\omega L\right)\left(G + j\omega L\right)} = 0.685 + j0.921 = \alpha + j\beta$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^7}{0.921} = \underline{\underline{6.823 \times 10^7\ \text{m/s}}}$$

(b) $\alpha = 0.685\ \text{Np/m} = 0.685 \times 8.686\ \text{dB/m} = 5.95\ \text{dB/m}$

$$\alpha l = 30 \rightarrow l = \frac{30}{5.95} = \underline{\underline{5.042\ \text{m}}}$$

**Prob.**

(a) $\gamma = \sqrt{(R+jwL)(G+jwC)} = \alpha + j\beta$ —(1)

Squaring both sides,

$\alpha^2 - \beta^2 + j2\alpha\beta = RG - w^2LC + jw(LG+RC)$ - (2)

Hence $\alpha^2 - \beta^2 = RG - w^2LC$ —(3)

$2\alpha\beta = w(LG+RC)$ —(4)

$|\gamma|^2 = \alpha^2 + \beta^2 = [(R^2+w^2L^2)(G^2+w^2C^2)]^{1/2}$ —(5)

Adding and subtracting (4) from (5) respectively give

$$\alpha = \left[\frac{RG - w^2LC + [(R^2+w^2L^2)(G^2+w^2C^2)]^{1/2}}{2}\right]^{1/2}$$

$$\beta = \left[\frac{-RG + w^2LC + [(R^2+w^2L^2)(G^2+w^2C^2)]^{1/2}}{2}\right]^{1/2}$$

(b) $V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$ — (1)

$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$ - (2)

$-\dfrac{dV_s}{dz} = (R+jwL)\,I_s$ — (3)

$-\dfrac{dI_s}{dz} = (G+jwC)\,V_s$ — (4)

Substituting (1) and (2) into (3),

$\gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} = (R+jwL)(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z})$

Equating components of $e^{-\gamma z}$ and $e^{+\gamma z}$,

$$\gamma V_o^+ = (R+j\omega L) I_o^+ \longrightarrow \frac{V_o^+}{I_o^+} = \frac{R+j\omega L}{\gamma}$$

$$-\gamma V_o^- = (R+j\omega L) I_o^- \longrightarrow -\frac{V_o^-}{I_o^-} = \frac{R+j\omega L}{\gamma}.$$

Similarly, substituting (1) and (2) into (4),

$$\gamma I_o^+ e^{-\gamma z} - \gamma I_o^- e^{\gamma z} = (G+j\omega C)\left[V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}\right]$$

Equating components of $e^{-\gamma z}$ and $e^{+\gamma z}$,

$$\gamma I_o^+ = (G+j\omega C) V_o^+ \longrightarrow \frac{V_o^+}{I_o^+} = \frac{\gamma}{G+j\omega C}$$

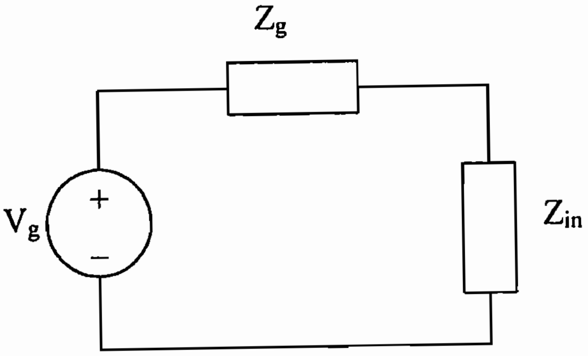
$$-\gamma I_o^- = (G+j\omega C) V_o^- \longrightarrow -\frac{V_o^-}{I_o^-} = \frac{\gamma}{G+j\omega C}$$

(c) $$Y_{in} = \frac{1}{Z_{in}} = \frac{1}{Z_o}\left[\frac{Z_o + jZ_L \tan\beta l}{Z_L + jZ_o \tan\beta l}\right]$$

Dividing every term in brackets by $Z_o Z_L$ gives

$$Y_{in} = Y_o\left[\frac{Y_L + jY_o \tan\beta l}{Y_o + jY_L \tan\beta l}\right].$$

**Prob.**



$$\beta l = \frac{2\pi}{\lambda}(0.64\lambda) = 230.4^o$$

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan\beta l}{Z_o + jZ_L \tan\beta l}\right] = 50\left[\frac{75 + j50\tan 230.4^o}{50 + j75\tan 230.4^o}\right]$$

$$= 50\left[\frac{75 + j50x1.2088}{50 + j75x1.2088}\right] = 43.05 - j17.62$$

$$I = \frac{V_g}{Z_{in} + Z_g} = \frac{12}{43.05 - j17.62 + 50} = 0.1245 + j0.02361$$

$$P_L = \frac{1}{2}|I|^2 R_{in} = \frac{1}{2}(0.1267)^2(43.05) = \underline{\underline{0.3456\text{ W}}}$$

**Prob.**

$$R = \frac{4.5\times10^3}{300} = 15\,\Omega/m, \quad L = \frac{0.15\times10^{-3}}{300} = 0.5\,\mu H/m,$$

$$G = \frac{60\times10^{-3}}{300} = 200\,\mu\mho/m, \quad C = \frac{12\times10^{-9}}{300} = 40\,pF/m.$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{(15+j2\pi\times6\times10^6\times0.5\times10^{-6})(200\times10^{-6}+j2\pi\times6\times10^6\times40\times10^{-12})}$$

$$= \sqrt{(15+j6\pi)(2+j4.8\pi)\times10^{-4}} = 0.1914\angle 66.97^\circ$$

$$\gamma = \underline{\underline{0.075 + j0.1761\ /m.}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times6\times10^6}{0.176} = \underline{\underline{2.142\times10^8\ m/s}}.$$

$$Z_o = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{15+j6\pi}{(2+j4.8\pi)\cdot10^{-4}}} = 54.16 - j15.03$$

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{30+j60-54.26+j15.03}{30+j60+54.26-j15.03}$$

$$= 0.841\angle 79.25^\circ$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1.841}{1-0.841} = 11.578.$$

**Prob.**

$$Z_o = \sqrt{\frac{L}{c}} = \sqrt{\frac{\mu d}{w}.\frac{d}{\varepsilon w}} = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_o = \eta_o \frac{d}{w} = 78$$

$$Z_o' = \eta_o \frac{d}{w'} = 75$$

$$\frac{78}{75} = \frac{w'}{w} \rightarrow w' = 1.04w$$

i.e. the width must be increased by 4%.

**Prob.**

(a) For a lossless line, R = 0 = G.

$$\gamma = j\omega\sqrt{LC} \quad \longrightarrow \quad \beta = \omega\sqrt{LC} = \omega\sqrt{\mu_o c_o} = \frac{\omega}{c}$$

$$u = \frac{\omega}{\beta} = c = \frac{1}{\sqrt{LC}}$$

(b) For lossless line, R = 0 =G

$$L = \frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a}, C = \frac{\pi\varepsilon}{\cosh^{-1}\frac{d}{2a}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\pi}.\frac{1}{\pi\varepsilon}}\cosh^{-1}\frac{d}{2a} = \frac{120\pi}{\pi\sqrt{\varepsilon_r}}\cosh^{-1}\frac{d}{2a}$$

$$= \frac{120}{\sqrt{\varepsilon_r}} \cosh^{-1} \frac{d}{2a}$$

Yes, true for other lossless lines.

**Prob.**

We apply the identities

$$\sinh(x+jy) = \sinh x \cos y + j \cosh x \sin y$$

$$= \left(\frac{e^{x}-e^{-x}}{2}\right) \cos y + j\left(\frac{e^{x}+e^{-x}}{2}\right) \sin y$$

$$\cosh(x+jy) = \cosh x \cos y + j \sinh x \sin y$$

$$= \left(\frac{e^{x}+e^{-x}}{2}\right) \cos y + j\left(\frac{e^{x}-e^{-x}}{2}\right) \sin y$$

$$\tanh(x+jy) = \frac{\sinh(x+jy)}{\cosh(x+jy)}.$$

For $\gamma = 0.02 + j0.05/m$, $l = 2m$, $\gamma l = 0.04 + j0.1$

$$\tanh \gamma l = 0.04038 + j0.1002$$

$$Z_{in} = Z_o \left(\frac{Z_L + Z_o \tanh \gamma l}{Z_o + Z_L \tanh \gamma l}\right)$$

$$= (400 - j150) \left[ (300 + j250) + (400 - j150)(0.04038 + j0.1002) \right.$$

$$\overline{400 - j150 + (300 + j250)(0.04038 + j0.1002)}$$

$$= \underline{\underline{375.2 + j271.6\ \Omega}}\ .$$

for $l = 10\text{m}$, $\gamma l = 0.2 + j0.5$, $\tanh\gamma l = 0.2533 + j0.519$

$$Z_{in} = (400 - j150)\frac{\left[(300 + j250) + (400 - j150)(0.2533 + j0.519)\right]}{400 - j150 + (300 + j250)(0.2533 + j0.519)}$$

$$= \underline{\underline{760.4 + j125.5\ \Omega}}\ .$$

**Prob.**

$$L = \frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a} = 4\times 10^{-7}\cosh^{-1}\frac{0.32}{0.12}$$

$$\underline{\underline{L = 0.655\ \mu\text{H/m}}}$$

$$C = \frac{\pi\varepsilon}{Cosh^{-1}\dfrac{d}{2a}} = \frac{\pi \times \dfrac{10^{-9}}{36\pi}\times 3.5}{Cosh^{-1}2.667}$$

$$\underline{\underline{C = 59.4\ \text{pF/m}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.655\times 10^{-6}}{59.4\times 10^{-12}}} = \underline{\underline{105\Omega}}$$

or

$$Z_o = \frac{120}{\sqrt{3.5}}\cosh^{-1}2.667 = \underline{\underline{105\Omega}}$$

**Prob.**

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}} = 75 \qquad (1)$$

$$\alpha = \sqrt{RG} = 0.06 \qquad (2)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = 2.8 \times 10^8 \qquad (3)$$

$$\alpha Z_o = R = 75 \times 0.06 = \underline{\underline{4.5\ \Omega/\text{m}}}$$

From (2),

$$\sqrt{RG} = 6 \times 10^{-2} \quad \longrightarrow \quad G = \frac{36 \times 10^{-4}}{4.5} = \underline{\underline{8 \times 10^{-4}\ \text{S/m}}}$$

From (1) and (3),

$$uZ_o = \frac{1}{C} = 75 \times 2.8 \times 10^8 \quad \longrightarrow \quad C = \frac{10^{-8}}{75 \times 2.8} = \underline{\underline{4.761 \times 10^{-11}\ \text{F/m}}}$$

$$L = \frac{Z_o}{u} = \frac{75}{2.8 \times 10^8} = \underline{\underline{2.678 \times 10^{-7}\ \text{H/m}}}$$

**Prob.**

$$Z_o = \sqrt{\frac{L}{c}}, \quad \gamma = j\beta = j\omega\sqrt{Lc}$$

$$Z_o\beta = \omega L \rightarrow \beta = \frac{\omega L}{Z_o} = \frac{2\pi \times 4.5 \times 10^9 \times 2.4 \times 10^6}{85}$$

$$= \underline{\underline{798.33\ \text{rad/m}}}$$

$$u = \frac{\omega}{\beta} = \frac{Z_o}{L} = \frac{85}{2.4 \times 10^{-6}} = \underline{\underline{3.542 \times 10^7\ \text{m/s}}}$$

**Prob.**

$$R + j\omega L = 6.5 + j2\pi \times 2 \times 10^{6} \times 3.4 \times 10^{-6} = 6.5 + j42.73$$

$$G + j\omega C = 8.4 \times 10^{-3} + j2\pi \times 2 \times 10^{6} \times 21.5 \times 10^{-12} = \left(8.4 + j0.27\right) \times 10^{-3}$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{6.5 + j42.73}{\left(8.4 + j0.27\right) \times 10^{-3}}}$$

$$Z_o = 71.71\angle 39.75^o = \underline{\underline{55.12 + j45.85\Omega}}$$

$$\gamma = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)} \qquad = \sqrt{\left(43.19\angle 81.34^o\right)\left(8.4 \times 10^{-3}\angle 1.84^o\right)}$$

$$= \underset{\underset{\alpha}{\uparrow}}{0.45} + \mathrm{j}\underset{\underset{\beta}{\uparrow}}{0.4}/\mathrm{m}$$

$$t = \frac{l}{u}, \text{ but } u = \frac{\omega}{\beta},$$

$$t = \frac{\beta l}{\omega} = \frac{0.39 \times 5.6}{2\pi \times 2 \times 10^{6}} = \underline{\underline{0.1783\mu s}}$$

**Prob.**

(a) $Z_{sc} = jZ_o \tan\beta l \longrightarrow j73 = j42 \tan\beta l$

$\tan\beta l = \frac{73}{42} = 1.738 \simeq \tan 60°$

---

$\frac{2\pi l}{\lambda} = \frac{\pi}{3} \longrightarrow l = \frac{\lambda}{6}$

But $\lambda = \frac{c}{f} = \frac{3\times10^8}{10^6} = 300m$, $\underline{\underline{l = 50m.}}$

(b) $Z_{oc} = -jZ_o \cot\beta l \longrightarrow j73 = -j42 \cot\beta l$

$\cot\beta l = -1.738 \simeq \cot 150°$

$\frac{2\pi l}{\lambda} = \frac{5\pi}{6} \longrightarrow l = \frac{5\lambda}{12} = \frac{5}{12} \times 300 = \underline{\underline{125 m.}}$

**Prob.**

(a) $$T_L = \frac{V_L}{V_o^+} = \frac{Z_L I_L}{\frac{1}{2}(V_L + Z_o I_L)} = \frac{2Z_L I_L}{Z_L I_L + Z_o I_L}$$

$$= \underline{\underline{\frac{2Z_L}{Z_L + Z_o}}}$$

$$1 + \Gamma_L = 1 + \frac{Z_L - Z_o}{Z_L + Z_o} = \underline{\underline{\frac{2Z_L}{Z_L + Z_o}}}$$

(b) (i) $\tau_L = \frac{2nZ_o}{nZ_o + Z_o} = \frac{2n}{n+1}$

(ii) $\tau_L =_{Y_L} \xrightarrow{\lim} {}_0 = \frac{2}{1 + Z_o/Z_L} = 2$

(iii) $\tau_L =_{Z_L} \xrightarrow{\lim} {}_0 = \frac{2Z_L}{Z_L + Z_o} = 0$

(iv) $\tau_L = \frac{2Z_o}{2Z_o} = 1$

**Prob.**

$Z_{in} = jZ_o \tan \beta l$

(a) $Z_{in} = j60 \tan 8.5 \times 0.15 = \underline{\underline{j196.89\ \Omega}}$ .

(b) $Z_{in} = j60 \tan 8.5 \times 1.5 = \underline{\underline{j11.14\ \Omega}}$ .

(c) $Z_{in} = j60 \tan \frac{2\pi}{\lambda} \frac{3\lambda}{4} = j60 \tan \frac{3\pi}{2} = \underline{\underline{-j\infty}}$ .

(d) $Z_{in} = j60 \tan \frac{2\pi}{\lambda} \frac{\lambda}{8} = j \quad \tan \frac{\pi}{4} = \underline{\underline{j60\ \Omega}}$

**Prob.**

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(3.5 + j2\pi x 400 x 10^6 x 2 x 10^{-6})(0 + j2\pi x 400 x 10^6 x 120 x 10^{-12})}$$
$$= \sqrt{(3.5 + j5026.55)(j0.3016)} = 0.0136 + j38.94$$

$$\alpha = \underline{\underline{0.0136\ \text{Np/m}}}, \qquad \beta = \underline{\underline{38.94\ \text{rad/m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi x 400 x 10^6}{38.94} = \underline{\underline{6.452 x 10^7\ \text{m/s}}}$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{3.5 + j5026.55}{j0.3016}} = \underline{\underline{129.1 - j0.045\ \Omega}}$$

**Prob.**

$$I_l = \frac{V_L}{Z_L}, \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{50e^{j30^o} - 50}{50e^{j30^o} + 50}$$

$$\approx j0.2679$$

From **eq. (**

$$V_o^+ = \frac{1}{2}(V_L + Z_o . \frac{V_L}{Z_L})e^{\gamma l} = \frac{V_L}{2Z_L}(Z_L + Z_o)e^{\gamma l}$$

$$V_o^- = \frac{V_L}{2Z_L}(Z_L - Z_o)e^{-\gamma l}$$

Substituting these in **eq. (**

$$I_s = \frac{V_L}{2Z_L Z_o}\left[(Z_L + Z_o)e^{\gamma l}e^{-\gamma z} - (Z_L - Z_o)e^{-\gamma l}e^{\gamma z}\right]$$

$$= \frac{V_L / \ Z_o}{1+\Gamma}\left[e^{-\gamma(z-l)} - \Gamma e^{\gamma(z-l)}\right]$$

But $l - z = \frac{\lambda}{8} \quad or \quad z - l = -\frac{\lambda}{8}$

$$I_s = \frac{10\angle 25^o}{1.035\angle 15^o}\left(\frac{1}{50}\right)\left(e^{j\pi/4} - j0.2679e^{-j\pi/4}\right)$$

$$= \underline{\underline{0.2\angle 40^o\,A}}$$

*or*

$$\beta z = \frac{2\pi}{\lambda}.\frac{\lambda}{8} = \frac{\pi}{4}, \quad I_L = \frac{V_L}{Z_L} = \frac{10e^{j25^o}}{50e^{j30^o}} = 0.2e^{-j5^o}$$

$$I\left(z = \frac{\pi}{8}\right) = I_L e^{j\beta l} = 0.2e^{-j5^o}e^{j45^o}$$

$$= \underline{\underline{0.2e^{j40^o}\,A}}$$

**Prob.**

(a) $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{120-50}{170} = \underline{\underline{0.4118}}$

For resistive load, $s = \frac{Z_L}{Z_o} = \underline{\underline{2.4}}$

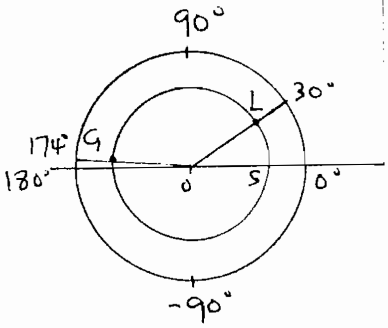
(b) $Z_{in} = Z_o \frac{Z_L + jZ_o \tan(\beta l)}{Z_o + jZ_L \tan(\beta l)}$

$$\beta l = \frac{2\pi}{\lambda}.\frac{\lambda}{6} = 60^o$$

$$Z_{in} = 50\left[\frac{120 + j50\tan(60^o)}{50 + j120\tan(60^o)}\right] = \underline{\underline{34.63\angle -40.65^o\,\Omega}}$$

**Prob.**

(a) $s = \frac{(Z_{in})_{max}}{Z_o} = \frac{180}{60} = \underline{\underline{3.}}$



(b) With center at O and radius OS, on the Smith chart, we draw a circle. Since $\frac{\lambda}{24} \longrightarrow \frac{720^\circ}{24} = 30^\circ$, the maximum $Z_{in}$ occurs $30^\circ\ (\cong \frac{\lambda}{24})$ from the load. from S, move 30° toward the load along the circle to reach L. Determine $z_L$ at L:

$$z_L = 1.96 + j1.3$$

$$Z_L = Z_o z_L = 60(1.96 + j1.3) = \underline{\underline{117.6 + j78\ \Omega.}}$$

(c) $0.3\lambda \longrightarrow 0.3 \times 720^\circ = 216^\circ$. Move along the circle from L through 216° toward the generator. Determine $z_{in}$ at G:

$$z_{in} = 0.35 + j0.053$$

$$Z_{in} = Z_o z_{in} = 60(0.35 + j0.053) = \underline{\underline{21 + j3.18\ \Omega.}}$$

**Prob.**

$$(a)\quad \beta l = \frac{1}{4} \times 100 = 25 rad = 1432.4^o = 352.4^o$$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 352.4^o}{60 - 40\tan 352.4^o}\right] = \underline{\underline{j29.375\Omega}}$$

$$V(z=0) = V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g = \frac{j29.375(10\angle 0^o}{j29.375 + 50 - j40}$$

$$= \frac{293.75\angle 90^o}{51.116\angle -12^o} = \underline{\underline{5.75\angle 102^o}}$$

(b) $Z_{in} = Z_L = \underline{\underline{j40\Omega.}}$

$$V_o^+ = \frac{V_g}{(e^{j\beta l} + \Gamma e^{-j\beta l})} \qquad \text{(l is from the load)}$$

$$V_L = \frac{V_g(1+\Gamma)}{(e^{j\beta l} + \Gamma e^{-j\beta l})} = \underline{\underline{12.62\angle 0^o}}$$

(c) $\beta l' = \frac{1}{4} \times 4 = 1rad = 57.3^o$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 57.3^o}{60 - 40\tan 57.3^o}\right] = \underline{\underline{-j3471.88\Omega.}}$$

$$V = \frac{V_g(e^{j} + \Gamma e^{-j})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{22.74\angle 0^o \ \text{V}}}$$

(d) 3m from the source is the same as 97m from the load., i.e.

$$l' = 100 - 3 = 97m, \quad \beta l' = \frac{1}{4} \times 97 = 24.25rad = 309.42^o$$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 309.42^o}{60 - 40\tan 309.42^o}\right] = \underline{\underline{-j18.2\Omega}}$$

$$V = \frac{V_g(e^{j97/4} + \Gamma e^{-j97/4})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{6.607\angle 180^o \ \text{V}}}$$

**Prob.**

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 - jX_c - 50}{60 - jX_c + 50}, \quad |\Gamma| = \frac{s-1}{s+1} = \frac{8}{10} = \frac{4}{5}$$

Hence $\frac{4}{5} = \left|\frac{10 - jX_c}{110 - jX_c}\right|$ or $\frac{16}{25} = \frac{100 + X_c^2}{110^2 + X_c^2}$

i.e. $9X_c^2 = 191100 \longrightarrow X_c = 145.72 = \frac{1}{wc} = \frac{1}{2\pi fc}$

$$C = \frac{1}{2\pi \times 5 \times 10^8 \times 145.72} = \underline{\underline{2.184\ pF}} .$$

$$\Gamma = \frac{10 - j145.72}{110 - j145.72} = \underline{\underline{0.8\angle -33.12^o}} .$$

**Prob.**

$$V_1 = V_s(z=0) = V_o^+ + V_o^- \qquad (1)$$

$$V_2 = V_s(z=l) = V_o^+ e^{-\gamma l} + V_o^- \mathrm{e}^{\gamma l} \qquad (2)$$

$$\mathrm{I}_1 = I_s(z=0) = \frac{V_o^+}{Z_o} - \frac{V_o^-}{Z_o} \qquad (3)$$

$$I_2 = -I_s(z=l) = -\frac{V_o^+}{Z_o} e^{-\gamma l} + \frac{V_o^-}{Z_o} \mathrm{e}^{\gamma l} \qquad (4)$$

$$(1) + (3) \rightarrow V_o^+ = \frac{1}{2}(V_1 + Z_o I_1)$$

$$(1) - (3) \rightarrow V_o^- = \frac{1}{2}(V_1 - Z_o I_1)$$

Substituting $V_o^+$ and $V_o^-$ in (2) gives

$$V_2 = \frac{1}{2}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})V_1 + \frac{1}{2}Z_o(e^{-\gamma l} - e^{\gamma l})I_1$$

$$V_2 = \cosh \gamma l V_1 - Z_o \sinh \gamma l I_1 \qquad (5)$$

Substituting $V_o^+$ and $V_o^-$ in (4),

$$I_2 = -\frac{1}{2Z_o}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2Z_o}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2Z_o}(e^{\gamma l} - e^{-\gamma l})V_1 + \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})I_1$$

$$I_2 = -\frac{1}{Z_o}\sinh \gamma l V_1 - \cosh \gamma l I_1 \qquad (6)$$

From (5) and (6)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & -Z_o \sinh \gamma l \\ -\frac{1}{Z_o}\sinh \gamma l & -\cosh \gamma l \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

But

$$\begin{bmatrix} \cosh \gamma l & -Z_o \sinh \gamma l \\ -\frac{1}{Z_o}\sinh \gamma l & -\cosh \gamma l \end{bmatrix}^{-1} = \begin{bmatrix} \cosh \gamma l & Z_o \sinh \gamma l \\ \frac{1}{Z_o}\sinh \gamma l & \cosh \gamma l \end{bmatrix}$$

Thus

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & Z_o \sinh \gamma l \\ \frac{1}{Z_o}\sinh \gamma l & \cosh \gamma l \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

**Prob.**

Using the Smith chart, $z_L = \dfrac{60 - j40}{75} = 0.8 - j0.533$

$$l = \frac{3}{4}\lambda \quad \longrightarrow \quad \frac{3}{4}x720^o = 540^o$$

At C, $Z_{in} = 75(0.8654 + j0.5769) = 65 + j43 \ \Omega$

$$z_{in} = \frac{65 + j43}{100} = 0.65 + j0.43$$

$$\frac{\lambda}{2} \longrightarrow \frac{720^o}{2} = 360^o$$

At B, $Z_{in} = 65 + j43$

$$z_{in} = \frac{65 + j43}{50} = 1.2981 + 0.8654$$

$$\frac{\lambda}{4} \longrightarrow \frac{720^o}{4} = 180^o$$

At A,

$$Z_{in} = 50(0.53 - j0.35) = \underline{\underline{26.7 - j17.8\ \Omega}}$$

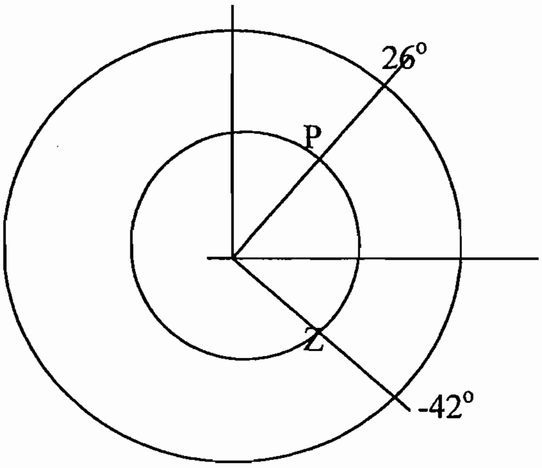
**Prob.**

$$z_L = \frac{Z_L}{Z_o} = \frac{100 + j150}{50} = 2 + j3$$

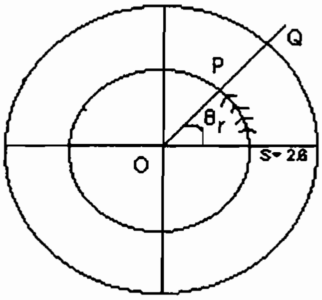
$$z_{in} = \frac{Z_{in}}{Z_o} = \frac{50 + j110}{50} = 1 + j2.2$$

$$\theta = 26^o - -42^o = 68^o$$

If $720 \rightarrow \lambda$, $68^o \rightarrow d = \frac{\lambda}{720^o} 68^o = \underline{\underline{0.0944\lambda}}$



**Prob.**



$$(a)\quad Z_{in} = \frac{Z_{in}}{Z_o} = \frac{120 + j80}{75} = 1.6 + j1.067$$

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.8\,\text{cm}}{8.7\,\text{cm}} = 0.4367, \quad \theta_\Gamma = 38^\circ$$

$$\Gamma = \underline{\underline{0.4367\angle 38^\circ}}, \quad s = \underline{\underline{2.6}}$$

($b$) The Load is purely resistive at s.

$$\theta_\Gamma = 38^\circ$$

But $720^\circ \to \lambda$, hence $38^\circ \to \frac{38\lambda}{720} = \underline{\underline{0.053\lambda}}$ from the load

**Prob.**

$$z_L = \frac{Z_L}{Z_O} = \frac{210}{100} = 2.1 = s$$

Or $\Gamma = \frac{Z_L - Z_O}{Z_L + Z_O} = \frac{110}{310}$,

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2.1$$

But $s = \frac{V_{\max}}{V_{\min}} \to V_{\max} = sV_{\min}$

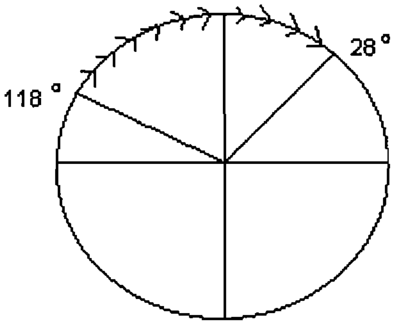
Since the line is $\frac{\lambda}{4}$ long, $\frac{\lambda}{4} \to \frac{720^o}{4} = 120^o$

Hence the sending end will be $V_{\min}$ , while the receiving end at $V_{\max}$

$V_{\min} = V_{\max} / s = 80/2.1 = 38.09$

$V_{sending} = \underline{\underline{38.09 < 90^o}}$

**Prob.**

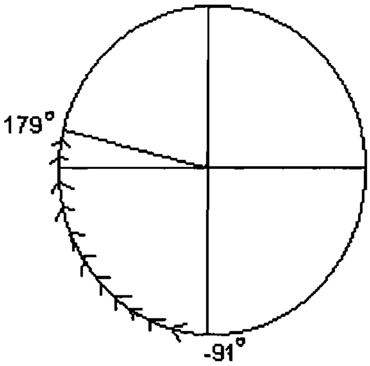


(a) If $\lambda \to 720°$, then $\frac{5\lambda}{8} \to \frac{5}{8} \times 720° = 450° \longrightarrow 90°$

$$z_L = \frac{Z_L}{Z_o} = \frac{j45}{75} = j0.6$$

$$z_{in} = 0 + j4, \quad Z_{in} = Z_o Z_{in} = 75(j4) = \underline{\underline{j300\Omega}}$$

(b) $z_L = \dfrac{25 - j65}{75} = 0.333 - j0.867$

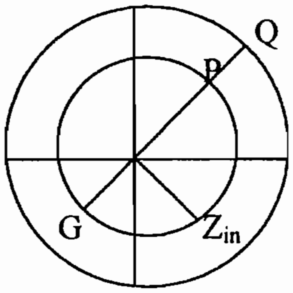


**Prob.**

Method1: $Z_{in} = \dfrac{80 - j60}{50} = 1.6 - j1.2$

$$\lambda = \frac{u}{f} = \frac{0.8 \times 3 \times 10^8}{3 \times 10^8} = 0.8m$$

$$l_1 = \frac{4.2}{2}m = 2.1m \to 720° \times \frac{2.1}{0.8} = 5 \text{ revolutions} + 90°$$



At G, $Z_{in} = 0.44 - j0.4$

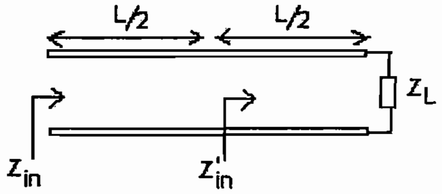
$$Z_{in} = Z_{in}Z_o = 50(0.44 - j0.4)$$

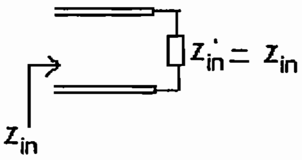
$$= \underline{\underline{22 - j20\Omega}}$$

$$|\Gamma| = \frac{OP}{OQ} = \frac{4.3cm}{9.3cm} = 0.4624, \theta_\Gamma = 50.5$$

$$\Gamma = \underline{\underline{0.4624\angle 50.5^\circ}}$$

<u>Method 2:</u>





$$\tan\beta l = \tan\frac{\omega l}{u} = \tan\frac{2\pi\times3\times10^8}{0.8\times3\times10^8}(2.1)$$

$$= \tan\left(21\times\frac{\pi}{4}\right) = 1$$

$$Z_{in} = Z_o\left[\frac{Z_L^{'} + jZ_o\tan\beta l}{Z_o + jZ_L^{'}\tan\beta l}\right] = 50\left[\frac{80 - j60 + j50\times1}{50 + j80 - j60\times1}\right]$$

$$= 29.6\angle -43.152^\circ = \underline{\underline{21.6 - 20.2\Omega}}$$

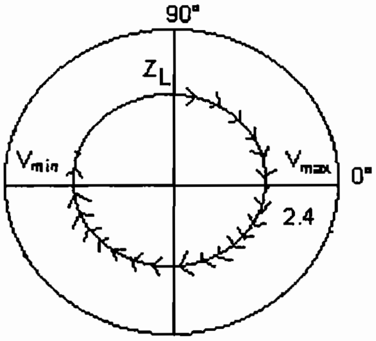
$$\Gamma^{'} = \frac{Z_L^{'} - Z_o}{Z_L^{'} + Z_o} = \frac{80 - j60 - 50}{80 - j60 + 50} = \frac{3 - j6}{13 - j6} = 0.4685\angle -38.66^\circ$$

$|\Gamma| = |\Gamma^{'}| = 0.4685$, but

$$\theta_\Gamma = \theta_{\Gamma^{'}} + 2\times\frac{\pi}{4} = -38.66^\circ + 90^\circ = 51.34^\circ$$

$$\Gamma = \underline{\underline{0.4685\angle 51.34^\circ}}$$

**Prob.**



(a) If $\lambda \to 720°$, then $\frac{\lambda}{8} \to 90°$

$Z_L = 0.7 + j0.68$

$Z_L = 50(0.7 + j0.68) = \underline{\underline{35 + j34\Omega}}$

(b) $l = \frac{\lambda}{4} + \frac{\lambda}{8} = \underline{\underline{0.375\lambda}}$

**Prob.**

(a) $Z_{in} = \frac{Z_{in}}{Z_o} = \frac{100 - j120}{80} = 1.25 - j1.5$

$\lambda = \frac{u}{f} = \frac{0.8 \times 3 \times 10^8}{12 \times 10^6} = 20m$

$l_1 = 22m = \frac{22\lambda}{20} = 1.1\lambda \to 720° + 72°$

$l_2 = 28m = \frac{28\lambda}{20} = 1.4\lambda \to 720° + 72° + 216°$

To locate P(the load), we move 2 revolutions plus 72° toward the load. At P,

$|\Gamma_L| = \frac{OP}{OQ} = \frac{5.1cm}{9.2cm} = 0.5543$

$\theta_\Gamma = 72° - 47° = 25°$

$\Gamma_L = \underline{\underline{0.5543\angle 25°}}$

(Exact value = 0.5624<25.15°)

$Z_{in}, \max = sZ_o = 3.7(80) = \underline{\underline{296\Omega}}$

(Exact value = 285.59 Ω)

$Z_{in}, \min = \frac{Z_o}{s} = \frac{80}{3.7} = \underline{\underline{21.622\Omega}}$

(Exact value =22.41 Ω)

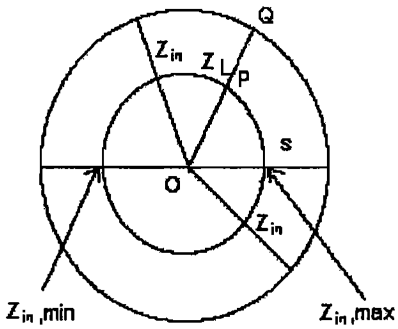
(b) Also, at P, $Z_L = 2.3 + j1.55$

$$Z_L = 80(2.3 + j1.55) = \underline{\underline{184 + j124\Omega}}$$

(Exact value = 183.45+j128.25 Ω)

At S, $s = \underline{\underline{3.7}}$

To Locate $Z'_{in}$, we move 216° from $Z_{in}$ toward the generator



At $Z'_{in}$,

$$z'_{in} = 0.48 + j0.76$$

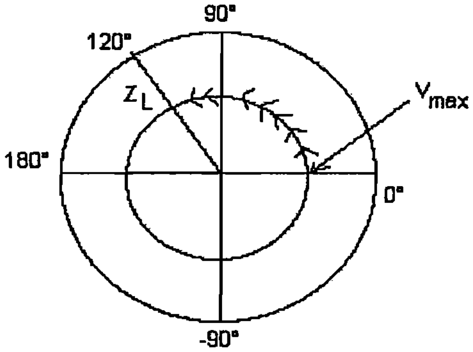
$$Z'_{in} = 80(0.48 + j0.76) = \underline{\underline{38.4 + j60.8\Omega}}$$

(Exact=37.56+j61.304 Ω)

(c) Between $Z_L$ and $Z_{in}$, we move 2 revolutions a the movement, we pass through $Z_{in,\,max}$ 3 times and Thus there are:

$$\underline{\underline{3\ Z_{in,\max} \text{ and } 2\ Z_{in,\min}}}$$

**Prob.**



(a) $\dfrac{\lambda}{2} = 120cm \to \lambda = 2.4m$

$$u = f\lambda \to f = \frac{u}{\lambda} = \frac{3\times10^8}{2.4} = \underline{\underline{125MHz}}$$

(b) $40cm \quad = \dfrac{40\lambda}{240} = \dfrac{\lambda}{6} \to \dfrac{720°}{6} = 120°$

$$Z_L = Z_o z_L = 150(0.48 + j0.48$$

$$= \underline{\underline{72 + j72\ \Omega}}$$

(Exact value = 73.308+j70.324 Ω)

