* Vectors: A $(\overline{A})$

in Cartasian (Rectangular) coordinates:

- a vector can be finite or infinite

* Representing vectors:

1- long format :-

$$\overline{A} = Ax\,\underline{\underline{\hat{ax}}} + Ay\,\underline{\underline{\hat{ay}}} + Az\,\underline{\underline{\hat{az}}}$$

where : Ax, Ay, Az are vector components.
And $\hat{ax}, \hat{ay}, \hat{az}$ are <u>unit vectors</u>.

2- short format: prefered to perform mathmetical operations.

$$\overrightarrow{A} = (Ax, Ay, Az)$$

- but always convert final answer to long format

<u>Point</u> (x, y, z) is different from a <u>vector</u>.

* Magnitude of vector :

$$A = |\overrightarrow{A}| = \sqrt{A_x^2 + Ay^2 + Az^2}$$

* unit vectors in the direction of $\overrightarrow{A}$:

$$\overrightarrow{a_A} = \frac{\overrightarrow{A}}{|\overrightarrow{A}|} = \frac{A_x a_x + Ay\,ay + A_z a z}{\sqrt{A_x^2 + Ay^2 + Az^2}} = \hat{a}_A$$

* Vector operations: $\overrightarrow{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$\overrightarrow{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z$

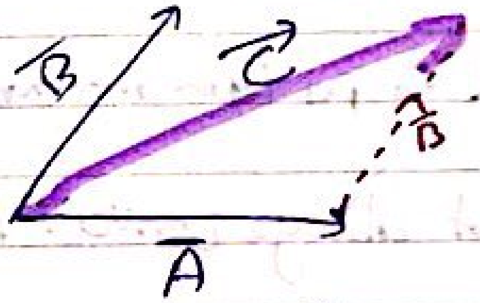
→ Addition & subtraction:

* $\overline{C} = \overline{A} + \overline{B} = (A_x + B_x)\hat{a}_x + (A_y + B_y)\hat{a}_y + (A_z + B_z)\hat{a}_z$

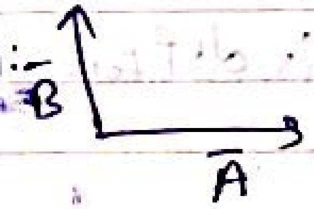
* $\overline{D} = \overline{A} - \overline{B} = (A_x - B_x)\hat{a}_x + (A_y - B_y)\hat{a}_y + (A_z - B_z)\hat{a}_z$

Graphically :-

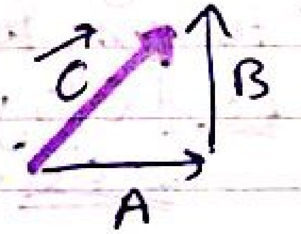
*(parallelgram method):



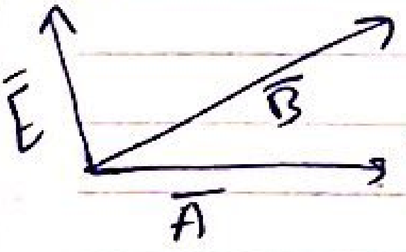
*(Trangular):-



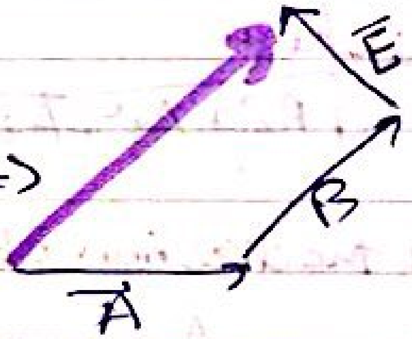
① ⇒



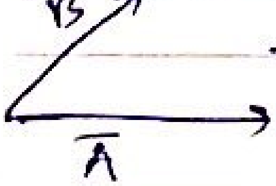
②



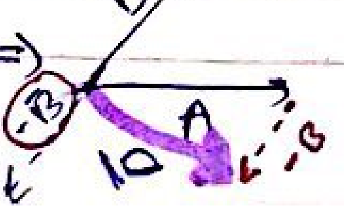
⇒



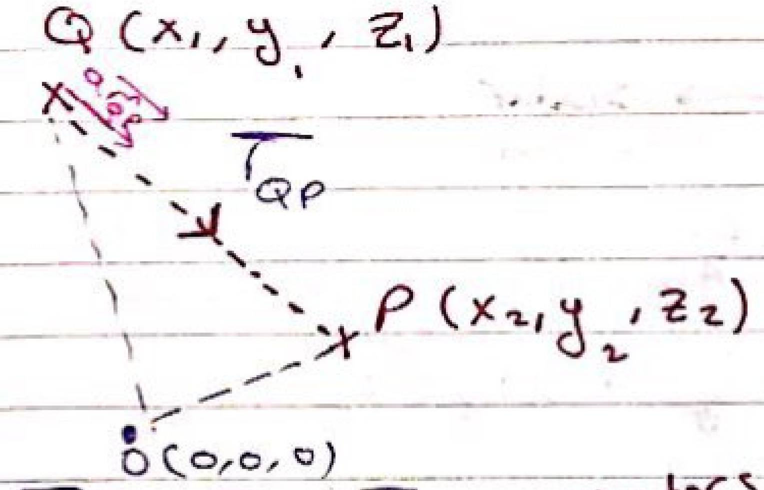
③ $\overline{D} = \overline{A} - B = \overline{A} + (-\overline{B})$



⇒



* Distance:



$\overline{r}_{QP} = \overline{r}_P - \overline{r}_Q$ vectors not points

$= (x_2, y_2, z_2) - (x_1, y_1 z_1)$

$= (x_2 - x_1)\hat{a}x + (y_2 - y_1)\hat{a}y + (z_2 - z_1)\hat{a}z$

$r_{QP} = |\overline{r}_{QP}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$$\hat{a}\, r_{QP} = \frac{\overline{r}_{QP}}{r_{QP}}$$

$\overline{QP_a} = -\overline{r}_{QP}$

$a_{P_Q} = -a\hat{r}_{QP}$

* Mutiplication:

1) Dot product ⟶ Scalar

$$\overrightarrow{A} \cdot \overrightarrow{B} = AB \cos \theta_{AB}$$

$$\overline{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

$$\overline{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z$$

$$\hat{a}_x \cdot \hat{a}_x = (1)(1) \cos 0^\circ = \boxed{1}$$

$$\hat{a}_x \cdot \hat{a}_y = (1)(1) \cos 90^\circ = \text{zero}$$

$$\overline{A} \cdot \overline{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\theta_{AB} = \cos^{-1}\left(\frac{\overline{A} \cdot \overline{B}}{|A||B|}\right)$$

$$\overline{A} \cdot \overrightarrow{B} = \overline{B} \cdot \overline{A}$$

$$A \cdot (\overrightarrow{B} + \overrightarrow{C}) = \overline{A} \cdot \overline{B} + \overline{A} \cdot \overline{C}$$

$$\overline{A} \cdot \overline{A} = A^2$$

2) cross product:

$$|\overline{A} \times \overline{B}| = AB \sin \theta_{AB}$$

$$\sin\theta_{AB} = \frac{|\overline{A}\times\overline{B}|}{AB}.$$

$$\overline{A}\times\overline{B} = 1\begin{vmatrix} \overset{1}{\hat{a}_x} & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

row → 1+1 ← coloum

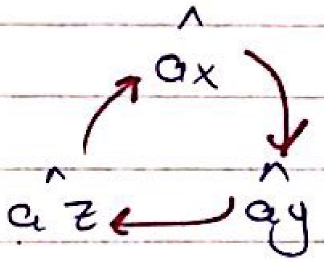
$$\overline{A}\times\overline{B} = (-1)^{1+1}(A_yB_z - A_zB_y)\overset{\hat{a}_x}{\uparrow} + (-1)^{1+2}(A_xB_z - A_zB_x)\overset{\hat{a}_y}{\uparrow}$$

$$+ (-1)^{1+3}(A_xB_y - A_yB_x)\,\hat{a}_z$$

$\overline{A}\times\overline{B} \perp \overline{A}$ and B

$|\hat{a}_x \times \hat{a}_x| = (1)(1)\sin(0) = $ zero

$|\hat{a}_x \times \hat{a}_y| = (1)(1)\sin(90) = \boxed{1}$



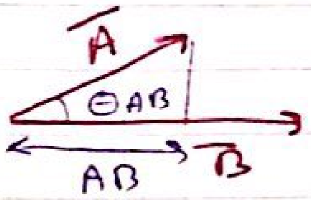
$\hat{a}_x \times \hat{a}_z = -(1)\,\hat{a}_y$

$\hat{a}_z \times \hat{a}_x = +\hat{a}_y$

$\overline{A}\times\overline{B} = -\overline{B}\times\overline{A}$

$A\times(\overline{B}+\overline{C}) = \overline{A}\times\overline{B} + \overline{A}\times\overline{C}$

* component of a vector along another vector.



A_B = scalar projection from A along B

$\overrightarrow{AB}$ = vector " " " " ".

$A_B = A\cos\theta_{AB}$

$\hat{a}_B = \dfrac{\overline{B}}{B}$

$\overline{A}\cdot\hat{a}_B = A(1)\cos\theta_{AB}$

* $\overline{A}\cdot\hat{a}_B = AB$ *

$\overline{AB} = A_B\,\hat{a}_B$

$\overline{A}_B = \underset{\underset{\text{scalar}}{\uparrow}}{(\overline{A}\cdot\hat{a}_B)}\,\hat{a}_B$

*Example: given $\overline{A} = 3\hat{a}_x + 4\hat{a}_y + 1\hat{a}_z$

$\overline{B} = 2\hat{a}_y + 5\hat{a}_z$

Find: a) The angle between A and B

b) A_B , $\overline{A_B}$

a)

Sol. $\overline{A}\cdot\overline{B} = AB\cos\theta_{AB}$

$\overline{A}\cdot\overline{B} = 0+8-5=3$

$A=\sqrt{26}$

$B=\sqrt{29}$ $\Longrightarrow \theta_{AB} = \cos^{-1}\left(\frac{3}{\sqrt{29}\cdot\sqrt{26}}\right)$

*try it using $\sin^{-1}\left(\frac{|\overline{A}\times\overline{B}|}{AB}\right)$

b) $A_B = A\cdot\hat{a}_B$

$a_B = \frac{(0,2,-5)}{\sqrt{29}}$

$A_B = \frac{8}{\sqrt{29}} - \frac{5}{\sqrt{29}} = \frac{3}{\sqrt{29}}$ or $A_B = A\cos\theta_{AB}$

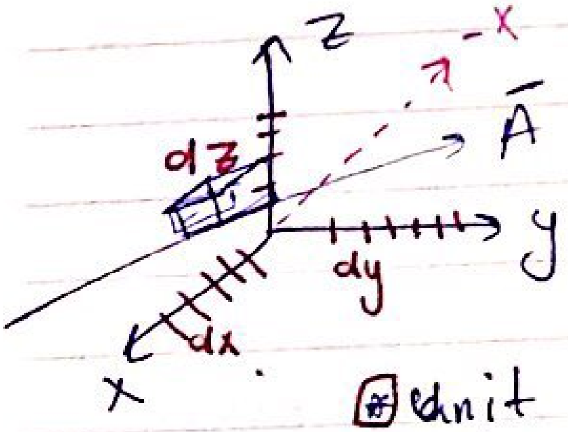
$= \frac{\cancel{\sqrt{26}}\cdot 3}{\sqrt{29}\cdot\cancel{\sqrt{26}}}$

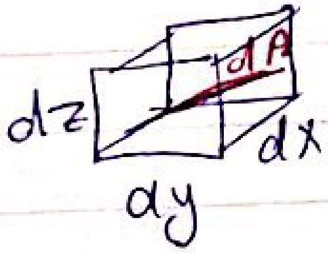
$\overline{AB} = A_B\,\hat{a}_B = \frac{3}{\sqrt{26}}\cdot\frac{1}{\sqrt{29}}(0,2,-5) = \frac{6}{29}\hat{a}_y - \frac{15}{29}\hat{a}_z$

CH.2 Coordinate Systems:

① Cartesian coordinates
(rectangular) 4

$$\left.\begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{array}\right\} \longrightarrow \text{3D object Infinite box.}$$





- Unit vectors : $\hat{a}_x, \hat{a}_y, \hat{a}_z$
- Differential elements: dx, dy, dz

- Differential length ($d\bar{L}$) → vector

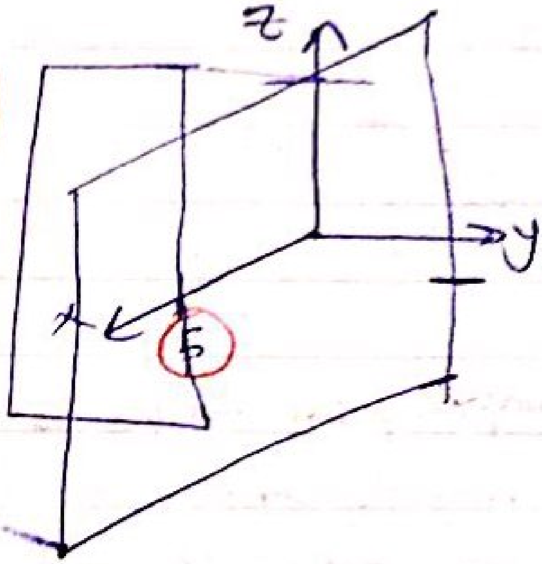
$$\bar{dL} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$

Defferential volume.