(c) $|\Gamma| = \dfrac{s-1}{s+1} = \dfrac{1.6}{3.9} = 0.444,$

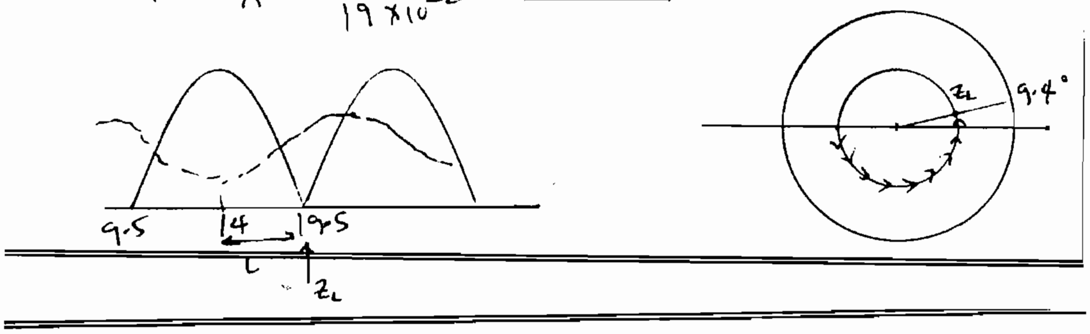
$$\Gamma = \underline{\underline{0.444\angle 120°}}$$

**Prob.**

$$s = \frac{V_{max}}{V_{min}} = \frac{0.8}{0.5} = \underline{\underline{1.6}}$$

$$\frac{\lambda}{2} = 19.0\,cm - 9.5 = 9.5\,cm \longrightarrow \lambda = 19\,cm$$

$$f = \frac{c}{\lambda} = \frac{3\times10^{8}}{19\times10^{-2}} = \underline{\underline{1.58\,GHz}}.$$

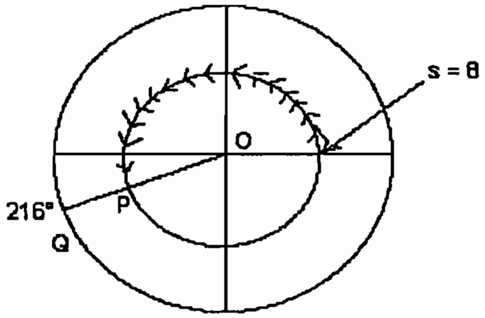


$$l = 19 - 14 = 5\,cm = \frac{5\lambda}{19} \longrightarrow 189.4°$$

$$Z_L = Z_o z_L = 50(1.59 + j0.14) = \underline{\underline{79.5 + j7\,\Omega}}.$$

$$|\Gamma| = \frac{s-1}{s+1} = \frac{0.6}{2.6} = 0.2307, \quad \Gamma = \underline{\underline{0.2307\angle 9.4°}}.$$

**Prob.**



$0.3\lambda \rightarrow 720° \times 0.3 = 216°$

At P, $z_L = 0.15 - j0.32$

$Z_L = Z_o z_L = \underline{\underline{15 - j32\Omega}}.$

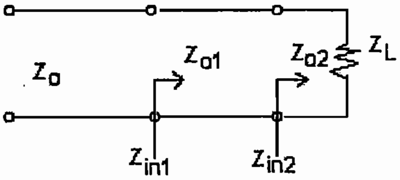
(Exact value = 13.7969 -j31.9316 Ω)

$$|\Gamma| = \frac{OP}{OQ} = \frac{7.2\text{ cm}}{9.3\text{ cm}} = 0.7742$$

$\Gamma = \underline{\underline{0.7742\angle 216°}}$

(Exact value = $0.7778\angle 216°$)

**Prob.**



**(a) From eq. (** $Z_{in2} = \frac{Z_{o2}^2}{Z_L}$

$$Z_{in1} = \frac{Z_{o1}^2}{Z_{in2}} = Z_o, \text{i.e. } Z_{in2} = \frac{Z_{o1}^2}{Z_o} = \frac{Z_{o2}^2}{Z_L}$$

$$Z_{o1} = Z_{o2}\sqrt{\frac{Z_o}{Z_L}} = 30\sqrt{\frac{50}{75}} = \underline{\underline{24.5\Omega.}}$$

(b) Also, $\frac{Z_o}{Z_{o1}} = \left(\frac{Z_{o2}}{Z_L}\right) \rightarrow Z_{o2} = \frac{Z_o Z_L}{Z_{o1}}$ (1)

Also, $\frac{Z_{o1}}{Z_{o2}} = \left(\frac{Z_{o2}}{Z_L}\right)^2 \rightarrow (Z_{o2})^3 = Z_{o1}Z_L^2$ (2)

From (1) and (2), $(Z_{o2})^3 = Z_{o1}Z_L^2 = \frac{Z_o^3 Z_L^3}{Z_{o1}^3}$ (3)

or $Z_{o1} = \sqrt[4]{Z_o^3 Z_L} = \sqrt[4]{(50)^3(75)} = \underline{\underline{55.33\Omega}}$

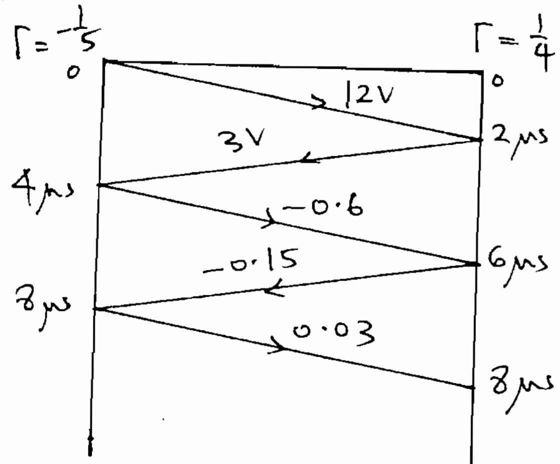
From (3), $Z_{o2} = \sqrt[3]{Z_{o1}Z_L^2} = \sqrt[3]{(55.33)(75)^2} = \underline{\underline{67.74\Omega.}}$

**Prob.**

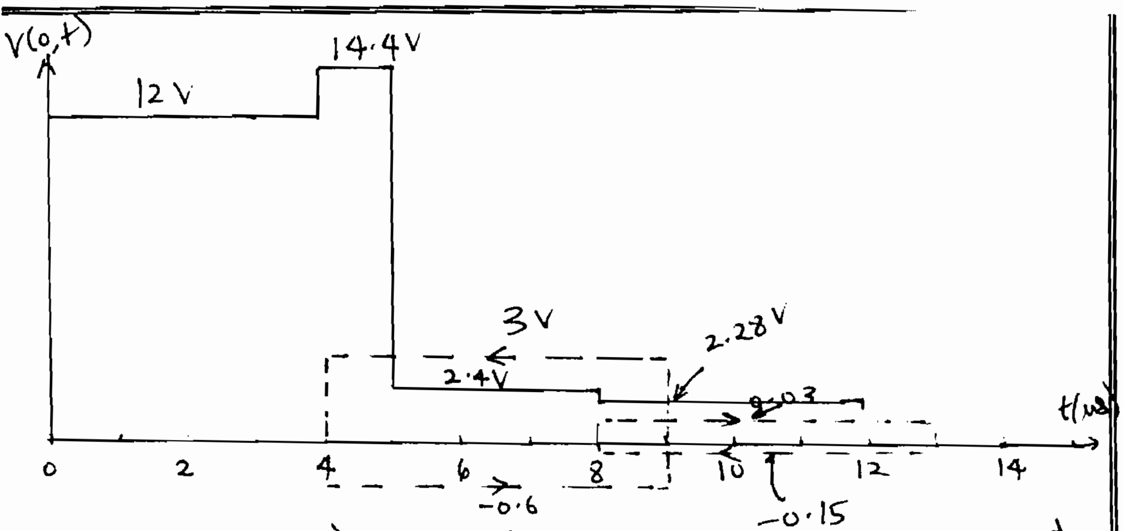
$t_1 = \frac{l}{u} = \frac{6m}{3\times10^8} = 2\mu s$, $V_0 = V_g \cdot \frac{Z_0}{Z_0+Z_g} = 20\left(\frac{60}{100}\right) = 12$

$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{40-60}{100} = -\frac{1}{5}$, $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100-60}{160} = \frac{1}{4}$.

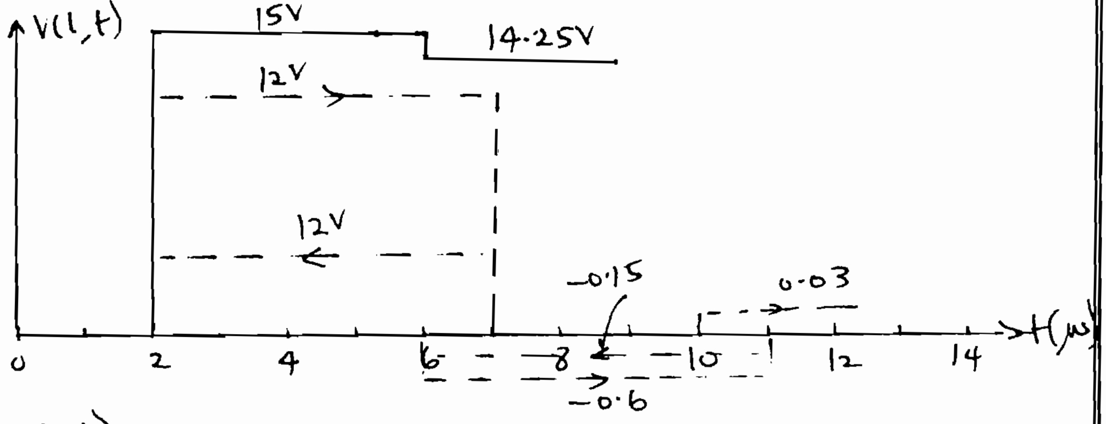
We only need the voltage bounce diagram because we can obtain $I(l,t)$ from $V(l,t)|Z_L$.

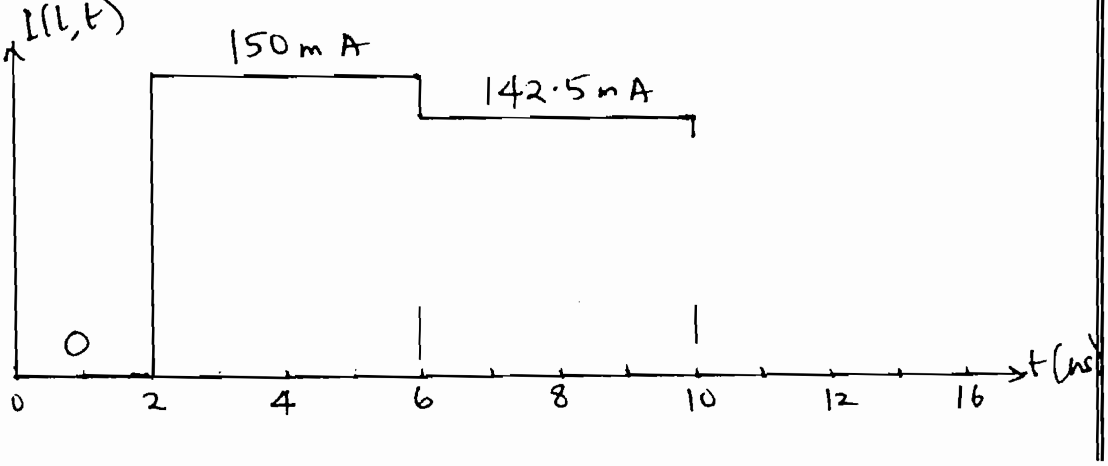


(voltage bounce diagram)



We obtain V(l,t) from the bounce diagram and divide by $Z_L = 100\Omega$ to obtain I(l,t).





**Prob.**

$$\frac{\lambda}{4} \to 180°, \quad z_L = \frac{74}{50} = 1.48, \quad \frac{1}{z_L} = 0.6756$$

This acts as the load to the left line. But there are two such loads in parallel due to the two lines on the right. Thus

$$Z_L' = 50\frac{\left(\frac{1}{Z_L}\right)}{2} = 25(0.6756) = 16.892$$

$$z_L' = \frac{16.892}{50} = 0.3378, \quad z_{in} = \frac{1}{z_L'} = 2.96$$

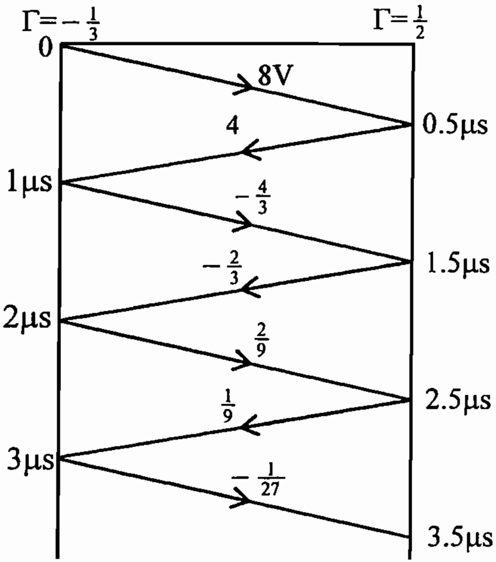
$$Z_{in} = 50(2.96) = \underline{\underline{148\Omega.}}$$

**Prob.**

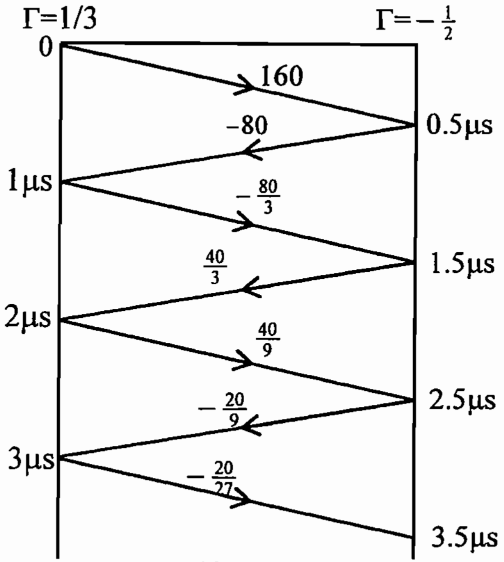
$$(a) \quad t_1 = \frac{l}{u} = \frac{150}{3\times10^8} = 0.5\mu s,$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{150-50}{150+150} = \frac{1}{2}, \quad \Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{25-50}{75} = -\frac{1}{3},$$

$$V_o = \frac{Z_o V_g}{Z_o + Z_g} = \frac{50(12)}{75} = 8V, \quad I_o = \frac{V_g}{Z_g + Z_o} = \frac{12}{75} = 160\,\text{mA}$$



(Voltage)



(Current)

The bounce diagrams are for the leading pulse. The bounce diagrams for the second pulse is delayed by 1 μs and negated because of -12V.

(b) For each time interval, we add the contributions of the two pulses together.

For $0 < t < 1\mu s$, $V(0,t) = 8V$

For $1 < t < 2\mu s$, $V(0,t) = -8 + 4 - 4/3 = -5.331V$

For $2 < t < 3\mu s$, $V(0,t) = -(4-4/3)-2/3 + 2/9 = -2.667 -0.444 = -3.11V$

For $3 < t < 4\mu s$, $V(0,t) = 0.444 + 1/9 -1/27 = 0.444+0.0741 = 0.518V$

For $4 < t < 5\mu s$, $V(0,t)= -0.0741 - 0.0124 = -0.0864V$

We do the same thing at the load end.

For $0 < t < 0.5\,\mu s$, $V(\ell,t) = 0$

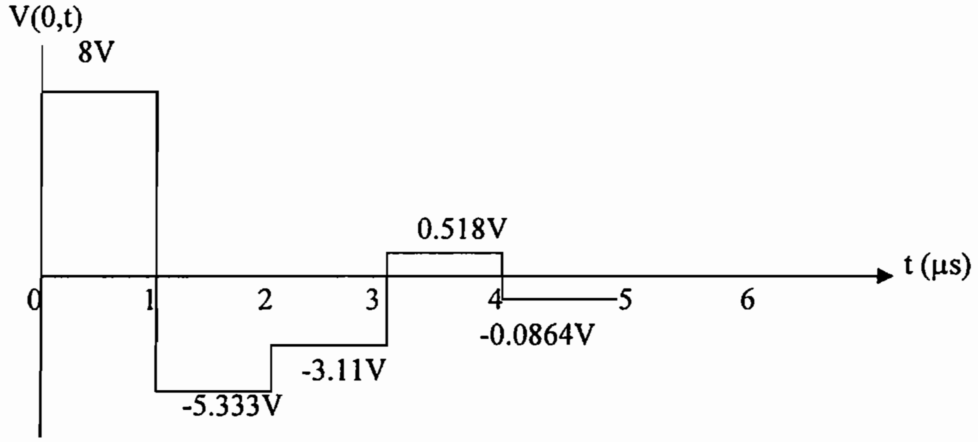
For $0.5 < t < 1.5\mu s$, $V(\ell,t)=8+4 = 12$

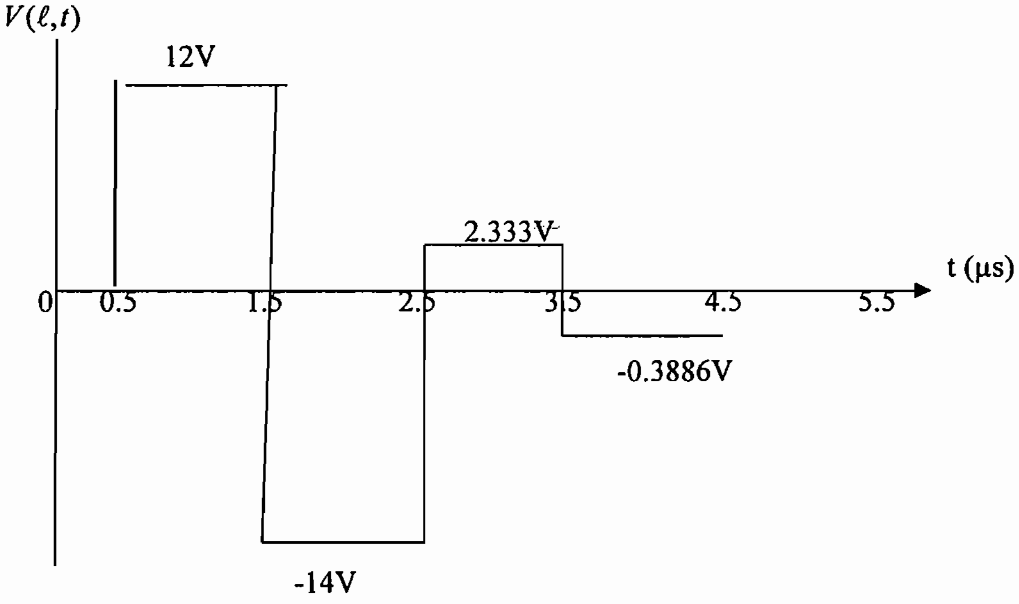
For $1.5 < t < 2.5\mu s$, $V(\ell,t)=-12 + (-4/3 -2/3)=-12-2=-14$

For $2.5 < t < 3.5\mu s$, $V(\ell,t)=2+2/9 +1/9=2.333$

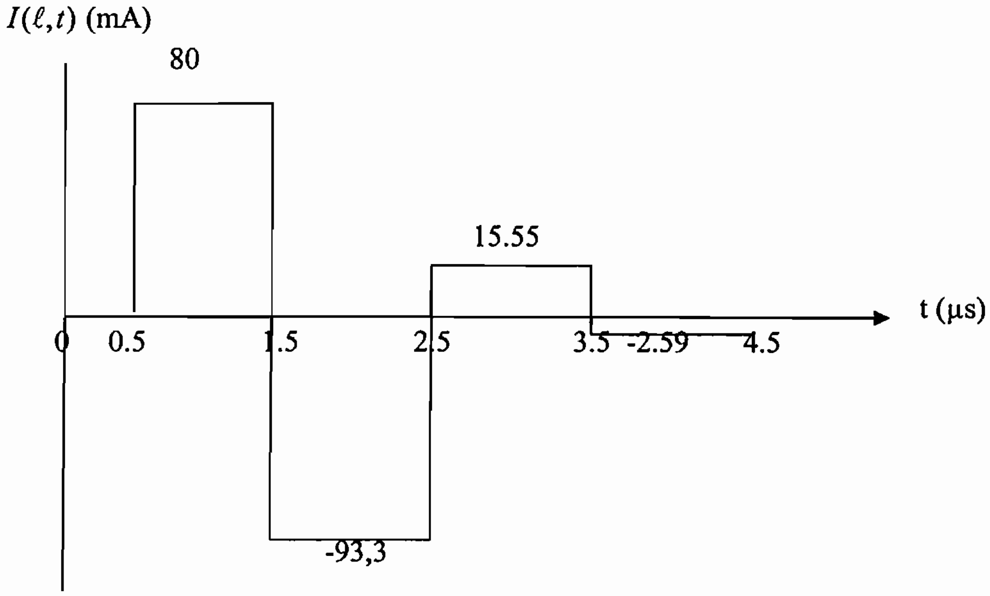
For $3.5 < t < 4.5\mu s$, $V(\ell,t)=-0.333-1/27-1/54=-0.3886V$

The results are shown below.





Since $I(\ell,t)=\frac{V(\ell,t)}{Z_L}=\frac{V(\ell,t)}{150}$, we scale $V(\ell,t)$ by a factor of 1/150 as shown below.



**Prob.**

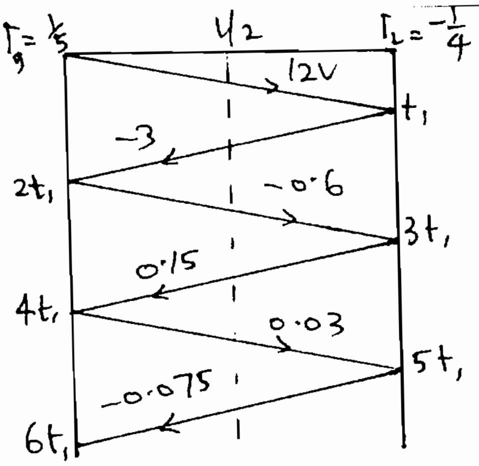
$$\Gamma_L=\frac{Z_L-Z_o}{Z_L+Z_o}=\frac{30-50}{80}=-\frac{1}{4},\quad \Gamma_g=\frac{Z_g-Z_o}{Z_g+Z_o}=\frac{75-50}{75+50}=\frac{1}{5},$$

$$t_1=\frac{l}{c}=\frac{600}{3\times10^8}=2\mu s,$$

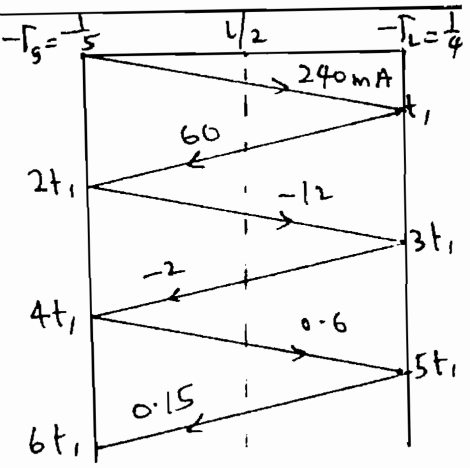
$$V_o=\frac{Z_o}{Z_o+Z_g}V_g=\frac{50(30)}{125}=12V,\quad I_o=\frac{V_o}{Z_o}=\frac{12}{50}=240\,mA.$$

$$V_\infty=\frac{Z_L}{Z_L+Z_g}V_g=\frac{30(30)}{75+30}=8.571V,\quad I_\infty=\frac{V_\infty}{Z_L}=\frac{30}{105}=285.7mA$$

(a) The bounce diagrams are as shown below.

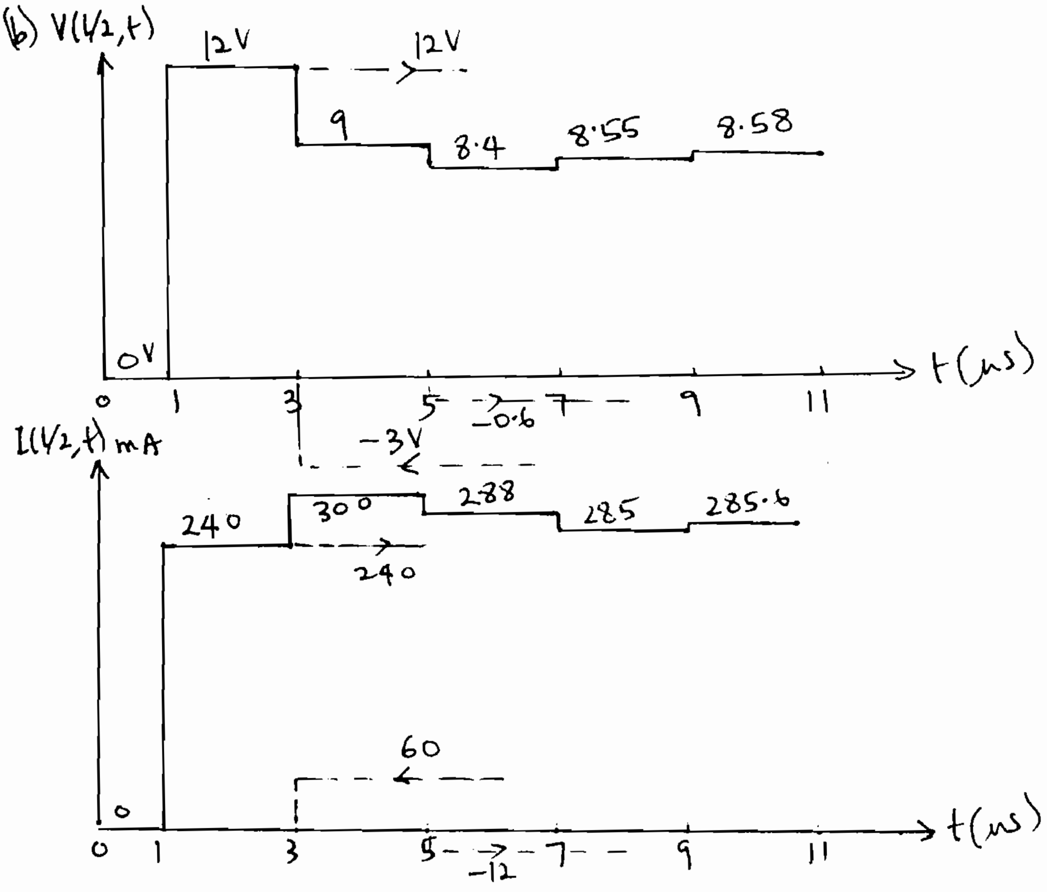


(voltage)



(current)

(b)



**Prob.**

(a) $\beta l = \frac{2\pi}{4} \cdot \frac{\lambda}{4} = \frac{\pi}{2}, \qquad \tan \beta l = \infty$

$$Z_{in} = Z_o \left( \frac{Z_L + jZ_o \tan \beta l}{Z_o + jZ_L \tan \beta l} \right) = Z_o \frac{\left( \frac{Z_L}{\tan \beta l} + jZ_o \right)}{\left( \frac{Z_o}{\tan \beta l} + jZ_L \right)}$$

As $\tan \beta l \to \infty$,

$$Z_{in} = \frac{Z_o^2}{Z_L} = \frac{(50)^2}{200} = \underline{\underline{12.5\Omega}}$$

(b) If $Z_L = 0$,

$$Z_{in} = \frac{Z_o^2}{0} = \underline{\underline{\infty}} \qquad \text{(open)}$$

(c) $Z_L = 25 // \infty = \frac{25 \times \infty}{25 + \infty} = \frac{25}{1 + \frac{25}{\infty}} = 25\Omega$

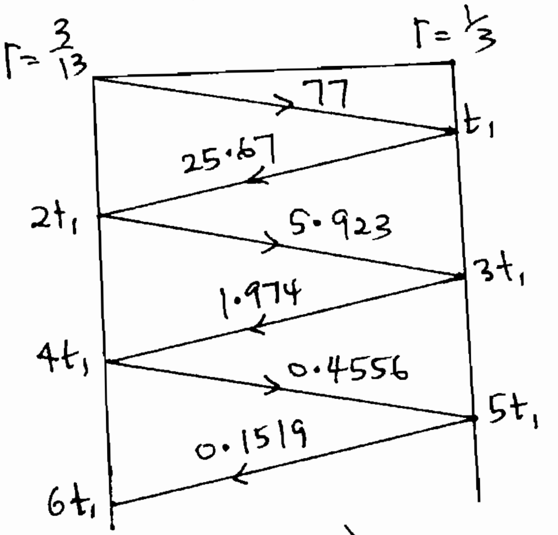
$$Z_{in} = \frac{(50)^2}{12.5} = \underline{\underline{200\Omega}}$$

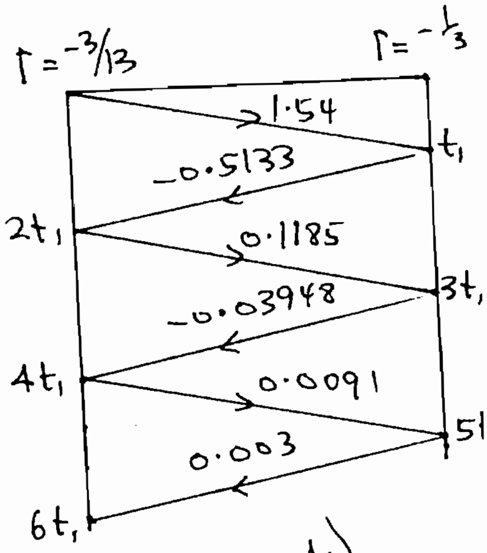
**Prob.**

$t_1 = \frac{l}{u} = \frac{300}{\frac{1}{4} \times 3 \times 10^8} = 4 \mu s,$

$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{80 - 50}{130} = \frac{3}{13}, \quad \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100 - 50}{150} = \frac{1}{3},$

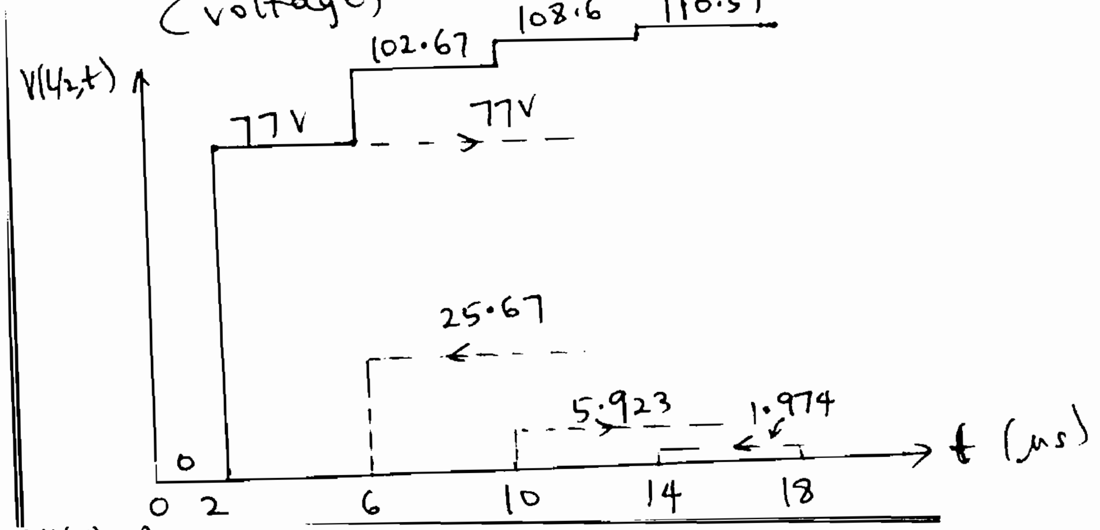
$V_o = \frac{Z_o}{Z_1 + Z_g} V_g = \frac{50(200)}{130} = 77V, \quad I_o = \frac{V_o}{Z_o} = \frac{77}{50} = 1.54 A$

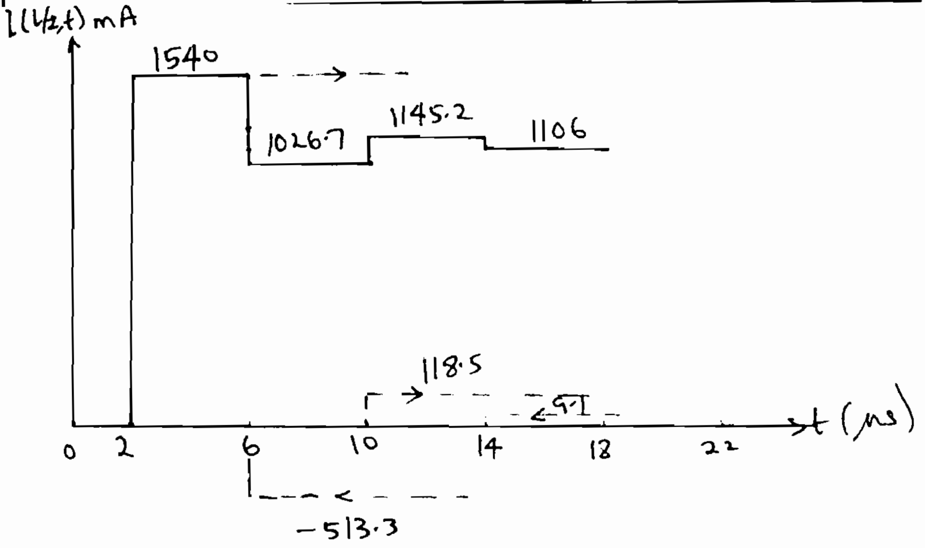




(voltage)

(current)



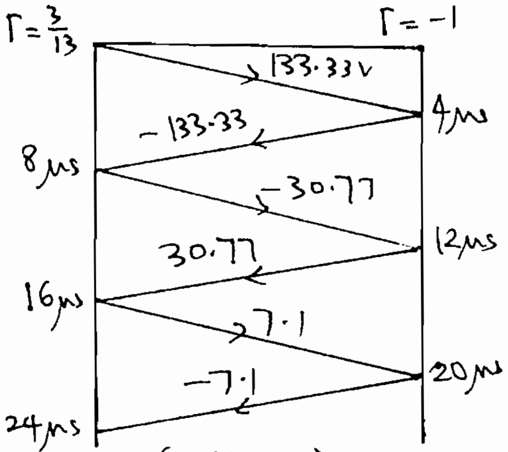


**Prob.**

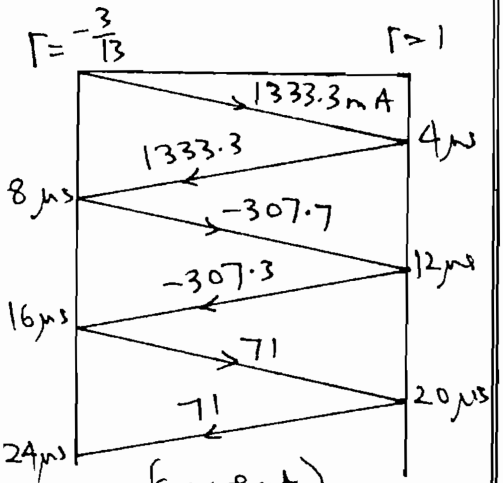
$V_\infty = \frac{Z_L}{Z_L+Z_g} V_g = \frac{100\,(200V)}{100+50} = 133.33$, $I_\infty = \frac{V_\infty}{Z_L} = 1333.33$ mA,

$t_1 = 4\mu s$. Set $V_0 = V_\infty$ and $I_0 = I_\infty$,

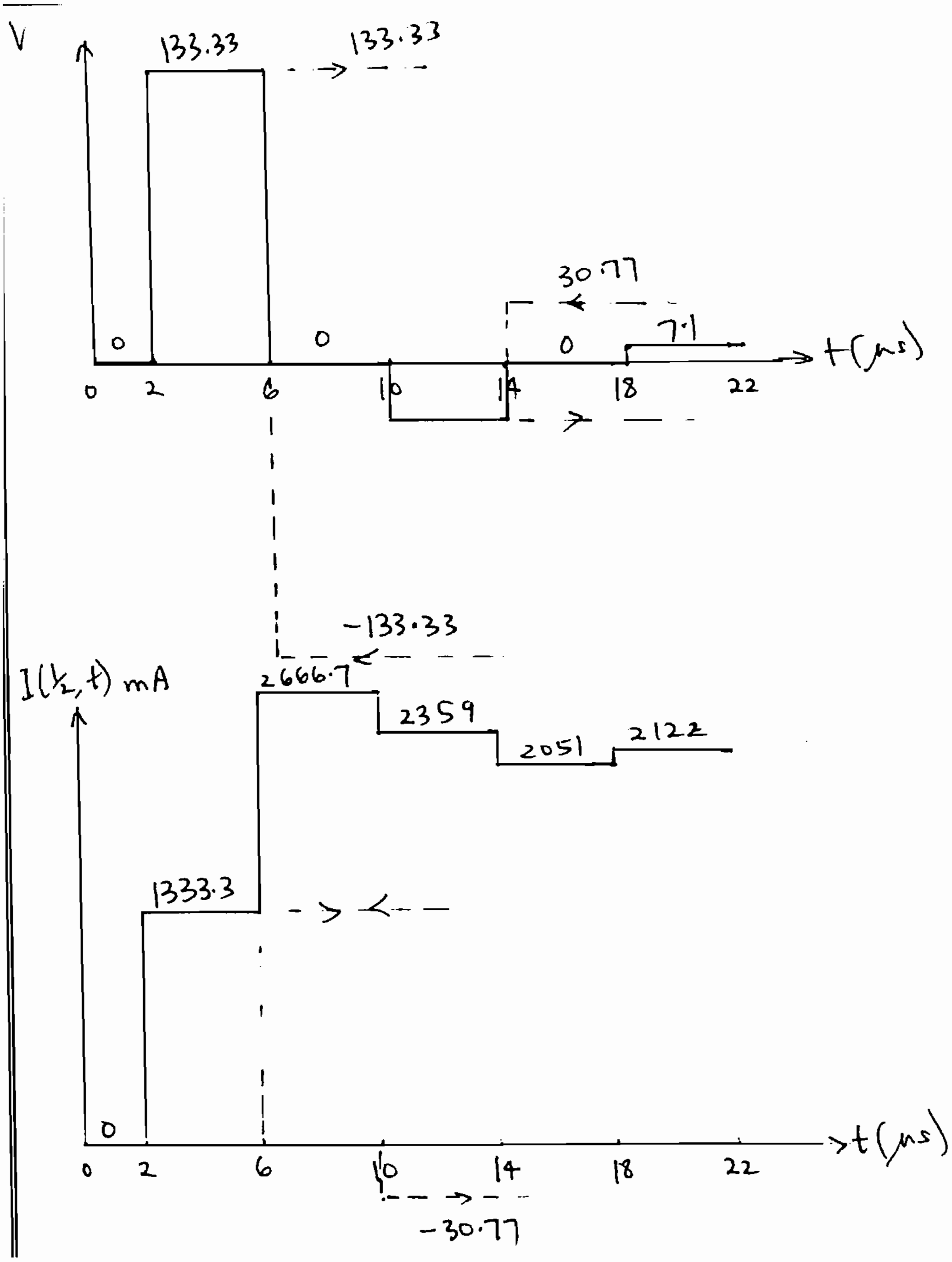
$\Gamma_L = \lim_{Z_L \to 0} \frac{Z_L - Z_o}{Z_L + Z_o} = -1$, $\Gamma_g = 3/13$



(voltage)



(current)


V
133.33
133.33
30.77
0
0
0
7.1
t (μs)
0
2
6
10
14
18
22
I(l/2, t) mA
-133.33
2666.7
2359
2051
2122
1333.3
0
t (μs)
0
2
6
10
14
18
22
-30.77

**Prob.**

$d_A = 0.12\lambda \quad \rightarrow \quad 0.12 \times 720^o = 86.4^o$

$l_A = 0.3\lambda \quad \rightarrow \quad 0.3 \times 720^o = 216^o$

(a) From the Smith Chart,

$z_L = 0.57 + j0.69$

$Z_L = 60(0.57 + j0.69)$

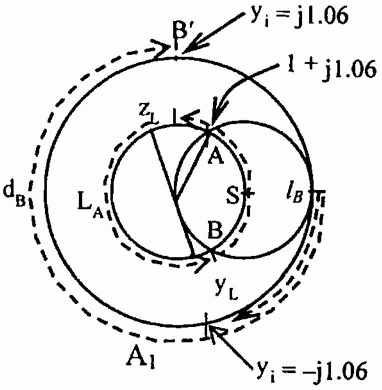
$= \underline{\underline{34.2 + j41.4\Omega}}$

(b) $d_B = \frac{360^o - 86.4^o}{720^o}\lambda \;=\; \underline{\underline{0.38\lambda}}$

$l_B = \frac{\lambda}{2} - \frac{(-62.4^o - -82^o)}{720^o}\lambda \;=\; \underline{\underline{0.473\lambda}}$

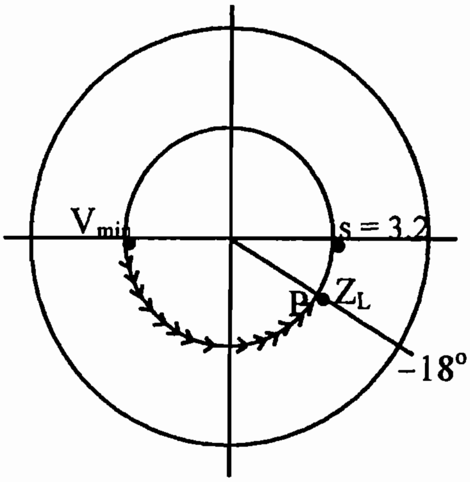
(c) $\underline{\underline{s\ = 2.65}}$

(Exact value = 2.7734 )



$$V(l,\infty) = V_o\left[1 + \Gamma_G\Gamma_L + (\Gamma_G\Gamma_L)^2 + (\Gamma_G\Gamma_L)^3 + ....\right]$$
$$+ V_o\Gamma_L\left[1 + \Gamma_G\Gamma_L + (\Gamma_G\Gamma_L)^2 + (\Gamma_G\Gamma_L)^3 + ....\right]$$

**Prob.**



$$\frac{\lambda}{2} = 32 - 12 = 20cm \rightarrow \lambda = 40\,\text{cm}$$

$$f = \frac{u}{\lambda} = \frac{3\times10^{8}}{40\times10^{-2}} = \underline{\underline{0.75\,\text{GHz}}}$$

$$l = 21 - 12 = 9cm = \frac{9\lambda}{40} \rightarrow \frac{9}{40}\times720^{o} = 162^{o}$$

At P, $z_L = 2.6 - j1.2$

$$Z_L = z_L Z_o = 50(2.6 - j1.2) = \underline{\underline{130 - j60\Omega}}$$

(Exact value = 130.49 – j58.219 Ω)

**Prob.**

(a) $\frac{W}{h} = 4 \longrightarrow$ wide strip, $D = \frac{60\pi}{\sqrt{9}} = 20\pi$

$$E = 2 + 0.4413 + 0.08226\times\frac{8}{21} + \frac{10}{18\pi}\left[1.452 + \ln 2.94\right]$$

$$= 2.8969$$

$$Z_o = \frac{D}{E} = \underline{\underline{21.69\,\Omega}}.$$

(b) $\epsilon_{eff} = 5 + 4\left(1 + \frac{10}{4}\right)^{-1/2} = \underline{\underline{7.138}}.$

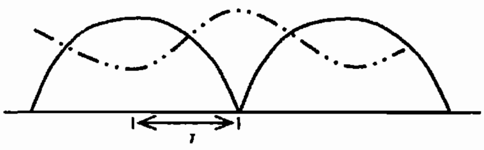
(c) $\alpha_d = \frac{27.3\times6.138}{8\times7.138} \times \frac{6\times10^{-4}\times3\times10^{8}}{10^{10}\sqrt{7.138}} = \underline{\underline{1.977\times10^{-5}\,dB/m.}}$

(d) $R_s = \sqrt{\frac{\pi\times10^{10}\times4\pi\times10^{-7}}{5.8\times10^{7}}} = 2.609\times10^{-12}$

$$\alpha_c = \frac{8.686\times2.609\times10^{-2}}{4\times10^{-3}\times21.69} = \underline{\underline{2.612\,dB/m}}.$$

e) $\alpha = \alpha_c + \alpha_d \simeq \alpha_c = \underline{\underline{2.612\times10^{-2}\,dB/cm}}.$

**Prob.**

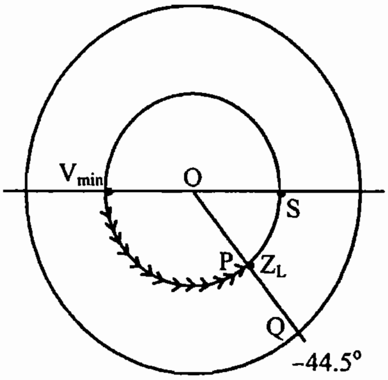


$$s = \frac{V_{\max}}{V_{\min}} = \frac{0.95}{0.45} = \underline{\underline{2.11}}$$

$$\frac{\lambda}{2} = 22.5 - 14 = 8.5 \quad \rightarrow \quad \lambda = 17\,\text{cm}$$

$$f = \frac{c}{\lambda} = \frac{3\times10^8}{0.17} = \underline{\underline{1.764\ \text{GHz}}}$$

$$l = 3.2\,\text{cm} = \frac{3.2}{17}\lambda \ \rightarrow \ 135.5^\circ$$



At P, $\quad z_L = 1.4 - j0.8$

$$Z_L = 50(1.4 - j0.8) = \underline{\underline{70 - j40\Omega}}$$

(Exact value = 70.606-j40.496 Ω)

$$|\Gamma| = \frac{s-1}{s+1} = \frac{1.11}{3.11} = 0.357, \qquad \theta_\Gamma = -44.5^o$$

$$\Gamma = \underline{\underline{0.357\angle -44.5^o}}$$

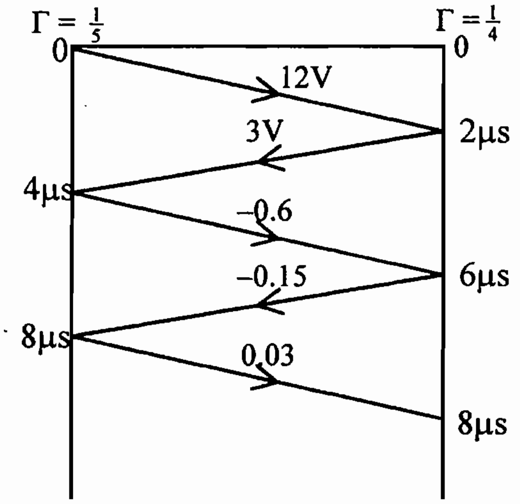
(Exact value = 0.3571∠-44.471°)

**Prob.**

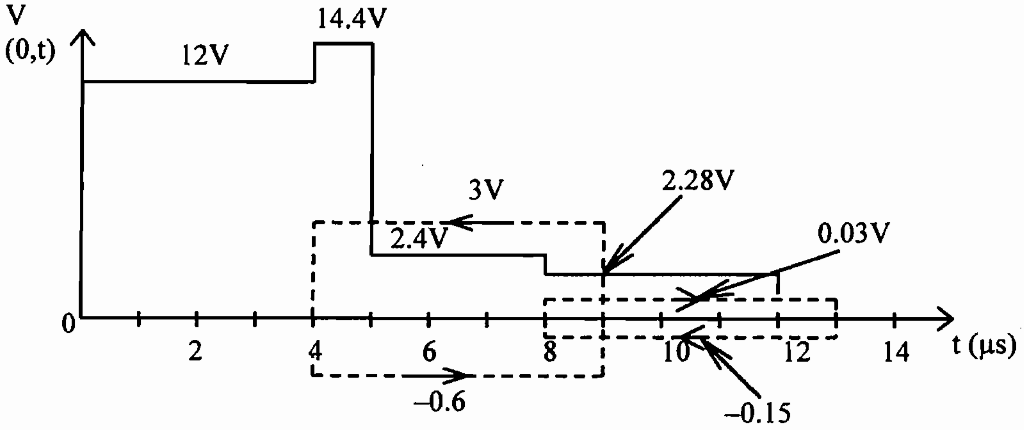
$$t_1 = \frac{l}{u} = \frac{6m}{3\times10^8} = 2\mu s, \quad V_o = V_g \cdot \frac{Z_o}{Z_L + Z_g} = 20\left(\frac{60}{100}\right) = 12V,$$

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{40-60}{100} = -\frac{1}{5}, \quad \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100-60}{160} = \frac{1}{4}.$$

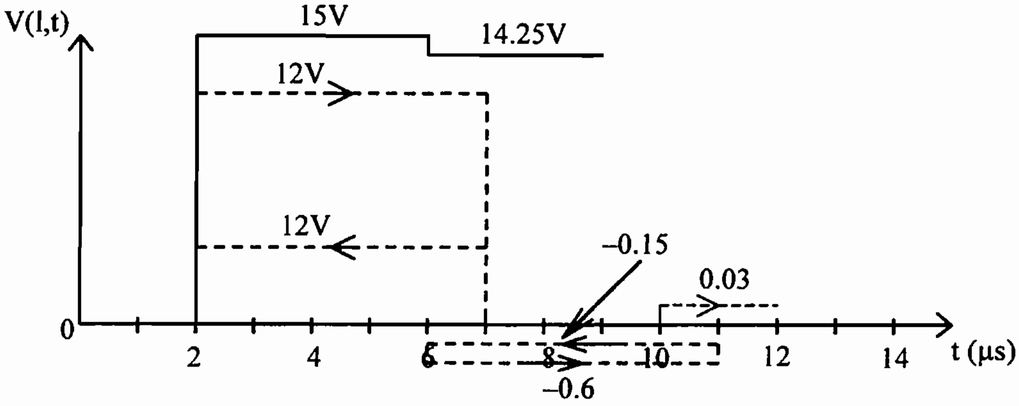
We only need the voltage bounce diagram because we can obtain *I(l, t)* from *V(l, t)/Z<sub>L</sub>*.

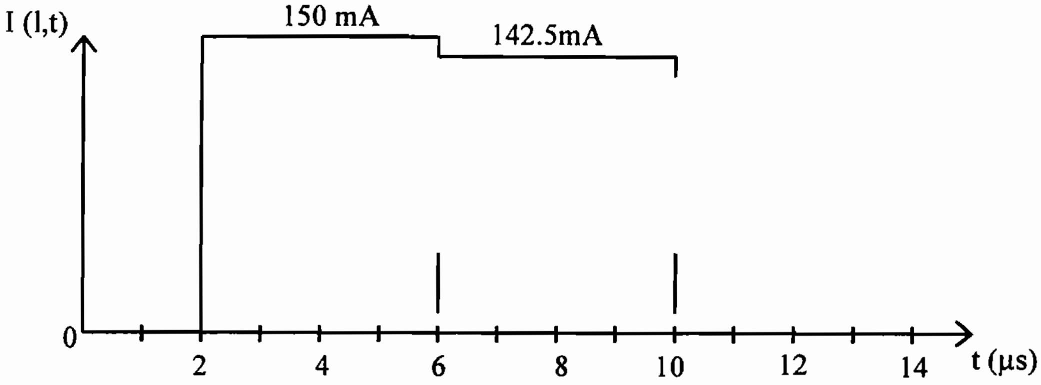


(Voltage bounce diagram)



We obtain *V(l, t)* from the bounce diagram and divide by $Z_L$ = 100Ω to obtain *I(l, t)*.





**Prob.**

$$A = \frac{120}{\sqrt{22}} = 25.258, \quad D = \frac{60\pi}{\sqrt{10}} = 59.64,$$

$$B = \frac{1}{2}\left(\frac{9}{11}\right)\left[\ln\frac{\pi}{2} + \frac{1}{10}\ln\frac{4}{\pi}\right] = 0.1946$$

$44 - 10 = 24 \longrightarrow Z_o = 20 < 24.$ Hence

$$\frac{w}{h} = \frac{2}{\pi}\left[9.364 - 1 - 2.875\right] + 0.2865\,(2.124 + 0.293 - 0.0517) = \underline{\underline{4.172}}$$

$$\varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\left(1 + \frac{10h}{w}\right)^{-1/2} = \frac{11}{2} + \frac{9}{2}(0.542)$$

$$= \underline{\underline{7.941}}\ .$$

**Prob.**

$w = 1.5cm$, h = 1cm, $\frac{w}{h} = 1.5$

$$\text{(a)} \quad \varepsilon_{\text{eff}} = \left(\frac{6\ +1}{2}\right) + \frac{\varepsilon_r\ -\ 1}{2\sqrt{1\ + 12h/w}} = 1.6\ + \frac{0.6}{\sqrt{1\ +\ 12/1.5}} = \underline{\underline{1.8}}$$

$$Z_0 = \frac{377}{\sqrt{1.8}\ \ (1.5\ + 1.393\ + 0.667\ \text{In}\ (2.944))} = \frac{281}{3.613} = \underline{\underline{77.77\Omega}}$$

(b) $\alpha_c = 8.686\dfrac{R_s}{wZ_o}$

$$R_s = \frac{1}{\sigma_c\ \sigma} = \sqrt{\frac{\mu\pi f}{\sigma_c}} = \sqrt{\frac{19 \times 2.5 \times 10^9 \times 4\pi \times 10^{-3}}{1.1 \times 10^7}}$$

$$= 2.995 \times 10^{-2}$$

$$\alpha_c = \frac{8.686 \times 2.995\times10^{-2}}{1.5\times10^{-2} \times 77.77} = \underline{\underline{0.223\text{dB/m}}}$$

$$u = \frac{c}{\sqrt{\varepsilon_{eff}}} \rightarrow \lambda = \frac{u}{f} = \frac{c}{f\sqrt{\varepsilon_{eff}}} = \frac{3 \times 10^8}{2.5\times10^9\ \sqrt{1.8}} = 8.944 \times 10^{-2}$$

$$\alpha_d = 27.3 \times \frac{0.8\,(2.2)}{1.2\ \sqrt{1.8}}\ \frac{2 \times 10^{-2}}{8.944\times10^{-2}} = \frac{96.096}{14.3996}$$

$$\alpha_d = \underline{\underline{6.6735\text{ dB/m}}}$$

(c) $\alpha = \alpha_c + \alpha_d = 6.8965$ dB/m

$$\alpha\ell = 20\text{dB} \rightarrow \ell = \frac{20}{\alpha} = \frac{20}{6.8965} = \underline{\underline{2.9\text{ m}}}$$

**Prob.**

(*a*) Let $x = w/h$. If $x < 1$,

$$50 = \frac{60}{\sqrt{4.6}} ln\left(\frac{8}{x} + x\right)$$

$$5\sqrt{4.6} - 6\text{In}\left(\frac{8}{x} + x\right) = 0$$

we solve for x (e.g using Maple) and get x = 2.027 or 3.945

which contradicts our assumptiom that x < 1. If x >1,

$$50 = \frac{120\pi}{\sqrt{4.6}\left[x + 1.393 + 0.667\,\ln(x + 1.444)\right]}$$

We solve this iteratively and obtain:

$$x = 1.4205, w = 1.4205h = 11.36 \text{ mm}$$

For this w and h,

$$\varepsilon_{eff} = 3.3856$$

(b) $\beta = \dfrac{\omega\varepsilon_{eff}}{c}$

$$\beta\ell = 45^0 = \frac{\pi}{4} = \frac{\omega\ell\varepsilon_{\text{eff}}}{c}$$

$$\ell = \frac{\pi\text{c}}{4\varepsilon_{\text{eff}}2\pi f} = \frac{3\times10^8}{8\times3.3856\times8\times10^9}$$

$$\underline{\underline{\ell = 0.00138 \text{ m}}}$$

**Prob.**

$$L = \frac{Z_o}{u} = \frac{Z_o\sqrt{\epsilon_r}}{c} = \frac{60\sqrt{5}}{3\times10^8} = \underline{\underline{1\,\mu\text{H/m.}}}$$

$$C = \frac{1}{Z_o u} = \frac{\sqrt{\epsilon_r}}{Z_o c} = \frac{\sqrt{5}}{60\times3\times10^8} = \underline{\underline{0.1242 \text{ nF/m.}}}$$

**Prob.**

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100-150}{250} = -0.2$$

$$RL = -20\log|\Gamma| = \underline{\underline{13.98 \text{ dB}}}$$

## Chapter —Practice Examples

### P.E.

(a) For TE$_{10}$, $f_c = 3$ GHz,

$$\sqrt{1-(f_c/f)^2} = \sqrt{1-(3/15)^2} = \sqrt{0.96}\,, \quad \beta_o = \omega/u_o = 4\pi f/c$$

$$\beta = \frac{4\pi f}{c}\sqrt{0.96} = \frac{4\pi\times 15\times 10^9}{3\times 10^8}\sqrt{0.96} = \underline{\underline{615.6}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times 15\times 10^9}{615.6} = \underline{\underline{1.531\times 10^8}} \text{ m/s}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = 60\pi, \quad \eta_{TE} = \frac{60\pi}{\sqrt{0.96}} = \underline{\underline{192.4\Omega}}$$

(b)For TM$_{11}$, $f_c = 3\sqrt{7.25}$ GHz, $\sqrt{1-(f_c/f)^2} = 0.8426$

$$\beta = \frac{4\pi f}{c}(0.8426) = \frac{4\pi\times 15\times 10^9(0.8426)}{3\times 10^8} = \underline{\underline{529.4}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times 15\times 10^9}{529.4} = \underline{\underline{1.78\times 10^8}} \text{ m/s}$$

$$\eta_{TM} = 60\pi(0.8426) = \underline{\underline{158.8\Omega}}$$

### P.E.

(a) Since $E_z \neq 0$, this is a TM mode

$$E_{zs} = E_o \sin(m\pi x/a)\sin(n\pi y/b)e^{-j\beta z}$$

$$E_o = 20, \quad \frac{m\pi}{a} = 40\pi \longrightarrow m=2, \quad \frac{n\pi}{b} = 50\pi \longrightarrow n=1$$

i.e. $\underline{\underline{\text{TM}_{21}\text{ mode}}}$.

(b) $f_c = \dfrac{u'}{2}\sqrt{(m/a)^2+(n/b)^2} = \dfrac{3x10^8}{2}\sqrt{40^2+50^2} = 1.5\sqrt{41}$ GHz

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2} = \frac{2\pi f}{c}\sqrt{f^2-f_c^2} = \frac{2\pi x10^9}{3x10^8}\sqrt{225-92.25} = \underline{\underline{241.3 \text{ rad/m.}}}$$

(c)

$$E_{xs} = \frac{-j\beta}{h^2}(40\pi)20\cos 40\pi x \sin 50\pi y e^{-j\beta z}$$

$$E_{vs} = \frac{-j\beta}{{}_{\prime}2}(50\pi)20\sin 40\pi x \cos 50\pi y e^{-j\beta z}$$

$$\frac{E_y}{E_x} = \underline{\underline{1.25 \tan 40\pi x \cot 50\pi y}}$$

**P.E.**

If $TE_{13}$ mode is assumed, $f_c$ and $\beta$ remain the same.

$\underline{\underline{f_c = 28.57 \text{ GHz}, \ \beta = 1718.81 \text{ rad/m}, \ \gamma = j\beta}}$

$$\eta_{TE13} = \frac{377/2}{\sqrt{1-(28.57/50)^2}} = \underline{\underline{229.69 \ \Omega}}$$

For m=1, n=3, the field components are:

$E_z = 0$

$$H_z = H_o \cos(\pi x / a)\cos(3\pi y / b)\cos(\omega t - \beta z)$$

$$E_x = -\frac{\omega\mu}{h^2}\left(\frac{3\pi}{b}\right)H_o \cos(\pi x / a)\sin(3\pi y / b)\sin(\omega t - \beta z)$$

$$E_y = \frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_o \sin(\pi x / a)\cos(3\pi y / b)\sin(\omega t - \beta z)$$

$$H_x = -\frac{\beta}{h^2}\left(\frac{\pi}{a}\right)H_o \sin(\pi x / a)\cos(3\pi y / b)\sin(\omega t - \beta z)$$

$$H_y = -\frac{\beta}{h^2}\left(\frac{3\pi}{a}\right)H_o \cos(\pi x / a)\sin(3\pi y / b)\sin(\omega t - \beta z)$$

Given that $H_{ox} = 2 = -\frac{\beta}{h^2}(\pi / a)H_o$,

$$H_{oy} = -\frac{\beta}{h^2}(3\pi / b)H_o = 6a / b = 6(1.5) / 8 = 11.25$$

$$H_{oz} = H_o = -\frac{2h^2 a}{\beta\pi} = \frac{-2\times 14.51\pi^2 \times 10^4 \times 1.5\times 10^{-2}}{1718.81\pi} = -7.96$$

$$E_{oy} = \frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_o = -\frac{2\omega\mu}{\beta} = 2\eta_{TE} = -459.4$$

$$E_{ox} = -E_{oy}\frac{3a}{b} = 459.4(4.5 / 0.8) = 2584.1$$

$E_x = 2584.1\cos(\pi x / a)\sin(3\pi y / b)\sin(\omega t - \beta z)$ V/m,

$E_y = -459.4\sin(\pi x / a)\cos(3\pi y / b)\sin(\omega t - \beta z)$ V/m,

$E_z = 0$,
$H_y = 11.25\cos(\pi x / a)\sin(3\pi y / b)\sin(\omega t - \beta z)$ A/m,

$H_z = -7.96\cos(\pi x / a)\cos(3\pi y / b)\cos(\omega t - \beta z)$ A/m

**P.E.**

$$f_{c11} = \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{3\times 10^8 \times 10^2}{2}\sqrt{1/8.636^2 + 1/4.318^2} = 3.883 \text{ GHz}$$

$$u_p = \frac{3\times 10^8}{\sqrt{1-(3.883/4)^2}} = \underline{\underline{12.5\times 10^8}} \text{ m/s,}$$

$$u_g = \frac{9\times 10^{16}}{12.5\times 10^8} = \underline{\underline{7.2\times 10^7}} \text{ m/s}$$

## P.E.

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}}$$

$$f_{TE101} = \frac{1.5\times10^{10}}{\sqrt{3}}\sqrt{1/25+0+1/100} = \underline{\underline{1.936}} \text{ GHz}$$

$$Q_{TE101} = \frac{1}{61\delta}, \text{ where}$$

$$\delta = \frac{1}{\sqrt{\pi f_{101}\mu\sigma_c}} = \frac{1}{\sqrt{\pi\times1.936\times10^9\times4\pi\times10^{-7}\times5.8\times10^7}} = 1.5\times10^{-6}$$

$$Q_{TE101} = \frac{10^6}{61\times1.5} = \underline{\underline{10,929}}$$

## P.E.

(a) By Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Thus
$\theta_2 = 90^\circ \longrightarrow \sin\theta_2 = 1$

$\sin\theta_1 = n_2/n_1$, $\theta_1 = \sin^{-1} n_2/n_1 = \sin^{-1} 1.465/1.48 = \underline{\underline{81.83^\circ}}$

(b) $NA = \sqrt{n_1^2 - n_2^2} = \sqrt{1.48^2 - 1.465^2} = \underline{\underline{0.21}}$

## P.E.

$\alpha l = 10 \log P(0)/P(l) = 0.2 \text{ X } 10 = 2$

$P(0)/P(l) = 10^{0.2}$, i.e. $P(l) = P(0)\ 10^{-0.2} = 0.631\ P(0)$

i.e. $\underline{\underline{63.1\ \%}}$

**CHAPTER —Problems**

**Prob.**

(a)

$$f_c = \frac{u'}{2}\sqrt{1/a^2 + 1/b^2} = \frac{3\times10^8}{2\sqrt{4}\times10^{-2}}\sqrt{1/2^2+1/3^2} = \underline{\underline{4.507}}\ \text{GHz}$$

(b)

$$\beta = \beta'\sqrt{1-(f_c/f)^2} = \frac{\omega}{u'}\sqrt{1\ -(f_c/f)^2} = \frac{2\pi\times20\times10^9\sqrt{4}}{3\times10^8}\sqrt{1-(4.508/20)^2}$$

$= \underline{\underline{816.2\ \text{rad/m}}}$

(c )

$$u = \omega/\beta = \frac{2\pi\times20\times10^9}{816.21} = \underline{\underline{1.54\times10^8}}\ \text{m/s}\ .$$

**Prob.**

(a) For $\text{TM}_{mn}$ modes, $H_z = 0$

$$E_{zs} = E_o\sin(\pi x/a)\sin(\pi y/b)e^{-\gamma z}$$

**Using eq. (** , all field components vanish for $\text{TM}_{01}$ and $\text{TM}_{10}$.

(b) See text.

**Prob.**

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} = \frac{3\times10^8}{2\sqrt{2.25}\times10^{-2}}\left[\left(\frac{m}{2.28}\right)^2+\left(\frac{n}{1.01}\right)^2\right]^{1/2}$$

$$= \frac{15}{\sqrt{2.25}}\left[\left(\frac{m}{2.28}\right)^2+\left(\frac{n}{1.01}\right)^2\right]^{1/2}\ \text{GHz}$$

Using this formula, we obtain the cutoff frequencies for the given modes as shown below.

| Mode | f c (GHz) |
|---|---|
| $TE_{01}$ | 9.901 |
| $TE_{10}$ | 4.386 |
| $TE_{11}$ | 10.829 |
| $TE_{02}$ | 19.802 |
| $TE_{22}$ | 21.66 |
| $TM_{11}$ | 10.829 |
| $TM_{12}$ | 20.282 |
| $TM_{21}$ | 13.23 |

**Prob.**

$$a/b = 3 \longrightarrow a = 3b$$

$$f_{c10} = \frac{u'}{2a} \longrightarrow a = \frac{u'}{2f_{c10}} = \frac{3x10^8}{2x18x10^9} \text{ m} = 0.833\text{cm}$$

A design could be $\underline{\underline{a = 9mm,\ b = 3mm.}}$

**Prob.**

For the dominant mode,

$$f_c = \frac{c}{2a} = \frac{3x10^8}{2x8} = 18.75\,MHz$$

(a) It $\underline{\underline{\text{will not pass}}}$ the AM signal, (b) it $\underline{\underline{\text{will pass}}}$ the FM signal.

**Prob.**

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2}$$

$$f_{c11} = f_{c03} \longrightarrow \frac{u'}{2}\sqrt{(1/a)^2 + (1/b)^2} = \frac{u'}{2}\sqrt{9/b^2}$$

$$\frac{9}{b^2} = \frac{1}{a^2} + \frac{1}{b^2} \longrightarrow a = \frac{b}{\sqrt{8}}$$

$$f_{c03} = \frac{3u'}{2b} \longrightarrow b = \frac{3c}{2f_{c03}} = \frac{9x10^8}{2x12x10^9} = 3.75 \text{ cm}$$

$\underline{\underline{a = 1.32 \text{ cm}, \ b = 3.75 \text{ cm}}}$

Since a < b, the dominant mode is $TE_{01}$

$$f_{c01} = \frac{c}{2b} = \frac{3x10^8}{2x3.75x10^{-2}} = 4 \text{ GHz} < f = 8 \text{ GHz}$$

Hence, the dominant mode $\underline{\underline{\text{will proprogate}}}$.

**Prob.**

$$f_c = \frac{c}{2}\sqrt{\frac{1}{7.2^2} + \frac{1}{3.4^2}} \times 10^2 = \frac{3\times10^8\times10^2}{2}\sqrt{\frac{1}{7.2^2} + \frac{1}{3.4^2}} = \underline{\underline{4.879 \text{ GHz}}}$$

$$\lambda_c = \frac{u'}{f_c} = \frac{3\times10^8}{4.879\times10^9} = \underline{\underline{6.149 \text{ cm}}}$$

**Prob.**

(a) This is $TM_{21}$ mode. Let $F = \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

$$\beta' = \frac{w}{u} = \frac{2\pi \times 5 \times 10^9}{3 \times 10^8} = \frac{100\pi}{3}$$

$$\beta = \beta' F \longrightarrow F = \frac{\beta}{\beta'} = 10 \times \frac{3}{100\pi} = \frac{3}{10\pi}$$

$$F^2 = \frac{9}{100\pi^2} = 1 - \left(\frac{f_c}{f}\right)^2 \longrightarrow f_c = f\sqrt{1 - \frac{9}{100\pi^2}}$$

$$= \underline{\underline{4.977 \text{ GHz}}}.$$

(b) $\underline{H_z = 0}$.

$$E_x = \frac{\beta}{h^2}\left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\sin(\omega t - 10z).$$

But $\beta = 10$, $\omega = 2\pi f = 10\pi \times 10^9$

$$h^2 = k_x^2 + k_y^2 = \gamma^2 + k^2 = \frac{\omega^2}{u^2} - \beta^2$$

$$= \frac{4\pi^2 \times 25 \times 10^{18}}{9 \times 10^{16}} - 100 = 10866$$

$$E_x = \underline{9.203 \times 10^{-4}\left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\sin(\omega t - 10z)\,V/}$$

**Prob.**

(a) $m = 0, n = 2$, i.e. $\underline{TE_{02} \text{ mode}}$.

(b) Since $a = b$, $f_c = \frac{u'}{a} = \frac{3\times 10^8}{1.2\times 10^{-2}} = 25$ GHz.

$$\lambda_c = \frac{u'}{f_c} = a = \underline{1.2\text{ cm}}.$$

(c) $$\beta = \frac{2\pi f}{c}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

or $$\sqrt{f^2 - f_c^2} = \frac{\beta c}{2\pi} = \frac{150 \times 3 \times 10^8}{2\pi} = 71.62 \times 10^8$$

$f^2 = f_c^2 + (7.162)^2$ all in GHz

$$f = \sqrt{25 + 7.162^2} = \underline{26\text{ GHz}}.$$

(d) $\gamma = j\beta = \underline{j150\,/m}$.

$$\eta_{TE_{02}} = \frac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1 - (25/26)^2}} = \underline{1372.55\,\Omega}.$$

**Prob.**

(a) Since m=2, n=3, the mode is $\underline{\underline{TE_{23}}}$.

(b) $\beta = \beta'\sqrt{1-(f_c/f)^2} = \frac{2\pi f}{c}\sqrt{1-(f_c/f)^2}$

But

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2+(n/b)^2} = \frac{3x10^8}{2x10^{-2}}\sqrt{(2/2.86)^2+(3/1.016)^2} = 46.19\text{ GHz},\ \ f = 50\text{ GHz}$$

$$\beta = \frac{2\pi x50x10^9}{3x10^8}\sqrt{1-(46.19/50)^2} = 400.68\text{ rad/m}$$

$$\gamma = j\beta = \underline{\underline{j400.7}}\text{ /m}$$

$$\text{(c)}\quad \eta = \frac{\eta'}{\sqrt{1-(f_c/f)^2}} = \frac{377}{\sqrt{1-(46.19/50)^2}} = \underline{\underline{985.3\Omega}}$$

**Prob.**

$$\gamma = \alpha = w\sqrt{\mu\epsilon}\sqrt{\left(\frac{f_c}{f}\right)^2-1} = w\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_r}\sqrt{\left(\frac{f_c}{\frac{3}{4}f_c}\right)^2-1}$$

$$= \frac{2w}{c}\sqrt{\frac{7}{9}} \quad \text{or} \quad \alpha = \frac{4\pi\sqrt{7}\,f}{3c} = \frac{\pi\sqrt{7}\,f_c}{c}$$

But $f_c = \frac{u'}{2a}\sqrt{m^2+n^2} = \frac{c}{4a}\sqrt{m^2+n^2}$

$$= \frac{3\times10^8}{0.1}\sqrt{m^2+n^2} = 3\sqrt{m^2+n^2}\text{ GHz.}$$

(a) for $TE_{10}$ mode, $f_c = 3$ GHz

$$\alpha = \pi\sqrt{7}\cdot\frac{3\times10^9}{3\times10^8} = 10\pi\sqrt{7}$$

$$\frac{E_o}{100} = E_o e^{-\alpha d} \longrightarrow d = \frac{1}{\alpha} \ln 100$$

$$d = \frac{1}{10\pi\sqrt{7}} \ln 100 = \underline{\underline{5.54 \text{ cm.}}}$$

(b) for $TE_{12}$ mode, $f_c = 3\sqrt{5}$ GHz,

$$\alpha = \pi\sqrt{7} \; \frac{3\sqrt{5} \times 10^9}{3 \times 10^8} = 10\pi\sqrt{35}$$

$$d = \frac{1}{10\pi\sqrt{35}} \ln 100 = \underline{\underline{2.475 \text{ cm.}}}$$

**Prob.**

**(a)**

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2} = \frac{3x10^8}{2x10^{-2}}\sqrt{1/1 + 4/9} = 18.03 \text{ GHz}$$

f = 1.2 $f_c$ = $\underline{\underline{21.63 \text{ GHz}}}$

(b) $\sqrt{1 - (f_c/f)^2} = \sqrt{1 - (1/1.2)^2} = 0.5528$

$$u_p = \frac{c}{\sqrt{1 - (f_c/f)}} = \frac{3x10^8}{0.5528} = \underline{\underline{5.427x10^8}} \text{ m/s}$$

$$u_g = u\sqrt{1 - (f_c/f)^2} = 3x10^8 x0.5528 = \underline{\underline{1.658x10^8}} \text{ m/s}$$

**Prob.**

(a) Since m=2 and n=1, we have $\underline{\underline{TE_{21} \text{ mode}}}$

(b) $\beta = \beta'\sqrt{1 - (f_c/f)^2} = \omega\sqrt{\mu_o \varepsilon_o}\sqrt{1 - (\omega_c/\omega)^2}$

$$\beta c = \sqrt{\omega^2 - \omega^2_c} \quad \longrightarrow \quad \omega_c^{\,2} = \sqrt{\omega^2 - \beta^2 c^2}$$

$$f_c = \frac{\omega_c}{2\pi} = \sqrt{f^2 - \frac{\beta^2 c^2}{4\pi^2}} = \sqrt{36x10^{18} - \frac{144x9x10^{16}}{4\pi^2}} = \underline{\underline{5.973}} \text{ GHz}$$

(c) $\eta_{TE} = \dfrac{\eta}{\sqrt{1-(f_c/f)^2}} = \dfrac{377}{\sqrt{1-(5.973/6)^2}} = \underline{\underline{3978\Omega}}$

(d)For TE mode,

$$E_y = \frac{\omega\mu}{h^2}(m\pi/a)H_o \sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t - \beta z)$$

$$H_x = \frac{-\beta}{h^2}(m\pi/a)H_o \sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t - \beta z)$$

$\beta = 12$, m = 2, n =1

$$E_{oy} = \frac{\omega\mu}{h^2}(m\pi/a)H_o, \quad H_{ox} = \frac{\beta}{h^2}(m\pi/a)H_o$$

$$\eta_{TE} = \frac{E_{oy}}{H_{ox}} = \frac{\omega\mu}{\beta} = \frac{2\pi\times 6\times 10^9\times 4\pi\times 10^{-7}}{12} = 4\pi^2\times 100$$

$$H_{ox} = \frac{E_{oy}}{\eta_{TE}} = \frac{5}{4\pi^2\times 100} = 1.267 \text{ mA/m}$$

$$\underline{\underline{H_x = -1.267\sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t - \beta z)}} \text{ mA/m}$$

**Prob.**

(a) Since m=2, n=3, the mode is $\underline{\underline{TE_{23}}}$.