$dv = dx\,dy\,dz$ } conserve order.

→ 2D surfaces :- produced by fixing the value of one variable and varying the other two.

graph of $\boxed{x=5}$



→ 1D segment → produced by fixing 2 variables

example: ① $x=2,\ z=-2$

an infinite line paralle to the y-axis

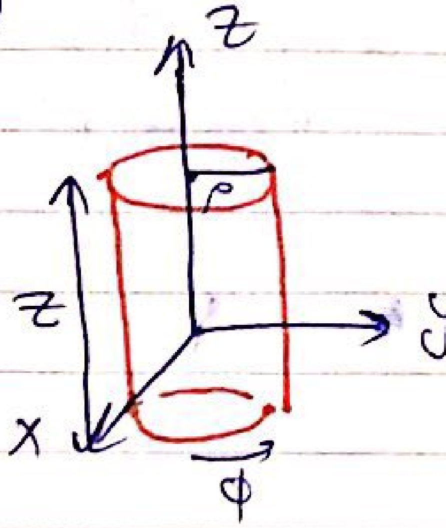
② $y=0,\ z=0$

an infinite line that lies on the x-axis

→ points: produced by fixing 3 variables.

$\boxed{2}$ cylindrical coordinates:

→ parameters: ρ, ϕ, z



→ $\begin{cases} 0 \leq \rho \leq \infty \\ 0 < \phi < 2\pi \\ -\infty < z \leq \infty \end{cases}$

→ unitvectors :- $\hat{a}_\rho$, $\hat{a}_\phi$, $\hat{a}_z$

→ Differential elements: $d\rho, d\phi, dz$

→ Differential length

$$\overline{dl} = d\rho \hat{a}_\rho + \rho d\phi \hat{a}_\phi + dz \hat{a}_z$$

* Differential Normal Surface Area :

$\overrightarrow{ds} = \rho \, d\phi \, dz \, \hat{a}_\rho$

$\overrightarrow{ds}_{Top} = \rho \, d\phi \, d\rho \, \hat{a}_z$

$\overrightarrow{ds}_{bottom} = \rho \, d\phi \, d\rho - \hat{a}_z$

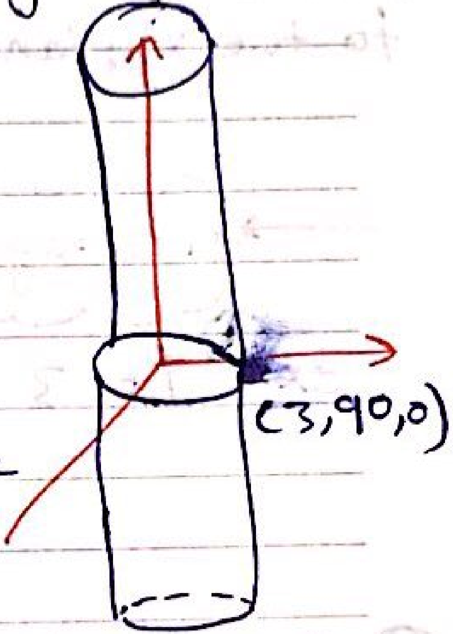
$\overrightarrow{ds}_{cut} = d\rho \, dz \, \hat{a}_\phi$

* Volume: (scalar) $\boxed{d\rho\ \rho d\phi\ dz}$

2D-Surfaces:-

produced by fixing one variable and varying the other two.

ex. *$\rho = 3$ → infinite hollow cylinder.



$\rho = 0$ → infinite line that is along the z axis.

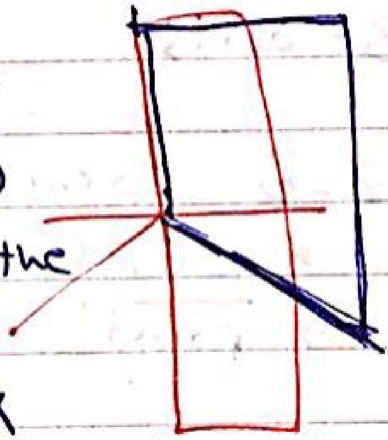
1D-segment :-

↳ when the radius is zero, the angles no longer effect graph.

ϕ = Constant

* $\phi = 90$

↳ an infinite plane that lies along the positive yz-plane.



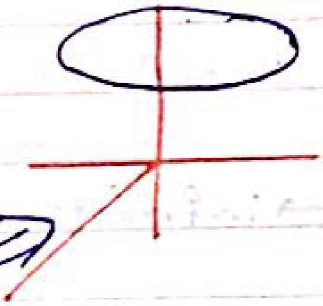
→ the normal to the 2D surface is always along the fixed variable.

→ ϕ increases from x towords y inthe counte clock wise direction- (normal will be with ϕ's movement).

* $\phi = 45$

↳ an infinite plane perpendicular to the xy plane.

$\boxed{z = \text{constant}}$

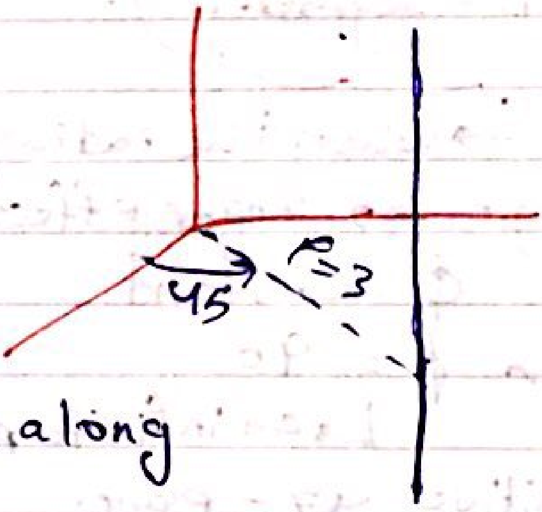


ex. $z=2$

infinite disk parallel to the xy plane

→ 1-D segments

ex. ① $\rho = 3$, $\phi = 45$



② $\rho=0$, $\phi = 70$

↳ an infinite line along the z-axis.

③ $\rho=5$, $z=5$ → circle parallel to the xy plane

if $z=0$ → a circle along the xy plane.

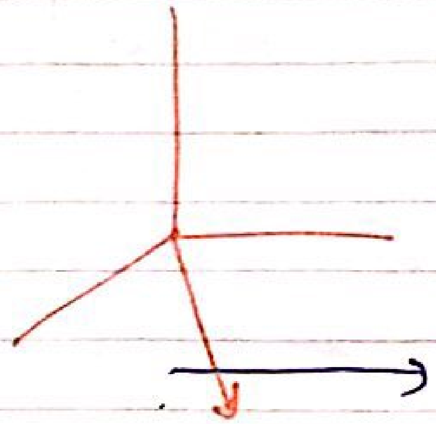
if $\rho=0$ → $\boxed{\text{point}}$ (when $\rho=0$)

④ $\phi = 45^\circ$, $z=0$ → a (semi-infinite) line along the xy plane.

$\phi = 90^\circ$, $z=?$

continue.

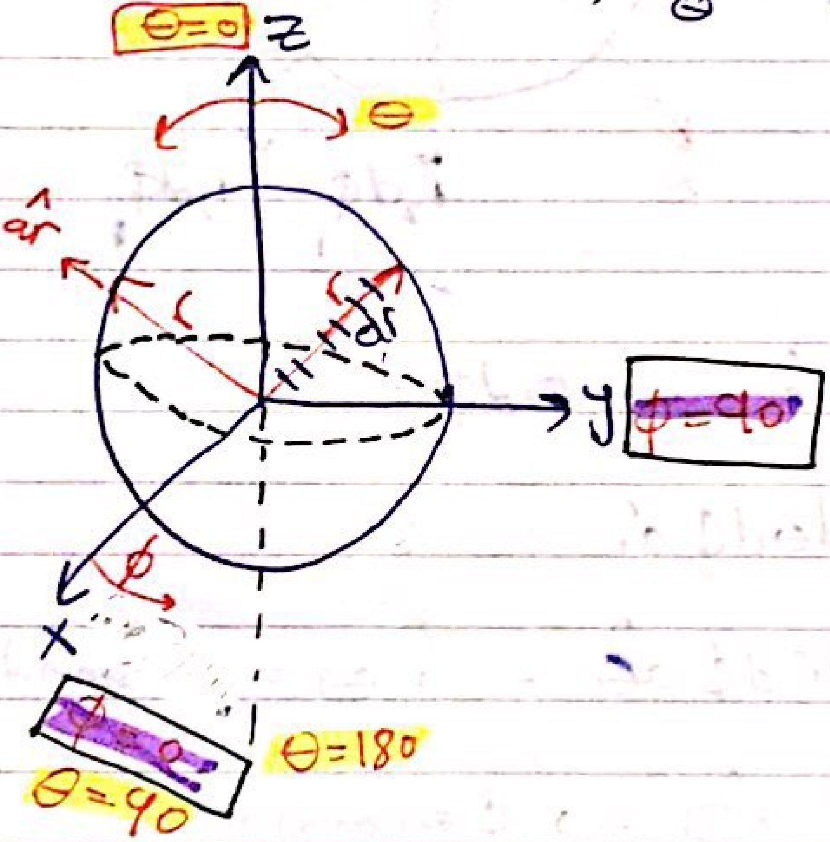
$\Longrightarrow$ semi-infinite line parallel to the xy plane.



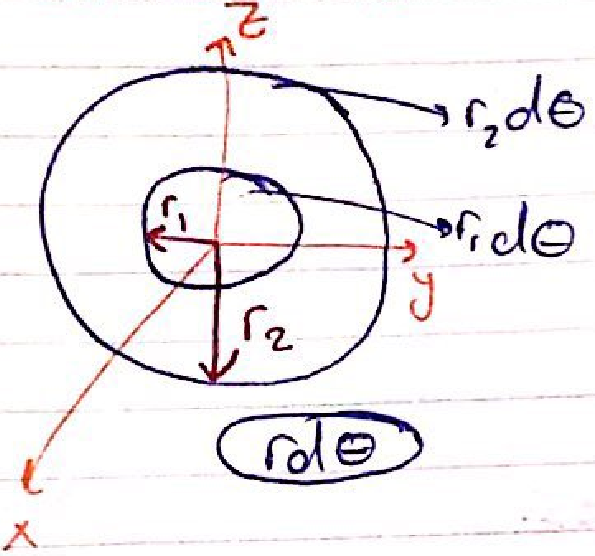
$\boxed{3}$ Spherical Coodinates.

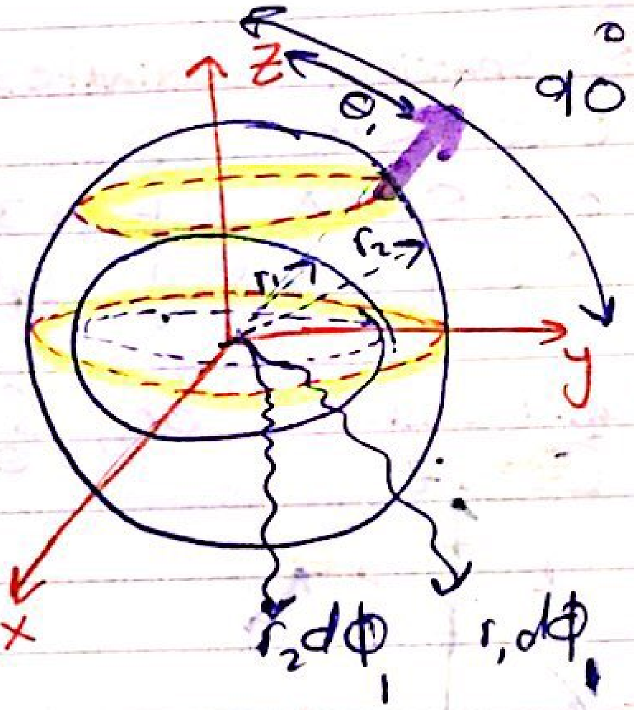
$$\left.\begin{array}{l} 0 \leq r < \infty \\ 0 \leq \theta \leq \pi \\ 0 \leq \phi \leq 2\pi \end{array}\right\} \longrightarrow$$ 3D object Inf. Solid splere.

*unit vectors: $\hat{a_r}$, $\hat{a}_\theta$, $\hat{a}_\phi$



*differential elements: dr, $rd\theta$, $r\sin\theta d\phi$





$\overline{dL} = dr\,\hat{ar} + rd\theta\,\hat{a\theta} + r\sin\theta\,d\phi\,\hat{a\phi}$

$\overline{ds} = r^2\sin\theta\,d\theta\,d\phi\,\hat{ar}$

$\overline{ds} = r\sin\theta\,dr\,d\phi\,\hat{a\theta} \longrightarrow \theta = \text{constant}$

$\overline{ds} = r\,dr\,d\theta\,\hat{a\theta} \longrightarrow \phi = \text{constant}$

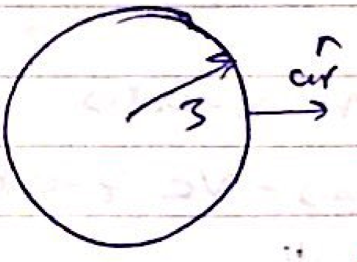
$dv = r^2\sin\theta\,dr\,d\theta\,d\phi \longrightarrow vol = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2\sin\theta\,dr\,d\theta\,d\phi$

2D Surfaces:

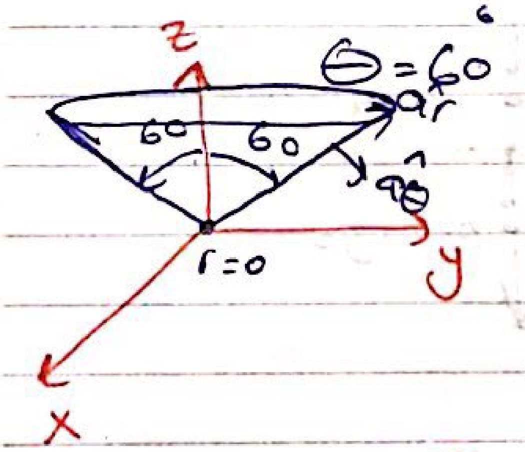
$\boxed{r = \text{constant}}$

- → $r =$ any value $\neq$ zero → hollow sphere
- → $r = 0$ → point.