(b) $\beta = \beta'\sqrt{1-(f_c/f)^2} = \dfrac{2\pi f}{c}\sqrt{1-(f_c/f)^2}$

But

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2} = \frac{3\times 10^8}{2\times 10^{-2}}\sqrt{(2/2.86)^2 + (3/1.016)^2} = 46.19\,\text{GHz}, \ \ f = 50 \text{ GHz}$$

$$\beta = \frac{2\pi\times 50\times 10^9}{3\times 10^8}\sqrt{1-(46.19/50)^2} = 400.68 \text{ rad/m}$$

$\gamma = j\beta = \underline{\underline{j400.7}}$ /m

(c) $\eta = \frac{\eta'}{\sqrt{1-(f_c/f)^2}} = \frac{377}{\sqrt{1-(46.19/50)^2}} = \underline{\underline{985.3\Omega}}$

**Prob.**

This is a hybrid mode. Let

$E_x = E_x' + E_x''$

where $E_x'$ and $E_x''$ are due to TM and TE components respectively.

For $E_x'$, $H_z = 0$ (TM mode). From Example 12.2,

if $E_z = E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos(\omega t - \beta z)$

$E_x' = \frac{\beta}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(\omega t - \beta z)$

---

$E_o = 50$, $\frac{m\pi}{a} = 40\pi$, $\frac{n\pi}{b} = 30\pi$

$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2}\times 50 = 25c = 7.5 \text{ GHz}$

$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{20\pi\times 10^9}{3\times 10^8}\sqrt{1-(7.5/10)^2}$

$= 138.53$

$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = 2500\pi^2$

$$\frac{\beta}{h^2}\left(\frac{m\pi}{a}\right)E_o = \frac{138.53}{2500\pi^2}\,40\pi(50) = 35.28$$

$$E_x' = 35.28\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(\omega t-\beta z)$$

for $E_x''$, $E_z = 0$ (TE mode)

Let $H_z = H_o \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos(\omega t-\beta z)$, $H_o = 0.1$,

$$E_x'' = \frac{-\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_o\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(\omega t-\beta z)$$

$$\frac{\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_o = \frac{20\pi\times10^9}{2500\pi^2}\,30\pi(0.1)\times4\pi\times10^{-7} = 30.16$$

$$E_x'' = -30.16\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(\omega t-\beta z)$$

Hence $E_x = E_x' + E_x''$

$$= \underline{\underline{5.12\cos 40\pi x\,\sin 30\pi y\,\sin(\omega t-\beta z)}}\,.$$

**Prob.**

**(a)**

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2},\ \beta = \beta'\sqrt{1-(f_c/f)^2}$$

$$u = \omega/\beta = \frac{u'}{\sqrt{1-(f_c/f)^2}},\ \lambda = 2\pi/\beta = \frac{\lambda'}{\sqrt{1-(f_c/f)^2}}$$

(b) If a = 2b = 2.5cm, $f_c = \frac{u'}{2a}\sqrt{m^2+4n^2}$. For TE$_{11}$,

$$f_c = \frac{3\times10^8}{2\times2.5\times10^{-2}}\sqrt{1+4} = 13.42 \text{ GHz},\quad u = \frac{3\times10^8}{\sqrt{1-(13.42/20)^2}} = \underline{\underline{4.06\times10^8}} \text{ m/s}$$

$$\lambda = u/f = \frac{4.046\times10^8}{200\times10^8} = \underline{\underline{2.023}}\ \text{cm}$$

For TE21,

$$f_c = \frac{3\times10^8}{2\times2.5\times10^{-2}}\sqrt{4+4} = 16.97\ \text{GHz},\quad u = \frac{3\times10^8}{\sqrt{1-(16.97/20)^2}} = \underline{\underline{5.669\times10^8}}\ \text{m/s}$$

$$\lambda = u/f = \frac{5.669\times10^8}{200\times10^8} = \underline{\underline{2.834}}\ \text{cm}$$

**Prob.**

$$\beta l = 240^\circ \longrightarrow \beta = \frac{240^\circ \pi}{180^\circ}\cdot\frac{1}{3\times10^{-2}} = \frac{400\pi}{9}\ \text{rad/m}$$

$$f_{cd} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2},\quad f_{ca} = \frac{1}{2\sqrt{\mu_o\epsilon_o}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

i.e. $f_{cd} = \dfrac{f_{ca}}{\sqrt{\epsilon_r}}$

$$\beta = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_{cd}}{f}\right)^2} = \frac{\omega}{c}\sqrt{\epsilon_r}\sqrt{1-\left(\frac{f_{ca}}{\sqrt{\epsilon_r}f}\right)^2}$$

or $$\epsilon_r = \left(\frac{\beta c}{2\pi f}\right)^2 + \left(\frac{f_{ca}}{f}\right)^2 = \left(\frac{400\pi\times3\times10^8}{9\times2\pi\times4\times10^9}\right)^2 + \left(\frac{10}{4}\right)^2$$

$$\epsilon_r = \underline{\underline{9.028}}$$

**Prob.**

Substituting $E_z = R\Phi Z$ into the wave equation,

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}(\rho R') + \frac{RZ}{\rho^2}\Phi'' + R\Phi Z'' + k^2 R\Phi Z = 0$$

Dividing by $R\Phi Z$,

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + k^2 = -\frac{Z''}{Z} = -k_z^2$$

i.e. $\underline{\underline{Z'' - k_z^2 Z = 0}}$

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + (k^2 + k_z^2) = 0$$

$$\frac{\rho}{R}\frac{d}{d\rho}(\rho R') + (k^2 + k_z^2)\rho^2 = -\frac{\Phi''}{\Phi} = k_\phi^2$$

or

$$\underline{\underline{\Phi'' + k_\phi^2 \Phi = 0}}$$

$\rho\frac{d}{d\rho}(\rho R') + (k_\rho^2\rho^2 - k_\phi^2)R = 0$, where $k_\rho^2 = k^2 + k_z^2$. Hence

$$\underline{\underline{\rho^2 R'' + \rho R' + (k_\rho^2\rho^2 - k_\phi^2)R = 0}}$$

**Prob.**

$$f_c = \frac{u'}{2a}$$

$$u_g = u'\sqrt{1-\left(\frac{f_c}{f}\right)^2} \longrightarrow \left(\frac{f_c}{f}\right)^2 = 1-\left(\frac{u_g}{u'}\right)^2 = 1-\left(\frac{1.8\times10^8}{\frac{3\times10^8}{\sqrt{2.2}}}\right)^2 = 0.2081$$

$$f_c = \sqrt{0.11}f = 2.053 \text{ GHz}$$

$$a = \frac{u'}{2f_c} = \frac{3\times10^8}{2\sqrt{2.2}\times2.053\times10^9} = \underline{\underline{4.926 \text{ cm}}}$$

**Prob.**

$$f_{c12} = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{4}{b^2}} = \frac{3\times10^8\times10^2}{2}\sqrt{\frac{1}{2.286^2}+\frac{4}{1.016^2}} = 30.25 \text{ GHz}$$

$$u_g = c\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 3\times10^8\sqrt{1-\left(\frac{30.25}{32}\right)^2} = \underline{\underline{9.791\times10^7 \text{ m/s}}}$$

**Prob.**

$$\boldsymbol{P}_{\text{ave}} = \frac{|E_{xs}|^2+|E_{ys}|^2}{2\eta}\boldsymbol{a}_z = \underline{\underline{\frac{\omega^2\mu^2\pi^2}{2\eta b^2h^4}H_o^{\ 2}\sin^2\pi y/b\boldsymbol{a}_z}}$$

where $\eta = \eta_{TE10}$.

$$\text{P}_{\text{ave}} = \int \boldsymbol{P}_{\text{ave}}.d\boldsymbol{S} = \frac{\omega^2\mu^2\pi^2}{2\eta b^2h^4}H_o^{\ 2}\int_{x=0}^{a}\int_{y=0}^{b} \sin^2\pi y/b dxdy$$

$$\text{P}_{\text{ave}} = \frac{\omega^2\mu^2\pi^2}{2\eta b^2h^4}H_o^2 ab/2$$

But $h^2 = (m\pi/a)^2+(n\pi/b)^2 = \dfrac{\pi^2}{b^2}$,

$$\underline{\underline{\text{P}_{\text{ave}} = \frac{\omega^2\mu^2 ab^3H_o^2}{4\pi^2\eta}}}$$

**Prob.**

This is TE mode for which

$$\frac{m\pi}{a} = 30\pi, \quad \frac{n\pi}{b} = 0$$

(a) $f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} = \frac{u'}{2}.30 = 15u'$

But $u' = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{5}}$, hence $f_c = \frac{15c}{\sqrt{5}} = \underline{\underline{2.015\ GHz}}$

$\lambda_c = \frac{u'}{f_c} = \frac{1}{15} = 0.06667m = \underline{\underline{6.667cm}}.$

(b) $\gamma = \sqrt{-\omega^2\mu\epsilon + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + j\omega\mu\sigma}$

$$\gamma = \sqrt{-(2\pi\times10^9\times4)^2\times4\pi\times10^{-7}\times\frac{10^{-9}}{36\pi}\times5 + (30\pi)^2 + 0 + j2\pi\times4\times10^9\times4\pi\times10^{-7}\times10^{-5}}$$

$$= \sqrt{-2.621\times10^4 + j0.3158} = 9.753\times10^{-4} + j161.9/m$$

$\underline{\underline{\alpha = 9.753\times10^{-4}\ Np/m, \quad \beta = 161.9\ rad/m}}.$

(c) $u = \frac{\omega}{\beta} = \frac{2\pi\times4\times10^9}{161.9} = \underline{\underline{1.551\times10^8\ m/s}}.$

**Prob.**

For TE$_{10}$ mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

But a = b, $R_s = \frac{1}{\sigma_c\delta} = \sqrt{\frac{\pi f\mu}{\sigma_c}}$

$$\alpha_c = \frac{2\sqrt{\frac{\pi f\mu}{\sigma_c}}}{a\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\left(\frac{f_c}{f}\right)^2\right] = \frac{k\sqrt{f}\left[\frac{1}{2}+\left(\frac{f_c}{f}\right)^2\right]}{\sqrt{1-(f_c/f)^2}}$$

where k is a constant.

$$\frac{d\alpha_c}{df} = \frac{k[1-(\frac{f_c}{f})^2]^{1/2}[\frac{1}{4}f^{-1/2}-\frac{3}{2}f_c^2 f^{-5/2}]-\frac{k}{2}[\frac{1}{2}f^{1/2}+f_c^2 f^{-3/2}](2f_c^2 f^{-3})[1-(\frac{f_c}{f})^2]^{-1/2}}{1-(f_c/f)^2}$$

For minimum value, $\frac{d\alpha_c}{df} = 0$. This leads to $\underline{\underline{f = 2.962\ f_c}}$.

**Prob.**

$$R_s = \sqrt{\frac{\pi \mu f}{\sigma_c}} = \sqrt{\frac{\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} = 2.858 \times 10^{-2}$$

$$f_{c10} = \frac{u'}{2a} = \frac{3 \times 10^8}{2\sqrt{2.6} \times 2 \times 10^{-2}} = 4.651 \text{ GHz}$$

$$f_{c11} = \frac{u'}{2}\left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = 10.4 \text{ GHz}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = \frac{377}{\sqrt{2.6}} = 233.81\Omega$$

(a) For TE$_{10}$ mode, ϵ eq. ( gives

$$\alpha_d + j\beta_d = \sqrt{-\omega^2 \mu\varepsilon + k_x^2 + k_x^2 + j\omega\mu\sigma_d}$$

$$= \sqrt{-\omega^2 / u^2 + \frac{\pi^2}{a^2} + j\omega\mu\sigma_d}$$

$$= \sqrt{-\left(\frac{2\pi \times 12 \times 10^9}{3 \times 10^8}\right)^2 (2.6) + \frac{\pi^2}{(2 \times 10^{-2})^2} + j2\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7} \times 10^{-4}}$$

$$= 0.012682 + j373.57$$

$$\underline{\underline{\alpha_d = 0.012682}}\ \text{Np/m}$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(\frac{f_c}{f})^2\right]$$

$$= \frac{2x2.858x10^{-2}}{10^{-2}(233.81)\sqrt{1-(4.651\ /12)^2}}\left[\frac{1}{2}+\frac{1}{2}(\frac{4.651}{12})^2\right] = \underline{\underline{0.0153}}\ \text{Np/m}$$

(b) For $TE_{11}$ mode,

$$\alpha_d + j\beta_d = \sqrt{-\omega^2/u^2 + 1/a^2 + 1/b^2 + j\omega\mu\sigma_d}$$

$$= \sqrt{-139556.21 + \frac{\pi^2}{(10^{-2})^2} + j9.4748} = 0.02344 + j202.14$$

$$\underline{\underline{\alpha_d = 0.02344}}\ \text{Np/m}$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{(b/a)^3+1}{(b/a)^2+1}\right] = \frac{2\times 2.858\times 10^{-2}}{10^{-2}(233.81)\sqrt{1-(10.4/12)^2}}\left[\frac{(1/8)+1}{(1/4)\ \ +1}\right]$$

$$\underline{\underline{\alpha_c = 0.0441}}\ \text{Np/m}$$

**Prob.**

$$\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\frac{\sigma}{\omega}$$

Comparing this with

$$\varepsilon_c = 16\varepsilon_o(1-j10^{-4}) = 16\varepsilon_o - j16\varepsilon_o \times 10^{-4}$$

$$\varepsilon = 16\varepsilon_o, \qquad \frac{\sigma}{\omega} = 16\varepsilon_o x10^{-4}$$

For $TM_{21}$ mode,

$$f_c = \frac{u'}{2}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} = 2.0963 \text{ GHz}, \quad f = 1.1 f_c = 2.3059 \text{ GHz}$$

$$\sigma = 16\varepsilon_o \omega \times 10^{-4} = 16 \times 2\pi \times 2.3059 \times 10^9 \times \frac{10^{-9}}{36\pi} \times 10^{-4} = 2.0525 \times 10^{-4}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = 30\pi$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{4.1\times10^{-4}\times30\pi}{2\sqrt{1-1/1.12}} = \underline{\underline{0.0231}} \text{ Np/m}$$

$$E_o e^{-\alpha_d z} = 0.8E_o \qquad \longrightarrow \qquad z = \frac{1}{\alpha_d}\ln(1/0.8) = \underline{\underline{9.66 \text{ m}}}$$

**Prob.**

This is similar to the previous problem. We take similar steps except that $H_{zs} = 0$ and

$$E_{zs} = E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right).$$

Thus we obtain other field components as

$$E_{xs} = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{c}\right)E_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{c}\right)$$

$$E_{ys} = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{c}\right)E_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{c}\right)$$

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{c}\right)$$

$$\underline{\underline{H_{ys} = \frac{-j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{c}\right)}}$$

**Prob.**

For $TE_{10}$ mode,

$$f_c = \frac{u'}{2a} = \frac{3\times 10^8}{2\sqrt{2.11}\times 4.8\times 10^{-2}} = 2.151 \text{ GHz}$$

(a) loss tangent $= \dfrac{\sigma}{\omega\varepsilon} = d$

$$\sigma = d\omega\varepsilon = 3\times 10^{-4}\times 2\pi\times 4\times 10^9\times 2.11\times\frac{10^{-9}}{36\pi} = 1.4086\times 10^{-4}$$

$$\eta' = \frac{120\pi}{\sqrt{2.11}} = 259.53$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{1.4067\times 10^{-4}\times 259.53}{2\sqrt{1-(2.151/4)^2}} = \underline{\underline{2.165\times 10^{-2} \text{ Np/m}}}$$

(b) $$R_s = \sqrt{\frac{\mu f\pi}{\sigma_c}} = \sqrt{\frac{\pi\times 4\times 10^9\times 4\pi\times 10^{-7}}{4.1\times 10^7}} = 1.9625\times 10^{-2}$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(f_c/f)^2\right] = \frac{3.925\times 10^{-2}(0.5+0.5\times 0.2892)}{2.4\times 10^{-2}\times 259.53\times 0.8431}$$

$$= \underline{\underline{4.818\times 10^{-3} \text{ Np/m}}}$$

**Prob.**

$$f_r = 1.5\times10^{10}\sqrt{\left(\frac{m}{3}\right)^2+\left(\frac{n}{2}\right)^2+\left(\frac{p}{4}\right)^2}\ \text{Hz}$$

$$f_{r\,TE011} = 15\sqrt{0+\frac{1}{4}+\frac{1}{16}} = 8.385\ \text{GHz, etc.}$$

The resonant frequencies are listed below.

| modes | Resonant frequency (GHz) |
|---|---|
| $TE_{101}$ | 6.25 |
| $TE_{011}$ | 8.38 |
| $TM_{110}$ | 9.01 |
| $TM_{111}$ | 9.76 |

**Prob.**

$$f_{c10} = \frac{u'}{2a} = \frac{3\times10^8}{2\times2\times10^{-2}} = 7.5\ \text{GHz}$$

Loss in dB/m = 8.69 $\alpha_{10}$ dB/m

For $TE_{10}$ mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right] = 0.0535\ \text{Np/m} = \underline{\underline{0.4647\ \text{dB/m}}}$$

**Prob.**

$$a < b < c \longrightarrow \frac{1}{a} > \frac{1}{b} > \frac{1}{c}$$

Hence the dominant mode is $TE_{011}$.

$$f_r = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2} = \frac{3\times10^8}{2\sqrt{4.5}}\sqrt{0 + \frac{1}{5^2} + \frac{1}{10^2}}\cdot 10^2$$

$$= \underline{\underline{1.581 \text{ GHz}}}.$$

**Prob.**

$$u' = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\epsilon_r}}$$

$$f_r = \frac{3\times10^8}{2\sqrt{2.6}}\,\frac{1}{10^{-2}}\sqrt{\frac{m^2}{16} + \frac{n^2}{9} + p^2} = 9.303\sqrt{\frac{m^2}{16} + \frac{n^2}{9} + p^2} \text{ GHz}$$

For TE modes, $m = 0, 1, 2, \ldots$, $n = 0, 1, 2 \ldots$, $p = 1, 2, 3, \ldots$

(no m = 0 = n). Thus for

$TE_{011} \rightarrow f_r = 9.805$ GHz

$TE_{101} \rightarrow f_r = 9.589$ GHz

$TE_{111} \rightarrow f_r = 10.07$ GHz, $TE_{201} \rightarrow f_r = 10.4$ GHz

for TM modes, $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$, $p = 0, 1, 2 \ldots$

Hence

$TM_{110} \rightarrow f_r = 3.876$ GHz, $TM_{111} \rightarrow f_r = 10.07$ GHz,

$TM_{210} \rightarrow f_r = 5.59$ GHz, $TM_{120} \rightarrow f_r = 6.623$ GHz,

$TM_{220} \rightarrow f_r = 7.752$ GHz, $TM_{310} \rightarrow f_r = 7.635$ GHz,

$TM_{320} \rightarrow f_r = 9.335$ GHz.

The modes whose resonant frequencies do not exceed 10 GHz are 2 TE and 8 TM modes listed below.

| modes | $f_r$ (GHz) |
|---|---|
| $TM_{110}$ | 3.876 |
| $TM_{210}$ | 5.59 |
| $TM_{120}$ | 6.623 |
| $TM_{220}$ | 7.752 |
| $TM_{310}$ | 7.635 |
| $TM_{320}$ | 9.335 |
| $TE_{101}$, $TM_{130}$ | 9.589 |
| $TE_{011}$, $TM_{410}$ | 9.805 |

**Prob.**

$$f_r = \frac{u'}{2}\sqrt{1/a^2 + 1/c^2}$$

For cubical cavity, a = b = c

$$f_r = \frac{u'}{2a}\sqrt{2} \longrightarrow a = \frac{u'}{\sqrt{2}f_r} = \frac{3x10^8}{\sqrt{2}x2x10^9} = 10.61 \text{ mm}$$

$\underline{\underline{a = b = c = 1.061 \text{ cm}}}$

**Prob.**

For $TE_{10}$ mode,

$$f_{c10} = \frac{u'}{2a} = \frac{3\times10^8}{2\times1.067\times10^{-2}} = 14.058 \text{ GHz}$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$= \frac{2\sqrt{\frac{\pi f\mu}{\sigma_c}}}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right] = \underline{\underline{0.0515 \text{ Np/m}}}$$

**Prob.**

(a) If $a = b = c$, $Q_{TE101} = \frac{2a^2 \cdot a^3}{\delta(4a^4+2a^4)} = \frac{a}{3\delta}$

(b) $f_{r101} = \frac{c\sqrt{2}}{2a} = \frac{c}{a\sqrt{2}}$

$$Q_{TE101} = \frac{a}{3\delta} = \frac{a}{3}\sqrt{\pi f_{r101}\,\mu_o\,\sigma_c} = \frac{1}{3}\sqrt{a^2\pi\frac{c}{a\sqrt{2}}\,\mu_o\,\sigma_c}$$

or $a = \frac{9Q^2_{TE101}\sqrt{2}}{\pi c\,\mu_o\,\sigma_c} = \frac{9\times 36\times 10^6\sqrt{2}}{\pi\times 3\times 10^8\times 4\pi\times 10^{-7}\times 1.5\times 10^7}$

$= \underline{\underline{2.579\text{ cm}}}$.

(c) $f_{r101} = \frac{c\sqrt{2}}{2a} = \frac{3\times 10^8\sqrt{2}}{2\times 2.579\times 10^{-2}} = \underline{\underline{8.225\text{ GHz}}}$.

**Prob.**

$\alpha = k\sqrt{\dfrac{f}{1-(f_c/f)^2}}$ , where k is a constant

$$\alpha = k\frac{f^{3/2}}{\sqrt{f^2-f_c^2}}$$

$$\frac{d\alpha}{df} = k\frac{\sqrt{f^2-f_c^2}\,\frac{3}{2}f^{1/2} - f^{3/2}\frac{1}{2}2f\frac{1}{\sqrt{f^2-f_c^2}}}{f^2-f_c^2}$$

For minimum $\alpha$, $\dfrac{d\alpha}{df} = 0$ which implies that

$$(f^2-f_c^2)\cdot\frac{3}{2}f^{1/2} - f^{5/2} = 0$$

or

$$\underline{\underline{f = \sqrt{3}f_c}}$$

**Prob.**

For the TE mode to z,

$E_{zs} = 0, H_{zs} = H_o\cos(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)$

$$E_{ys} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} \;\; \underline{\underline{= -\frac{j\omega\mu}{h^2}(m\pi/a)H_o\sin(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)}}$$

as required.

$$E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} = \frac{j\omega\mu}{h^2}(n\pi / b)H_o \cos(m\pi x / a)\sin(n\pi y / b)\sin(p\pi z / c)$$

From Maxwell's equation,

$$-j\omega\mu \boldsymbol{H}_s = \nabla \times \boldsymbol{E}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & E_{ys} & 0 \end{vmatrix}$$

$$H_{xs} = \frac{1}{j\omega\mu}\frac{\partial E_{ys}}{\partial z} = \underline{\underline{-\frac{1}{h^2}(m\pi / a)(p\pi / c)H_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)}}$$

**Prob.**

Maxwell's equation can be written as

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x}$$

For a rectangular cavity,

$$h^2 = k_x{}^2 + k_y{}^2 = (m\pi / a)^2 + (n\pi / b)^2$$

For TM mode, $H_{zs} = 0$ and

$$E_{zs} = E_o \sin(m\pi x / a)\sin(n\pi y / b)\cos(p\pi z / c)$$

Thus

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} = \underline{\underline{\frac{j\omega\varepsilon}{h^2}(n\pi / b)E_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)}}$$

as required.

$$H_{xs} = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y}$$

$$= -\frac{j\omega\varepsilon}{h^2}(m\pi / a)E_o \cos(m\pi x / a)\sin(n\pi y / b)\cos(p\pi z / c)$$

From Maxwell's equation,

$$j\omega\varepsilon \boldsymbol{E}_s = \nabla \times \boldsymbol{H}_s = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_{xs} & H_{ys} & 0 \end{vmatrix}$$

$$E_{ys} = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{\partial z} \quad \underline{\underline{= \frac{1}{h^2}(n\pi / b)(p\pi / c)E_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)}}$$

**Prob.**

$$f_r = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2 + (p/c)^2}$$

where for TM mode to z, m = 1, 2, 3,…, n=1, 2, 3, …., p = 0, 1, 2, ….

and for TE mode to z, m = 0,1, 2, 3,…, n=0,1, 2, 3, …., p = 1, 2, 3, … , $(m+n) \neq 0$.

(a) If $a < b < c$, $1/a > 1/b > 1/c$,

The lowest TM mode is $TM_{110}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is $TE_{011}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{b^2} + \frac{1}{c^2}} > \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is $\underline{\underline{TE_{011}}}$.

(b) If $a > b > c$, $1/a < 1/b < 1/c$,

The lowest TM mode is $TM_{110}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is $TE_{101}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{c^2}} < \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is TM$_{110}$.

(c) If a = c > b, 1/a = 1/c < 1/b,

The lowest TM mode is TM$_{110}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$

The lowest TE mode is TE$_{101}$ with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{c^2}} < \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$

Hence the dominant mode is TE$_{101}$.

**Prob.**

b = 2a, c = 3a

$$f_r = \frac{u'}{2}\sqrt{(m/a)^2 + (n/2a)^2 + (p/3a)^2}, u' = \frac{c}{\sqrt{2.5}}$$

$$\frac{u'}{2a} = \frac{3\times10^8}{2\sqrt{2.5}\times3\times10^{-2}} = 3.162\times10^9$$

$$f_r = 3.162\sqrt{m^2 + n^2/4 + p^2/9} \text{ GHz}$$

| TE/TM | m n p | f$_r$ GHz |
|---|---|---|
| TE | 0 1 1 | 1.9003 |
| TE | 0 1 2 | 2.6352 |
| TE | 0 2 1 | 3.3333 |
| TE | 1 0 1 | |
| TM | 1 1 0 | 3.5355 |
| Both | 1 1 1 | 3.6893 |

Thus the lowest five modes have resonant frequencies at

1.9, 2.6352, 3.333, and 3.535 GHz

**Prob.**

**(a)**

$$f_r = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2 + (p/c)^2}$$

a = b = c = 3.2 cm, m=1, n=0, p=1, u' = c

$$f_r = \frac{3\times10^8}{2\times3.2\times10^{-2}}\sqrt{1^2+0^2+1^2} \quad = \underline{\underline{6.629\ \text{GHz}}}$$

(b)

$$Q = \frac{a}{3}\sqrt{\pi f_{r101}\,\mu_o\sigma_c} \quad = \frac{3.2\times10^{-2}}{3}\sqrt{\pi\times6.629\times10^9\times4\pi\times10^{-7}\times1.37\times10^7}$$

$$= \underline{\underline{6387}}$$

**Prob.**

$$f_r = \frac{c}{2a}\sqrt{m^2+n^2+p^2}$$

The lowest possible modes are TE$_{101}$, TE$_{011}$, and TM$_{110}$. Hence

$$f_r = \frac{c}{2a}\sqrt{2} \quad \longrightarrow \quad a = \frac{c}{f_r\sqrt{2}} = \frac{3\times10^8}{\sqrt{2}\times3\times10^9} = 7.071\ \text{cm}$$

$\underline{\underline{a = b\ = c = 7.071\ \text{cm}}}$

**Prob.**

(a) For a = b = c,

$$f_r = \frac{u'}{2a}\sqrt{m^2+n^2+p^2}$$

The lowest possible modes are TE$_{101}$, TE$_{011}$, and TM$_{110}$. Hence,

$$f_r = \frac{u'}{2a}\sqrt{2} \quad \longrightarrow \quad a = \frac{u'\sqrt{2}}{2f_r} = \frac{3\times10^8\sqrt{2}}{2\times6.2\times10^9} = 0.0342$$

$\underline{\underline{a = b = c = 3.42\ \text{cm}}}$

(b) $a = \frac{u'\sqrt{2}}{2f_r} = \frac{3\times10^8\sqrt{2}}{2\times\sqrt{2.25}\times6.2\times10^9} = 0.0228$

$\underline{\underline{a = b = c = 2.28 \text{ cm}}}$

**Prob.**

This is a TM mode to z. From Maxwell's equations,

$$\nabla x E_s = -j\omega\mu H_s$$

$$H_s = -\frac{1}{j\omega\mu}\nabla x E_s = \frac{j}{\omega\mu}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_{zs}(x,y) \end{vmatrix} = \frac{j}{\omega\mu}\left(\frac{\partial E_{zs}}{\partial y}a_x - \frac{\partial E_{zs}}{\partial x}a_y\right)$$

But

$$E_{zs} = 200\sin 30\pi x \sin 30\pi y, \quad \frac{1}{\omega\mu} = \frac{1}{6\times10^9\times4\pi\times10^{-7}} = \frac{10^{-2}}{24\pi}$$

$$H_s = \frac{j10^{-2}}{24\pi}\times200\times30\pi\left\{\sin 30\pi x\cos 30\pi y a_x - \cos 30\pi x \sin 30\pi y a_y\right\}$$

$$\mathbf{H} = \text{Re}\,(\mathbf{H}_s\, e^{j\omega t})$$

$$\underline{\underline{H = 2.5\left\{-\sin 30\pi x\cos 30\pi y a_x + \cos 30\pi x \sin 30\pi y a_y\right\}\sin 6\times10^9\pi t}} \quad \text{A/m}$$

**Prob.**

$$\text{n} = \text{c/u}_m = \frac{3x10^8}{2.1x10^8} = \underline{\underline{1.4286}}$$

**Prob.**

When an optical fiber is used as the transmission medium, cable radiation is eliminated. Thus, optical fibers offer total EMI isolation because they neither emit nor pick up EM waves.

**Prob.**

(a) $\mathrm{NA} = \sqrt{n_1^{\,2} - n_2^{\,2}} = \sqrt{1.62^2 - 1.604^2} = \underline{\underline{0.2271}}$

(b) $\mathrm{NA} = \sin\theta_a = 0.2271$ or $\theta_a = \sin^{-1} 0.2271 = \underline{\underline{13.13^\circ}}$

(c) $\mathrm{V} = \dfrac{\pi d}{\lambda}\,\mathrm{NA} = \dfrac{\pi\times 50\times 10^{-6}\times 0.2271}{1300\times 10^{-9}} = 27.441$

$\mathrm{N} = \mathrm{V}^2/2 = \underline{\underline{376 \text{ modes}}}$

**Prob.**

(a) $\mathrm{V} = \dfrac{\pi d}{\lambda}\sqrt{n_1^{\,2} - n_2^{\,2}} = \dfrac{\pi\times 2.5\times 10^{-6}\times 2}{1.3\times 10^{-6}}\sqrt{1.45^2 - 1^2} = \underline{\underline{12.69}}$

(b) $\mathrm{NA} = \sqrt{n_1^{\,2} - n_2^{\,2}} = \sqrt{1.45^2 - 1^2} = \underline{\underline{1.05}}$

(c ) $\mathrm{N} = \mathrm{V}^2/2 = \underline{\underline{80 \text{ modes}}}$

**Prob.**

(a) $\mathrm{NA} = \sin\theta_a = \sqrt{n_1^{\,2} - n_2^{\,2}} = \sqrt{1.53^2 - 1.45^2} = 0.4883$

$\theta_a = \sin^{-1} 0.4883 = \underline{\underline{29.23^\circ}}$

(b) $\mathrm{P(l)/P(0)} = 10^{-\alpha l/10} = 10^{-0.4\mathrm{X}5/10} = 0.631$

i.e. $\underline{\underline{63.1\ \%}}$

**Prob.**

$\mathrm{P(l)} = \mathrm{P(0)}\,\mathrm{e}^{-\alpha\ l/10} = 10\mathrm{x}\ 10^{-0.5\ \mathrm{x}\ 0.85}\ \mathrm{mW} = \underline{\underline{3.758\ \mathrm{mW}}}$

**Prob.**

**As shown in eq. (** $\log_{10} P_1/P_2 = 0.434 \ln P_1/P_2$ ,

$1 \text{ Np} = 20 \log_{10} e = 8.686 \text{ dB}$ or $1 \text{ Np/km} = 8.686 \text{ dB/km}$,

or $1\text{Np/m} = 8686 \text{ dB/km}$. Thus,

$$\alpha_{12} = \underline{\underline{8686\alpha_{10}}}$$

**Prob.**

$$P(0) = P(l)\ 10^{\alpha\, l/10} = 0.2 \text{ x } 10^{0.4 \text{ x } 30/10} \text{ mW} = \underline{\underline{3.1698 \text{ mW}}}$$

**Prob.**

See text.

## Chapter Practice Examples

### P.E.

(a) For this case, r is at near field.

$$H_{\phi s} = \frac{I_o dl \sin\theta}{4\pi}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)e^{-j\beta r}, \quad \beta r = \frac{2\pi}{\lambda}.\frac{\lambda}{5} = 72^{\circ}$$

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi \times 3\times 10^8}{10^8} = 6\pi, \qquad \beta = \frac{2\pi}{\lambda} = \frac{1}{3}$$

$$H_{\phi s} = \frac{(0.25)\dfrac{6\pi}{100}\sin 30^o}{4\pi}\left(\frac{j1/3}{6\pi/5} + \frac{1}{(6\pi/5)^2}\right)e^{-j72^o} = 0.2119\angle -20.511^o \text{mA/m}$$

$\boldsymbol{H} = \text{Im}\ \left(H_{\phi s}e^{j\omega t}\ \boldsymbol{a}_{\phi}\right)$ Im is used since $I = I_o \sin \omega t$

$$= \underline{\underline{0.2119\sin(10^8 - 20.5^o)\boldsymbol{a}_{\phi}}} \text{ mA/m}$$

(b)For this case, r is at far field. $\beta = \frac{2\pi}{\lambda}\times 200\lambda = 0^o$

$$H_{\phi s} = \frac{j(0.25)(\frac{2\pi}{\lambda})\frac{\lambda}{100} Sin60^o e^{-j0^o}}{4\pi(6\pi \times 200)} = 0.2871e^{j90^o} \quad \mu Am$$

$$\boldsymbol{H} = \text{Im}\ (H_{\phi s}\boldsymbol{a}_{\phi}e^{j\omega t}) \quad = \underline{\underline{0.2871\sin(10^8 + 90^o)\boldsymbol{a}_{\phi}}} \quad \mu Am.$$

### P.E.

(a) $l = \frac{\lambda}{4} = \underline{\underline{1.5m}},$

(b) $I_o = \underline{83.3\text{mA}}$

(c) $R_{rad} = 36.56\,\Omega$, $P_{rad} = \frac{1}{2}(0.0833)^2 36.56$

$= \underline{\underline{126.8 \text{ mW}}}.$

(d) $Z_L = 36.5 + j21.25,$

$$\Gamma = \frac{36.5 + j21.25 - 75}{36.5 + j21.25 + 75} = 0.3874\angle 140.3^o$$

$$s = \frac{1+0.3874}{1-0.3874} = \underline{\underline{2.265}}$$

**P.E.**

$$D = \frac{4\pi U_{\max}}{P_{rad}}$$

(a) For the Hertzian monopole

$$U(\theta,\phi) = \sin^2\theta\,,\ 0<\theta<\pi/2, \qquad 0<\phi<2\pi,\ \mathrm{U}_{\max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin^2 \sin\theta\, d\theta\, d\phi = \frac{4\pi}{3}$$

$$D = \frac{4\pi(1)}{4\pi/3} = \underline{\underline{3}}$$

(b) For the $\frac{\lambda}{4}$ monopole,

$$U(\theta,\phi) = \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta},\ \mathrm{U}_{\max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \sin\theta\, d\theta\, d\phi = 2\pi(0.609)$$

$$D = \frac{4\pi(1)}{2\pi(0.609)} = \underline{\underline{3.28}}$$

**P.E.**

(a) $$\text{P}_{\text{rad}} = \eta_r P_{in} = 0.95(0.4)$$

$$D = \frac{4\pi U_{\max}}{P_{rad}} = \frac{4\pi(0.5)}{0.4 \times 0.95} = \underline{\underline{16.53}}$$

(b) $$D = \frac{4\pi(0.5)}{0.3} = \underline{\underline{20.94}}$$

**P.E.**

$$P_{rad} = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin\theta \sin\theta \, d\theta \, d\phi = \frac{\pi^2}{2}, \ \text{U}_{\max} = 1$$

$$D = \frac{4\pi(1)}{\pi^2/2} = \underline{\underline{2.546}}$$

**P.E.**

(a) $$f(\theta) = |\cos\theta| \cos\left[\frac{1}{2}(\beta d \cos\theta + \alpha)\right]$$

where $\alpha = \pi$, $\beta d = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \pi$

$$\underbrace{|\cos\theta|}_{\text{unit pattern}} \ \underbrace{\cos\left[\frac{1}{2}(\pi\cos\theta + \pi)\right]}_{\text{group pattern}} = f(\theta)$$

For the group pattern, we have nulls at

$$\pi/2(\cos\theta + 1) = \pi/2 \longrightarrow \theta = \pm\pi/2$$

and maxima at

$$\frac{\pi}{2}(\cos\theta + 1) = 0 \quad\longrightarrow\quad \cos\theta = -1$$

Thus the group pattern and the resultant patterns are as shown i **Fig.**

(b) $f(\theta) = |\cos\theta|\cos\left[\frac{1}{2}(\beta d\cos\theta + \alpha)\right]$

where $\alpha = -\frac{\pi}{2}$, $\beta d = \pi/2$

$$f(\theta) = \underset{\substack{\Downarrow \\ \text{unit pattern}}}{|\cos\theta|}\;\cos\left[\frac{1}{2}\underset{\substack{\Downarrow \\ \text{group pattern}}}{\left(\frac{\pi}{2}\cos\theta - \frac{\pi}{2}\right)}\right]$$

For the group pattern, the nulls are at

$$\frac{\pi}{4}(\cos\theta - 1) = -\frac{\pi}{2} \quad\longrightarrow\quad \theta = 180^o$$

and maxima at

$$\cos\theta - 1 = 0 \quad\longrightarrow\quad \theta = 0$$

Thus the group pattern and the resultant patterns are as shown in **Fig.**

## P.E.

$$A_e = \frac{\lambda^2}{4\pi}G_d, \quad \lambda = \frac{c}{f} = \frac{3\times10^8}{10^8} = 3 \text{ m}$$

For the Hertzian dipole,

$$G_d = 1.5\sin^2\theta$$

$$A_e = \frac{\lambda^2}{4\pi}(1.5\sin^2\theta)$$

$$A_{e,\max} = \frac{1.5\lambda^2}{4\pi} = \frac{1.5\times9}{4\pi} = \underline{\underline{1.074 \text{ m}^2}}$$

By definition,

$$P_r = A_e P_{ave} \longrightarrow P_{ave} = \frac{P_r}{A_e} = \frac{3\times10^{-6}}{1.074}$$

$$= \underline{\underline{2.793\ \mu\text{W}/\text{m}^2}}$$

## P.E.

(a) $$G_d = \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 \frac{1}{2}\frac{E^2}{\eta}}{P_{rad}} = \frac{2\pi r^2 E^2}{\eta P_{rad}}$$

$$= \frac{2\pi\times400\times10^6\times144\times10^{-6}}{120\pi\times100\times10^3} = 0.0096$$

$$G = 10\log_{10} G_d = \underline{\underline{-20.18 \text{ dB}}}$$

(b) $$G = \eta_r G_d = 0.98\times0.0096 = \underline{\underline{9.408\times10^{-3}}}$$

**P.E.**

$$r = \left[ \frac{\lambda^2 G_d{}^2 \sigma}{(4\pi)^3} \frac{P_{rad}}{P_r} \right]^{1/4}$$

where $\lambda = \frac{c}{f} = \frac{3 \times 10^8}{6 \times 10^9} = 0.05m$

$$A_e = 0.7\pi a^2 = 0.7\pi(1.8)^2 = 7.125m^2$$

$$G_d = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi(7.125)}{25 \times 10^{-4}} = 3.581 \times 10^4$$

$$r = \left[ \frac{25 \times 10^{-4} \times (3.581)^2 \times 10^8 \times 5 \times 60 \times 10^3}{(4\pi)^3 \times 0.26 \times 10^{-3}} \right]^{1/4}$$

$$= 1168.4\text{m} = \underline{\underline{0.631 \text{ nm}}}$$

At $r = r_{\max}/2 = 584.2m,$

$$P = \frac{G_d P_{rad}}{4\pi r^2} = \frac{3.581 \times 10^4 \times 60 \times 10^3}{4\pi(584.2)^2} = \underline{\underline{501 \text{ W/m}^2}}$$

## CHAPTER Problems

**Prob.**

(a) $|E|_{max} \propto \frac{1}{r} \longrightarrow \frac{|E|_{max}}{5} = \frac{10}{r}$ or $|E|_{max} = \frac{50}{r}$

At $r = 20$, $|E|_{max} = \frac{10}{20}(5) = \underline{\underline{2.5 \text{ V/m}}}$.

(b) At $r = 30$, $|E|_{max} = \frac{50}{30} = 1.67$ V/m

$$|H|_{max} = \frac{|E|_{max}}{\eta} = \frac{1.67}{120\pi} = \underline{\underline{4.43 \text{ mA/m}}}$$

(c) $|E|_{max} = \frac{50}{40} = \frac{5}{4}$ V/m

$$\mathcal{P}_{ave} = \frac{|E|^2_{max}}{2\eta} = \frac{1}{240\pi} \times \left(\frac{5}{4}\right)^2 = \underline{\underline{2.072 \text{ mW/m}^2}}.$$

**Prob.**

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^7} = 30 \text{ m}$$

$$|E_{\theta s}| = \frac{\eta I_o \beta dl}{4\pi r}\sin\theta$$

At (100,0,0), r = 100m, $\theta = \frac{\pi}{2}$, $\eta = 120\pi$

$$|E_{\theta s}| = \frac{120(10)}{4(100)}\frac{2\pi}{30}(0.2)(1) = \underline{\underline{0.126}} \text{ V/m}$$

**Prob.**

**From eq. (** , $R_{rad} = \frac{2P_{rad}}{I_o^2} = \frac{2\pi\eta}{3}\left(\frac{dl}{\lambda_w}\right)^2$

where $\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{120\pi}{9}$, $\lambda_w$ = wavelength in sea water

But $\lambda_w = \frac{u}{f} = \frac{1}{f\sqrt{\mu_o\epsilon_o\mu_r\epsilon_r}} = \frac{c}{9f} = \frac{\lambda}{9}$

where $\lambda$ = wavelength in air.

$$R_{rad} = \frac{2\pi}{3}\left(\frac{120\pi}{9}\right)\left(\frac{\lambda}{15}\cdot\frac{9}{\lambda}\right)^2 = \underline{\underline{31.58\,\Omega}}.$$

**Prob.**

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{3\times10^8} = 1m \qquad \beta = \frac{2\pi}{\lambda} = 2\pi$$

$r = 10$, $\theta = 30^o$, $\phi = 90^o$

$$H_{\phi s} = \frac{j(2)(2\pi)5\times10^{-3}}{4\pi x10}\sin 30e^{-j20\pi} = \underline{\underline{j0.25}}\text{ mA/m}$$

$$\eta = 120\pi = 377\Omega$$

$$E_{\theta s} = \eta H_{\phi s} = \underline{\underline{j94.25}}\,\text{mV/m}$$

**Prob.**

**From eq. (** $, A_{zs} = \mu \frac{I_o dl e^{-j\beta r}}{4\pi r} = \frac{C_o}{r} e^{-j\beta r}$

$$(\nabla^2 + k^2) A_{zs} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[ r^2 \frac{\partial}{\partial r}\left(\frac{C_o}{r} e^{-j\beta r}\right)\right] + \frac{k^2 C_o}{r} e^{-j\beta r}$$

$$= \frac{C_o}{r^2}\frac{\partial}{\partial r}\left[-j\beta r e^{-j\beta r} - e^{-j\beta r}\right] + \frac{k^2 C_o}{r} e^{-j\beta r}$$

$$= \frac{C_o}{r} e^{-j\beta r}(-\beta^2 + k^2) = 0 \text{ if } \underline{\underline{k = \beta}}.$$

**Prob.**

(a) $f = 1.5 \text{ MHz}, \ \lambda = \frac{c}{f} = \frac{3\times10^8}{1.5\times10^6} = 200 m >> dl = 1m$

i.e. Hertzian dipole.

$$P_{rad} = 40\pi^2 \left(\frac{dl}{\lambda}\right)^2 I_o^2 = 40\left(\frac{1}{200} \times 200 \times 10^{-3} \pi\right)^2$$

$$= \underline{\underline{394.78 \ \mu W}}.$$

(b) $\lambda = \frac{3\times10^8}{150\times10^6} = 2m, \ l = \frac{\lambda}{2}$, i.e. half-wave dipole

$$P_{rad} = 36.56 I_o^2 = 36.56(0.04) = \underline{\underline{1.462 W}}.$$

**Prob.**

Using vector transformation,

$A_{rs} = A_{xs}\sin\theta\cos\phi$ , $A_{\theta s} = A_{xs}\cos\theta\cos\phi$ , $A_{\phi s} = -A_{xs}\sin\phi$

$$\boldsymbol{A}_s = \frac{50e^{-j\beta r}}{r}(\sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi)$$

$$\frac{\nabla\times\boldsymbol{A}_s}{\mu} = \boldsymbol{H}_s = \frac{100\cos\theta\sin\phi}{\mu r^2\sin\theta}e^{-j\beta r}\boldsymbol{a}_r - \frac{50}{\mu r^2}(1-j\beta r)\sin\phi e^{-j\beta r}\boldsymbol{a}_\theta$$
$$-\frac{50}{\mu r^2}\cos\theta\cos\phi(1+j\beta r)e^{-j\beta r}\boldsymbol{a}_\phi$$

At far field, only $\frac{l}{r}$ term remains. Hence

$$\boldsymbol{H}_s = \frac{j50}{\mu r}\beta e^{-j\beta r}(\sin\phi\boldsymbol{a}_\theta - \cos\theta\cos\phi\boldsymbol{a}_\phi)$$
$$\boldsymbol{E}_s = -\eta\boldsymbol{a}_r\times\boldsymbol{H}_s = \frac{-j50\beta\eta e^{-j\beta r}}{\mu r}(\sin\phi\boldsymbol{a}_\phi + \cos\theta\cos\phi\boldsymbol{a}_\theta)$$
$$\boldsymbol{H} = \mathrm{Re}\left[\boldsymbol{H}_s e^{jwt}\right] = \underline{\underline{\frac{-50}{\mu r}\beta\sin(\omega t-\beta r)}}\underline{\underline{(\sin\phi\boldsymbol{a}_\theta - \cos\theta\cos\phi\boldsymbol{a}_\phi)}}\text{ A/m}$$
$$\boldsymbol{E} = \mathrm{Re}\left[\boldsymbol{E}_s e^{jwt}\right] = \underline{\underline{\frac{50\eta\beta}{\mu r}\sin(\omega t-\beta r)}}\underline{\underline{(\sin\phi\boldsymbol{a}_\phi + \cos\theta\cos\phi\boldsymbol{a}_\theta)}}\text{ V/m}$$

**Prob.**

$$\nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} = \varepsilon\frac{\partial\boldsymbol{E}}{\partial t} \longrightarrow j\omega\varepsilon\boldsymbol{E}_s = \nabla\times\boldsymbol{H}_s$$
$$\boldsymbol{E}_s = \frac{1}{j\omega\varepsilon}\nabla\times\boldsymbol{H}_s$$

But $\boldsymbol{B}_s = \mu\boldsymbol{H}_s = \nabla\times\boldsymbol{A}_s \longrightarrow \boldsymbol{H}_s = \frac{1}{\mu}\nabla\times\boldsymbol{A}_s$

$$\boldsymbol{E}_s = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\boldsymbol{A}_s = \frac{1}{j\omega\mu\varepsilon}\left[\nabla(\nabla\cdot\boldsymbol{A}_s) - \nabla^2\boldsymbol{A}_s\right]$$

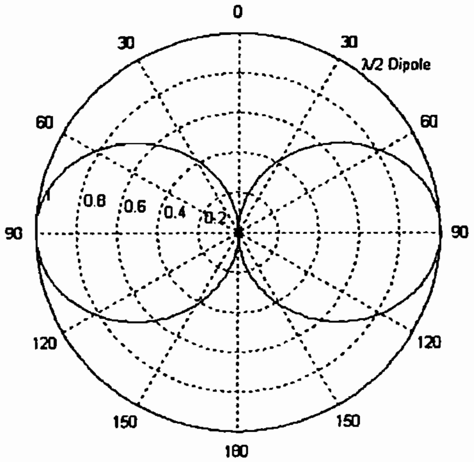
But $\nabla^2\boldsymbol{A}_s + \omega^2\mu\varepsilon\boldsymbol{A}_s = -\mu\boldsymbol{J}_s = 0 \longrightarrow \nabla^2\boldsymbol{A}_s = -\omega^2\mu\varepsilon\boldsymbol{A}_s$

$$\boldsymbol{E}_s = \frac{1}{j\omega\mu\varepsilon}\omega^2\mu\varepsilon\boldsymbol{A}_s + \frac{\nabla(\nabla\bullet\boldsymbol{A}_s)}{j\omega\mu\varepsilon} = -j\omega\boldsymbol{A}_s + \frac{\nabla(\nabla\bullet\boldsymbol{A}_s)}{j\omega\mu\varepsilon}$$

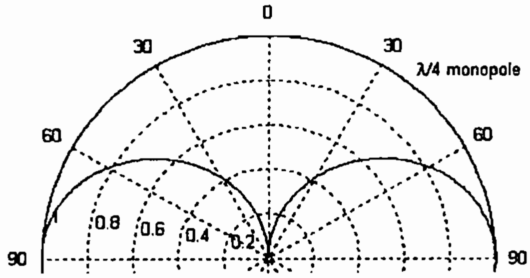
as required.

**Prob.**

(a) $$f(\theta) = \left| \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right|$$



(b) The same as for $\frac{\lambda}{2}$ dipole except that the fields are zero for $\theta \rangle \frac{\pi}{2}$ as shown.



**Prob.**

(a) $I_o = 60, \quad \theta = \pi/2, \quad \lambda = \frac{3 \times 10^8}{2 \times 10^6} = 150m, \quad \beta = \frac{2\pi}{\lambda} = \frac{\pi}{75},$

$\beta r = 4\pi/3.$

$$E_{\theta s} = \frac{j120\pi \times 60 e^{-j4\pi/3} \cos 0}{2\pi\, 100 \sin \pi/2} = j36 e^{-j4\pi/3}$$

$$= \underline{\underline{-31.2 - j18 \ \text{V/m}}}.$$

(b) $\beta r = \pi$, $H_{\phi s} = \dfrac{j60 e^{-j\pi}}{2\pi(300)} = \underline{\underline{j31.83 \ \text{mA/m}}}.$

**Prob.**

This is a monopole antenna

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1.5 \times 10^6} = 200$$

$l \langle\langle \lambda$ ,hence it is a Hertzian monopole.

$$R_{rad} = \frac{1}{2} 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 = 40\pi^2 \left(\frac{1}{200}\right)^2 = 9.87 \text{ m}\Omega$$

$$P_{rad} = P_t = \frac{1}{2} I_o^2 R_{rad}$$

$$I_o^2 = \frac{2P_t}{R_{rad}} = \frac{8}{9.87 \times 10^{-3}} = 810.54$$

$$\underline{\underline{I_o = 28.47 A}}$$

**Prob.**

$I_o = 5$, $w = 8\pi \times 10^8 \longrightarrow f = 400 \text{ mHz}$,

$\lambda = \frac{c}{f} = \frac{3\times 10^8}{4\times 10^8} = 0.75 m$, $\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \pi/2$.

from the previous problem,

$$E_{\theta s} = \frac{j\eta_o I_o e^{-j\beta r}}{2\pi r} f(\theta), \text{ where } f(\theta) = \frac{\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\frac{\beta l}{2}}{\sin\theta}$$

(a) $r = 10 m$, $\theta = 30°$,

$$f(\theta) = \frac{\cos\left(\frac{\pi}{4}\cos 30°\right) - \cos\pi/4}{\sin 30°} = 0.1408$$

$$E_{\theta s} = \frac{j 60 \times 5 e^{-j\frac{2\pi \times 10}{0.75}}}{10}(0.1408) = \underline{\underline{4.224 \angle -30° \text{ V/m}}}.$$

(b) $r = 100$, $\theta = 60°$, $f(\theta) = \frac{\cos(\pi/4 \cos 60°) - \cos\pi/4}{\sin 60°} = 0.2503$

$$E_{\theta s} = \frac{j 60 \times 5 e^{-j\frac{2\pi}{0.75}\times 100}}{100}(0.2503) = \underline{\underline{0.751 \angle -30° \text{ V/m}}}.$$

**Prob.**

Let us model this as a short Hertzian dipole.

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 80\pi^2(1/8)^2 = \underline{\underline{12.34\ \Omega}}$$

**Prob.**

$\partial l = 5m$, $\lambda = \frac{c}{f} = \frac{3\times 10^8}{3\times 10^6} = 100$

$\frac{\partial l}{\lambda} = \frac{5}{100} = \frac{1}{20}\langle\frac{1}{10}$

$$R_{rad} = 80\pi^2\left(\frac{\partial l}{\lambda}\right)^2 = \frac{80\pi^2}{400} = \underline{\underline{1.974\Omega}}$$

**Prob.**

**Change the limits in eq.(** ) to $\pm \frac{l}{2}$ i.e.

$$A_s = \frac{\mu I_o e^{j\beta z Cos\theta}}{4\pi r} \frac{(j\beta \cos\theta \cos\beta t + \beta \sin\beta t)}{-\beta^2 \cos^2\theta + \beta^2} \Big|_{-l/2}^{l/2}$$

$$= \frac{\mu I_o e^{j\beta r}}{2\pi r} \frac{1}{\beta \sin^2\theta} \left[ \sin\frac{\beta l}{2} \cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta \cos\frac{\beta l}{2} \sin\left(\frac{\beta l}{2}\cos\theta\right) \right]$$

But $\boldsymbol{B} = \mu \boldsymbol{H} = \nabla \times \boldsymbol{A}$

$$H_{\phi s} = \frac{1}{\mu r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right],$$

where $A_o = -A_z \sin\theta, A_r = A_z \cos\theta$

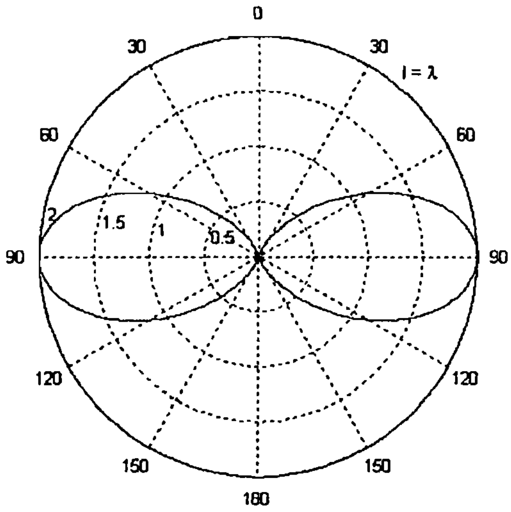
$$H_{\phi s} = \frac{I_o}{2\pi r} \frac{e^{-j\beta r}}{\beta}\left(\frac{j\beta}{\sin\theta}\right)\left[ \sin\frac{\beta l}{2} \cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta \cos\frac{\beta l}{2} \sin\left(\frac{\beta l}{2}\cos\theta\right) \right] + \frac{I_o}{2\pi r^2} e^{-j\beta r}(......)$$

For far field, only the $\frac{1}{r}$-term remains. Hence

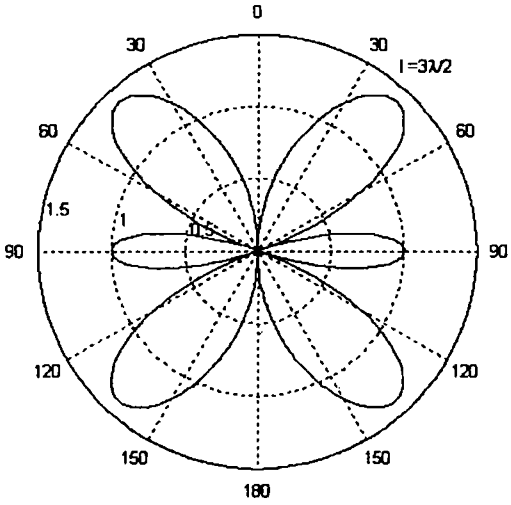
$$H_{\phi s} = \frac{jI_o}{2\pi r} e^{-j\beta r} \frac{\left[ \sin\frac{\beta l}{2} \cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\theta \cos\frac{\beta l}{2} \sin\left(\frac{\beta l}{2}\cos\theta\right) \right]}{\sin\theta}$$

(b) $f(\theta) = \dfrac{\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\frac{\beta l}{2}}{\sin\theta}$

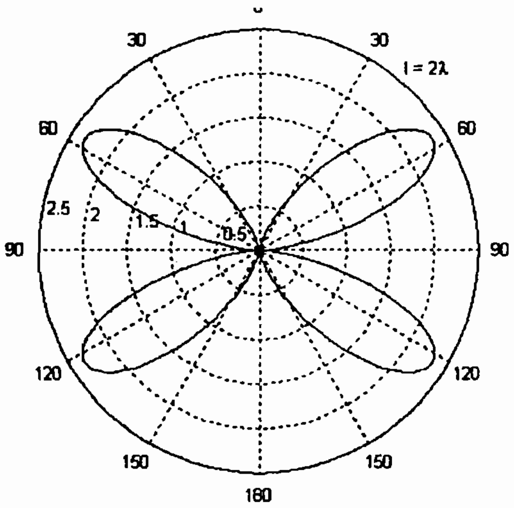
For $l = \lambda, f(\theta) = \dfrac{\cos(\pi \cos\theta) + 1}{\sin\theta}$



For $l = \frac{3\lambda}{2}, f(\theta) = \frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}$



For $l = 2\lambda,\ f(\theta) = \frac{\cos\theta\sin\left(2\pi\cos\theta\right)}{\sin\theta}$



**Prob.**

$Z_{in} = 73 + j42.5$

$$\Gamma = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{23 + j42.5}{123 + j42.5} = \underline{\underline{0.3713\angle 42.52^o}}$$

$$s = \frac{1 + |r|}{1 - |r|} = \frac{1.3713}{1 - 0.3713} = \underline{\underline{2.181}}$$

**Prob.**

$$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4}, \quad S = \pi \rho_o^2 N$$

$$= 320\pi^6 \left(\frac{\rho_o}{\lambda}\right)^4 N^2$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{10^8} = 3, \quad \rho = 0.1m, \quad \frac{\rho_o}{\lambda} = \frac{1}{30}.$$

(a) If $N = 1$, $R_{rad} = 320\pi^6 \dfrac{1}{81\times10^4} = \underline{\underline{0.3798\,\Omega}}.$

(b) If $N = 20$, $R_{rad} = 0.3798 \times 400 = \underline{\underline{152\,\Omega}}.$

**Prob.**

(a) $$\lambda = \frac{c}{f} = \frac{3\times10^8}{100\times10^6} = 30 \text{ m}$$

$$E_{\max} = \frac{\eta\pi I_o S}{r\lambda^2} \xrightarrow{\qquad} I_o = \frac{E_{\max} r\lambda^2}{\eta\pi S}$$

$$I_o = \frac{50\times10^{-3}\times3\times30^2}{120\eta^2\pi(0.2)^2 100} = \underline{\underline{9.071 \text{ mA}}} \quad (\text{S}=\text{N}\pi r^2)$$

(b) $$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4} = \frac{320\pi^4\pi^2(0.2)^4\times10^4}{30^4} = 6.077\Omega$$

$$P_{rad} = \frac{1}{2}I_o^{\,2}R_{rad} = \frac{1}{2}(9.071)^2\times10^{-6}\times6.077$$

$$= \underline{\underline{0.25 \text{ mW}}}$$

**Prob.**

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{80\times10^6} = 3.75 \text{ m}$$

$$S = N\pi\rho_o^2$$

$$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4} = \frac{320\pi^4 N^2\pi^2\rho_o^4}{\lambda^4} \longrightarrow N^2 = \frac{\lambda^4 R_{rad}}{320\pi^6(1.2\times10^{-2})^4}$$

$$N^2 = \frac{(3.75)^4\times8}{320\pi^6(1.2\times10^{-2})^4} = 248006 \longrightarrow N \simeq \underline{\underline{498}}$$

**Prob.**

(a) Let $\boldsymbol{H}_s = \dfrac{\cos 2\theta}{\eta_o r} e^{-j\beta r} \boldsymbol{a}_H$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \quad \longrightarrow \quad \boldsymbol{a}_\theta \times \boldsymbol{a}_H = \boldsymbol{a}_r \quad \longrightarrow \quad \boldsymbol{a}_H = \boldsymbol{a}_\phi$$

$$\underline{\underline{\boldsymbol{H}_s = \frac{\cos 2\theta}{120\pi r} e^{-j\beta r} \boldsymbol{a}_\phi}}$$

(b) $\boldsymbol{P}_{ave} = \dfrac{|E_s|^2}{2\eta} \boldsymbol{a}_r = \dfrac{\cos^2(2\theta)}{2\eta r^2} \boldsymbol{a}_r$

$$P_{rad} = \frac{1}{2\eta} \iint \frac{\cos^2 2\theta}{r^2} r^2 \sin\theta d\theta d\phi = \frac{1}{240\pi}(2\pi) \int_0^\pi \cos^2 2\theta \sin\theta d\theta$$

But $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$$P_{rad} = -\frac{1}{120} \int_0^\pi (2\cos^2\theta - 1)^2 d(\cos\theta)$$

$$= -\frac{1}{120} \int_0^\pi (4\cos^4\theta - 4\cos^2\theta + 1) d(\cos\theta)$$

$$= -\frac{1}{120} \left( \frac{4\cos^5\theta}{5} - \frac{4\cos^3\theta}{3} + \cos\theta \right) \Bigg|_0^\pi$$

$$= -\frac{1}{120}[-\frac{4}{5} + \frac{4}{3} - 1 - \frac{4}{5} + \frac{4}{3} - 1] = \frac{1}{120}(\frac{14}{15})$$

$$= \underline{\underline{7.778 \text{ mW}}}$$

(c )

$$P_{rad} = -\frac{1}{120} \int_{60^o}^{120^o} (2\cos^2\theta - 1)^2 d(\cos\theta)$$

$$= -\frac{1}{120} \left( \frac{4\cos^5\theta}{5} - \frac{4\cos^3\theta}{3} + \cos\theta \right) \Bigg|_{60^o}^{120^o}$$

$$= -\frac{1}{120}[\frac{4}{5}(-\frac{1}{32}) - \frac{4}{3}(-\frac{1}{8}) - \frac{1}{2} - \frac{4}{5}(\frac{1}{32}) + \frac{4}{3}(\frac{1}{8}) - \frac{1}{2}] = \frac{1}{60}[\frac{1}{40} + \frac{1}{2} - \frac{1}{6}]$$

$$= 5.972 \text{ mW}$$

which is $\dfrac{5.972}{7.778} = 0.7678$ or $\underline{\underline{76.78\%}}$

**Prob.**

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(a) $G_d = \dfrac{U(\theta,\phi)}{U_{ave}}$

$H_{\phi s} = j\dfrac{I_o \beta\, dl}{4\pi r} \sin\theta\, e^{-j\beta r}$

$P_{ave} - \left|\frac{1}{2} Re\left[\vec{E}_s \times \vec{H}_s^*\right]\right| = \frac{1}{2}\eta\, |H_{\phi s}|^2 = \frac{k}{r^2} \sin^2\theta$

$U(\theta,\phi) = r^2 P_{ave} = k\sin^2\theta$

$U_{ave} = \frac{1}{4\pi}\int U\, d\Omega = \frac{k}{4\pi}\iint \sin^3\theta\, d\theta\, d\phi$

$= \frac{k}{4\pi}(2\pi)\left(\frac{4}{3}\right) = \frac{2k}{3}$

$G_d = \dfrac{k\sin^2\theta}{2k/3} = 1.5\sin^2\theta$

$G_d(\theta = 40^\circ) = \underline{\underline{0.6198}}$

$G_d(\theta = 60^\circ) = \underline{\underline{1.125}}$

(b) for a half-wave dipole,

$G_d = \dfrac{1.64\cos^2(\pi/2 \cos\theta)}{\sin^2\theta}$

$G_d(\theta = 40^\circ) = \underline{\underline{0.5124}}$.

$G_d(\theta = 60^\circ) = \underline{\underline{1.5462}}$.

(c) The same as for Hertzian dipole,

$G_d(\theta = 40^\circ) = \underline{\underline{0.6198}}$

$G_d(\theta = 60^\circ) = \underline{\underline{1.125}}$.

**Prob.**

(a) $$E_{\theta s} = \frac{j\eta I_o \beta\, dl}{4\pi r} \sin\theta\, e^{-j\beta r}$$

$$R_{rad} = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2$$

$$G_d = \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 . \frac{1}{2\eta} |E_{\theta s}|^2}{\frac{1}{2} I_o^{\,2} R_{rad}}$$

$$= \frac{4\pi r^2}{I_o^2} . \frac{1}{80\pi^2} \left(\frac{\lambda}{dl}\right)^2 . \frac{1}{\eta} \frac{\eta^2 I_o^{\,2} \beta^2 (dl)^2 \sin^2\theta}{16\pi^2 r^2}$$

$$G_d = \underline{\underline{1.5 \sin^2\theta}}$$

(b) $$D = G_{d,\max} = \underline{\underline{1.5}}$$

(c) $$A_e = \frac{\lambda^2}{4\pi} G_d = \underline{\underline{\frac{1.5\lambda^2 \sin^2\theta}{4\pi}}}$$

(d) $$R_{rad} = 80\pi^2 \left(\frac{1}{16}\right)^2 = \underline{\underline{3.084\ \Omega}}$$

**Prob.**

$$R_{dc} = \frac{l}{\sigma S}, \quad S = \pi a^2$$

$$R_{dc} = \frac{l}{\sigma \pi a^2}$$

$$R_l = R_{ac} = \frac{a}{2\delta} R_{dc} = \frac{a}{2\delta} \frac{l}{\sigma \pi a^2}$$

$$\text{Now } \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 15 \times 10^6 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 1.7 \times 10^{-5}\text{m}$$

Alternatively, since $\delta \langle\langle a$, current is confined to a cylindrical shell of thickness $\delta$. Hence

$$R_l = R_{ac} = \frac{l}{\sigma(2\pi a)\delta}$$

$$l = \frac{\lambda}{2} = \frac{c}{2f} = \frac{3 \times 10^8}{2 \times 15 \times 10^6} = 10 \text{ m}$$

Substituting yields

$$R_l = 1.2371\Omega$$

$$R_{rad} = 73\Omega$$

$$\eta_r = \frac{R_{rad}}{R_{rad} + R_l} = \frac{73}{73 + 1.2371} = \underline{\underline{98.33\%}}$$

**Prob.**

(a) $P_{rad} = \int P_{rad} . \partial s = P_{ave} . 2\pi r^2$ (hemisphere)

$$P_{ave} = \frac{P_{rad}}{2\pi r^2} = \frac{200 \times 10^3}{2\pi(2500 \times 10^6)} = 12.73\mu W / m^2$$

$$\boldsymbol{P}_{ave} = \underline{\underline{12.73\boldsymbol{a}_r \ \mu\text{W/m}^2}}.$$

(b) $$P_{ave} = \frac{(E_{\max})^2}{2\eta}$$

$$E_{\max} = \sqrt{2\eta P_{ave}} = \sqrt{240\pi \times 12.73 \times 10^{-6}}$$

$$= \underline{\underline{0.098 \text{ V/m}}}$$

**Prob.**

$$G_d = \frac{u}{u_{ave}} = \frac{4\pi r^2 \mathcal{P}_{ave}}{\int \vec{\mathcal{P}}_{ave} \cdot d\vec{S}} = \frac{8\pi \sin\theta \cos\phi}{\int \vec{\mathcal{P}}_{ave} \cdot d\vec{S}} .$$

But $\int \vec{\mathcal{P}}_{ave} \cdot d\vec{S} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi/2} 2\sin\theta \cos\phi \sin\theta \, d\theta \, d\phi$

$$= 2\int_0^{\pi/2} \cos\phi \, d\phi \int_0^{\pi} \sin^2\theta \, d\theta = 2\sin\phi \Big|_0^{\pi/2} (\pi/2) = \pi$$

$$G_d = \underline{\underline{8 \sin\theta \cos\phi}} \cdot , \; D = G_{d,max} = 8$$

**Prob.**

$$P_{rad} = \frac{I_o^2 \eta \beta^2}{32\pi^2} (2\pi) \frac{4}{3} = \frac{I_o^2 \eta \beta^2 (dl)^2}{12\pi}$$

$$P_{ave} = \frac{I_o^2 \eta \beta^2 (dl)^2 \sin^2\theta}{32\pi^2 r^2}$$

$$\frac{P_{ave}}{P_{rad}} = \frac{\sin^2\theta}{32\pi^2 r^2} 12\pi = \frac{1.5 \sin^2\theta}{4\pi r^2}$$

$$P_{ave} = \frac{1.5 \sin^2\theta}{4\pi r^2} P_{rad}$$

**Prob.**

(a) $U_{max} = 1$

$$U_{ave} = \frac{P_{rad}}{4\pi} = \frac{\int u d\Omega}{4\pi}$$

$$= \frac{1}{4\pi} \int\int \sin^2 2\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi}(2\pi)\int_0^{\pi}(2\sin\theta\cos\theta)^2 d(-\cos\theta)$$

$$= 2\int_0^{\pi}\left(\cos^4\theta - \cos^2\theta\right)d(\cos\theta)$$

$$= 2\left[\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right]\Bigg|_0^{\pi}$$

$$= 2\left[-\frac{2}{5} + \frac{2}{3}\right] = \frac{8}{15}$$

$U_{ave} = \underline{\underline{0.5333}}$

$$D = \frac{U_{\max}}{U_{ave}} = \underline{\underline{1.875}}$$

(b) $U_{\max} = 4$

$$U_{ave} = \frac{1}{4\pi}\int u d\Omega = \frac{4}{4\pi}\int\int\frac{\sin\theta}{\sin^2\theta}d\theta d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}d\phi\int_{\pi/3}^{\pi/2}\cos ec\theta d\theta = \frac{\pi}{\pi}\ln\sqrt{3}$$

$U_{ave} = \underline{\underline{0.5493}}$

$$D = \frac{U_{\max}}{U_{ave}} = \frac{16}{3\ln\sqrt{3}} = \underline{\underline{9.7092}}$$

(c ) $U_{\max} = 2$

$$U_{ave} = \frac{1}{4\pi}\int u d\Omega = \frac{1}{4\pi}\int\int 2\sin^2\theta\sin^2\phi\sin\theta d\theta d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\sin^2\phi d\phi\int_0^{\pi}\left(1 - \cos^2\theta\right)d(-\cos\theta)$$

$$= \frac{1}{2\pi}\cdot\frac{\pi}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Bigg|_0^{\pi} = \frac{1}{4}\left[-\frac{2}{3} + 2\right] = \frac{1}{3}$$

$U_{ave} = \underline{\underline{0.333}}$

$$D = \frac{U_{\max}}{U_{ave}} = \underline{\underline{6}}$$

**Prob.**

(a) $$U_{ave} = \frac{1}{4\pi}\int u d\Omega$$

$$= \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \sin^2\theta \sin\theta d\theta d\phi$$

$$= \frac{1}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Bigg|_0^{\pi} = \frac{1}{2}\left(-\frac{2}{3}+2\right) = \frac{1}{2}.\frac{4}{3} = \frac{2}{3}$$

$$U_{ave} = 0.6667$$

$$G_{\phi} = \frac{U}{U_{ave}} = \underline{\underline{1.5 Sin^2\theta}}$$

$$D = G_{\phi,\max} = \underline{\underline{1.5}}$$

(b) $$U_{ave} = \frac{1}{4\pi}\iint 4\sin^2\theta\cos^2\phi \sin\theta d\theta d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\cos^2\phi d\phi \int_0^{\pi}\left(1-\cos^2\theta\right)d(-\cos\theta)$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\left(1+\cos 2\phi\right)d\phi\left(\frac{\cos^3\theta}{3}-\cos\theta\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\phi+\frac{sin2\phi}{2}\right)\Bigg|_0^{\pi}\left(\frac{4}{3}\right)$$

$$= \frac{1}{2\pi}(\pi)\left(\frac{4}{3}\right) = \frac{2}{3}$$

$$U_{ave} = 0.6667$$

$$G_{d,\max} = \frac{U}{U_{ave}} = \underline{\underline{6\sin^2\theta\cos^2\phi}}$$

$$D = G_{d,\max} = \underline{\underline{6}}$$

(c) $$U_{ave} = \frac{10}{4\pi}\iint \cos^2\theta \sin^2\frac{\phi}{2} \sin\theta \, d\theta \, d\phi$$

$$= \frac{10}{4\pi}\int_0^{\pi/2} \sin^2\frac{\phi}{2} d\phi \int_0^{\pi} \left(\cos^2\theta\right) d(-\cos\theta)$$

$$= \frac{10}{4\pi}\int_0^{\pi/2} \frac{1}{2}\left(1 - \cos\phi\right) d\phi \left(-\frac{\cos^3\theta}{3}\right)\Bigg|_0^{\pi}$$

$$= \frac{10}{4\pi}\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\phi + \sin^2\phi\right)\Big|_0^{\pi} = \frac{10}{12\pi}\left(\frac{\pi}{2} - 1\right)$$

$U_{ave} = 0.1514$, $\qquad U_{\max} = 5$

$$G_{d,\max} = \frac{U_{\max}}{U_{ave}} = \underline{\underline{33.02\cos^2\theta \sin^2\frac{\phi}{2}}}$$

$$D = G_{d,\max} = \underline{\underline{33.02}}$$

**Prob.**

$$\boldsymbol{P}_{ave} = \frac{|E_r|^2}{2\eta}\boldsymbol{a}_r = \frac{I_o^2 \sin^2\theta}{2\eta r^2}\boldsymbol{a}_r$$

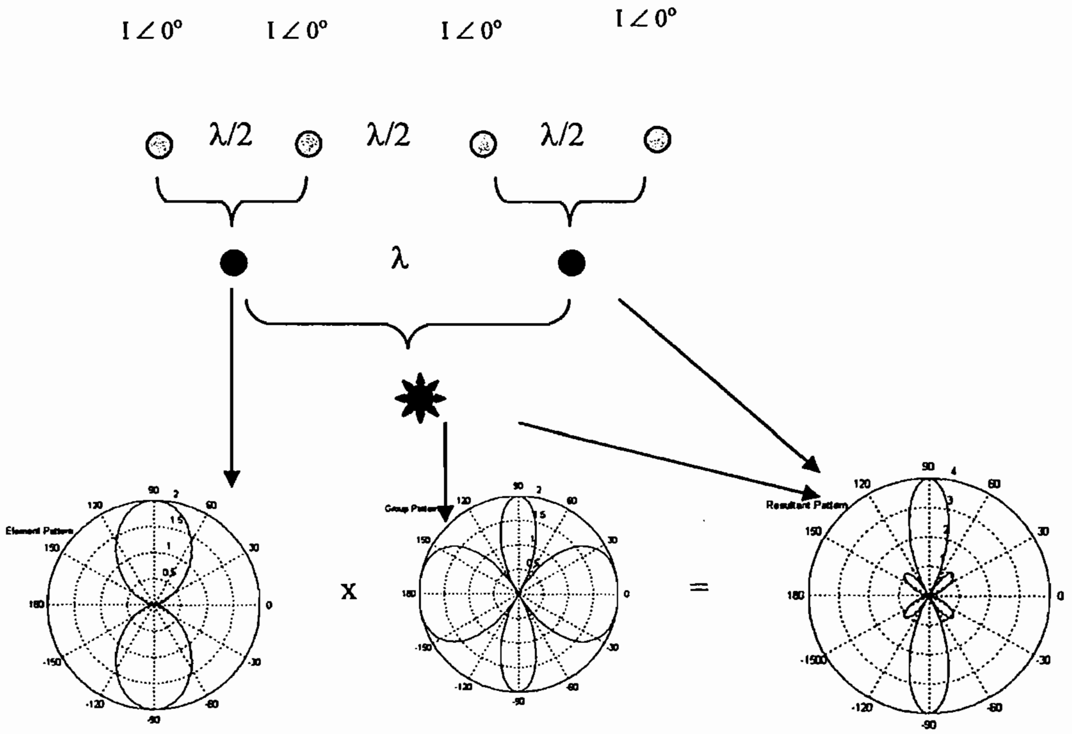
$$P_{rad} = \frac{I_o^2}{2\eta}\iint \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi = \frac{I_o^2}{240\pi}(2\pi)\int_0^{2\pi}(1 - \cos^2\theta) d(-\cos\theta)$$

$$= \frac{I_o^2}{120}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Bigg|_0^{\pi} = \frac{I_o^2}{120}(-1/3 + 1 - 1/3 + 1) = \frac{I_o^2}{90}$$

$$I_o^2 = 90 P_{ave} = 90 \times 50 \times 10^{-3} \longrightarrow I_o = \underline{\underline{2.121 \text{ A}}}$$

**Prob.**

(a) The resultant pattern is obtained as follows.