$r = 3$



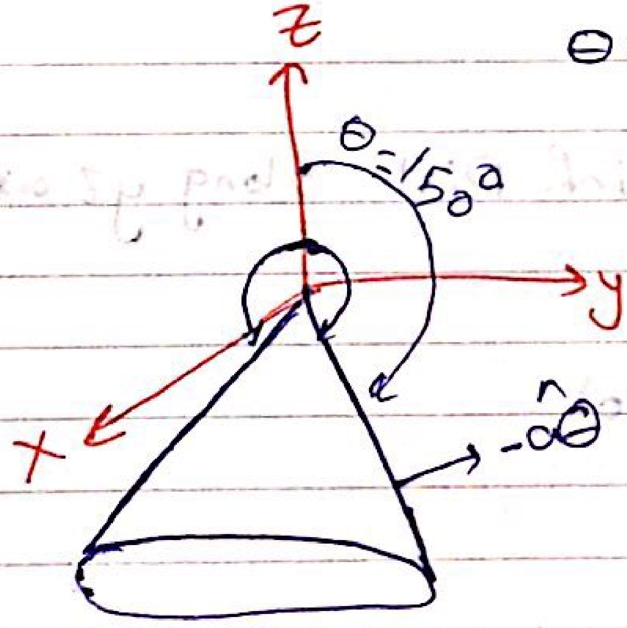
$\boxed{\theta = \text{constant}}$ * if $0 < \theta < 90°$



inf. hollow cone.

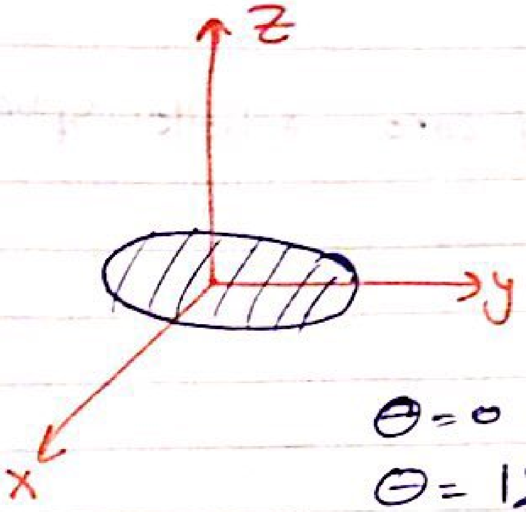
* if $90° < \theta < 180°$

$\theta = 150°$



inf. hollow cone.

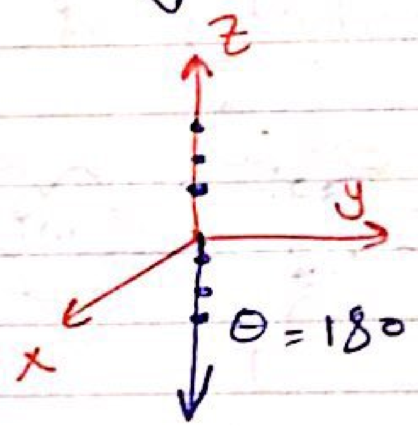
$\Theta = 90^\circ$ → inf. Disk along xy plane.



$\Theta = 0 \Rightarrow$ +ve z-axis

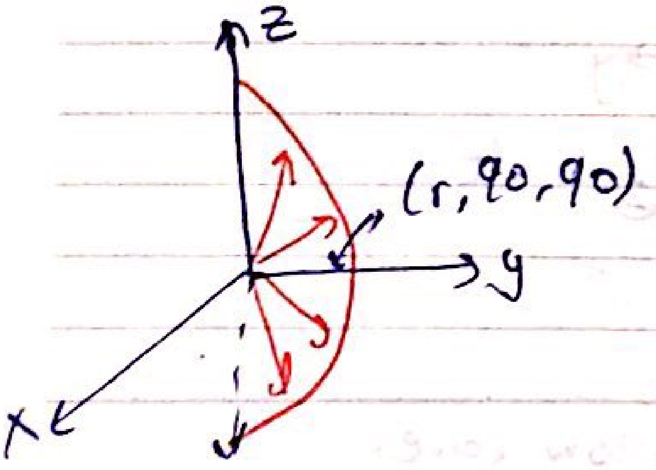
$\Theta = 180^\circ \Rightarrow$ -ve z-axis.

$\Downarrow$



$$\boxed{\phi = \text{constant}}$$

$\phi = 90$ → semi inf. Disk along yz axis



1-D segment :

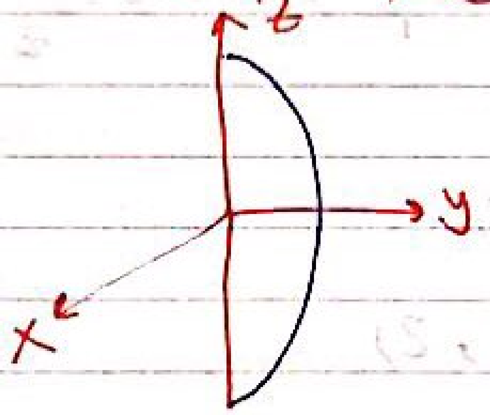
$\boxed{*}$ r, θ are constants

- circle $\begin{pmatrix} \text{if } r \neq 0 \\ \theta = 0, 180 \end{pmatrix}$ Parallel. xy plane.
- circle along xyplane. if $\begin{pmatrix} \theta = 90^\circ \\ r \neq 0 \end{pmatrix}$

$\boxed{*}$ r, ϕ are constants.

- half circle $(r \neq 0)$

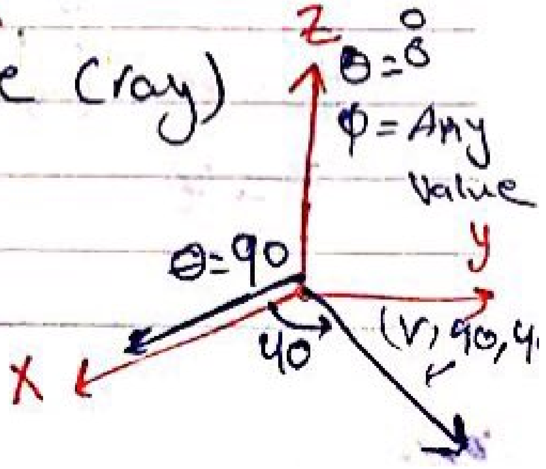
$\underset{\sim}{r=3}$, $\underset{\sim}{\phi = \frac{\pi}{2}}$



$\boxed{*}$ θ, ϕ are constants.

- Semi-inf line (ray)

$\theta = 90^\circ$, $\phi = 40$



* Transformation between coordinates :-

(x, y, z)

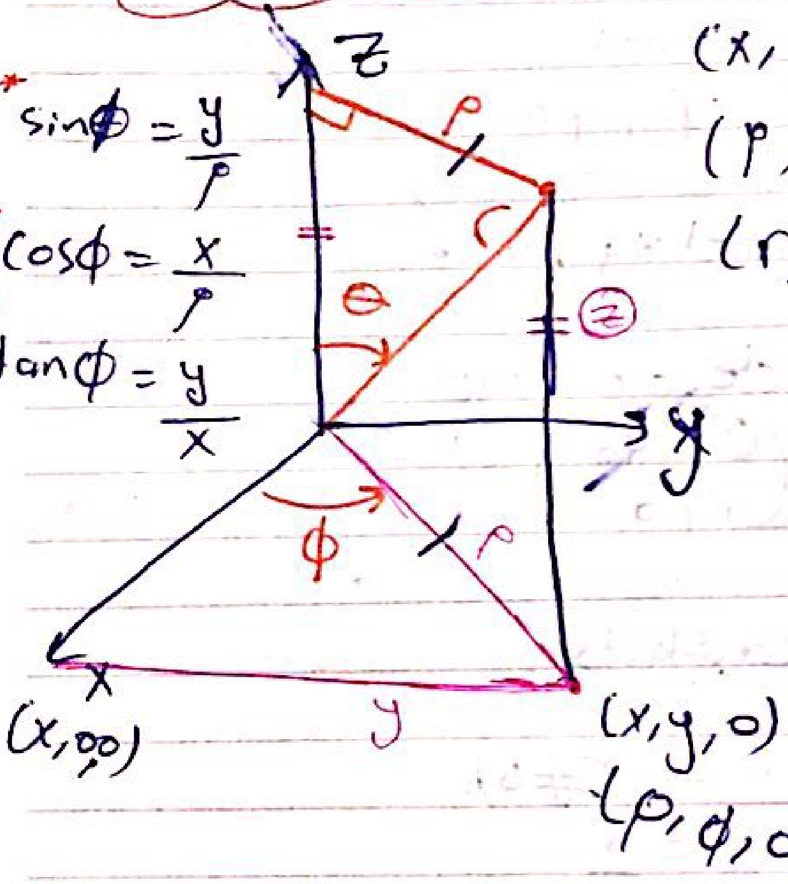
(ρ, φ, z)

(r, θ, φ)

* $\sin\phi = \frac{y}{\rho}$

* $\cos\phi = \frac{x}{\rho}$

* $\tan\phi = \frac{y}{x}$



* $\sin\theta = \frac{\rho}{r}$, * $\cos\theta = \frac{z}{r}$, * $\tan\theta = \frac{\rho}{z}$

* {Cart ⟶ cyl.

$(x, y, z) \longrightarrow (\rho, \phi, z)$

$\rho = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$z = z$

Cyl → cart :-

$x = \rho \cos\phi$

$y = \rho \sin\phi$

$z = z$

Cart → sph.

$(x, y, z) \longrightarrow (r, \theta, \phi)$

$r = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$.

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

sph → cart.

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Cyl → sph.

$(\rho, \phi, z) \longrightarrow (r, \theta, \phi)$

$r = \sqrt{\rho^2+z^2}$

$\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$

$\phi = \phi$

sph → cyl.

$\rho = r\sin\theta$

$\phi = \phi$

$z = r\cos\theta$

* Transformation of vectors.

Cart ⇄ cyl.

Cart. $\bar{A} = A_x \hat{a_x} + A_y \hat{a_y} + A_z \hat{a_z}$

cyl. $\bar{A} = A_\rho \hat{a_\rho} + A_\phi \hat{a_\phi} + A_z \hat{a_z}$

$$\underbrace{\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}}_{3X1} = \underbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3x\boxed{3}} \underbrace{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}}_{\boxed{3}x1} \xleftarrow{} \text{given}$$

$\Longrightarrow A_\rho = \cos\phi A_x + \sin\phi A_y$

i.e $A_x = x^2 y$

replace any x, y by it's en.

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$x^2y = (\rho\cos(\phi))^2(\rho\sin\phi)$

* transforming from cyl ⟶ cart we replace each coulm with row in the matrix.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

cart ⇌ sph.

cart. $\overline{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

sph. $\overline{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$

$$\begin{matrix}\hat{a}_r\\ \hat{a}_\theta\\ \hat{a}_\phi\end{matrix}\quad \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} \hat{a}_x \\ \hat{a}_y \\ \hat{a}_z \end{bmatrix}$$

we can also use the matrix to get the unit vectors.

→ replace any x, y, z by:-

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

ex. $\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$

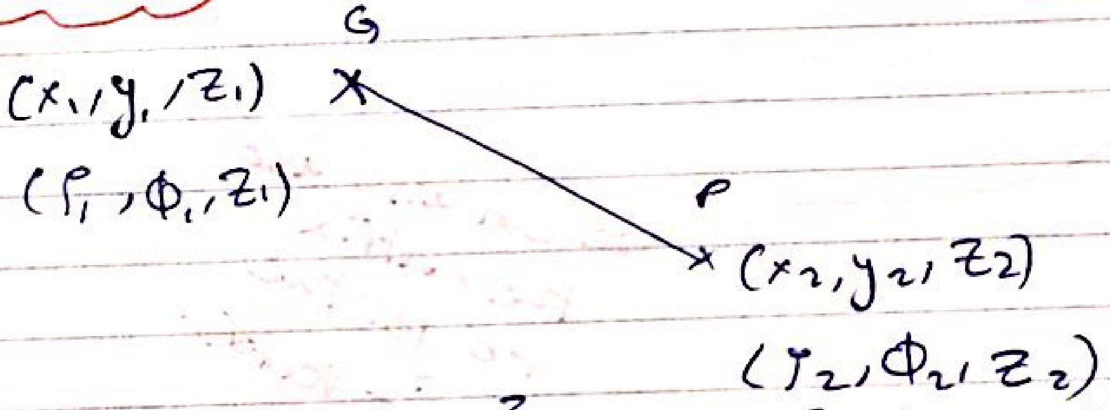
Cyl ⇄ sph $\overline{A} = (A_\rho, A_\phi, A_z) \rightarrow \overline{A} = (A_r, A_\theta, A_\phi)$

$\Longrightarrow$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$* \boxed{A_\phi = A_\phi}$$

*Distance :-



$$\text{cart} \Rightarrow d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$d^2 = (\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2)\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2$$

$$\Rightarrow \text{if } \underbrace{\phi = \phi_2}_{\cos(\phi_2 - \phi_1) = \boxed{1}} \Rightarrow (\rho_2 - \rho_1)^2 + (z_2 - z_1)^2$$

*CH.3

*Integrals :

→ line Integral → $d\bar{L}$

$$\int_{L} A \cdot d\bar{L} = \int_{x} A_x dx + \int_{y} A_y dy + \int_{z} A_z dz$$

المسار لأنه line integral

→ any vector / cart / sph / cyl...