(b) The array is replaced by by + $\frac{\lambda}{4}$ +

∠0° ∠π /2

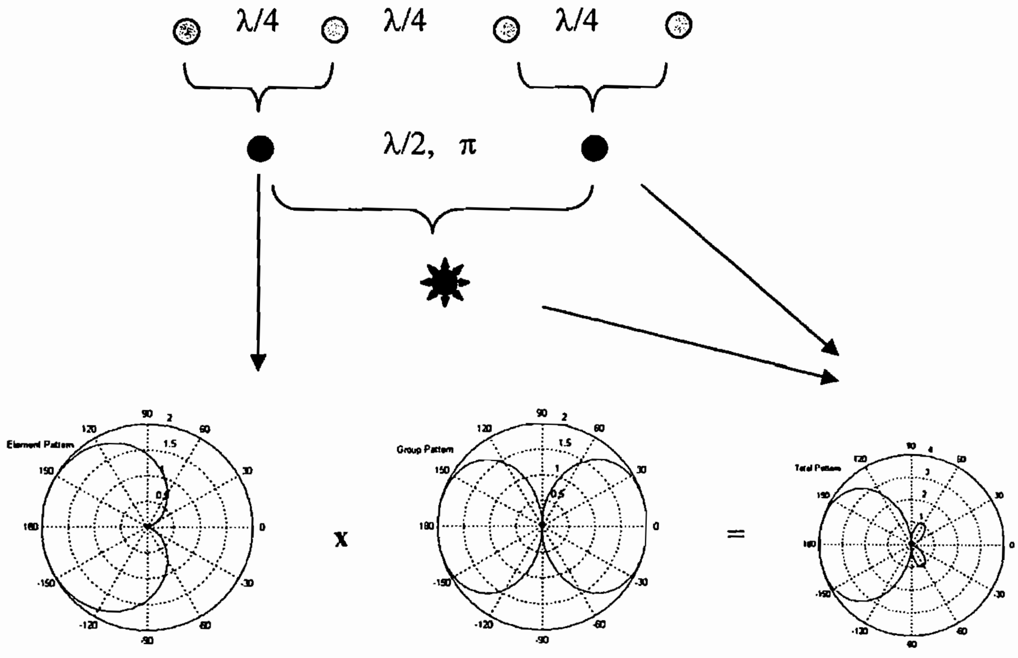
where + stands for ⟷

∠0° ∠π

Thus the resultant pattern is obtained as shown.

I∠0° I∠90° I∠180° I∠270°

=



**Prob.**

**This is similar to Fig. (** except that the elements are z-directed.

$$\boldsymbol{E}_s=\boldsymbol{E}_{s1}+\boldsymbol{E}_{s2}=\frac{j\eta\beta I_o dl}{4\pi}\left[\sin\theta_1\frac{e^{-j\beta r_1}}{r_1}\boldsymbol{a}_{\theta 1}+\sin\theta_2\frac{e^{-j\beta r_2}}{r_2}\boldsymbol{a}_{\theta 2}\right]$$

where $r_1\cong r-\frac{d}{2}\cos\theta,\qquad r_2\cong r+\frac{d}{2}\cos\theta,\qquad \theta_1\cong\theta_2\cong\theta,\qquad \boldsymbol{a}_{\theta 1}\cong\boldsymbol{a}_{\theta 2}=\boldsymbol{a}_\theta$

$$\boldsymbol{E}_s=\frac{j\eta\beta I_o dl}{4\pi}\sin\theta\,\boldsymbol{a}_\theta\left[e^{j\beta d\cos\theta/2}+e^{-j\beta d\cos\theta/2}\right]$$

$$\underline{\underline{\boldsymbol{E}_s=\frac{j\eta\beta I_o dl}{2\pi}\sin\theta\ \cos(\frac{1}{2}\beta d\cos\theta)\boldsymbol{a}_\theta}}$$

**Prob.**

$$P_{ave} = \frac{|E_s|^2}{2\eta} a_r = \frac{25 \sin^2 2\theta}{2\eta r^2} a_r$$

$$P_{rad} = \frac{25}{2\eta} \iint (2\sin\theta \cos\theta)^2 \sin\theta \, d\theta \, d\phi$$

$$P_{rad} = \frac{25}{240\pi}(2\pi) \int_0^\pi 4\sin^2\theta \cos^2\theta \, d(-\cos\theta)$$

$$= \frac{25}{120} \int_0^\pi (\cos^4\theta - \cos^2\theta) d(-\cos\theta)$$

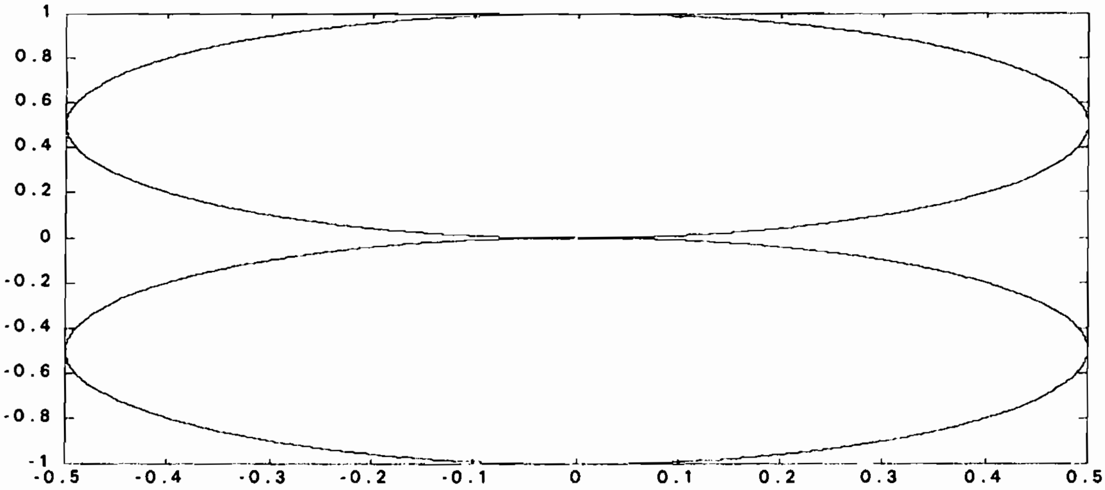
$$= \frac{25}{120} \left( \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} \right) \Bigg|_0^\pi = \frac{25}{120} \left( -\frac{2}{5} + \frac{2}{3} \right)$$

$$P_{rad} = \underline{\underline{55.55 \text{ mW}}}$$

**Prob.**

$f(\theta) = |\cos\theta \cos\phi|$

For the vertical pattern, $\phi = 0 \longrightarrow$ $f(\theta) = |\cos\theta|$ which is sketched below.



**Prob.**

(a) $$P_{rad} = \int P_{ave}.dS = \frac{1}{2\eta}\int \left|E_{\phi s}\right|^2 \partial S$$

$$= \frac{0.04}{16\pi^2}\left(\frac{1}{2\pi}\right)\int\int \frac{\cos^4\theta}{r^2} r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{0.04}{16\pi^2}\left(\frac{1}{240\pi}\right)(2\pi)\int_0^{\pi} \cos\theta\, d(-\cos\theta).10^6$$

$$= \frac{0.04}{16\pi^2}\frac{10^6}{120}\left(-\frac{\cos^5\theta}{5}\right)\Bigg|_0^{\pi} = \frac{10^4}{480\pi^2}.\frac{2}{5}$$

$$P_{rad} = \underline{\underline{0.8443 \text{ W}}}$$

(b) $$G_d = \frac{4\pi U(\theta,\phi)}{P_{rad}} = \frac{4\pi r^2 P_{ave}}{P_{rad}}$$

$$= 4\pi r^2.\frac{0.04\cos^4\theta}{16\pi^2 r^2}.\frac{10^6}{240\pi}.\frac{12\pi^2}{100}$$

$$G_d = 5\cos^4\theta$$

Since $\cos 60^o = \frac{1}{2}$,

$$G_d = 5\left(\frac{1}{2}\right)^4 = \underline{\underline{0.625}}$$

**Prob.**

(a) $\text{AF} = 2\cos\left[\frac{1}{2}\left(\beta d\cos\theta + \alpha\right)\right], \quad \alpha = 0, \quad \beta d = \frac{2\pi}{\lambda}\lambda = 2\pi$

$\text{AF} = \underline{\underline{2\cos(\pi\cos\theta)}}$

(b)Nulls occur when

$\cos(\pi\cos\theta) = 0 \longrightarrow \pi\cos\theta = \pm\pi/2, \pm 3\pi/2, \ldots$

or

$$\theta = \underline{\underline{60^o, 120^o}}$$

(c) Maxima and minima occur when

$\frac{df}{d\theta} = 0 \longrightarrow \sin(\pi\cos\theta)\pi\sin\theta = 0$

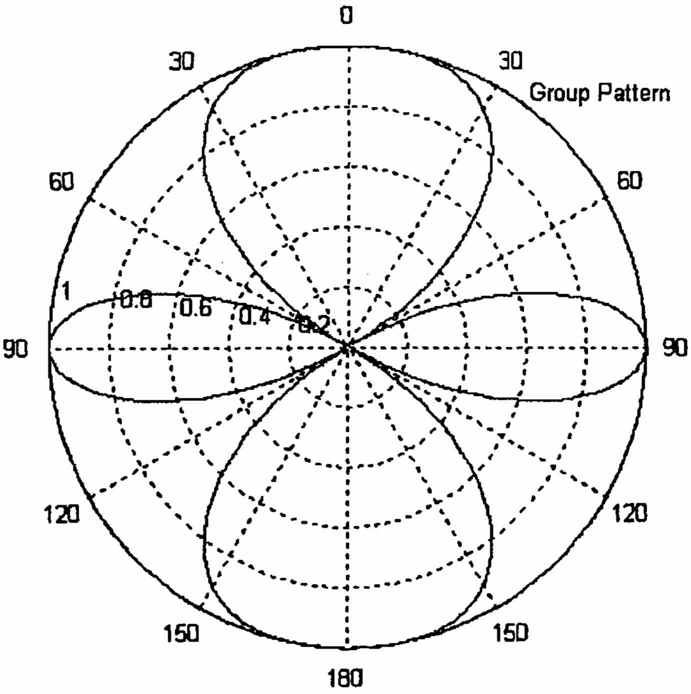
i.e. $\sin\theta = 0 \longrightarrow \theta = 0^o, 180^o$

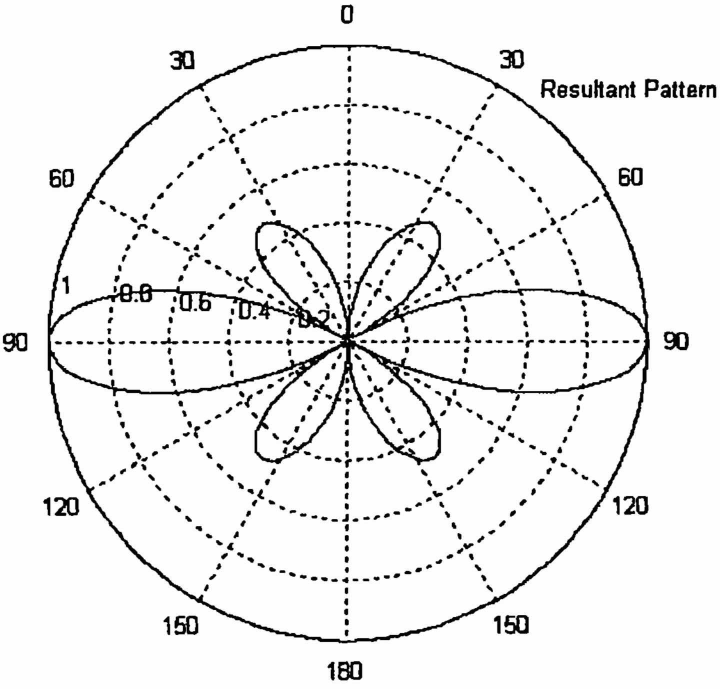
$\cos\theta = 0 \longrightarrow \theta = 90^o$

or

$$\theta = \underline{\underline{0^o, 90^o, 180^o}}$$

(d ) The group pattern is sketched below.


0
30
30
Group Pattern
60
60
1
0.8
0.6
0.4
0.2
90
90
120
120
150
150
180


0
30
30
Resultant Pattern
60
60
1
0.8
0.6
0.4
0.2
90
90
120
120
150
150
180

**Prob.**

(a) The group pattern is

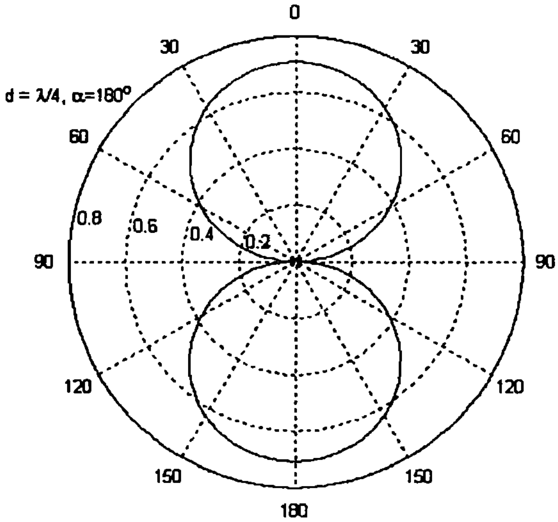
$$f(\theta) = \cos\left[\frac{1}{2}(\beta d\cos\theta + \alpha)\right]$$

$$f(\theta) = \cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\cos\theta + \pi\right)\right]$$

$$= \cos\left(\frac{\pi}{4}\cos\theta + \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{4}\cos\theta\right)$$

Maxima occur when $\theta = 0, \pi$ and nulls occur for $\theta = \pm\pi/2$

The group pattern is shown below.

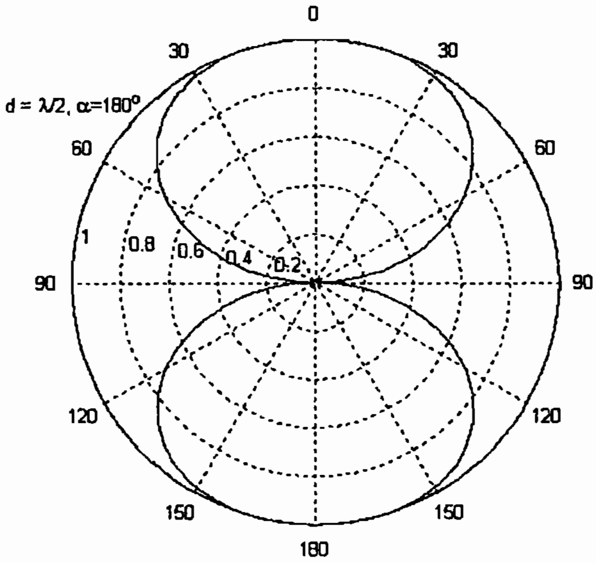


(b) For $d = \frac{\lambda}{2}, f(\theta) = \cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}.\frac{\lambda}{2}\cos\theta + \pi\right)\right]$

$$= \cos\left(\frac{\pi}{2}\cos\theta + \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\cos\theta\right)$$

Maxima occur when $\theta = 0, \pi$ and nulls occur for $\theta = \pm\pi/2$

The group pattern is sketched below.



**Prob.**

$$A_e = \frac{P_r}{P_{ave}} = \frac{P_r}{|E_r|^2 \big/ 2\eta} = \frac{2\eta P_r}{|E_r|^2}$$

$$= \frac{2 \times 120\pi \times 2 \times 10^{-6}}{25 \times 10^2 \times 10^{-6}} = \frac{48\pi}{250} = \underline{\underline{0.6031}}$$

**Prob.**

$$A_e = \frac{\lambda^2}{4\pi} G_d$$

where $G_d = \frac{4\pi u}{U_{ave}} = \frac{4\pi U}{P_{rad}}$

But $E_{\phi s} = \frac{\eta \pi I_o S}{r\lambda^2} \sin\theta \, e^{-j\beta r}$

$$U = r^2 P_{ave} = \frac{r^2 \left|E_{\phi s}\right|^2}{2\eta} = \frac{\eta \pi^2 I_o{}^2 S^2 \sin^2\theta}{2\lambda^4}$$

$$P_{rad} = \int P_{ave} dS = \frac{\eta \pi^2 I_o^2 S^2}{2\lambda^4} \int\int \sin^3 \theta \partial\theta \partial\phi$$

$$= \frac{\eta \pi^2 I_o^2 S^2}{\lambda^4} .(2\pi)\left(\frac{4}{3}\right)$$

$$G_d = 4\pi \frac{\dfrac{\eta \pi^2 I_o^2 S^2 \sin^2 \theta}{2\lambda^4}}{\dfrac{\eta \pi^2 I_o^2 S^2}{2\lambda^4} . \dfrac{8\pi}{3}} = \frac{3}{2} \sin^2 \theta$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^8} = 3m,$$

$$A_e = \frac{3\lambda^2}{8\pi} \sin^2 \theta = \frac{3 \times 9}{8\pi} \left(\frac{1}{2}\right)^2 = \underline{\underline{0.2686}}$$

**Prob.**

(a) For N = 2, $f(\theta) = \cos\left[\frac{1}{2}(\beta d \cos\theta + \alpha)\right]$

$$\alpha = 0, d = \frac{\lambda}{4}$$

$$f(\theta) = \cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda} . \frac{\lambda}{4} \cos\theta + 0\right)\right] = \cos\left(\frac{\pi}{4} \cos\theta\right)$$

Maxima and minima occur at

$$\frac{d}{d\theta}\left[\cos\left(\frac{\pi}{4} \cos\theta\right)\right] = 0$$

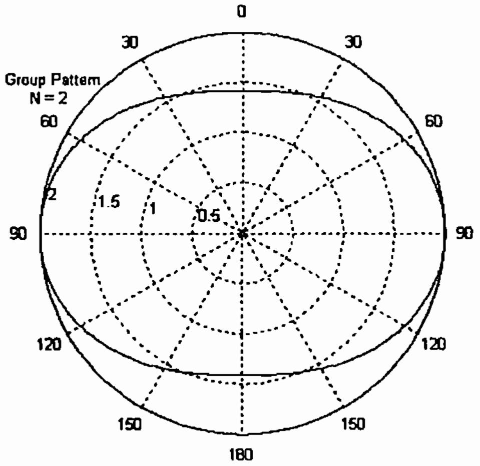
$$\sin\theta \sin\left(\frac{\pi}{4} \cos\theta\right) = 0$$

$$\sin\theta = 0 \rightarrow \theta = \pi, 0 \text{ and } f(\theta) = 0.707$$

$$\sin\left(\frac{\pi}{4} \cos\theta\right) \rightarrow \cos = 0 \rightarrow \theta = 90^o, f(\theta) = 1$$

Nulls occur as $\frac{\pi}{4} cos\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ (No Solution)

The group pattern is sketched below.



(b) For N = 4,

$$AF = \frac{\sin 2(\beta d \cos\theta + 0)}{\sin \frac{1}{2}(\beta d \cos\theta + 0)}$$

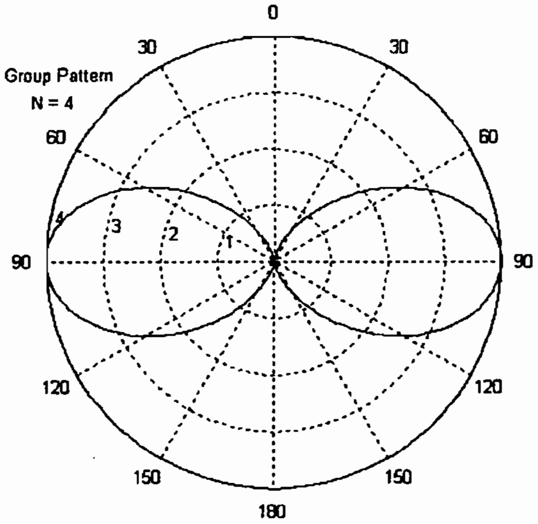
$$\text{Now,}\quad \frac{\sin 4\theta}{\sin\theta} = \frac{2\sin 2\theta \cos 2\theta}{\sin\theta} = 4\cos 2\theta \cos\theta$$

$$AF = 4\cos(\beta d \cos\theta)\cos\left(\frac{1}{2}\beta d \cos\theta\right)$$

$$f(\theta) = \cos\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\cos\theta\right)\cos\left(\frac{1}{2}\frac{2\pi}{\lambda}\frac{\lambda}{4}\right)\cos\theta$$

$$= \cos\left(\frac{\pi}{2}\cos\theta\right)\cos\left(\frac{\pi}{4}\cos\theta\right)$$

The plot is shown below.



**Prob.**

$$P_r = P_i A_e = P_i \frac{\lambda^2}{4\pi} G_d$$

$$P_{r,\max} = P_i \frac{\lambda^2}{4\pi} G_{d,\max}$$

But $G_{d,\max} = D = 1.64$ and

$$P_i = \frac{E^2}{2\eta}, \ \lambda = \frac{c}{f} = \frac{3 \times 10^8}{60 \times 10^6} = 5m$$

$$P_{r,\max} = \frac{E^2 \lambda^2 D}{8\pi\eta} = \frac{9 \times 10^{-6} \times 25 \times 1.64}{8\pi(120\pi)}$$

$$= \underline{\underline{38.9 \text{ nW}}}$$

**Prob.**

$$G_{dt} = 10^4, G_{dr} = 10^{3.2} = 1585$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{15\times 10^9} = 0.02m = \frac{1}{50}$$

$$P_r = G_{dr}G_{dt}\left(\frac{\lambda}{4\pi r}\right)^2 P_t = 10^4(1585)\left(\frac{0.02}{4\pi\times 2.456741\times 10^7}\right)^2 320$$

$$= 2.129\times10^{-11}\ \text{W} = \underline{\underline{21.29\ \text{pW}}}$$

**Prob.**

$$G_d = \frac{U}{U_{ave}} = \frac{4\pi r^2 P_{ave}}{P_{rad}} \text{ or } P_{ave} = \frac{G_d P_{rad}}{4\pi r^2}$$

$$G_d = 10^{3.4} = 2511.9$$

$$P_{ave} = \frac{2511.9\times 7.5\times 10^3}{4\pi\left(40\times 10^3\right)^2}$$

$$= \underline{\underline{0.937\ \text{mW}/\text{m}^2}}$$

**Prob.**

(a) $A_{er} = \frac{\lambda^2}{4\pi}G_{dr}, \qquad A_{et} = \frac{\lambda^2}{4\pi}G_{dt}$

$$P_r = G_{dr}G_{dt}\left(\frac{\lambda}{4\pi r}\right)^2 P_t = \left(\frac{4\pi}{\lambda^2}A_{er}\right)\left(\frac{4\pi}{\lambda^2}A_{et}\right)\left(\frac{\lambda}{4\pi r}\right)^2 P_t$$

$$\text{or } \frac{P_r}{P_t} = \frac{A_{er}A_{et}}{\lambda^2 r^2}$$

(b) $P_{r,\max} = \frac{A_{cr}A_{ct}}{\lambda^2 r^2}P_t \quad , \quad A_{cr} = A_{ct} = \frac{\lambda^2}{4\pi}(1.68)$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{100\times 10^6} = 3m,$$

$$P_{r,\max} = \frac{\left(0.13\lambda^2\right)^2(80)}{\lambda^2\left(10^3\right)^2} = \underline{\underline{12.8\mu W}}$$

**Prob.**

(a) $$P_i = \frac{|E|^2}{2\eta_o} = \frac{P_{rad}G_d}{4\pi r^2} \to |E_i| = \sqrt{\frac{240\pi P_{rad}G_d}{4\pi r^2}}$$

$$|E_i| = \frac{1}{r}\sqrt{60P_{rad}G_d} = \frac{1}{120\times10^3}\sqrt{60\times200\times10^3\times3500}$$

$$= \underline{\underline{1.708 \text{ V/m}}}$$

(b) $$|E_s| = \sqrt{\frac{|E_i|^2\sigma}{4\pi r^2}} = \sqrt{\frac{1.708^2\times8}{4\pi\times14400\times10^6}} = \underline{\underline{11.36\ \mu\text{V/m}}}$$

(c) $$P_c = P_i\sigma = \frac{1.708^2}{240\pi}(8) = \underline{\underline{30.95 \text{ mW}}}$$

(d) $$P_i = \frac{|E|^2}{2\eta_o} = \frac{(11.36)^2\times10^{-12}}{240\pi} = 1.712\times10^{-13} \quad \text{W/m}^2$$

$$\lambda = \frac{3\times10^8}{15\times10^8} = 0.2m, A_{2r} = \frac{\lambda^2G}{4\pi} = \frac{0.04\times3500}{4\pi}$$

$$P_r = P_aA_{er} = 1.712\times10^{-13}\times11.14 = 1.907\times10^{-12}$$

or $$P_r = \frac{(\lambda G_d)^2\sigma P_{rad}}{(4\pi)^3r^4} = \frac{(0.2\times3500)^2\times8\times2\times10^5}{(4\pi)^3\times12^4\times10^{16}}$$

$$= \underline{\underline{1.91\times10^{-12} \text{ W}}}$$

**Prob.**

$$G_d(dB) = 40dB = 10\log_{10} G_d \quad \longrightarrow \quad G_d = 10^{40/10} = 10^4$$

$$\lambda = c/f = \frac{3x10^8}{6x10^9} = \frac{1}{20} \text{ m}$$

$$r^4 = \frac{(\lambda G_d)^2 \sigma P_{rad}}{(4\pi)^3 P_r} = \frac{\left(\frac{10^4}{20}\right)^2 (1)8\times10^3}{(4\pi)^3 10\times10^{-12}} = \frac{10^{20}}{992.2}$$

$$r = \frac{10^5}{5.6124} = \underline{\underline{17.82 \text{ km}}}$$

**Prob.**

$$P_r = \frac{(\lambda G_d)^2 \sigma P_{rad}}{(4\pi)^3 r^4} \to P_{rad} = \frac{(4\pi)^3 r^4 P_r}{(\lambda G_d)^2 \sigma}$$

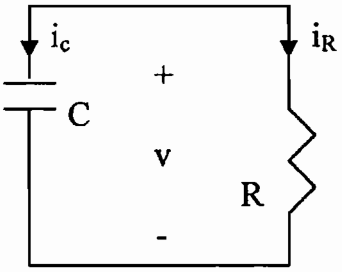
$$\lambda = \frac{c}{f} = \frac{3\times10^8}{6\times10^9} = \frac{1}{20} \langle\langle r = 250m$$

$$40 = \log_{10} G_d \to G_d = 10^4$$

---

$$P_{rad} = \frac{(4\pi)^3 (0.25\times10^3)^4 \times 2\times10^{-6}}{\left(\frac{1}{20}\times10^4\right)^2 \times 0.8} = \underline{\underline{77.52 \text{ W}}}$$

**Prob.**



$i_c + i_R = 0$; hence $C dv/dt + v/R = 0$

or $dv/v = - dt/RC$

so that $\ln v = - t/\tau + \ln v_o$, $\tau = RC = 125 \text{ x } 10^{-12} \text{ x } 2 \text{ X } 10^3 = 0.25 \ \mu \text{ s}$

$v = v_o e^{-t/\tau}$ , $v(0) = v_o = 1500$

$$i_c = C\, dv/dt = C\,( -1/\tau )\, v_o e^{-t/\tau} = \frac{-125\times10^{-12}}{0.25\times10^{-6}} \times 1500\, e^{-t/\tau}$$

$$= \underline{\underline{-0.75e^{-t/\tau} \text{ A}, \quad \tau = 0.25\ \mu\text{s}}}$$

**Prob.**

$$30dB = \log\frac{P_t}{P_r} \to \frac{P_t}{P_r} = 10^3 = 1000$$

But $P_r = (G_d)^2\left(\frac{3}{50\times4\pi\times12}\right)^2 P_t = P_t\left(\frac{G_d}{800\pi}\right)^2$

$$\left(\frac{G_d}{800\pi}\right)^2 = \frac{P_r}{P_t} = \frac{1}{1000} = \left(\frac{1}{10\sqrt{10}}\right)^2$$

or $G_d = \frac{800\pi}{10\sqrt{10}} = 79.476$

$$G_d = 10\log 79.476 = \underline{\underline{19 \text{ dB}}}$$

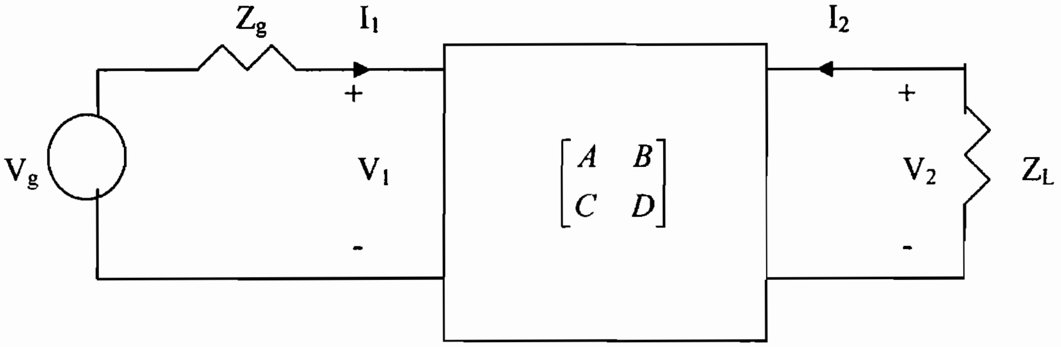
**Prob.**

$$G_{dt} = 25 = 10\log_{10} G_{dt} \rightarrow G_{dt} = 10^{2.5} = 316.23$$

$$G_{dr} = 10^{3} = 1000$$

$$P_r = 316.23\times10^{3}\left(\frac{1}{4\pi\times1.5\times10^{3}}\cdot\frac{3\times10^{8}}{1.5\times10^{9}}\right)^{2}\times200 = \underline{\underline{7.12\text{ mW}}}$$

**Prob.**



By definition,

$$V_1 = AV_2 - BI_2 \qquad (1)$$
$$I_1 = CV_2 - DI_2 \qquad (2)$$

Let $V_2$ and $\overline{V}_2$ be respectively the load voltages when the filter circuit is present and when it is absent.

$$V_2 = -I_2 Z_L = \frac{I_1 Z_L}{CZ_L + D}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{V_1}{I_1}\right)(CZ_L + D)} = \frac{V_g Z_L}{\left(Z_g + \frac{AV_2 - BI_2}{CV_2 - DI_2}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{AZ_L + B}{CZ_L + D}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g\left(CZ_L + D\right) + AZ_L + B\right)}$$

$$\bar{V}_2 = \frac{V_g Z_L}{\left(Z_g + Z_L\right)}$$

Ratio and modulus give

$$\left|\frac{\bar{V}_2}{V_2}\right| = \frac{\left(Z_g\left(CZ_L + D\right) + AZ_L + B\right)}{Z_g + Z_L}$$

Insertion loss =

$$\mathrm{IL} = 20\log_{10}\left|\frac{\bar{V}_2}{V_2}\right| = 20\log_{10}\left|\frac{\left(Z_g\left(CZ_L + D\right) + AZ_L + B\right)}{Z_g + Z_L}\right|$$

which is the required result

**Prob.**

(a) $R_{dc} = \dfrac{l}{\sigma S} = \dfrac{10^3}{0.96\times10^{-4}\times6.1\times10^7} = \underline{\underline{17.1\ \mathrm{m}\Omega\ /\mathrm{km}}}$

(b) $R_{ac} = \dfrac{l}{\delta w \sigma}$, $\pi\, a^2 = 0.8 \times 1.2 = 0.96$ or $a = 0.5528$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi\times6\times10^6\times4\pi\times10^{-7}\times6.1\times10^7}} = \frac{1}{12.1\times\pi\times10^3}$$

$$R_{ac} = \frac{1000\times12.1\times\pi\times10^3}{1.2\times10^{-2}\times6.1\times10^7} = \underline{\underline{51.93\ \Omega}}$$

## Chapter Practice Examples

### P.E.

For the exact solution,

$(D^2 + 1)\,y = 0 \quad \longrightarrow \quad y = A\cos x + B\sin x$

$y\,(0) = 0 \longrightarrow \quad A = 0$

$y(1) = 1 \longrightarrow 1 = B\sin 1$ or $B = 1/\sin 1$

Thus, $y = \sin x/\sin 1$

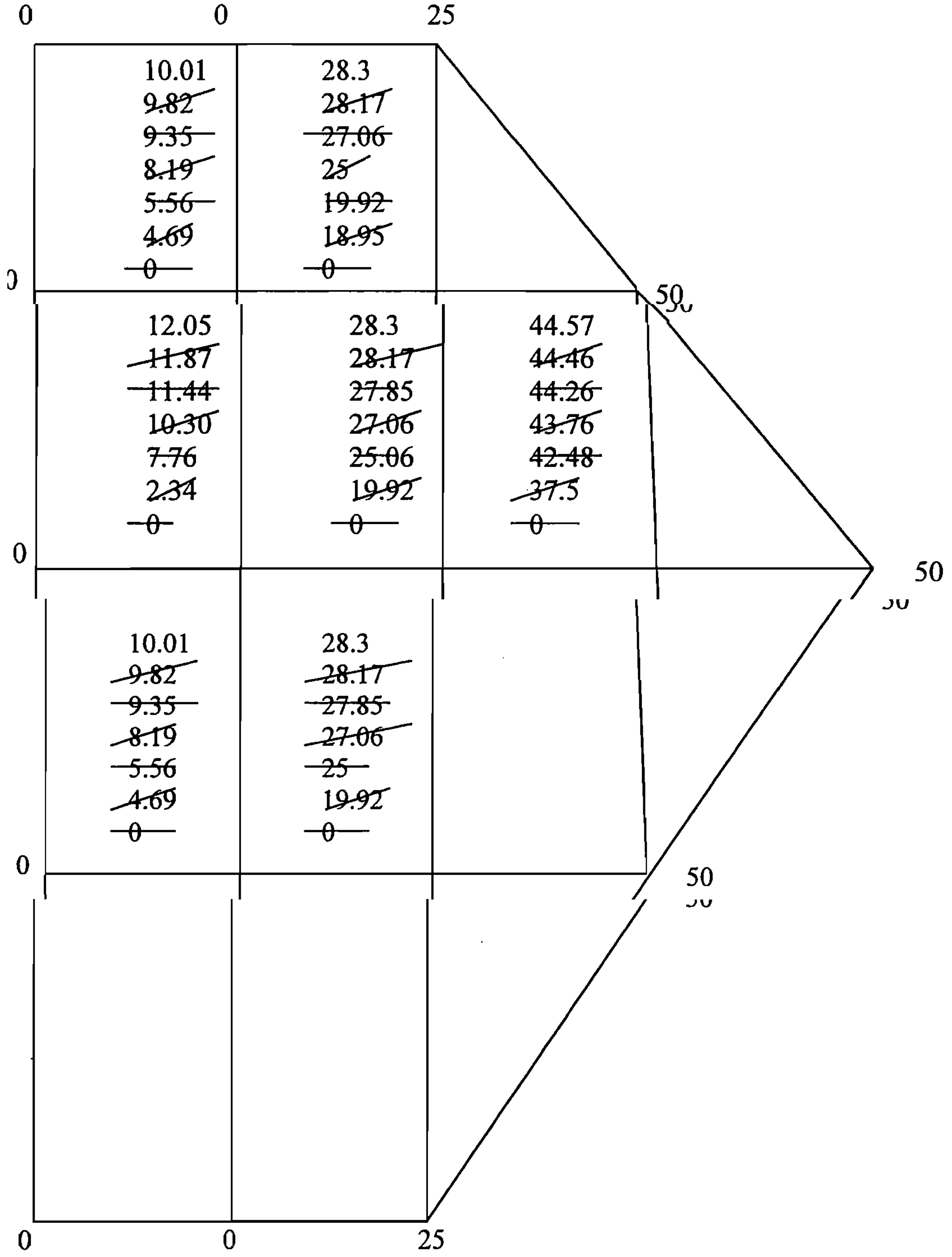
For the finite difference solution,

$$y'' + y = 0 \quad \longrightarrow \quad \frac{y(x+\Delta) - 2y(x) + y(x-\Delta)}{\Delta^2} + y = 0$$

or

$$y(x) = \frac{y(x+\Delta) + y(x-\Delta)}{2 - \Delta^2}, y(0) = 0, y(1) = 1, \Delta = 1/4$$

**P.E.** By applying eq. ( to each node as shown below, we obtain the following results after 5 iterations.



**P.E.**

(a) Using the program in Fig. with NX = 4 and NY =8, we obtain the potential at center as $V(2,4) = \underline{\underline{31.08\ V}}$

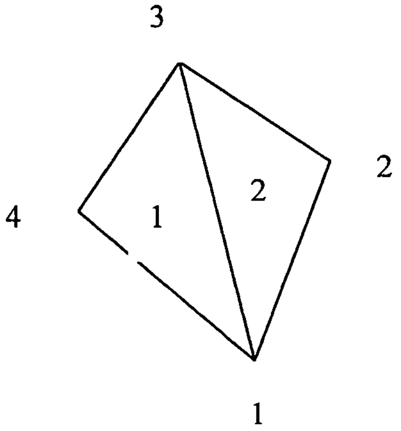
(b) Using the same program with NX = 12 and NY = 24, the potential at the center is

$$V(6,12) = \underline{\underline{42.86\ V}}$$

**P.E.**

By combining the ideas in Figs. and and dividing each wire into N segments, the results listed in Table is obtained.