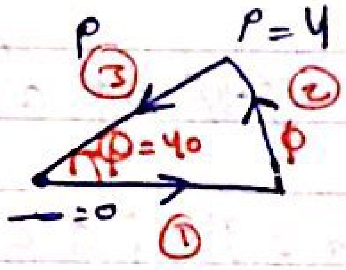
ex. $\bar{A} = (A_x, A_y, A_z)$

$d\bar{L} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

$\boxed{A \cdot d\bar{L}} = A_x dx + A_y dy + A_z dz$

⟹ Closed line integral:

$$\oint_{L} A \cdot d\bar{L}$$



$$\oint_{l} A \cdot dL = \oint_{L_1} \bar{A}\, dL_1 + \oint_{L_2} \bar{A}\, dL_2 + \oint_{L_3} \bar{A}\, dL_3$$

$dL_1 = d\rho\, a\hat{\varphi}$

$dL_2 = \rho\, d\phi\, \hat{d\phi}$

$dL_3 = d\bar{L}_1$

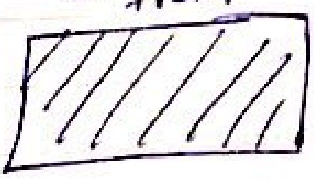
* dL دائماً +ve ، وحدود التكامل حسب اتجاه السهم .

→ Surface integral:

$\int_s \bar{A}\cdot\bar{ds}$

$\int_z\int_y A_x\, dy\, dz$

$\bar{ds}_{front} = dy dz\, \hat{a}x$



$\bar{Ax}\cdot\bar{ds}$

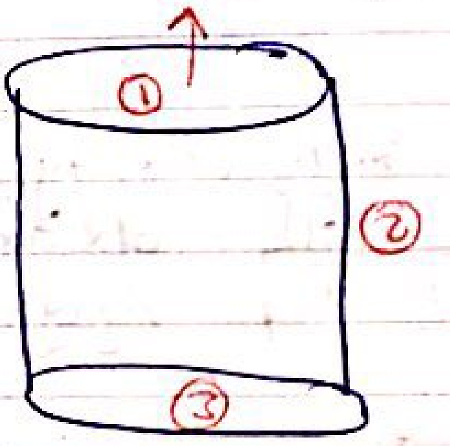
$(Ax\hat{ax} \cdot dydz\hat{ax} + (0) + (0)$

→ closed surface integral:

$\bar{A} = (A_\rho, A\phi, Az)$

$\oint_s \bar{A}\cdot\bar{ds}$

$= \oint_{s_1} \bar{A}\,\bar{ds}_1 + \int_{s_2} \bar{A}\,\bar{ds}_2 + \int_{s_3} \bar{A}.ds_3$



$\bar{ds}_1 = \rho d\rho\, d\phi\ \hat{az}$

$\bar{ds}_2 = \rho d\phi dz\ \hat{a\rho}$

$\bar{ds}_3 = \rho d\rho\, d\phi\ \hat{az}$

→ volume integral: dv

$\int_v |\hat{A}|_s dv \longrightarrow \sqrt{Ax^2 + Ay^2 + Az^2}$

$dv = r^2 \sin\theta dr d\theta d\phi$

$$\int_r A\,dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{d} A r^2 \sin\theta \, dr\, d\theta\, d\phi$$

** Del operator ∇ => vector.

→ cart :- $\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$

→ cyl :- $\nabla = \frac{\partial}{\partial \rho} + \hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial z}\hat{a}_z$

→ sph :- $\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{a}_\phi$

* uses:

① Gradient : → vector

∇v = Gradient of v

Cart $\nabla v = \frac{\partial v}{\partial x}\hat{a}_x + \frac{\partial v}{\partial y}\hat{a}_y + \frac{\partial v}{\partial z}\hat{a}_z$

ex:- $V = \rho^2\cos\phi\, e^z$ → cyl

$$\nabla V = 2\rho\cos\phi\, e^z\, \hat{a}_\rho - \rho\sin\phi\, e^z\, \hat{a}_\phi + \rho^2\cos\phi\, e^z\, \hat{a}_z$$

② divergence → scalar: $\nabla\cdot\bar{A}$ → Divergence of A

** cart : $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(A_x, A_y, A_z)$

given

*Δ cyl : $= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) \oplus \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}$

*sph: $\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} \oplus \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} \oplus \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$\boxed{3}$ Curl → vector.

$\nabla \times \bar{A} \equiv$ curl of $\bar{A}$.

(given)

in Cart : $\nabla \times \bar{A} = \begin{bmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$

in Cyl : $\nabla \times \bar{A} = \frac{1}{\rho} \begin{bmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{bmatrix}$

in Sph : $\frac{1}{r^2 \sin\theta} \begin{bmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{bmatrix}$

$\boxed{4}$ luplacian ⇒ scalar.

$\nabla \cdot \nabla V \Rightarrow \nabla^2 V \Rightarrow$ luplacian of V

→ used later in chapter 6

* Ch. 4 :- :- electrostatics fields :-

⟹ to have a static electric field. the charge should be static (notmoving, $V=0$)

*types of charges:-

1) point charge Q or q.

continuos charge:
- 2) line charge (ρ_L)
- 3) surface charge (ρ_s)
- 4) volume charge (ρ_v)

5) electric dipole.

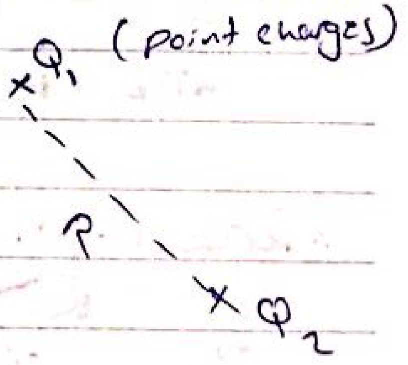
6) polarized dielectric.

⟶ two main equations deal with electrostatics:

1) Coulomb's law → general

2) Gauss law → Special case from coulomb's law.

** Coulomb's law :- (discovered by experiment).



f = force.

$F \propto Q_1, Q_2$

$f \propto \frac{1}{R^2}$ } ⟶ $F \propto \frac{Q_1 Q_2}{R^2}$

$$\Rightarrow F = \boxed{K}\frac{Q_1 Q_2}{R^2}$$

↳ constant $= \frac{1}{4\pi \epsilon_0}$ ⟹ depends on: 1) channel used 2) units used

$\boxed{\epsilon_0}$ ⟶ freespace permibility by $= \frac{10^{-9}}{36\pi} = 8.853\times10^{-12}$ F/m

$$\tilde{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 . R^2} \quad , N .$$

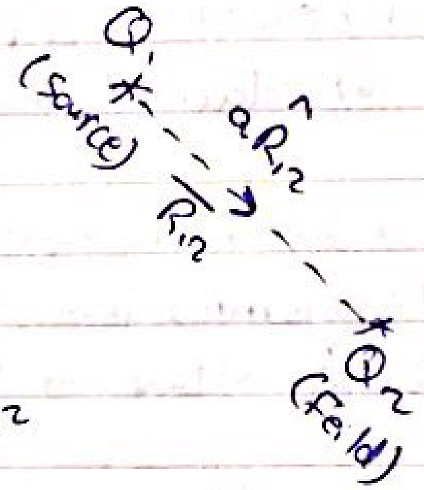
↳ magnitude only.

→ todetermine the direction of the force

the force on Q_2 due to Q_1 → $\overline{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 . R_{12}^2} \hat{aR}_{12}$

$\downarrow$ $\boxed{\text{vector}}$

→ Distance = feild - source



$\overline{f}_{12} = -\overline{F}_{12}$

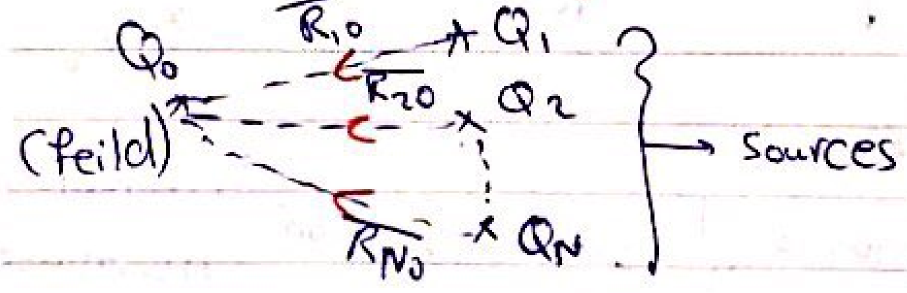
$|\overline{F}_{12}| = |\overline{F}_{21}|$

→ $\hat{aR}_{12} = -\hat{aR}_{21}$

→ we know:- $\frac{\overline{R}_{12}}{|R_{12}|} = \hat{aR}_{12}$

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 * R_{12}^2} * \frac{\overline{R}_{12}}{R_{12}} = \frac{Q_1 Q_2 \overline{R}_{12}}{4\pi\epsilon_0 . R_{12}^3}$$

* Force on point charge due to N charges:



The force on Q_0 → $\overline{F} = \overline{F}_{10} + \overline{F}_{20} + \cdots + \overline{F}_{N0}$

$$\longrightarrow F = \frac{Q_1 Q_0 (\bar{r}_0 - \bar{r}_1)}{4\pi\epsilon_0 |\bar{r}_0 - \bar{r}_1|^3} + \frac{Q_0 Q_2 (\bar{r}_0 - \bar{r}_2)}{4\pi\epsilon_0 |\bar{r}_0 - \bar{r}_2|^3} + \dots +$$

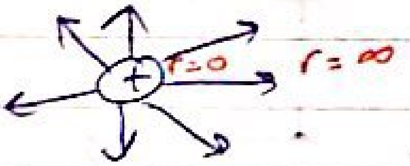
$$\frac{Q_0 Q_1 (\bar{r}_0 - \bar{r}_n)}{4\pi\epsilon_0 |\bar{r}_0 - \bar{r}_n|^3}$$

$$\Longrightarrow F = \frac{Q_0}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k (\bar{r}_0 - r_k)}{|r_0 - \bar{r}_k|^3}$$

* Electric feild Intinsity :-

$$\bar{E} = \frac{F}{Q_{feild}} = N/C \text{ or } V/m$$

$\longrightarrow$ flux lines :-



$E \propto \frac{1}{R^2}$ at $r=0 \rightarrow \bar{E} = \infty$

$r = \infty \rightarrow \bar{E} = 0$

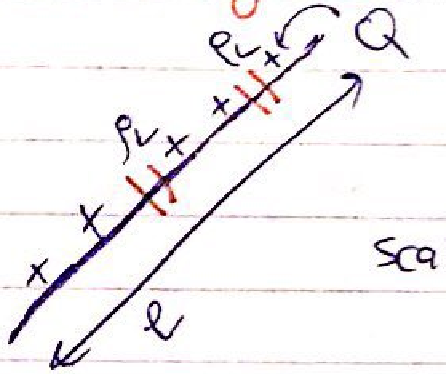
*Example: point charges 1mC and 2mC located at $\overset{S_1}{(3,2,-1)}$ and $\overset{S_2}{(-1,-1,4)}$. find F and E on along charge located at (0,3,1).

vector feild

$$\underset{S_1 \text{ on feild}}{f_1} = \frac{1 \times 10^{-3} \times 10 \times 10^{-9} (-3,1,2)}{4\pi \times \frac{10^{-9}}{36\pi} (1.6)^{\frac{3}{2}}} + f_2 \text{ feild}$$

* continuous charge distribution:

1) Line Charge dist $\longrightarrow \rho_L$



$$Q = \int \rho_L \, d_L$$

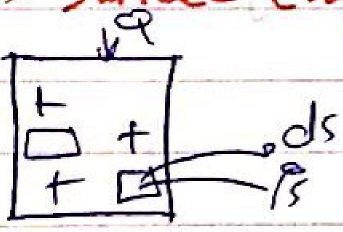
scalar scalar scalar

$$Q = \rho_L \ell$$

$$\rho_L = \frac{Q}{\ell}$$

$\hookrightarrow$ c/m

2) Surface charge dist.



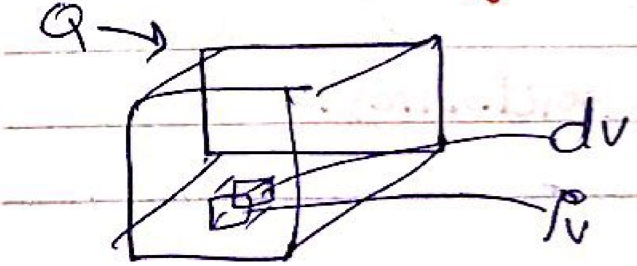
$$Q = \int_s \rho_s \, ds$$

$$Q = \rho_s S$$

$$\rho_s = \frac{Q}{S}$$

$\hookrightarrow$ c/m²

3) volume charge dist.