**P.E.** (a)



For element 1, local 1-2-3 corresponds with global 1-3-4 so that $A_1 = 0.35$,

$P_1 = 0.8$, $P_2 = 0.6$, $P_3 = -1.4$, $Q_1 = -0.5$, $Q_2 = 0.5$, $Q_3 = 0$

$$C^{(1)} = \begin{bmatrix} 0.6357 & 0.1643 & -0.8 \\ 0.1643 & 0.4357 & -0.6 \\ -0.8 & -0.6 & 1.4 \end{bmatrix}$$

For element 2, local 1-2-3 corresponds with global 1-2-3 so that $A_2 = 0.7$,

$P_1 = 0.1,\ P_2 = 1.4,\ P_3 = -1.5,\ Q_1 = -1,\ Q_2 = 0, Q_3 = 1$

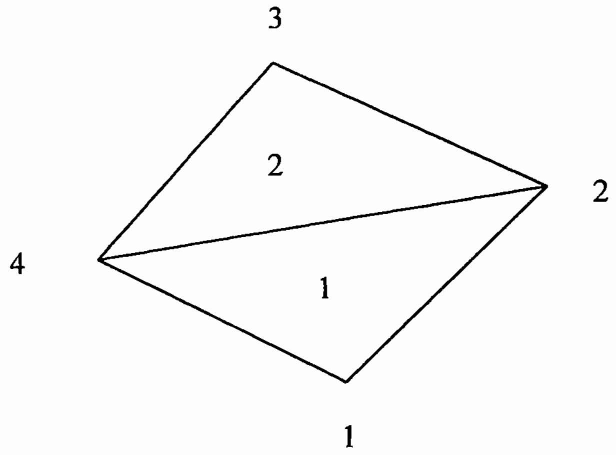
$$C^{(2)} = \begin{bmatrix} 0.3607 & 0.05 & -0.4107 \\ 0.05 & 0.7 & -0.75 \\ -0.4107 & -0.75 & 1.1607 \end{bmatrix}$$

The global coefficient matrix is given by

$$C = \begin{bmatrix} C^{(1)}{}_{11} + C_{11}{}^{(2)} & C_{12}{}^{(2)} & C_{12}{}^{(1)} + C_{13}{}^{(2)} & C_{13}{}^{(1)} \\ C_{21}{}^{(2)} & C_{22}{}^{(2)} & C_{23}{}^{(2)} & 0 \\ C_{21}{}^{(1)} + C_{31}{}^{(2)} & C_{32}{}^{(2)} & C_{22}{}^{(1)} + C_{33}{}^{(2)} & C_{23}{}^{(1)} \\ C_{31}{}^{(1)} & 0 & C_{32}{}^{(2)} & C_{33}{}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9964 & 0.05 & -0.2464 & -0.8 \\ 0.05 & 0.7 & -0.75 & 0 \\ -0.2464 & -0.75 & 1.596 & -0.6 \\ -0.8 & 0 & -0.75 & 1.4 \end{bmatrix}$$

(b)



For element 1, local 1-2-3 corresponds with global 1-2-4 .

$P_1 = 0.9000$ ; $P_2 = 0.6000$ ; $P_3 = -1.5000$

$Q_1 = -1.5000$ ; $Q_2 = 0.5000$; $Q_3 = 1$;

$A_1 = 0.6750$;

$$C^{(1)} = \begin{bmatrix} 1.1333 & -0.0778 & -1.0556 \\ -0.0778 & 0.2259 & -0.1481 \\ -1.0556 & -0.1481 & 1.2037 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds with global numbering 2-3-4.

$P_1 = 0.8000$; $P_2 = -0.9000$ ; $P_3 = 0.1000$ ;

$Q_1 = -0.5000$ ; $Q_2 = 1.5000$ ; $Q_3 = -1$ ;

$A_2 = 0.3750$ ;

$$C^{(2)} = \begin{bmatrix} 0.5933 & -0.9800 & 0.3867 \\ -0.9800 & 2.0400 & -1.0600 \\ 0.3867 & -1.0600 & 0.6733 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C^{(1)}{}_{11} & C_{12}{}^{(1)} & 0 & C_{13}{}^{(1)} \\ C_{21}{}^{(1)} & C_{22}{}^{(1)} + C_{11}{}^{(2)} & C_{12}{}^{(2)} & C_{23}{}^{(1)} + C_{13}{}^{(2)} \\ 0 & C_{12}{}^{(2)} & C_{22}{}^{(2)} & C_{23}{}^{(2)} \\ C_{31}{}^{(1)} & C_{32}{}^{(1)} + C_{31}{}^{(2)} & C_{32}{}^{(2)} & C_{33}{}^{(1)} + C_{33}{}^{(2)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1.1333 & -0.0778 & 0 & -1.0556 \\ -0.0778 & 0.8193 & -0.9800 & 0.2385 \\ 0 & -0.9800 & 2.0400 & -1.0600 \\ -1.0556 & 0.2385 & -1.0600 & 1.8770 \end{bmatrix}$$

**P.E.**

We use the MATLAB program in Fig. The input data for the region in Fig is follows:

```
 NE = 32;  ND =  26;  NP = 18;
 NL = [ 1  2  4
     2  5  4
     2  3  5
     3  6  5
     4  5  9
     5  10  9
     5  6  10
     6  11  10
     7  8  12
      8  13  12
     8  9  13
     9  14  13
      9  10  14
     10  15  14
     10  11  15
     11  16  15
     12  13  17
      13  18  17
     13  14  18
   14  19  18
   14  15  19
   15  20  19
   15  16  20
   16  21  20
   17  18  22
   18  23  22
   18  19  23
   19   24  23
    19  20  24
    20  25  24
   20  21  25
   21  26  25];
X = [ 1.0 1.5 2.0 1.0 1.5 2.0 0.0  0.5 1.0 1.5  2.0 0.0 0.5 1.0 1.5 2.0 0.0 0.5  1.0 1.5
     2    0.0  0.5 1.0 1.5 2.0];
 Y = [ 0.0  0.0  0.0 0.5 0.5  0.5 1.0 1.0  1.0 1.0  1.0  1.5  1.5  1.5  1.5  1.5  2.0  2.0  2.0
   2.0  2.0  2.5  2.5  2.5  2.5  2.5  ];
 NDP = [ 1   2  3  6  11  16  21  26  25  24  23  22  17  12  7  8  9  4];
 VAL = [0.0  0.0  15.0  30.0  30.0  30.0  30.0  25.0  20.0  20.0  20.0 10.0  0.0  0.0 0.0
        0.0  0.0  0.0];
```

With this data, the finite element (FEM) solution is compared with the finite difference (FD) solution as shown in the table below.

| Node # | X | Y | FEM | FD |
|---|---|---|---|---|
| 5 | 1.5 | 0.5 | 11.265 | 11.25 |
| 10 | 1.5 | 1.0 | 15.06 | 15.02 |
| 13 | 0.5 | 1.5 | 4.958 | 4.705 |
| 14 | 1.0 | 1.5 | 9.788 | 9.545 |
| 15 | 1.0 | 1.5 | 18.97 | 18.84 |
| 18 | 0.5 | 2.0 | 10.04 | 9.659 |
| 19 | 1.0 | 2.0 | 15.32 | 15.85 |
| 20 | 1.5 | 2.0 | 21.05 | 20.87 |

## CHAPTER Problems

### Prob.

Exact solution:

$$(D^2+4)y=0 \longrightarrow y(x)=A\cos 2x+B\sin 2x$$

$$y(0)=0 \longrightarrow 0=A$$

$$y(1)=10 \longrightarrow 10=B\sin 2 \longrightarrow B=\frac{10}{\sin 2}$$

$$y(x)=10\frac{\sin 2x}{\sin 2}$$

$$y(0.25)=10\frac{\sin 0.5}{\sin 2}=\underline{\underline{5.272}}$$

Finite difference solution:

$$\frac{y(x+\Delta)-2y(x)+y(x-\Delta)}{\Delta^2}+4y(x)=0$$

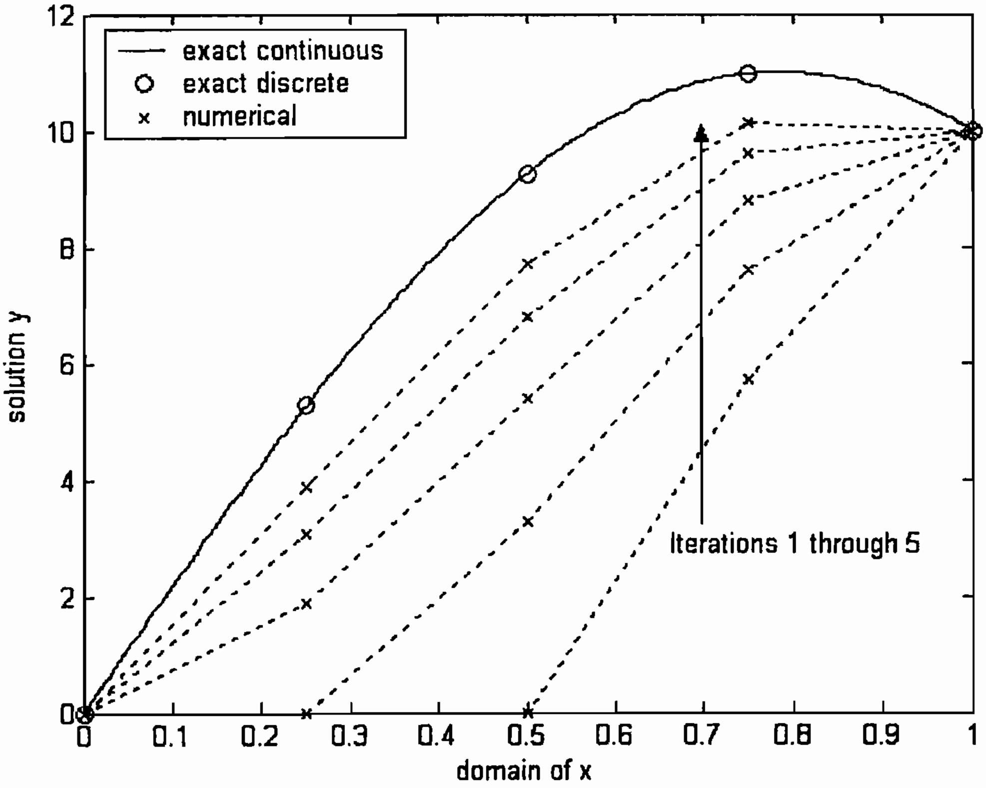
$$y(x+\Delta)+y(x-\Delta)=2y(x)-4\Delta^2 y(x)=(2-4\Delta^2)y(x)$$

or

$$y(x)=\frac{y(x+\Delta)+y(x-\Delta)}{(2-4\Delta^2)}, \quad \Delta=0.25$$

Using this scheme, we obtain the result shown below. The number of iterations is not enough to get accurate result. The numerical results are compared with the exact solution as shown in the figure below.

| Iteration | 0 | 0.25 | 0.5 | 0.75 | 1.0 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 10 |
| 1 | 0 | 0 | 0 | 5.7143 | 10 |
| 2 | 0 | 0 | 3.2653 | 7.5802 | 10 |
| 3 | 0 | 1.8659 | 5.398 | 8.7987 | 10 |
| 4 | 0 | 3.0844 | 6.7904 | 9.5945 | 10 |
| 5 | 0 | 3.8802 | 7.7904 | 10.1142 | 10 |



**Prob.**

**(a)**

$$\frac{dV}{dx} = \frac{V(x+\Delta x) - V(x - \Delta x)}{2\Delta x}$$

For $\Delta x = 0.05$ and at x = 0.15,

$$\frac{dV}{dx} = \frac{2.0134 - 1.00}{0.05X2} = \underline{\underline{10.117}}$$

$$\frac{d^2V}{dx^2}=\frac{V(x+\Delta x)-2V(x)+V(x-\Delta x)}{(\Delta x)^2}=\frac{2.0134+1.0017-2x1.5056}{(0.05)^2}=\underline{\underline{1.56}}$$

(b) V = 10 sinh x, dV/dx = 10 cosh x. At x = 0.15, dV/dx = $\underline{\underline{10.113}}$

which is close to the numerical estimate.

$d^2V/dx^2$ = 10 sinh x. At x = 0.15, $d^2V/dx^2$ = $\underline{\underline{1.5056}}$

which is slightly lower than the numerical value.

**Prob.**

$$\nabla^2 V=\frac{\partial^2 V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{\partial^2 V}{\partial z^2}=0$$

The equivalent finite difference expression is

$$\frac{V(\rho_o+\Delta\rho,z_o)-2V(\rho_o,z_o)+V(\rho_o-\Delta\rho,z_o)}{(\Delta\rho)^2}+\frac{1}{\rho_o}\frac{V(\rho_o+\Delta\rho,z_o)-V(\rho_o-\Delta\rho,z_o)}{2\Delta\rho}$$

$$+\frac{V(\rho_o,z_o+\Delta z)-2V(\rho_o,z_o)+V(\rho_o,z_o-\Delta z)}{(\Delta z)^2}=0$$

If $\Delta z=\Delta\rho=h$, rearranging terms gives

$$V(\rho_o,z_o)=\frac{1}{4}V(\rho_o,z_o+h)+\frac{1}{4}V(\rho_o,z_o-h)+(1+\frac{h}{2\rho_o})V(\rho+h,z_o)$$

$$+(1-\frac{h}{2\rho_o})V(\rho-h,z_o)$$

as expected.

**Prob.**

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0\,, \tag{1}$$

$$\frac{\partial^2 V}{\partial \rho^2} = \frac{V_{m+1}{}^{n} \; - 2V_m{}^{n} + V_{m+1}{}^{n}}{(\Delta \rho)^2}\,, \tag{2}$$

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{V_m{}^{n+1} \; - 2V_m{}^{n} + V_m{}^{n-1}}{(\Delta \phi)^2}\,, \tag{3}$$

$$\left.\frac{\partial V}{\partial \rho}\right|_{m,n} = \frac{V^n{}_{m+1} - V^n{}_{m-1}}{2\Delta \rho}\,. \tag{4}$$

Substituting (2) to (4) into (1) gives

$$\nabla^2 V = \frac{V^n{}_{m+1} - V^n{}_{m-1}}{m\Delta\rho(2\Delta\rho)} + \frac{V_{m+1}{}^{n} \; - 2V_m{}^{n} + V_{m+1}{}^{n}}{(\Delta\rho)^2} + \frac{V_m{}^{n+1} \; - 2V_m{}^{n} + V_m{}^{n-1}}{(m\Delta\rho\Delta\phi)^2}$$

$$= \frac{1}{(\Delta\rho)^2}\left[(1 - \frac{1}{2m})V_{m-1}{}^{n} \; - 2V_m{}^{n} \; + (1 + \frac{1}{2m})V_{m-1}{}^{n} \; + \frac{1}{(m\Delta\phi)^2}(V_m{}^{n+1} \; - 2V_m{}^{n} \; + V_m{}^{n-1} \;)\right]$$

as required.

**Prob.**

$$V_o = \frac{V_1 + V_2 + V_3 + V_4}{4} = \frac{10 - 40 + 50 + 80}{4} = \underline{\underline{25\text{V}}}$$

**Prob.**

$V_1 = 0.25\,(V_2 + 30 + 0 - 20) = V_2/4 + 2.5$ (1)

$V_2 = 0.25\,(V_1 + 20 + 0 + 30) = V_1/4 + 12.5$ (2)

Substituting (2) into (1),

$V_1 = 2.5 + V_1/16 + 3.125 \longrightarrow V_1 = \underline{\underline{6\text{ V}}}$

$V_2 = V_1/4 + 12.5 = \underline{\underline{14\text{ V}}}$

**Prob.**

$$k = \frac{h^2 \rho_o}{\varepsilon_o} = \frac{10^{-2} \times \dfrac{100}{\pi} \times 10^{-9}}{\dfrac{10^{-9}}{36\pi}} = 36$$

$V_1 = \frac{1}{4}\,(V_2 + 30 + 0 - 20 + k) = V_2/4 + 11.5$ (1)

$V_2 = \frac{1}{4}\,(V_1 + 20 + 0 + 30 + k) = V_1/4 + 21.5$ (2)

Substituting (2) into (1) gives

$V_1 = 11.5 + V_1/16 + 5.375 \longrightarrow V_1 = \underline{\underline{18\text{ V}}}$

$V_2 = V_1/4 + 12.5 = \underline{\underline{26\text{ V}}}$

**Prob.**

It can be done in two ways.

Iterative method:

$$V(i,j)=\frac{1}{4}\left[V(i+1,j)+V(i-1,j)+V(i,j+1)+V(i,j-1)\right]$$

| Iteration | $V_1$ | $V_2$ | $V_3$ | $V_4$ |
|---|---|---|---|---|
| 1 | 25 | 56.25 | 6.25 | 40.625 |
| 2 | 40.625 | 70.3125 | 20.31 | 47.656 |
| 3 | 47.66 | 73.83 | 28.83 | 49.41 |
| 4 | 49.41 | 74.707 | 24.71 | 49.85 |
| 5 | 49.85 | 74.93 | 24.93 | 49.96 |

Band matrix method:

At node 1,

$$-4V_1+V_2+V_3=-100-0$$

At node 2,

$$-4V_2+V_1+V_4=-100-100$$

At node 3,

$$-4V_3+V_1+V_4=0$$

At node 4,

$$-4V_4+V_3+V_2=-100-0$$

In matrix form,

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -100 \\ -200 \\ 0 \\ -100 \end{bmatrix}$$

Using calculator or MATLAB, we obtain

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} -100 \\ -200 \\ 0 \\ -100 \end{bmatrix} = \begin{bmatrix} 50 \\ 75 \\ 25 \\ 50 \end{bmatrix}$$

**Prob.**

$$V_1 = \frac{1}{4}(0-10+0+V_2) \longrightarrow 4V_1 = -10+V_2 \qquad (1)$$

$$V_2 = \frac{1}{4}(0+V_1+V_3) \longrightarrow 4V_2 = V_1+V_3 \qquad (2)$$

$$V_3 = \frac{1}{4}(0+V_2+10) \longrightarrow 4V_3 = V_2+10 \qquad (3)$$

From (1),

$$V_2 = 4V_1+10 \qquad (4)$$

Substituting (3) and (1) into (2),

$$4V_2 = \frac{-10+V_2}{4} + \frac{V_2+10}{4} \longrightarrow 16V_2 = -10+2V_2+10$$

$$V_2 = 0$$

From (4), $V_1 = -10/4 = -2.5$

$$V_3 = 2.5$$

$$\underline{\underline{V_1 = -2.5, \quad V_2 = 0, \quad V_3 = 2.5}}$$

**Prob.**

$$k=\frac{h^2\rho_s}{\varepsilon}=10^{-4}\times\frac{50\times10^{-9}}{\frac{10^{-9}}{36\pi}}=0.18\pi=0.5655$$

At node 1,

$$V_1=\frac{1}{4}[0+V_2+V_3+k] \longrightarrow 4V_1-V_2-V_3=k \qquad (1)$$

At node 2,

$$V_2=\frac{1}{4}[0+V_1+V_4+k] \longrightarrow 4V_2-V_1-V_4=k \qquad (2)$$

At node 3,

$$V_3=\frac{1}{4}[0+2V_1+V_4+k] \longrightarrow 4V_3-2V_1-V_4=k \qquad (3)$$

At nde 4,

$$V_4=\frac{1}{4}[0+2V_2+V_3+k] \longrightarrow 4V_4-2V_2-V_3=k \qquad (4)$$

Putting (1) to (4) in matrix form,

$$\begin{bmatrix}4 & -1 & -1 & 0\\ -1 & 4 & 0 & -1\\ -2 & 0 & 4 & -1\\ 0 & -2 & -1 & 4\end{bmatrix}\begin{bmatrix}V_1\\ V_2\\ V_3\\ V_4\end{bmatrix}=\begin{bmatrix}0.5655\\ 0.5655\\ 0.5655\\ 0.5655\end{bmatrix}$$

Using calculator or MATLAB, we obtain

$$\underline{\underline{V_1=V_2=0.3231\text{ V},\quad V_3=V_4=0.4039\text{ V}}}$$

**Prob.**

**(a)**

$$V_1 = \frac{1}{4}\,(0 + 100 + V_3 + V_2),\quad V_2 = \frac{1}{4}\,(0 + 100 + V_1 + V_4),$$

$$V_3 = \frac{1}{4}\,(0 + 0 + V_1 + V_4),\quad V_4 = \frac{1}{4}\,(0 + 0 + V_2 + V_3)$$

We apply these iteratively n=5 times and obtain the result below.

| N | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $V_1$ | 0 | 25 | 34.375 | 36.72 | 37.305 | 37.45 |
| $V_2$ | 0 | 31.25 | 35.937 | 37.11 | 37.403 | 37.475 |
| $V_3$ | 0 | 6.25 | 10.937 | 12.11 | 12.403 | 12.475 |
| $V_4$ | 0 | 9.375 | 11.719 | 12.305 | 12.45 | 12.487 |

(b) By band matrix method,

$$4V_1 - V_2 - V_3 = 100$$

$$-V_1 + 4V_2 - V_4 = 100$$

$$-V_1 + 4V_3 - V_4 = 0$$

$$-V_2 - V_3 + 4V_3 = 0$$

In matrix form,

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \end{bmatrix}$$

$$AV = B \longrightarrow V = A^{-1}B$$

which yields $\underline{V_1 = 37.5 = V_2,\quad V_3 = 12.5 = V_4.}$

These values are more accurate than those obtained in part(a). Why? The average of the values should give 25 V which is the potential at the center of the region. The values in part(a) give 24.96 V while the value in part (b) gives 25 V.

**Prob.**

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ V_f \end{bmatrix} = \begin{bmatrix} -200 \\ -100 \\ -100 \\ -100 \\ 0 \\ 0 \end{bmatrix}$$

$$\qquad\qquad [A] \qquad\qquad\qquad\qquad [B]$$

(b)

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -30 \\ -15 \\ -30 \\ -7.5 \\ 0 \\ -7.5 \\ 0 \\ 0 \end{bmatrix}$$

$$\qquad\qquad [A] \qquad\qquad\qquad\qquad [B]$$

**Prob.**

(a) Matrix [A] remains the same. To each term of matrix [B], we add $-h^2 \rho_s / \varepsilon$.

(b) Let $\Delta x = \Delta y = h = 0.25$ so that $\mathrm{NX} = 5 = \mathrm{NY}$.

$$\frac{\rho_v}{\varepsilon} = \frac{x(y-1)10^{-9}}{10^{-9}/36\pi} = 36\pi x(y-1)$$

Modify the program **in Fig. 13.26, as follows.**

```
  H=0.1;
for  I=1:nx –1
  for  J=1: ny-1
  X = H*I;
  Y=H*J;
  RO = 36.0*pi*X*(Y-1);
  V(I,J) = 0.25*( V(I+1,J) + V(I-1,J) + V(I,J+1) + V(I,J-1) + H*H*RO );
   end
 end
```

This is the major change. However, in addition to this, we must set

```
v1 = 0.0;
v2 = 10.0;
v3 = 20.0;
v4 = -10.0;
nx = 5;
ny = 5;
```

The results are:

$V_a =$ 4.6095   $V_b =$ 9.9440   $V_c =$ 11.6577
$V_d =$ -1.5061   $V_e =$ 3.5090   $V_f =$ 6.6867
$V_g =$ -3.2592   $V_h =$ 0.2366   $V_i =$ 3.3472

**Prob.**

$$\frac{1}{c^2}\frac{\Phi^{j+1}{}_{m,n} + \Phi^{j-1}{}_{m,n} - 2\Phi^{j}{}_{m,n}}{(\Delta t)^2} = \frac{\Phi^{j}{}_{m+1,n} + \Phi^{j}{}_{m-1,n} - 2\Phi^{j}{}_{m,n}}{(\Delta x)^2}$$
$$+\frac{\Phi^{j}{}_{m,n+1} + \Phi^{j}{}_{m,n-1} - 2\Phi^{j}{}_{m,n}}{(\Delta z)^2}$$

If $h = \Delta x = \Delta z$, then after rearranging we obtain

$$\Phi^{j+1}{}_{m,n} = 2\Phi^{j}{}_{m,n} - \Phi^{j-1}{}_{m,n} + \alpha(\Phi^{j}{}_{m+1,n} + \Phi^{j}{}_{m-1,n} - 2\Phi^{j}{}_{m,n})$$
$$+\alpha(\Phi^{j}{}_{m,n+1} + \Phi^{j}{}_{m,n-1} - 2\Phi^{j}{}_{m,n})$$

where $\alpha = (c\Delta t / h)^2$.

**Prob.**

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} \quad \longrightarrow \quad \frac{V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t)}{(\Delta x)^2} =$$
$$\frac{V(x,t+\Delta t) - 2V(x,t) + V(x,t-\Delta t)}{(\Delta t)^2}$$

$$V(x,t+\Delta t) = \left(\frac{\Delta t}{\Delta x}\right)^2 \left[V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t\right] + 2V(x,t) - V(x,t-\Delta t)$$

or

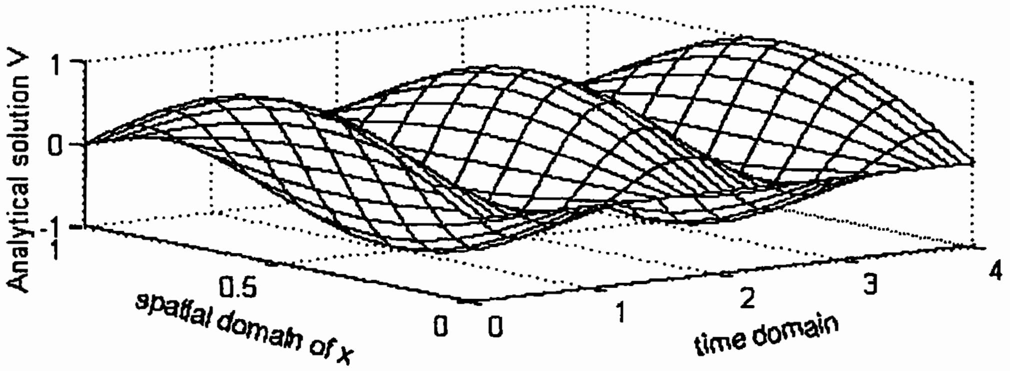
$$V(i,j+1) = \alpha\left[V(i+1,j) + v(i-1,j)\right] + 2(1-\alpha)V(i,j) - V(i,j-1)$$

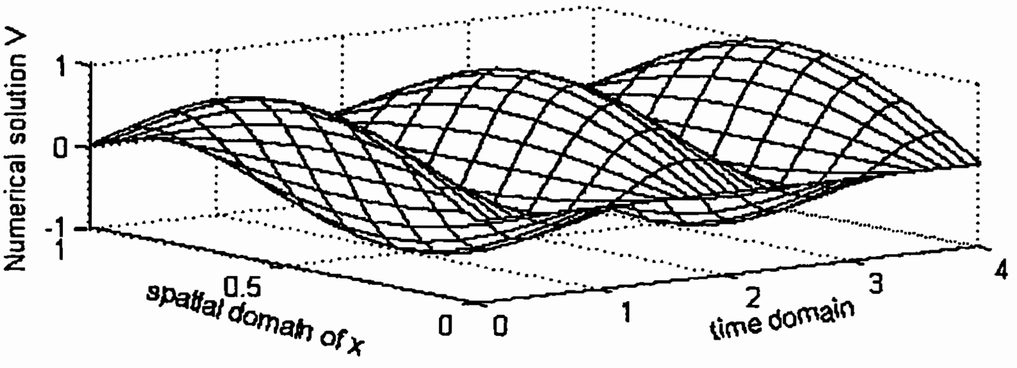
where $\alpha = \left(\frac{\Delta t}{\Delta x}\right)^2$. Applying the finite difference formula derived above, the followir

programs was developed.

```
xd=0:.1:1;td=0:.1:4;
[t,x]=meshgrid(td,xd);
Va=sin(pi*x).*cos(pi*t);%Analytical result
subplot(211) ;mesh(td,xd,Va);colormap([0 0 0])
%%%% Numerical result
N=length(xd);M=length(td);
v(:,1)=sin(pi*xd');
v(2:N-1,2)=(v(1:N-2,1)+v(3:N,1))/2;
for k=2:M-1
   v(2:N-1,k+1)=-v(2:N-1,k-1)+v(1:N-2,k)+v(3:N,k);
end
```

The results of the finite difference algorithm agree perfectly with the exact solution as shown below.





**Prob.**

(a)Points 1, 3, 5, and 7 are equidistant from O. Hence

$$V_o = \frac{1}{4}(V_1 + V_3 + V_5 + V_7) \qquad (1)$$

Also points 2, 4, 6, and 8 are equidistant from O so that

$$V_o = \frac{1}{4}(V_2 + V_4 + V_6 + V_8) \qquad (2)$$

Adding (1) and (2) gives

$$2V_o = \frac{1}{4}(V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8)$$

or

$$V_o = \frac{1}{8}\sum_{i=1}^{8} V_i$$

as required.

**Prob.**

Combining the ideas in the programs in **Figs.** **and** **we develop** a Matlab code which gives

$$N = 20 \longrightarrow C = 19.4 \text{ pF/m}$$

$$N = 40 \longrightarrow C = 13.55 \text{ pF/m}$$

$$N = 100 \longrightarrow \underline{\underline{C = 12.77 \text{ pF/m}}}$$

For the exact value, d/2a = 50/10 = 5

$$C = \frac{\pi\varepsilon}{\cosh^{-1}\dfrac{d}{2a}} = \frac{\pi \times 10^{-9}/36\pi}{\cosh^{-1} 5} = \underline{\underline{12.12}} \text{ pF/m}$$

**Prob.**

To determine V and **E** at (-1,4,5), we use the program in **Fig.**

$$V = \int_0^L \frac{\rho_L dl}{4\pi\varepsilon_o R}, \text{ where } R = \sqrt{26 + (4 - y')^2}$$

$$V = \frac{\Delta}{4\pi\varepsilon}\sum_{k=1}^{N} \frac{\rho_k}{\sqrt{26 + (y - y_k)^2}}$$

$$E = \int_0^L \frac{\rho_L dl R}{4\pi\varepsilon_o R^3}$$

where **R** = **r** – **r'** = (-1, 4-y', 5), R = |**R**|

$$E_x \cong \frac{\Delta}{4\pi\varepsilon}\sum_{k=1}^{N}\frac{(-1)\rho_k}{[26+(4-y_k)^2\ ]^{3/2}}$$

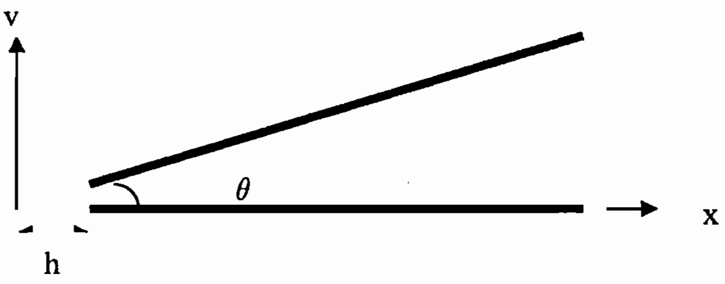
$$E_y \cong \frac{\Delta}{4\pi\varepsilon}\sum_{k=1}^{N}\frac{(4-y_k)\rho_k}{[26+(4-y_k)^2\ ]^{3/2}}$$

$$E_z = -5E_x$$

For N = 20, $V_o$ = 1V, L = 1m, a = 1mm, the program in **Fig. 14.20 is modified. The** result is:

$V = 12.47$ mV, $\mathbf{E} = -0.3266\,\mathbf{a}_x + 1.1353\mathbf{a}_y + 1.6331\mathbf{a}_z$ mV/m

**Prob.**



To find C, take the following steps:

(1)Divide each line into N equal segments. Number the segments in the lower conductor as 1, 2, …, N and segments in the upper conductor as N+1, N+2, …, 2N,

(2) Determine the coordinate ($x_k$, $y_k$) for the center of each segment.

For the lower conductor, $y_k = 0$, k=1, …, N, $x_k = h + \Delta\,(k-1/2)$, k = 1,2,… N

For the upper conductor, $x_k = [h + \Delta\,(k-1/2)]\sin\theta$, k=N+1, N+2, …,2 N,

$x_k = [h + \Delta\,(k-1/2)]\cos\theta$, k = N+1,N+2,… 2N

where h is determined from the gap g as

$$h = \frac{g}{2\sin\theta/2}$$

(3)Calculate the matrices [V] and [A] with the following elements

$$V_k = \begin{cases} V_o, k = 1, \ldots, N \\ -V_o, k = N + 1, \ldots 2N \end{cases}$$

$$A_{ij} = \begin{cases} \dfrac{\Delta}{4\pi\varepsilon R_{ij}}, i \neq j \\ 2\ln\Delta / a, i = j \end{cases}$$

where $R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

(4) Invert matrix [A] and find $[\rho] = [A]^{-1}$ [V].