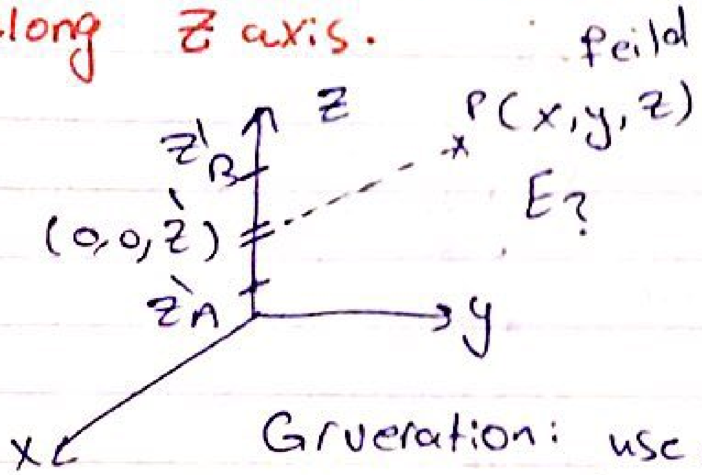


$$Q = \int_v \rho_v \, dv$$

$$\rho_v = \frac{Q}{\text{volume}}$$

$\hookrightarrow$ c/m³

E-field due to a finite line of charge along z axis.



Grueration: use dashes to represent the source - and without dashes to represent the field.

$$\overline{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r$$

$$\overline{E} = \int_l \frac{\rho_l \, dl}{4\pi\epsilon_0 r^2}\hat{a}_r \qquad \rho = \sqrt{x^2+y^2}$$

$$dl = dz'$$

$$\overline{r} = x\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z$$

$$r = \sqrt{x^2+y^2+(z-z')^2}$$

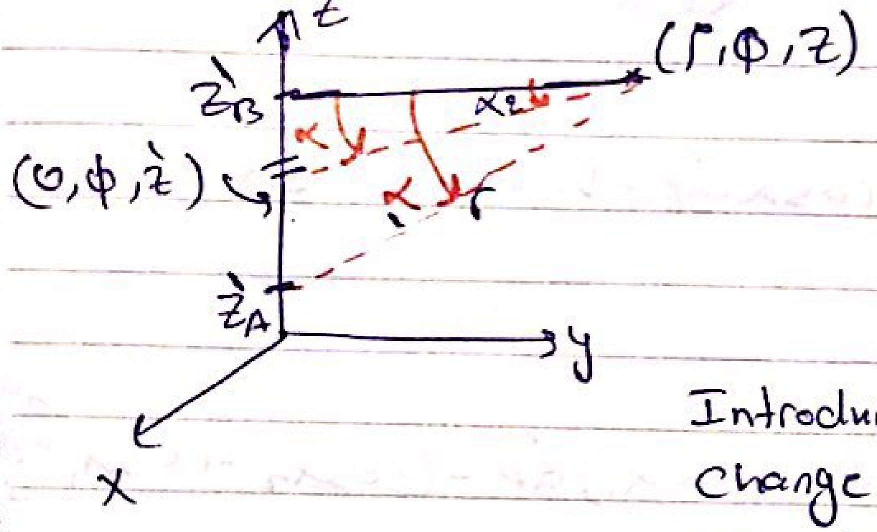
$$\overline{E} = \frac{\rho_l}{4\pi\epsilon_0}\int_{z'_A}^{z'_B} \frac{(x,y,z-z')}{[x^2+y^2+(z-z')^2]^{\frac{3}{2}}}\, dz$$

* converting to cylindrical coordinates:

$$\overline{R} = \rho\hat{a}_\rho + (z-z')\hat{a}_z$$

$$r = R = \sqrt{\rho^2+(z-z')^2}$$

$$\overline{E} = \frac{\rho_L}{4\pi} \int_{z'_A}^{z'_B} \frac{\rho \hat{a}_\rho + (z-z')\, \hat{a}_z}{\left[\rho^2 + (z-z')^2\right]^{\frac{3}{2}}}\, dz$$



Introduce angles α, α_1, α

change from $dz' \longrightarrow d\alpha$

$$\sin\alpha = \frac{z-z'}{r}$$

$$\cos\alpha = \frac{\rho}{r}$$

$$\tan\alpha = \frac{z-z'}{\rho}$$

$$\Rightarrow z - z' = \rho \tan\alpha$$

$$-dz' = \rho \sec^2\alpha \, d\alpha$$

$$\boxed{dz' = -\rho \sec^2\alpha \, d\alpha}$$

$$r^2 = \rho^2 + (z-z')^2$$

$$= \rho^2 + \rho^2 \tan^2\alpha$$

$$= \rho^2 \left(1 + \frac{\sin^2\alpha}{\cos^2\alpha}\right)$$

$$r = \rho^2 \sec^2\alpha$$

$$r^3 = \rho^3 \sec^3\alpha$$

$$* \rho = r\cos\alpha = \rho \sec\alpha \cdot \cos\alpha$$

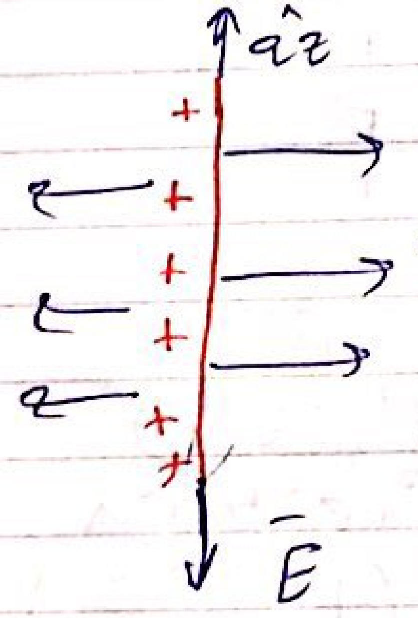
$$z - z' = r\sin\alpha = \rho \sec\alpha \sin\alpha$$

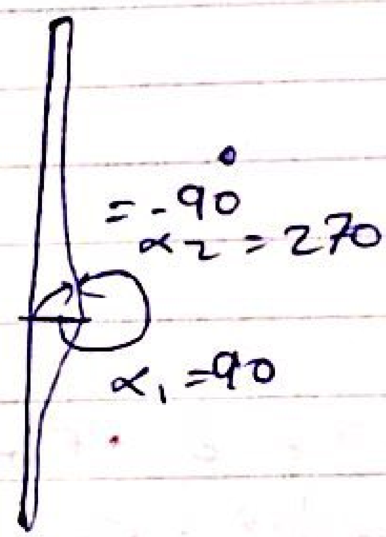
$$\Longrightarrow$$

$$\overline{E} = \frac{\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec\alpha\left(\cos\alpha\,\hat{a_\rho} + \sin\alpha\,\hat{a_z}\right)}{\rho^3 \sec^3\alpha} \left(-\rho\sec^2\alpha\right) d\alpha$$

$$\overline{E} = \frac{-\rho_L}{4\pi\epsilon_0 \rho} \int_{\alpha_1}^{\alpha_2} \left(\cos\alpha\,\hat{a_\rho} + \sin\alpha\,\hat{a_z}\right) d\alpha$$

$$\Rightarrow \frac{-\rho_L}{4\pi\epsilon_0 \rho}\left[\left(\sin\alpha_2 - \sin\alpha_1\right)\hat{a_\rho} - \left(\cos\alpha_2 - \cos\alpha_1\right)\hat{a_z}\right] \quad V/m$$

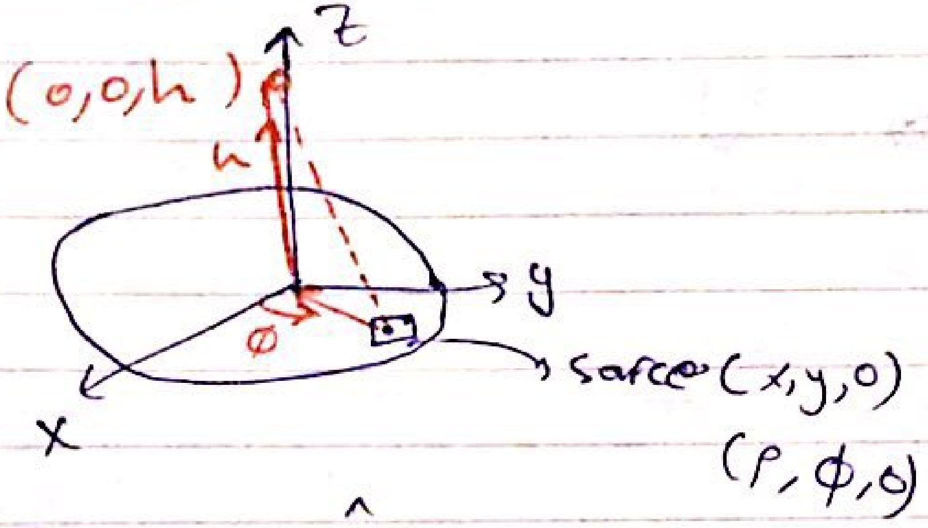




for Any inf. line

$$\overline{E} = \frac{\rho_L}{2\pi\epsilon_0 \cdot \rho}\,\hat{a_\rho}$$

Ex: for an inf. sheet of chargealon xy plane. find $\bar{E}$ at (0,0,h) or (0,0,-h).



$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{ar} \quad , \quad Q = \int_s \rho_s \, ds$$

$$\bar{E} = \int_s \frac{\rho_s \, ds \, \bar{r}}{4\pi\epsilon_0 \, r^2}$$

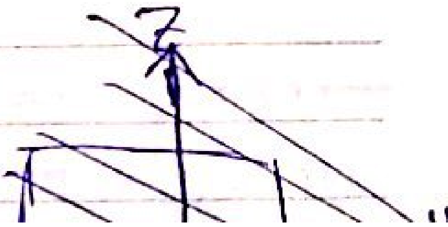
ds = dxdy cart.
ds = ρdφdz cyl.

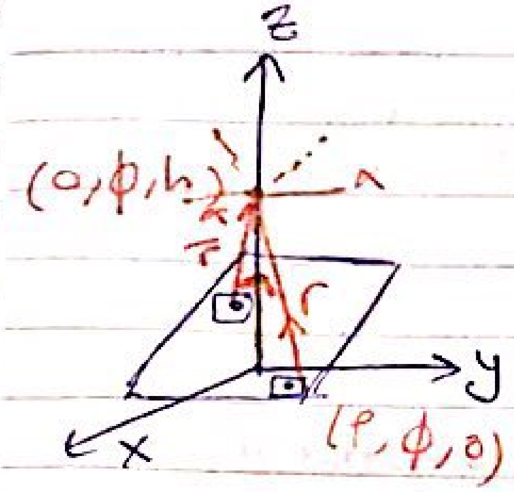
$$\bar{r} = -\rho\, a r\hat{\rho} + h a\hat{z}$$

$$r = \sqrt{\rho^2 + h^2}$$

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{-\rho a\hat{\rho} + h a\hat{z} \; \rho d\rho d\phi}{[\rho^2+\alpha^2]^{\frac{3}{2}}}$$

E - for inf. Sheet along xy plane.





$$\bar{E} = \int_s \frac{\rho_s \, ds}{4\pi\epsilon_\circ r^2} \, \hat{ar}$$

$$\bar{r} = -\rho a\hat{\rho} + h a\hat{z}$$

$$r = \sqrt{\rho^2 + h^2}$$

$$ds = \rho \, d\rho \, d\phi$$

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_\circ} \int_0^{2\pi} \int_0^{\pi} \frac{(\cancel{-\rho a\hat{\rho}} + h a\hat{z})}{[\rho^2 + h^2]^{\frac{3}{2}}} \, \rho \, d\rho \, d\phi$$

→ The ρ_ component will be cancelled due to symmetry.

$$\bar{E} = \frac{\rho_s h}{2\epsilon_\circ} \int_0^{\infty} \frac{\rho \, d\rho}{\ }$$

let $u = \rho^2 + h^2$

$du = 2\rho \, d\rho$

$\rho \, d\rho = \frac{du}{2}$

$$\bar{E} = \frac{\rho_s h}{2\epsilon_\circ} \int \frac{\frac{du}{2}}{u^{\frac{3}{2}}} \, a\hat{z}$$

$$= \frac{-\rho_s h}{2\epsilon_\circ} \, \frac{1}{2} \, \frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \Big| \, a\hat{z}$$

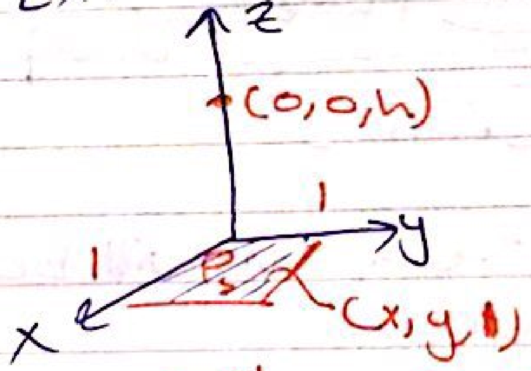
$$= \frac{-\rho_s h}{2\epsilon_\circ} \, \frac{1}{\sqrt{\rho^2 + h^2}} \Big|_0^{\infty} \, a\hat{z}$$

$$\overline{E} = \frac{-\rho_s h}{2\epsilon_0}\left[0 + \frac{1}{h}\right]\hat{a}_z$$

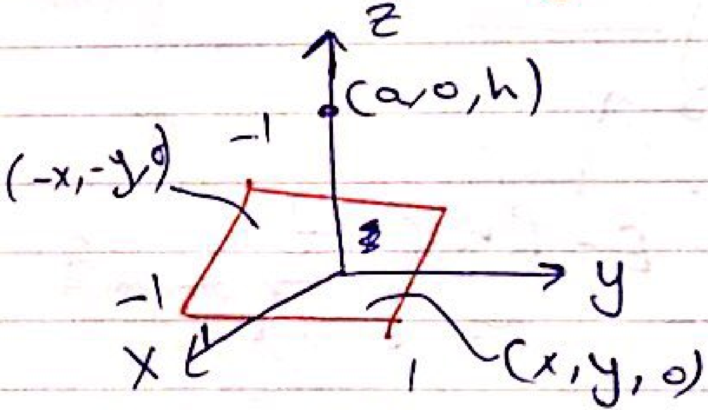
$$E = \frac{\rho_s}{2\epsilon_0}\hat{a}_z \quad V/m$$

$$\boxed{\overline{E} = \frac{\rho_s}{2\epsilon}\hat{a}_n \quad V/m} \rightarrow \text{in general}$$

* Ex.

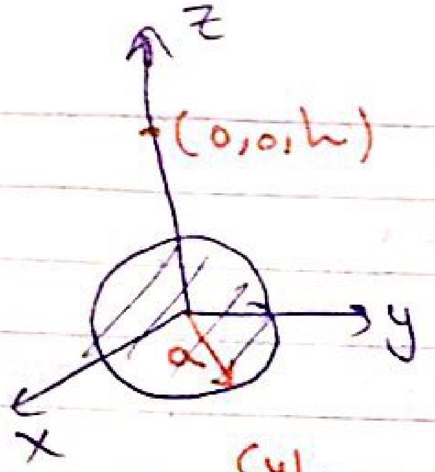


cart, no symmetry.

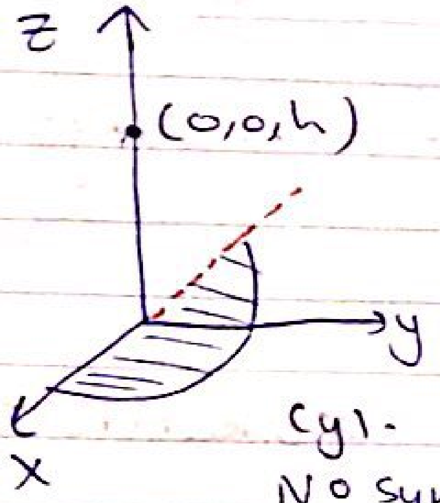


cart.

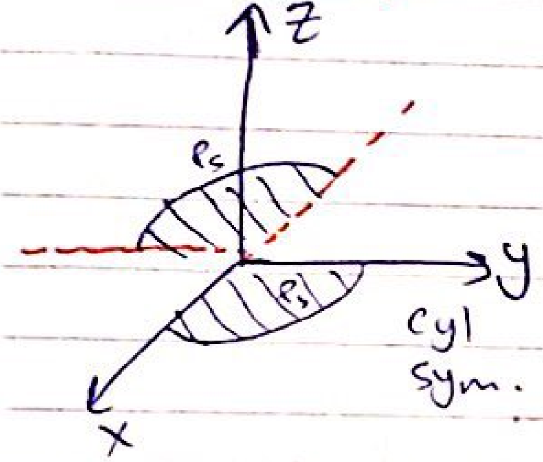
symm → x,y X



Cyl.
Sym

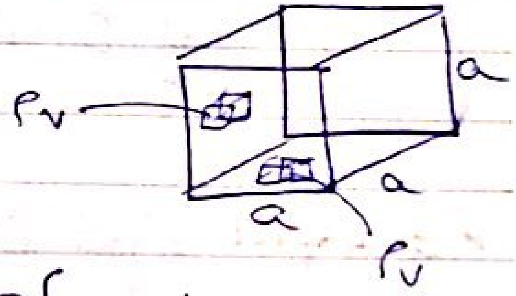


Cyl.
No syn.



cyl
sym.

$\bar{E}$-field for a uniform volume.



$$Q = \int_v \rho_v dV$$

$$\Longrightarrow \bar{E} = \int_v \frac{\rho_v dV}{4\pi\epsilon_0 r^2} \hat{a_r}$$

$$\bar{D} = \int_v \frac{\rho_v dV}{4\pi r^2} \hat{a_r}$$

* Electric flux Density ($\bar{D}$) :-

$$\boxed{\bar{D} = \epsilon_0 \bar{E}}$$ → constitutive Relation

$$\bar{E} = \frac{\bar{D}}{\epsilon_0}$$

$$\frac{F}{m} \cdot \frac{V}{m} = \frac{c}{m^2}$$

- point charge $\overline{E} = \frac{Q}{4\pi\epsilon_0 r^2} a\hat{r}$

$\overline{D} = \frac{Q}{4\pi r^2} a\hat{r}$ $\boxed{C/m^2}$

* Electric flux (ψ)
or (ψ_e)
* scalar

$\psi = \int_s \overline{D} \cdot \overline{ds} = Q$

or $\psi = \int_s \overline{E} \cdot ds = \frac{Q}{\epsilon_0}$

field

* Example: find $\overline{D}$ at (4,0,3) if there is a point charge -5π mC at (4,0,0) S1 and a line charge 3π mC/m along y-axis S2

$\overline{D} = \overline{D}_Q + \overline{D}_\rho$

$\overline{D}_Q = \frac{Q\overline{r}}{Q\pi r^3} = \frac{-5\pi * 10^{-3}(3a\hat{z})}{4\pi(3)^3} = -0.138\, a\hat{z}$ mC/m²

$\overline{D}_l = \frac{\rho_L}{2\pi\rho} a\hat{\rho} = \frac{\rho_L \overline{\rho}}{2\pi\rho^2}$

$\overline{\rho} = (4,0,3) - (0,y,0)$.

↳ shortest distance ($y=0$)

$\overline{\rho} = (4,0,3)$

$\rho = 5$

$$\bar{D}_{\rho} = \frac{3\pi \times 10^{-3} (4,0,3)}{2\pi \times 25}$$

$$= 0.24\hat{ax} + 0.18\hat{az} \quad mc/m^2$$

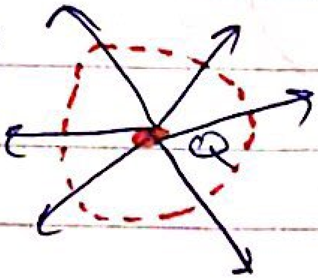
$$\bar{D} = \bar{D}_{Q} + \bar{P}_{\rho} = 240\hat{ax} + 42\hat{az} \quad \mu_c/m^2$$

$$\bar{E} = \frac{\bar{D}}{\epsilon_{\circ}}$$

* Gauss's law:-

$$\Phi = \oint \underset{\hookrightarrow C/m^2}{\bar{D}} \cdot \bar{ds} = Q_{enclosed} = Q$$

$\oint$



$$= \int_L \rho_L dL = \int_s \rho_s ds = \int_V \rho_v dv$$

$$\bar{D} = \frac{Q}{4\pi r^2} \hat{a}_r$$

$$S = 4\pi r^2$$

*→ Electric flux (scalar)

$$\Phi = \oint_s \bar{D} \cdot d\bar{s}$$

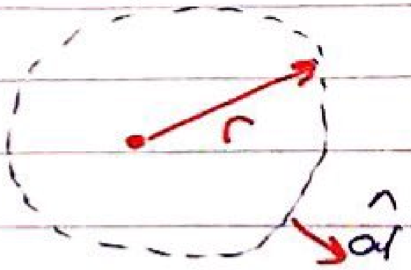
$\oint_s \bar{D} \cdot \bar{ds} = \int_v \rho_v dv$ ⟹ first Maxwell's eq. (in integral form).

$\nabla \cdot \bar{D} = \rho_v$ ⟹ 1st Maxwell's eq. int (differential form).

* Application on Gauss's law:

(*) $\boxed{1}$ To find $\bar{E}, \bar{D}$ due to a point charge.

$$\oint_s \bar{D} \cdot \bar{ds} = Q_{enc} = Q$$



$\bar{D} = D_r \underset{\sim}{\hat{a_r}}$

$\bar{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \underset{\sim}{\hat{a_r}}$

$$\int_0^{2\pi} \sin\theta \, d\theta = 2$$

$\rightarrow 4\pi$

$$\int_0^{2\pi} d\theta = 2\pi$$

$\rightarrow D_r (4\pi, 2) = Q$

$$D_r = \frac{Q}{4\pi r^2}$$

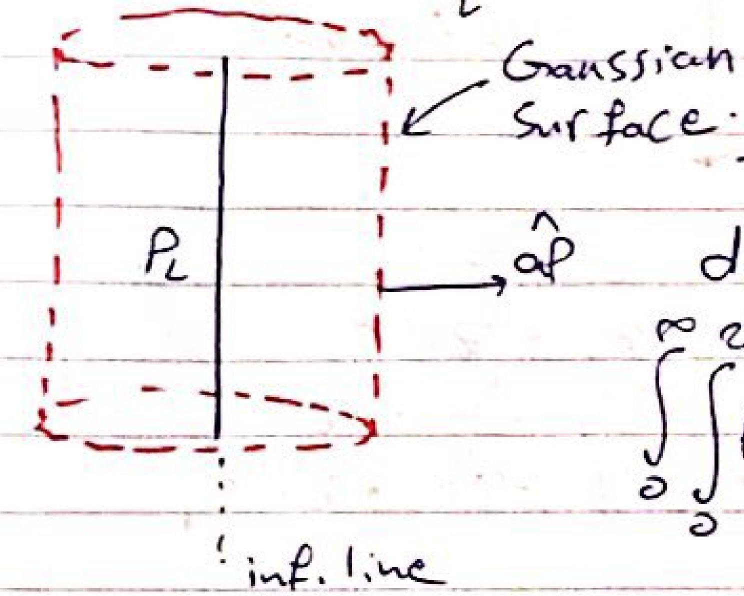
$\boxed{1}$ $\bar{D}_r = \frac{Q}{4\pi r^2} \hat{a_r}$ C/m^2

$\boxed{2}$ $\bar{E} = \frac{\bar{D}}{\epsilon_0}$

$$\bar{E} = \frac{Q}{4\pi \epsilon_0 r^2} \hat{a_r} \quad V/m$$

(*) ② To find $\bar{E}$ or $\bar{D}$ due to an infinite line of charges.

$$\oint_s \bar{D}\cdot d\bar{s} = \int_L \rho_L \, d_L$$



$\bar{D} = \rho_\rho \hat{a}_\rho$

$d_s = \rho \, d\phi \, dz \, \hat{a}_\rho$

$$\int_0^\infty \int_0^{2\pi} \rho D \rho \, d\phi \, dz = \int_{-\infty}^{\infty} \rho_L \, d_L$$

$$\int_{-L}^{L} \int_0^{2\pi} D_\rho \rho \, d\phi \, dz = \int_{-L}^{L} \rho_L \, dL, \boxed{L \to \infty}$$

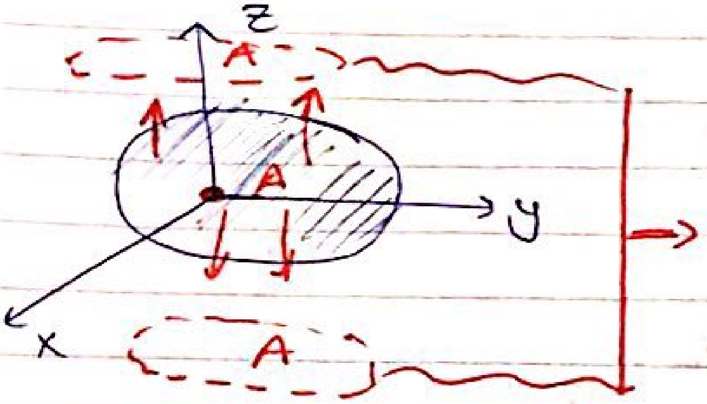
$D_\rho \rho (2\pi) = \rho_L$

$D_\rho = \frac{\rho_L}{2\pi\rho}$

$\bar{D} = \frac{\rho_L}{2\pi\rho} \hat{a}_\rho$, $\epsilon = \frac{\rho_L}{2\pi\epsilon\cdot\rho} \hat{a}_\rho$ V/m

C/m^2

$\boxed{3}$ $\overline{E}$ or $\overline{D}$ due to an inf. sheet.



$$\overline{D} = D_z(a_z) \;,\; z>0$$
$$D_z(-\hat{a}_z) \;,\; z<0$$
$$\overline{ds}_{top} = \rho d\rho d\phi \; \hat{a}_z \;,\; z>0$$
$$\overline{ds}_{bot} = \rho d\rho d\phi(-\hat{a}_z) \;,\; z<0$$

$$\int_{s_{top}} \overline{D} \cdot ds_{top} + \int_{s_{bot}} \overline{D} \; \overline{ds}_{bot}$$