(5) Find the charge Q on one conductor

$$Q = \sum \rho_k \Delta = \Delta \sum_{k=1}^{N} \rho_k$$

(6) Find C = |Q|/2V$_o$

Taking N= 10, V$_o$ = 1.0, a program was developed to obtain the following result.

| $\theta$ | C (in pF) |
|---|---|
| 10 | 8.5483 |
| 20 | 9.0677 |
| 30 | 8.893 |
| 40 | 8.606 |
| 50 | 13.004 |
| 60 | 8.5505 |
| 70 | 9.3711 |
| 80 | 8.7762 |
| 90 | 8.665 |
| 100 | 8.665 |
| 110 | 10.179 |
| 120 | 8.544 |
| 130 | 9.892 |
| 140 | 8.7449 |
| 150 | 9.5106 |
| 160 | 8.5488 |
| 170 | 11.32 |
| 180 | 8.6278 |

**Prob.**

We may modify the program in I **in Fig.** **and obtain the result in the table** below. $Z_o \cong \underline{\underline{100\ \Omega}}$.

| N | $Z_o$, in Ω |
|---|---|
| 10 | 97.2351 |
| 20 | 97.8277 |
| 30 | 98.0515 |
| 40 | 98.1739 |
| 50 | 98.2524 |

**Prob.**

We make use of the formulas in Problem

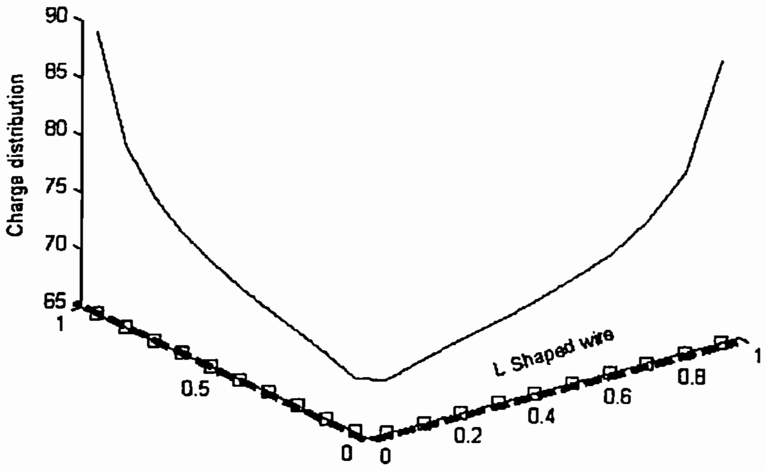
$$V_i = \sum_{j=1}^{2N} A_{ij} \rho_i$$

where N is the number of divisions on each arm of the conductor.

The MATLAB codes is as follows:

```
aa=0.001;
L=2.0;
N=10;  %no.of divisions on each arm
NT=N*2;
delta=L/(NT);
x=zeros(NT,1);
y=zeros(NT,1);
%Second calculate the elements of the coefficient matrix
for i=1:N-1
   y(i)=0;
   x(i)=delta*(i-0.5)
end
for i=N+1:NT
   x(i)=0;
   y(i)=delta*(i-N-0.5);
end
for  i=1:NT
   for j=1:NT
      if (i ~=j)
         R=sqrt( (x(i)-x(j))^2 + (y(i)-y(j))^2 )
      A(i,j)=-delta*R;
     else
      A(i,j)=-delta*(log(delta)-1.5);
     end
   end
end
%Determine the matrix of constant vector B and find rho
B=2*pi*eo*vo*ones(NT,1);
rho=inv(A)*B;
```

The result is presented  below.



| Segment | X | y | ρ in pC/m |
|---|---|---|---|
| 1 | 0.9500 |  | 89.6711 |
| 2 | 0.8500 | 0 | 80.7171 |
| 3 | 0.7500 | 0 | 77.3794 |
| 4 | 0.6500 | 0 | 75.4209 |
| 5 | 0.5500 | 0 | 74.0605 |
| 6 | 0.4500 | 0 | 73.0192 |
| 7 | 0.3500 | 0 | 72.1641 |
| 8 | 0.2500 | 0 | 71.4150 |
| 9 | 0.1500 | 0 | 70.6816 |
| 10 | 0.0500 | 0 | 69.6949 |
| 11 | 0 | 0 | 69.6949 |
| 12 | 0 | 0.0500 | 70.6816 |
| 13 | 0 | 0.1500 | 71.4150 |
| 14 | 0 | 0.2500 | 72.1641 |
| 15 | 0 | 0.3500 | 73.0192 |
| 16 | 0 | 0.4500 | 74.0605 |
| 17 | 0 | 0.5500 | 75.4209 |
| 18 | 0 | 0.6500 | 77.3794 |
| 19 | 0 | 0.7500 | 80.7171 |
| 20 | 0 | 0.8500 | 89.6711 |

**Prob.**

**(a) Exact Solution Yields**

$C = 2\pi\varepsilon / ln(\Delta / a) = 8.02607x10^{-11}$ F/m and $\underline{\underline{Z_o = 41.559\Omega}}$

where a = 1cm and Δ = 2cm. The numerical solution is shown below.

| N | C (pF/m) | $Z_o(\Omega)$ |
|---|---|---|
| 10 | 82.386 | 40.486 |
| 20 | 80.966 | 41.197 |
| 40 | 80.438 | 41.467 |
| 100 | 80.025 | 41.562 |

(b)For this case, the numerical solution is shown below.

| N | C (pF/m) | $Z_o(\Omega)$ |
|---|---|---|
| 10 | 109.51 | 30.458 |
| 20 | 108.71 | 30.681 |
| 40 | 108.27 | 30.807 |
| 100 | 107.93 | 30.905 |

**Prob.**

We modify the MATLAB code i
changing the input data and matrices [A] and [B]. We let

$x_i = h + \Delta\ (i\text{-}1/2),\ i = 1,2,\ldots N, \qquad \Delta = L/N$

$y_i = h/2,\ j = 1,2,\ldots N, \quad z_k = t/2,\ k = 1,2,\ldots N$

and calculate

$$R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

We obtain matrices [A] and [B]. Inverting [A] gives

$[q] = [A]^{-1}\,[B], \quad [\rho_v] = [q]/(ht\Delta\ ), \quad C = \dfrac{\sum_{i=1}^{N} q_i}{10}$

The computed values of $[\rho_v]$ and C are shown below.

| i | $\rho_{vi}(\times 10^{-6})C/m^3$ |
|---|---|
| 1, 20 | 0.5104 |
| 2, 19 | 0.4524 |
| 3, 18 | 0.4324 |
| 4, 17 | 0.4215 |
| 5, 16 | 0.4144 |
| 6, 15 | 0.4096 |
| 7, 14 | 0.4063 |
| 8, 13 | 0.4041 |
| 9, 12 | 0.4027 |
| 10,11 | 0.4020 |

C = 17.02 pF

**Prob.**

**From the given figure, we obtain**

$$\alpha_1 = \frac{A_1}{A} = \frac{1}{2A}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

as expected. The same applies for $\alpha_2$ and $\alpha_3$.

**Prob.**

(a) For the element in (a),

A = ½ ( 1 – 0.5x0.25 ) = 0.4375

$$\alpha_1 = \frac{1}{2A}[0.875 - 0.75x - 0.5y] = 1 - 0.8571x - 0.5714y$$

$$\alpha_2 = \frac{1}{2A}[0 + x - 0.5y] = 1.1429x - 0.5714y$$

$$\alpha_3 = \frac{1}{2A}[0 - 0.25x + y] = -0.2857x + 1.1429y$$

For the element in (b),

A = ½ [0.5x1.6 – (-1)x1.6 ] = 1.2

$$\alpha_1 = 1.25 - 0.625y$$

$$\alpha_2 = -1.5 + 0.667x + 0.4167y$$

$$\alpha_3 = 1.25 - 0.667x + 0.2083y$$

(b)For the element in (a),

$P_1$ = -0.75, $P_2$ = 1.0, $P_3$ = -0.25, $Q_1$ = -0.5 = $Q_2$, $Q_3$ = 1.0

$$C_{ij} = \frac{1}{4A}[P_jP_i + Q_jQ_j] = (\nabla\alpha_1.\nabla\alpha_2)A$$

Hence,

$$C^{(1)} = \begin{bmatrix} 0.4643 & -0.2857 & -0.1786 \\ -0.2857 & 0.7143 & -0.4286 \\ -0.1786 & -0.4286 & 0.6071 \end{bmatrix}$$

For the element in (b),

$P_1 = 0, P_2 = 1.6, P_3 = -1.6, Q_1 = -1.5,\ Q_2 = 1.0, Q_3 = 0.5,\ A=1.2$

Hence,

$$C^{(2)} = \begin{bmatrix} 0.4687 & -0.3125 & -0.1562 \\ -0.3125 & 0.7417 & -0.4292 \\ -0.1562 & -0.4292 & 0.5854 \end{bmatrix}$$

**Prob.**

**(a)**

$$2A = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1 & 3 & 1/2 \\ 1 & 2 & 2 \end{vmatrix} = 15/4$$

$$\alpha_1 = \frac{4}{15}[(6-1)+(-1\frac{1}{2})x+(-1)y] = \frac{4}{15}(5-1.5x-y)$$

$$\alpha_2 = \frac{4}{15}[(1-1)+\frac{3}{2}x-\frac{3}{2}y] = \frac{4}{15}(1.5x-1.5y)$$

$$\alpha_3 = \frac{4}{15}[(1/4-3/2)+0x+\frac{5}{2}y] = \frac{4}{15}(-1.25+2.5y)$$

$$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

Substituting V=80, $V_1 = 100, V_2 = 50, V_3 = 30$, $\alpha_1$, $\alpha_2$, and $\alpha_3$ leads to

20 = 7.5x + 10y + 3.75

Along side 12, y=1/2 so that

20 = 15x/2 + 5 + 15/4 $\longrightarrow$ x=3/2, i.e (1.5, 0.5)

Along side 13, x =y

20 = 15x/2 + 10x + 15/4 $\longrightarrow$ x=13/4, i.e. (13/14, 13/14)

Along side 23, y = -3x/2 + 5

20 = 15x/2 – 15 + 50 + 15/4 $\longrightarrow$ x=-5/2 (not possible)

Hence intersection occurs at

$\underline{\underline{\text{(1.5, 0.5) along 12 and (0.9286, 0.9286) along 13}}}$

(b) At (2,1),

$$\alpha_1 = \frac{4}{15}, \quad \alpha_2 = \frac{6}{15}, \quad \alpha_3 = \frac{5}{15}$$

$$V(2,1) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = (400 + 300 + 150)/15 = \underline{\underline{56.67\text{ V}}}$$

**Prob.**

$$2A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 4 \end{vmatrix} = 9$$

$$\alpha_1 = \frac{1}{9}[(0-0) + (4-0)x + (0-1)y] = \frac{1}{9}(4x - y)$$

$$\alpha_2 = \frac{1}{9}[(0-0) + (0+1)x + (2-0)y] = \frac{1}{9}(x + 2y)$$

$$\alpha_3 = \frac{1}{9}[(8+1) + (-1-4)x + (1-2)y] = \frac{1}{9}(9 - 5x - y)$$

$$V_e = \alpha_1 V_{e1} + \alpha_2 V_{e2} + \alpha_{31} V_{e3}$$

$V(1,2) = 8(4\text{-}2)/9 \ + 12(1+4)/9 \ + \ 10(9\text{-}5\text{-}1)/9 \ = 96/9 \ = \ \underline{\underline{10.667\ \text{V}}}$

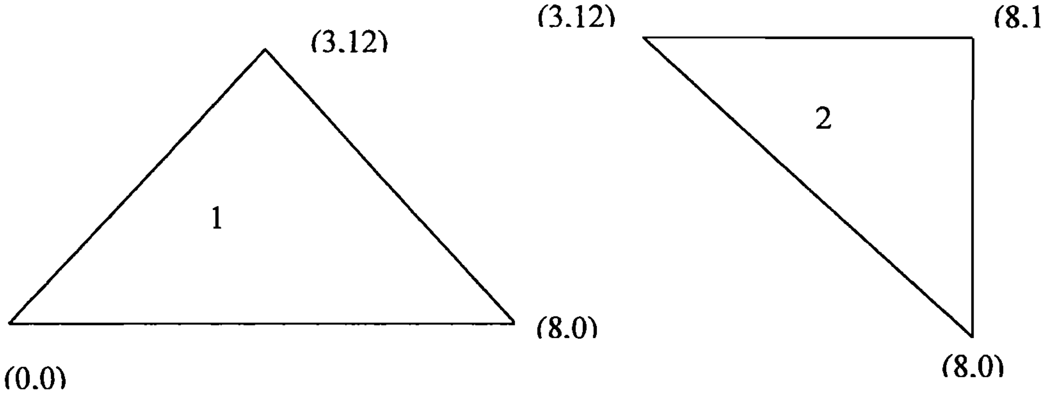
At the center $\alpha_1 = \ \alpha_2 = \alpha_3 = 1/3$ so that

V(center) = (8 + 12 + 10)/3 = 10

Or at the center, (x, y) = (0 + 1 + 2, 0 + 4 –1)/3 = (1,1)

$V(1,1) = 8(3)/9 \ + 12(3)/9 \ + 10(3)/9 \ = \ \underline{\underline{10\ \text{V}}}$

**Prob.**



For element 1, local numbering 1-2-3 corresponds to global numbering 4-2-1.

$P_1 = 12, P_2 = 0, P_3 = \text{-}12, Q_1 = \text{-}3, \ Q_2 = 8, Q_3 = \text{-}5,$

A = (0 + 12 x 8)/2 = 48

$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_j]$$

$$C^{(1)} = \begin{bmatrix} 0.7969 & -0.125 & -0.6719 \\ -0.125 & 0.3333 & -0.2083 \\ -0.6719 & -0.2083 & 0.8802 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 2-4-3.

$P_1 = -12, P_2 = 0, P_3 = 12, Q_1 = 0, Q_2 = -5, Q_3 = 5,$

$A = (0 + 60)/2 = 30$

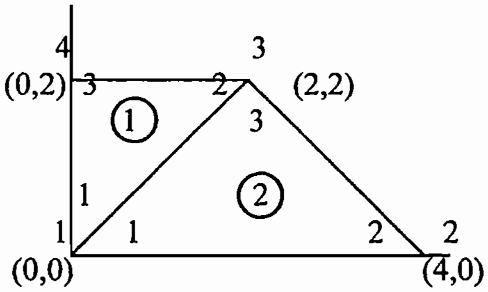
$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_j]$$

$$C^{(2)} = \begin{bmatrix} 1.2 & 0 & -1.2 \\ 0 & 0.208 & -0.208 \\ -1.2 & -0.208 & 1.408 \end{bmatrix}$$

$$C = \begin{bmatrix} C^{(1)}{}_{33} & C_{23}{}^{(1)} & 0 & C_{31}{}^{(1)} \\ C_{23}{}^{(1)} & C_{22}{}^{(1)} + C_{11}{}^{(2)} & C_{13}{}^{(2)} & C_{21}{}^{(1)} + C_{12}{}^{(2)} \\ 0 & C_{31}{}^{(2)} & C_{33}{}^{(2)} & C_{32}{}^{(2)} \\ C_{13}{}^{(1)} & C_{21}{}^{(1)} + C_{21}{}^{(2)} & C_{23}{}^{(2)} & C_{22}{}^{(2)} + C_{11}{}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8802 & -0.2083 & 0 & -0.6719 \\ -0.2083 & 1.533 & -1.2 & -0.125 \\ 0 & -1.2 & 1.4083 & -0.2083 \\ -0.6719 & -0.125 & -0.2083 & 1.0052 \end{bmatrix}$$

**Prob.**



For element 1,

$$P_1 = 0, P_2 = 2, P_3 = -2, Q_1 = -2, Q_2 = 0, Q_3 = 2$$
$$A = \frac{1}{2}(4-0) = 2, \quad 4A = 8$$

$$C^{(1)}=\frac{1}{8}\begin{bmatrix}4 & 0 & -4\\ 0 & 4 & -4\\ -4 & -4 & 8\end{bmatrix}=\begin{bmatrix}0.5 & 0 & -0.5\\ 0 & 0.5 & -0.5\\ -0.5 & -0.5 & 1\end{bmatrix}$$

For element 2,

$$P_1=-2,P_2=2,P_3=0,Q_1=-2,Q_2=-2,Q_3=4$$

$$A=\frac{1}{2}(8-0)=4,\qquad 4A=16$$

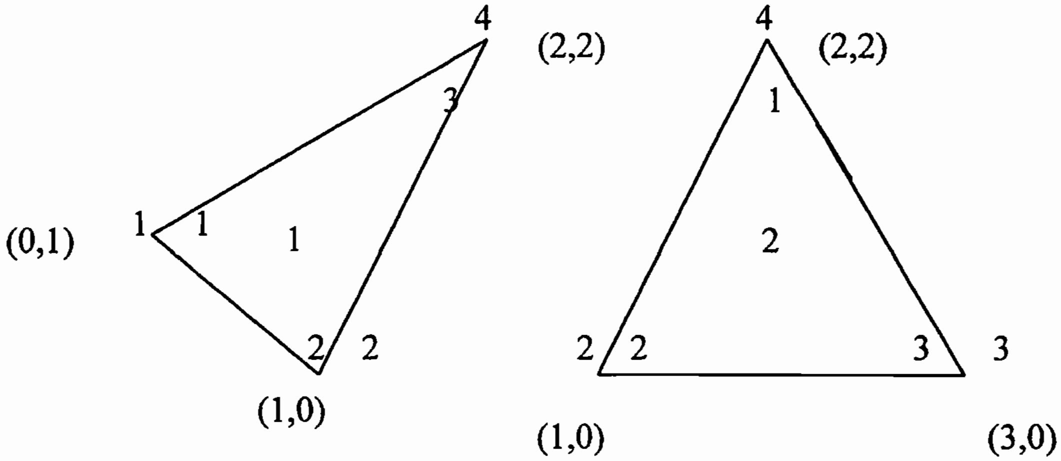
$$C^{(2)}=\frac{1}{16}\begin{bmatrix}8 & 0 & -8\\ 0 & 8 & -8\\ -8 & -8 & 16\end{bmatrix}=\begin{bmatrix}0.5 & 0 & -0.5\\ 0 & 0.5 & -0.5\\ -0.5 & -0.5 & 1\end{bmatrix}$$

The global coefficient matrix is

$$C=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14}\\ C_{21} & C_{22} & C_{23} & C_{24}\\ C_{31} & C_{32} & C_{33} & C_{34}\\ C_{41} & C_{42} & C_{43} & C_{44}\end{bmatrix}=\begin{bmatrix}C_{11}^{(1)}+C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)}+C_{13}^{(2)} & C_{13}^{(1)}\\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0\\ C_{21}^{(1)}+C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)}+C_{33}^{(2)} & C_{23}^{(1)}\\ C_{31}^{(1)} & 0 & C_{32}^{(1)} & C_{33}^{(1)}\end{bmatrix}$$

$$=\underline{\underline{\begin{bmatrix}1 & 0 & -0.5 & -0.5\\ 0 & 0.5 & -0.5 & 0\\ -0.5 & -0.5 & 1.5 & -0.5\\ -0.5 & 0 & -0.5 & 1\end{bmatrix}}}$$

**Prob.**



For element 1, local numbering 1-2-3 corresponds to global numbering 1-2-4.

$P_1 = -2, P_2 = 1, P_3 = 1, Q_1 = 1, Q_2 = -2, Q_3 = 1,$

$A = (P_2 Q_3 - P_3 Q_2)/2 = 3/2$, i.e. $4A = 6$

$$C_{ij} = \frac{1}{4A}[P_j P_i + Q_j Q_j]$$

$$C^{(1)} = \frac{1}{6}\begin{bmatrix} 5 & -4 & -1 \\ -4 & 5 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 4-2-3.

$P_1 = 0, P_2 = -2, P_3 = 2, Q_1 = 2, Q_2 = -1, Q_3 = -1,$

$A = 2$, $4A = 8$

$$C^{(2)} = \frac{1}{8}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C^{(1)}{}_{11} & C_{12}{}^{(1)} & 0 & C_{13}{}^{(1)} \\ C_{12}{}^{(1)} & C_{22}{}^{(1)} + C_{22}{}^{(2)} & C_{23}{}^{(2)} & C_{23}{}^{(1)} + C_{21}{}^{(2)} \\ 0 & C_{23}{}^{(2)} & C_{33}{}^{(2)} & C_{31}{}^{(2)} \\ C_{13}{}^{(1)} & C_{23}{}^{(1)} + C_{21}{}^{(2)} & C_{31}{}^{(2)} & C_{33}{}^{(1)} + C_{11}{}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8333 & -0.667 & 0 & -0.1667 \\ -0.6667 & 1.4583 & -0.375 & -0.4167 \\ 0 & -0.375 & 0.625 & -0.25 \\ -0.1667 & -0.4167 & -0.25 & 0.833 \end{bmatrix}$$

**Prob.**

We can do it by hand as in **Ex..** However, it is easier to prepare an input file and use the program in **Fig.** The MATLAB input data is

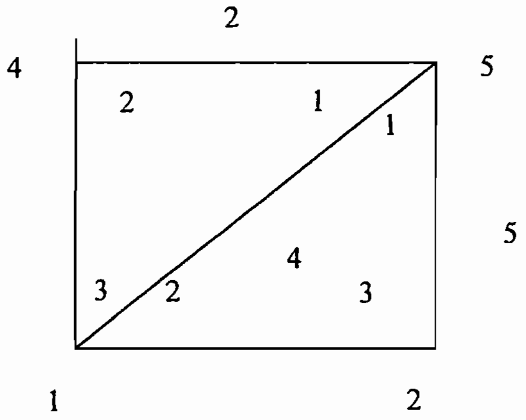
```
NE = 2;
ND = 4;
NP = 2;
NL = [1  2  4
     4   3  2];
X = [ 0.0   1.0  3.0  2.0];
Y = [ 1.0  0.0  0.0  2.0];
NDP= [ 1  3 ];
VAL = [ 10.0  30.0]
```

The result is $V = \begin{bmatrix} 10 \\ 18 \\ 30 \\ 20 \end{bmatrix}$

From this,

$$\underline{\underline{V_2 = 18\text{ V},\quad V_4 = 20\text{ V}}}$$

**Prob.**



The local numbering 1-2-3 in element 3 corresponds with the global numbering 5-4-1, while the local number 1-2-3 in element 4 corresponds with the global numbering 5-1-2.

$C_{5,5} = C_{11}^{(2)} + C_{11}^{(3)} + C_{11}^{(4)} + C_{11}^{(5)}$, $A = 2$,

$C_{11}^{(2)} = (2 \text{ x } 2 + 2 \text{ x } 2)/8 = 1 = C_{11}^{(5)}$

$C_{11}^{(3)} = (2 \text{ x } 2 + 0)/8 = ½ = C_{11}^{(4)}$

$C_{5,5} = 1 + 1 + ½ + ½ = \underline{\underline{3}}$

$C_{5,1} = C_{31}^{(3)} + C_{21}^{(4)}$

But $C_{31}^{(3)} = \frac{1}{8}(P_3P_1 + Q_3 Q_1) = 0$ since $P_3 = 0 = Q_3$

$C_{21}^{(4)} = \frac{1}{8}(P_2P_1 + Q_2 Q_1) = 0$ since $P_3 = 0 = Q_3$

$\underline{\underline{C_{5,1} = 0}}$

**Prob.**

**As in P.E. we use the program in Fig. The input data based on Fig. is as follows.**

```
NE =50;  ND= 36;  NP= 20;
 NL = [1     8      7
       1     2      8
       2     9      8
       2     3      9
       3     10     9
       3     4      10
       4     11     10
       4     5      11
       5     12     11
       5     6      12
       7     14     13
       7     8      14
       8     15     14
       8     9      15
       9     16     15
       9     10     16
       10    17     16
       10    11     17
       11    18     17
       11    12     18
       13    20     19
       13    14     20
       14    21     20
       14    15     21
       15    22     21
       15    16     22
       16    23     22
       16    17     23
       17    24     23
       17    18     24
       19    26     25
       19    20     26
       20    27     26
       20    21     27
       21    28     27
       21    22     28
       22    29     28
       22    23     29
       23    30     29
       23    24     30
       25    32     31
       25    26     32
```

```
            26      33      32
            26      27      33
            27      34      33
            27      28      34
            28      35      34
            28      29      35
            29      36      35
            29      30      36];
    X = [0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6 0.8  1.0
        0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6  0.8  1.0  0.2  0.4  0.6  0.8  1.0];
    Y = [0.0  0.0   0.0  0.0  0.0  0.0  0.2  0.2  0.2  0.2  0.2  0.2  0.4  0.4  0.4  0.4  0.4
    0.4 0.6  0.6  0.6  0.6  0.6  0.6  0.8  0.8  0.8  0.8  0.8  0.8  1.0 1.0  1.0  1.0  1.0 1.0];
    NDP = [ 1   2  3  4  5  6  12  18  24  30  36  35  34  33  32  31  25  19  13  7];
    VAL = [ 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  50.0  100.0  100.0  100.0
        100.0  50.0  0.0  0.0  0.0  0.0];
```

With this data, the potentials at the free nodes are compared with the exact values as shown below.

| Node no. | FEM Solution | Exact Solution |
|---|---|---|
| 8 | 4.546 | 4.366 |
| 9 | 7.197 | 7.017 |
| 10 | 7.197 | 7.017 |
| 11 | 4.546 | 4.366 |
| 14 | 10.98 | 10.60 |
| 15 | 17.05 | 16.84 |
| 16 | 17.05 | 16.84 |
| 17 | 10.98 | 10.60 |
| 20 | 22.35 | 21.78 |
| 21 | 32.95 | 33.16 |
| 22 | 32.95 | 33.16 |
| 23 | 22.35 | 21.78 |
| 26 | 45.45 | 45.63 |
| 27 | 59.49 | 60.60 |
| 28 | 59.49 | 60.60 |
| 29 | 45.45 | 45.63 |

**Prob.**

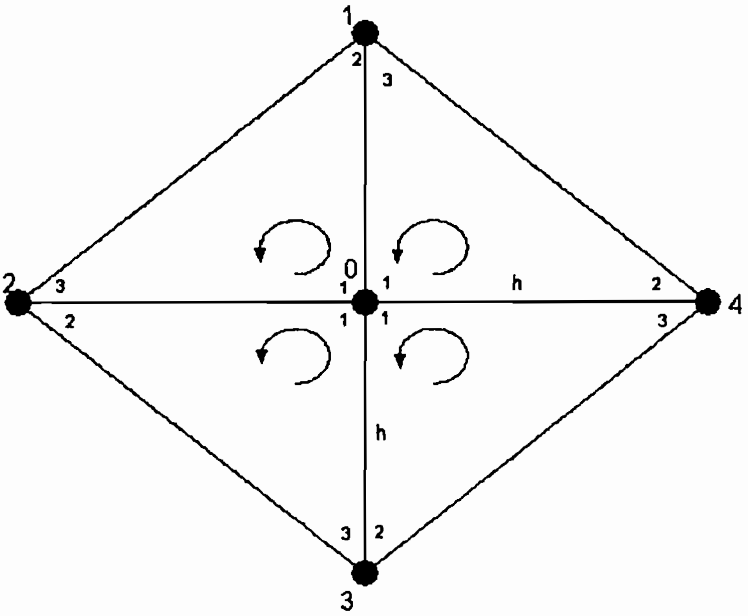
We use exactly the same input data as in the previous problem except that the last few lines are replaced by the following lines.

```
VAL = [ 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  29.4  58.8  95.1  95.1
       58.8  29.4  0.0  0.0  0.0  0.0];
```

The potential at the free nodes obtained with the input data are compared with the exact solution as shown below.

| Node no. | FEM Solution | Exact Solution |
|---|---|---|
| 8 | 3.635 | 3.412 |
| 9 | 5.882 | 5.521 |
| 10 | 5.882 | 5.521 |
| 11 | 3.635 | 3.412 |
| 14 | 8.659 | 8.217 |
| 15 | 14.01 | 13.30 |
| 16 | 14.01 | 13.30 |
| 17 | 8.659 | 8.217 |
| 20 | 16.99 | 16.37 |
| 21 | 27.49 | 26.49 |
| 22 | 27.49 | 26.49 |
| 23 | 16.99 | 16.37 |
| 26 | 31.81 | 31.21 |
| 27 | 51.47 | 50.5 |
| 28 | 51.47 | 50.5 |
| 29 | 31.81 | 31.21 |

**Prob.**



For element 1, the local numbering 1-2-3 corresponds with nodes with $V_1$ , $V_2$ , and $V_3$.

$$V_o = -\frac{1}{C_{oo}}\sum_{i=1}^{4} V_i C_{io}$$

$$C_{oo} = \sum_{j=1}^{4} C_{oj}^{(e)} = \frac{1}{4h^2/2}(hh+hh)\times 2 + \frac{1}{4h^2/2}(hh+0)\times 4 = 4$$

$$C_{o1} = \frac{2\times 1}{2h^2}[P_3P_1 + Q_3Q_1] = \frac{2}{2h^2}[-hh-0] = -1$$

$$C_{o2} = \frac{2\times 1}{2h^2}[P_1P_2 + Q_1Q_2] = \frac{2}{2h^2}[-h\times 0 + h\times(-h)] = -1$$

Similarly, $C_{03} = -1 = C_{04}$. Thus

$V_o = ( V_1 + V_2 + V_3 + V_4 )/4$

which is the same result obtained using FDM.

**Prob.**

The MATLAB code is similar to the one in Fig.14.40. When the program is run, it gives $Z_o = \underline{\underline{40.587\ \Omega}}$.

**Prob.**

The finite difference solution is obtained by following the same steps as in **Ex.**
We obtain $\underline{\underline{Z_o = 43\ \Omega}}$

**Prob.**

$$V_1 = \frac{1}{4}(V_2 + 100 + 100 + 100) = \frac{1}{4}V_2 + 75$$

$$V_2 = \frac{1}{4}(V_1 + V_4 + 2V_3)$$

$$V_3 = \frac{1}{4}(V_2 + V_5 + 200) = \frac{1}{4}(V_2 + V_5) + 50$$

$$V_4 = \frac{1}{4}(V_2 + V_7 + 2V_5)$$

$$V_5 = \frac{1}{4}(V_3 + V_4 + V_6 + V_8)$$

$$V_6 = \frac{1}{4}(V_5 + V_9 + 200) = \frac{1}{4}(V_5 + V_9) + 50$$

$$V_7 = \frac{1}{4}(V_4 + 2V_8 + 0) = \frac{1}{4}(V_4 + 2V_8)$$

$$V_8 = \frac{1}{4}(V_5 + V_7 + V_9)$$

$$V_9 = \frac{1}{4}(V_6 + V_8 + 100 + 0) = \frac{1}{4}(V_6 + V_8) + 25$$

Using these equations, we apply iterative method as shown below.

| | 1st | 2nd | 3rd | 4th | 5th |
|---|---|---|---|---|---|
| $V_1$ | 75 | 79.687 | 87.11 | 89.91 | 92.01 |
| $V_2$ | 18.75 | 48.437 | 59.64 | 68.06 | 74.31 |
| $V_3$ | 54.69 | 65.82 | 73.87 | 79.38 | 82.89 |
| $V_4$ | 4.687 | 19.824 | 34.57 | 46.47 | 53.72 |
| $V_5$ | 14.687 | 35.14 | 49.45 | 57.24 | 61.78 |
| $V_6$ | 53.71 | 68.82 | 74.2 | 77.01 | 78.6 |
| $V_7$ | 1.172 | 6.958 | 18.92 | 26.08 | 30.194 |
| $V_8$ | 4.003 | 20.557 | 28.93 | 33.53 | 36.153 |
| $V_9$ | 39.43 | 47.34 | 50.78 | 52.63 | 53.69 |

## Appendix A—Practice Examples

**P.E. A.1**

(a) $\boldsymbol{A}+\boldsymbol{B}=(1,0,3)+(5,2,-6)=(6,2,-3)$

$$|\boldsymbol{A}+\boldsymbol{B}|=\sqrt{36+4+9}=\underline{\underline{7}}$$

(b) $5\boldsymbol{A}-\boldsymbol{B}=(5,0,15)-(5,2,-6)=\underline{\underline{(0,-2,21)}}$

(c) The component of $\boldsymbol{A}$ along $\mathbf{a}_y$ is $A_y=\underline{\underline{0}}$

(d) $3\boldsymbol{A}+\boldsymbol{B}=(3,0,9)+(5,2,-6)=(8,2,3)$

A unit vector parallel to this vector is

$$\boldsymbol{a}_{11}=\frac{(8,2,3)}{\sqrt{64+4+9}}$$
$$=\underline{\underline{\pm(0.9117\boldsymbol{a}_x+0.2279\boldsymbol{a}_y+0.3419\boldsymbol{a}_z)}}$$

**P.E. A.2** (a) $\boldsymbol{r}_p=\underline{\underline{\boldsymbol{a}_x-3\boldsymbol{a}_y+5\boldsymbol{a}_z}}$

$$\boldsymbol{r}_R=\underline{\underline{3\boldsymbol{a}_y+8\boldsymbol{a}_z}}$$

(b) The distance vector is

$$\boldsymbol{r}_{QR}=\boldsymbol{r}_R-\boldsymbol{r}_Q=(0,3,8)-(2,4,6)=\underline{\underline{-2\boldsymbol{a}_x-\boldsymbol{a}_y+2\boldsymbol{a}_z}}$$

(c) The distance between Q and R is

$$|\,r_{QR}\,|=\sqrt{4+1+4}=\underline{\underline{3}}$$

**P.E. A.3** Consider the figure shown below:

$$\mathbf{u}_Z = \mathbf{u}_P + \mathbf{u}_W = -350\mathbf{a}_x + \frac{40}{\sqrt{2}}\left(-\mathbf{a}_x + \mathbf{a}_y\right)$$

$$= -378.28\mathbf{a}_x + 28.28a_y \text{ km/hr}$$

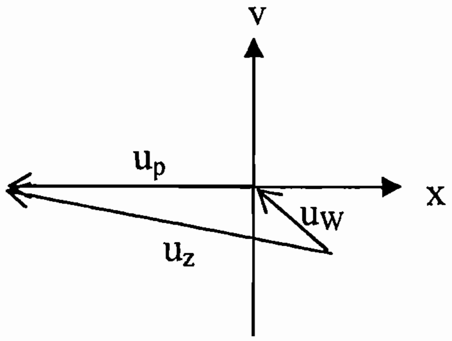
or

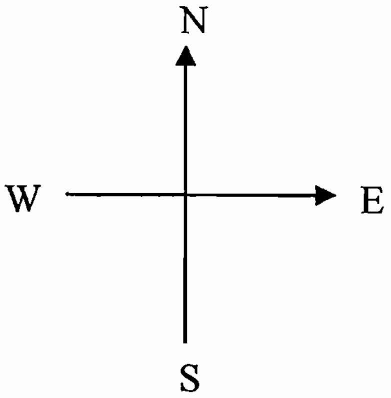
$$\mathbf{u}_z = 379.3\angle 175.72^\circ \text{ km/hr}$$

Where $\mathbf{u}_p$ = velocity of the airplane in the absence of wind

$\mathbf{u}_w$ = wind velocity

$\mathbf{u}_z$ = observed velocity





**P.E. A.4**

Using the dot product,

$$\cos\theta_{AB} = \frac{\mathbf{A} \bullet \mathbf{B}}{AB} = \frac{-13}{\sqrt{10}\sqrt{65}} = -\sqrt{\frac{13}{50}}$$

or using the cross product,

$$\sin\theta_{AB} = \frac{|\mathbf{A} x\, \mathbf{B}|}{AB} = \sqrt{\frac{481}{650}}$$

Either way,

$$\underline{\underline{\theta_{AB} = 120.66^\circ}}$$

**P.E. A.5**

(a) $\boldsymbol{E}_F = (\boldsymbol{E} \cdot \boldsymbol{a}_F)\boldsymbol{a}_F = \dfrac{(\boldsymbol{E} \cdot \boldsymbol{F})\boldsymbol{F}}{|\boldsymbol{F}|^2} = \dfrac{-10(4,-10,5)}{141}$

$$= \underline{\underline{-0.2837\boldsymbol{a}_x + 0.7092\boldsymbol{a}_y - 0.3546\boldsymbol{a}_z}}$$

(b) $\boldsymbol{E} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55,16,-12)$

$$\boldsymbol{a}_{E\times F} = \underline{\underline{\pm(0.9398,0.2734,-0.205)}}$$

**P.E. A.6** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$ showing that ***a***, ***b***, and ***c*** form the sides of a triangle.

$\boldsymbol{a} \cdot \boldsymbol{b} = 0$,
hence it is a right angle triangle.

$$\text{Area} = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}|\boldsymbol{b} \times \boldsymbol{c}| = \frac{1}{2}|\boldsymbol{c} \times \boldsymbol{a}|$$

$$\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}\begin{vmatrix} 4 & 0 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \frac{1}{2}|(3,-17,12)|$$

$$\text{Area} = \frac{1}{2}\sqrt{9 + 289 + 144} = \underline{\underline{10.51}}$$