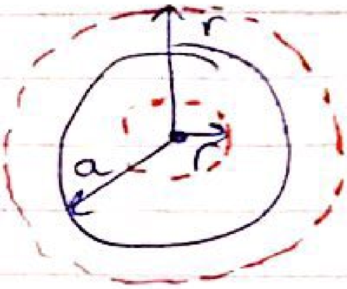
$$D_z \cancel{A} + D_z \cancel{A} = \rho_s \cancel{A}$$

$$D_z = \frac{\rho_s}{2}$$

$$D = \frac{\rho_s}{2} \hat{a}_n \begin{cases} \hat{a}_z \\ -\hat{a}_z \end{cases}$$

$$\overline{E} = \frac{\rho_s}{2\epsilon_0} \hat{a}_n \quad v/m$$

$\boxed{4}$ $\overline{E}$ or $\overline{D}$ for a uniform volume charge dist. On a sphere of radius (a). (find $\overline{E}, \overline{D}$ everywhere).



for $0<r<a$

$$\oint_s \bar{D}\cdot\bar{ds} = Q_{enc} = \int_v \rho_v dv$$

$$\bar{D} = D_r \hat{a_r}$$

$$ds = r^2 \sin\theta d\theta d\phi \hat{a_r}$$

$$\int_0^{2\pi}\int^{\pi} D_r r^2 \sin\theta d\theta d\phi = \int_0^{2\pi}\int^{\pi}\int_0^{r} \rho_v r^2 dr d\theta d\phi$$

$$D_r \cdot \cancel{4\pi} \cancel{r^2} = \rho_v \cancel{4\pi} \cancel{r^3}$$

$$\bar{D}$$

$$D_r = \frac{\rho_v \cancel{r^2}}{3} = \frac{\rho_v r \hat{a_r}}{3} \text{ c/m}^2$$

$$\bar{E} = \frac{\rho_v r}{3\epsilon_0} \hat{a_r}$$

for $a<r<\infty$

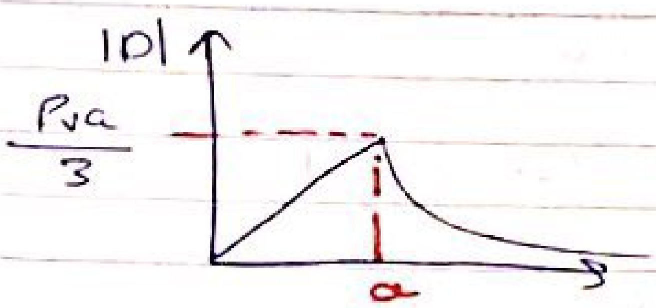
$$4\pi r^2 D_r = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_v r^2 \sin\theta \, dr d\theta d\phi$$

$$\oplus \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_v r^2 \sin\theta \, dr d\theta d\phi$$

$$4\pi r^2 D_r = \underline{\rho_v 4\pi a^3}$$

$$\bar{D} = \frac{\rho_v a^3}{3r^2} \hat{a_r} , a<r \; 3$$

$$\bar{E} = \frac{\rho_v a^3}{3\epsilon_0 r^2} , a<r$$



* example: given $\overline{D} = z\rho\cos^2\phi\,\hat{a}_z$ c/m²,

- calculate: a) The charge density at $(1, \frac{\pi}{4}, 3)$

b) the total charge enclosed by the cylinder of radius 1m and $-2 \leq z \leq 2$m

sol.

a) ρ_L ×

$\rho_s = \overline{D} \cdot \hat{a}_n$

✓ $\rho_v = \nabla \cdot \overline{D}$

$$\rho_v = \nabla \cdot \overline{D} = \frac{\partial dz}{\partial z} \Rightarrow \rho_v = \rho\cos^2\phi \text{ c/m}^3$$

$$\rho_v \Big|_{(1,\frac{\pi}{4},3)} = 0.5 \text{ c/m}^3$$

b) $Q = \int_v \rho_v \, dv$

$$Q = \oint_s \rho_s \, ds = \oint_s \overline{D} \cdot \overline{ds}$$

$$= \int_v \nabla \cdot \overline{D} \, dv$$

$$\boxed{Q = \int_v \rho_v \, dV}$$

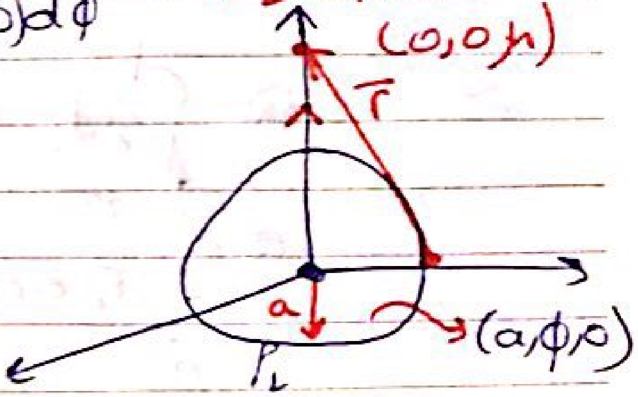
$\rightarrow$ b) $$Q \int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{1} \rho \cos^2\phi \, \rho \, d\rho \, d\phi \, dz$$

$$\left(\frac{1}{3}\right)(2 - -2) \int_0^{2\pi} \cos^2\phi \, d\phi$$

$$\Rightarrow \frac{4}{3}\int_0^{2\pi} \frac{1}{2}(1+\cos 2\phi)\, d\phi$$

$$\boxed{= \frac{4}{3}\pi \, C} \ast$$

* Exaple ②



a) Find $\bar{E}$ at (0,0,h)?

$$\bar{E} = \int_L \frac{\rho_L \, d_L}{4\pi\epsilon_0} \hat{a_r} = \int \frac{\rho_L \, d_L}{4\pi\epsilon_0 r^2} \cdot \frac{\bar{r}}{r}$$

~~① $d_L = \rho d\phi$~~ ~~$\int \rho_L \, d_L$~~

① $d_L = \rho \, d\phi = a \, d\phi$ ② $\bar{r} = -a\hat{a_\rho} + h\hat{a_z}$

$r = \sqrt{a^2 + h^2}$

③ $$\bar{E} = \frac{\rho_L}{4\pi\epsilon_0}\int_0^{2\pi} \frac{-a\hat{a_\rho} \, h\hat{a_z}}{(a^2+h^2)^{\frac{3}{2}}}$$

$$\Rightarrow \bar{E} = \frac{\rho_L * h * a}{4\pi\epsilon_0 (a^2+h^2)^{\frac{3}{2}}} \int_0^{2\pi} d\phi = 2\pi$$

$$\bar{E} = \frac{\rho_L \, h a}{2\epsilon_0 (a^2+h^2)^{\frac{3}{2}}} \hat{a_z} \quad V/m$$

$\Longrightarrow$

b) proof: if a → 0 then the $\bar{E}$ -field will be as a point Charge

(*) $$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r}$$

Sol. $Q = \int \rho_L \, d_L$

$$Q = \int_{d}^{2\pi} \rho_L \, a \, d\phi$$

$$Q = 2\pi a \rho_L$$

$$\rho_L = \frac{Q}{2\pi a}$$

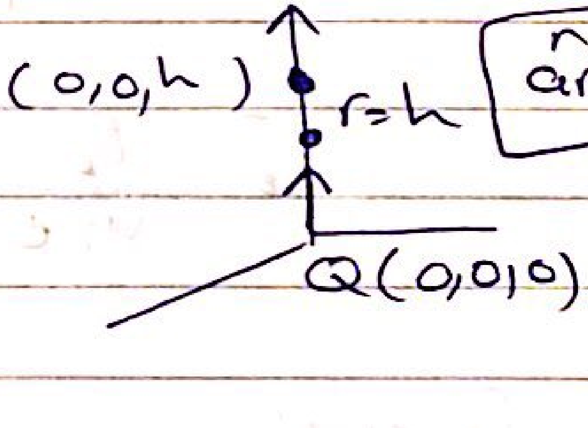
$$\xrightarrow{} \quad E = \frac{\frac{Q}{2\pi a}\, ha}{2\epsilon \cdot (a^2+h^2)^{\frac{3}{2}}}$$

$$\xrightarrow{} \quad \bar{E} = \frac{Q}{4\pi\epsilon \cdot h^2}\,\hat{a}_z$$

$$\Downarrow$$

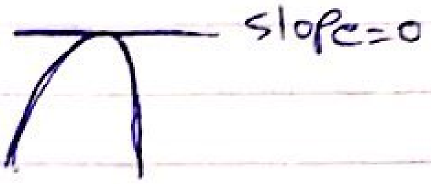
$$= \frac{Q}{4\pi\epsilon_0}\,\hat{a}_r \quad \#$$

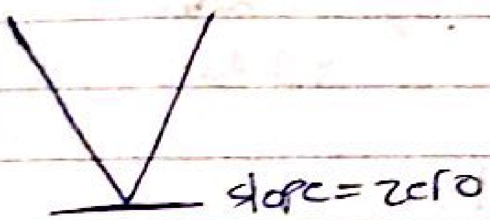
(*) because $r^2 = h^2$



$$\boxed{\hat{a}_r = \hat{a}_z}$$

c) determine the value of (h) to have max(E)





$$\frac{dE}{dh}=0$$

$$\frac{d \cdot \dfrac{P_L h a \, a\hat{z}}{2\varepsilon \cdot (a^2+h^2)^{\frac{3}{2}}}}{dh} = \boxed{\text{zero}}$$

$$\frac{P_L a}{2\varepsilon \cdot}\left(\frac{(a^2+h^2)^{\frac{3}{2}}(1)-(h)(\tfrac{3}{2})(a^2+h^2)^{\frac{1}{2}} * (2h))}{(a^2+h^2)^3}\right)$$

$$\Longrightarrow (a^2+h^2)^{\frac{3}{2}} - 3h^2(a^2+h^2)^{\frac{1}{2}} = 0$$

$$(a^2+h^2)^{\frac{1}{2}}(a^2+h^2-3h^2)=0$$

$$\underset{\neq 0}{\downarrow} \qquad \underset{a^2-2h^2=0}{\downarrow}$$

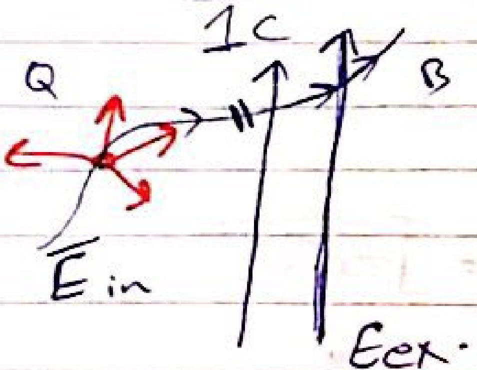
$$\hookrightarrow \boxed{h=\frac{\pm a}{\sqrt{2}} \text{ m}}$$

* Electric potential:

(v) → scalar

$$\text{voltage} = \frac{\text{work}}{\text{charge}}$$

$$1\text{v} = \frac{1\text{J}}{1\text{c}}$$



$$W = \overset{\text{the work is done.}}{-}\bar{F}\cdot\bar{l} \text{ by } \bar{E}\,\text{ext.}$$

$$= -\int_l \bar{F}\cdot\bar{dl} \quad , \bar{F} = Q\bar{E}$$

$$W = -Q\int_l \bar{E}\cdot\bar{dl}$$

$$\underset{\curvearrowleft}{V_{AB}} = \frac{W}{Q} = -\int_l \bar{E}\cdot\bar{dl}$$

$$= V_B - V_A$$

* if V_{AB} is +ve → gain in potential

→ $\int_l \bar{E}\cdot\bar{dl}$ is −ve

→ The work is done by E ext.

*if V_{AB} is -ve → drop in potential.

↳ $\int_{l} \bar{E}\cdot\bar{dl}$ is +ve.

↳ The work is done by the field itself.

* V for a point charge:

$$V = -\int_{l} \bar{E}\cdot\bar{dl} \, , \quad \bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a_r} \longrightarrow * \ \boxed{\bar{dl} = dr\,\hat{a_r} \cdots x \ x}$$

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0 r}\Bigg|_{r_A}^{r_B} = \frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_A}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$= V_B - V_A.$$

$$V = \frac{Q}{4\pi\epsilon_0 r} \quad (V)$$

* for line charge ⟹ $V = \int_{l} \frac{\rho_L \, dl}{4\pi\epsilon_0 r}$

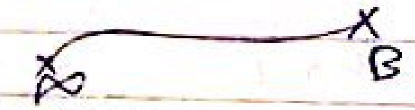
* for surface charge ⟹ $V = \int \frac{\rho_s \, ds}{4\pi\epsilon_0 r}$

* for volume charge $\Rightarrow V = \int_{v} \frac{\rho_v \, dv}{4\pi\epsilon_0 r}$

* for point charges:

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k}{|\vec{r} - \vec{r}_k|} \quad (V)$$

- Assume point a is at ∞



$r_A \simeq \infty$

$$V_{\infty B} = V_B - V_\infty$$

$$= \frac{Q}{4\pi\epsilon_0 r_B} - \cancelto{0}{\frac{Q}{4\pi\epsilon_0 r_\infty}}$$

$$= \frac{Q}{4\pi\epsilon_0 r_B} = V_B$$

* Example : Two point charges $-4\mu c$ and $5\mu c$ are located at $(2,-1,2)$ and $(0,4,-2)$

Find the potential at $A(1,0,1)$

ref. ⟶ $V_\infty = 0$

$$* \; V_A = \frac{-4 * 10^{-6}}{4\pi \frac{*10^{-9}}{36\pi} * \underbrace{|(-1,1,-2)|}_{\nearrow \sqrt{6}}}$$

$$\oplus \; \frac{5 * 10^{-6}}{4\pi \frac{10^{-9}}{36\pi} |(1,-4,3)|} \longleftarrow \sqrt{26}$$

$\Rightarrow V_A - V_\infty = V_A - \text{zero} = V_A$

$$\boxed{V_A = -5.872\ \text{kV}}$$

*Example: A point charge of 5nc located at (−3, 4, 0) and a line $y=1$, $z=1$ carries a uniform charge of 2nc/m

b) If V= 100 V at B(1,2,1) find V at C(−2,5,3). ← field & ref.

↖ field

ref is point B

$V_{BC} = V_C - V_B$

$V_C = V_{BC} + V_B$

$V_{AB} = V_B - V_A$

$V_{\infty B} - V_{\infty A}$

$V_B - V_\infty - V_A + V_\infty$

$= V_B - V_A$

$V_{BC} = V_{BCQ} + V_{BC\ell}$

$$V_{BCQ} = -\int_{\ell} \bar{E}\cdot\bar{dL} = -\int_{r_B}^{r_C} \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r \cdot dr\,\hat{a}_r$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_C} - \frac{1}{r_B}\right)$$

$r_C = |(-2,5,3)-(-3,4,0)| = |(1,1,3)| = \sqrt{11}$

$r_B = |(1,2,1)-(-3,4,0)| = |(4,-2,1)| = \sqrt{21}$

$$V_{BC\ell} = \int_{s} \frac{\rho_L \, dL}{4\pi\epsilon . r} \times$$

$$V_{BC\ell} = -\int_{\rho} \bar{E} \cdot \bar{dL}$$

$$= -\int_{\rho_B}^{\rho_C} \frac{\rho_L}{2\pi\epsilon \cdot \rho} \hat{a\rho} \cdot d\rho \, \hat{a\rho} \ldots$$

$$= \frac{-\rho_L}{2\pi\epsilon .} (\ln \rho_C - \ln \rho_B)$$

$$\frac{-\rho_L}{2\pi\epsilon} \ln\left(\frac{\rho_C}{\rho_B}\right)$$

$$\rho_C = |(-2,5,3)-(-2,1,1)| = |(0,4,2)| = \sqrt{20}$$

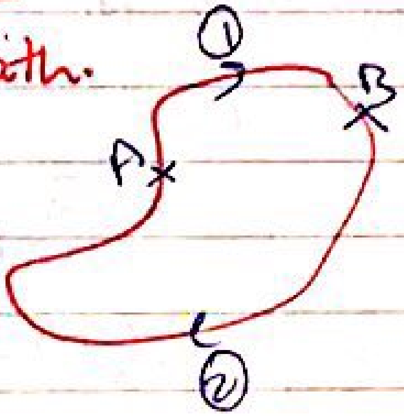
$$\rho_B = |(1,2,1)-(1,1,1)| = |(0,1,0)| = 1$$

$$V_{BC} = -50.175 \text{ V}$$

$$V_C = 49.825 \text{ V} < V_B = 100 \text{ V}$$

* Relation between $\bar{E}$ and V :-

→ voltage over a closed path.



$$V = \oint_{\ell} \bar{E} \cdot \bar{dL}$$

$$= -\int_{r_A}^{r_B} \bar{E} \cdot \bar{dL} + -\int_{r_B}^{r_A} \bar{E} \cdot \bar{dL}$$

$$= -\int_{r_A}^{r_B} \bar{E} \cdot \bar{dL} + \int_{r_A}^{r_B} \bar{E} \cdot \bar{dL} = 0$$

$\oint_{l} \overline{E} \cdot dl = 0 \Rightarrow$ equi-potential surface.

$v = 0$

(2nd Maxwell's eq.)
in integral form.

$\boxed{\nabla \times \overline{E} = 0} \rightarrow$ 2nd Maxwell's eq
(in a diff form).

$\oint_{l} \bar{E}\cdot dL = 0 \implies$ equi-potential surface.

$V = 0$

(2nd Maxwell's eq.) integral form.

$\boxed{\nabla \times \bar{E} = 0} \longrightarrow$ 2nd Maxwell's eq (in a diff form).

E is irrational

$\nabla \cdot \bar{D} = \rho v \implies D \cdot \epsilon_o \bar{E} = \rho v$

$\nabla \cdot \bar{E} = \frac{\rho v}{\epsilon_o E} \neq 0$

⊛ To find $\bar{E}$ from V

$V = -\int_{l} \bar{E}\cdot dL \implies dv = -E\cdot \bar{dL}$

in cart: $\bar{E} = E_x dx + E_y dy + E_z a\hat{z}$

$\bar{dl} = dx\, a\hat{x} + dy\, a\hat{z} + dz\, a\hat{z}$

$E\cdot \bar{dl} = E_x dx + E_y dy + E_z dz$

$dv = \frac{dv}{\partial x} dx + \frac{dv}{\partial y} dy + \frac{dv}{\partial z} dz$

- equating the x component.

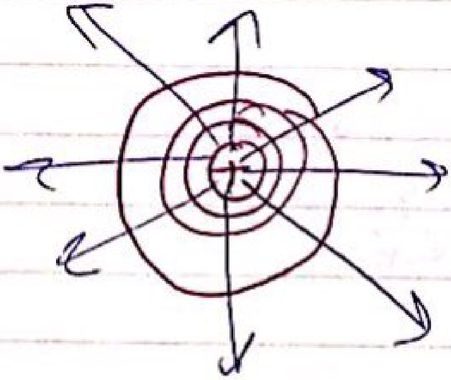
$\frac{dv}{\partial x} dx = -E_x dx \longrightarrow \boxed{E_x = \frac{-\partial v}{\partial x}}$

→ equating the y-componet:

$$\frac{\partial V}{\partial y} dy = -E_y dy \longrightarrow \boxed{E_y = \frac{-\partial V}{\partial y}}$$

→ equating the z-component:

$$\frac{\partial V}{\partial z} dz = -E_z dz \longrightarrow E_z = \frac{-\partial V}{\partial z}$$



So :-

$$\overline{E} = -\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z} a_z$$

$$\boxed{\overline{E} = -\nabla V} \rightarrow \text{gradient}$$

$\nabla \times \overline{E} = \text{zero} \rightarrow E$ is irrotational

$\nabla \times (-\nabla V) = \text{zero} \longrightarrow$ Always

* **Example:** Given $V = \underline{10}$ $\sin\theta\cos\phi$

Find: a) $\overline{D}$ at $(2, \frac{\pi}{2}, 0)$ r^2

b) Calculate the work done in moving a $10 \mu C$ charge from $A(1, 30^\circ, 120^\circ)$ to $B(4, 90^\circ, 60^\circ)$

Sol → next

$\overline{E} = -\nabla V \quad , \overline{D} = \epsilon . E$

$\overline{D} = \epsilon . \nabla V$

$\overline{E} = \left(\frac{\partial V}{\partial r} \hat{ar} + \frac{\partial V}{r d\theta} a\hat{\theta} + \frac{\partial V}{r\sin\theta d\phi} a\hat{\phi} \right).$

$\overline{E} = \left(\frac{20\sin\theta}{r^3} \cos\phi \hat{ar} - \frac{10}{r^3} \cos\theta\cos\phi \hat{a\phi} + \right.$

$\left. \frac{10}{r^3} \sin\phi \hat{a\phi} \right).$

$\overline{D}\Big| = \epsilon . \overline{E} = \epsilon . \left(\frac{20}{8} \hat{ar} \right) = 22.1 \hat{ar} \ c/m^2$

$(2, \frac{\pi}{2}, 0)$

b) $W = QV_{AB}$

$= Q(V_B - V_A)$

$= 10 * 10^{-6} \left(\left(\frac{10}{10} * \frac{1}{2} \right) - \left(10 * \frac{1}{2} * -\frac{1}{2} \right) \right)$

$= 10 * 10^{-6} \left(\frac{10}{5} + \frac{5}{2} \right) = 28.125 \ MJ$

Method 2.

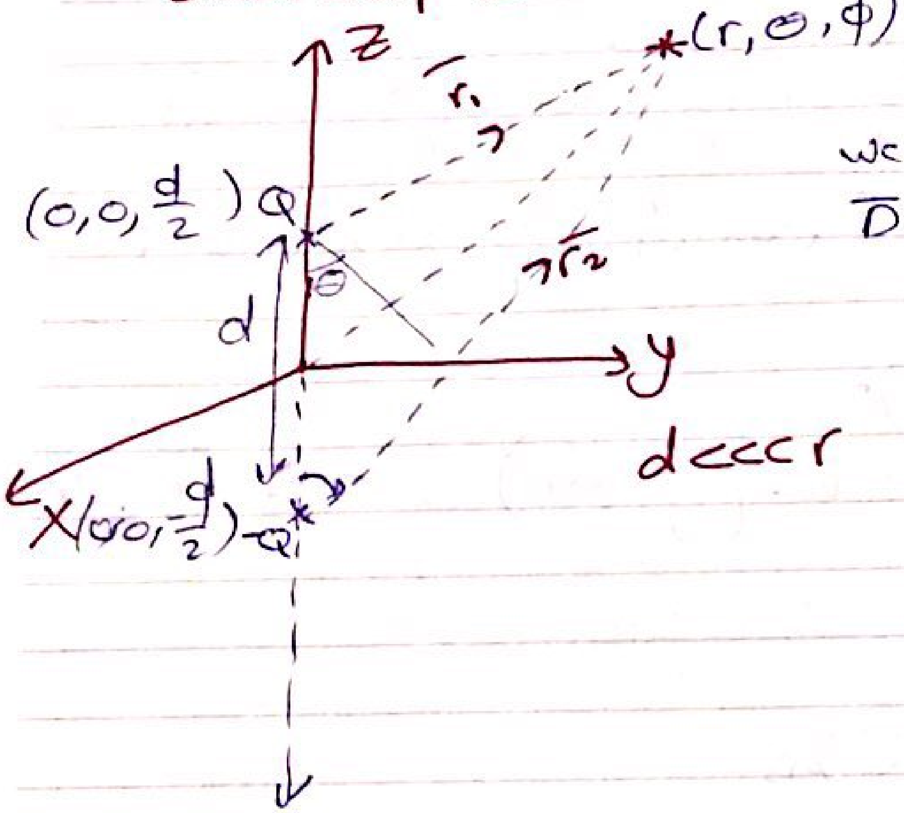
$W = -Q \int_{r_A}^{r_B} \overline{E} \cdot \overline{dl}$

$W = -Q \int_{r_A}^{r_b} (E_r dr + r E_\theta d\theta + r\sin\theta E_\phi d\phi)$

$W = -Q, \int_{r_A}^{4} E_r dr \Big|_{\substack{\theta = 30° \\ \phi = 120°}} \oplus \int_{30°}^{90°} r E_\theta d\theta \Big|_{\substack{r=4 \\ \phi = 120}} \oplus \int_{120}^{60°} r\sin\theta E_\phi d\phi \Big|_{\substack{r=4 \\ \theta = 90}}$

$= 28.125 MJ$

* Electric dipole:



we need to find $\bar{D}$, $\bar{E}$ & V.

$d \lll r$

$V = V_{+} \oplus V_{-}$

$= \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$

$$\boxed{V = \frac{Q(r_2 - r_1)}{4\pi\epsilon_0 r_1 r_2} \text{ volt}} \Longrightarrow ①$$

Q, d is not given

to simplify the equation

$\cos\theta = \frac{r_2 - r_1}{d} \Longrightarrow d\cos\theta = r_2 - r_1$

$\Longrightarrow r_1 r_2 \approx r^2$

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} \text{ volt} \Longrightarrow ②$$

define the dipole momentz — $(\bar{P})$:

$P = Qd$. m(c.m)

$$\boxed{V = \frac{P\cos\theta}{4\pi\epsilon_\circ r^2}} \Rightarrow ③$$

$P\cos\theta = P \cdot \hat{a}_r =$

$$V = \frac{\bar{P} \cdot \hat{a}_r}{4\pi\epsilon_\circ r^2} \text{ volt}$$

→ for point charge : $V \propto \frac{1}{r}$, $E \propto \frac{1}{r^2}$

→ for two point charge dipole: $V \propto \frac{1}{r^2}$, $E \propto \frac{1}{r^3}$

$$E = \bar{E}_- \oplus \bar{E}_+ \quad \boxed{\text{or}} \quad \bar{E} = -\nabla V$$

$$\bar{E} = -\left(\frac{\partial V}{\partial r}\hat{a}_r + \frac{\partial V}{r\partial\theta}\hat{a}_\theta + \cancel{\frac{\partial V}{r\sin\theta\,\partial\phi}\hat{a}_\phi} \ldots\right) \rightarrow \text{zero}$$

$$\bar{E} = \frac{-P}{4\pi\epsilon_\circ}\left(\frac{-2}{r^3}\cos\theta\,\hat{a}_r + \frac{-\sin\theta}{r^3}\hat{a}_\theta\right)$$

$$= \frac{P}{4\pi\epsilon_\circ r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

$$\bar{D} = \bar{E} \cdot \epsilon_\circ$$

*Example: Two dipoles with dipole moments $-5\hat{a_z}$ nC·m and $9\hat{a_z}$ nC·m are located at (0,0,-2) and (0,0,3). Find V and $\bar{E}$ at the origin.

$$V = V_1 + V_2$$

$$= \frac{\bar{p_1} \cdot \bar{r_1}}{4\pi\epsilon \cdot r_1^3} + \frac{\bar{P_2} \cdot \bar{r_2}}{4\pi\epsilon \cdot r_2^3}$$

$$= -20.25V$$

(*)

$\bar{r_1} = (0,0,0) - (0,0,-2) = \underline{2\hat{a_z}}$

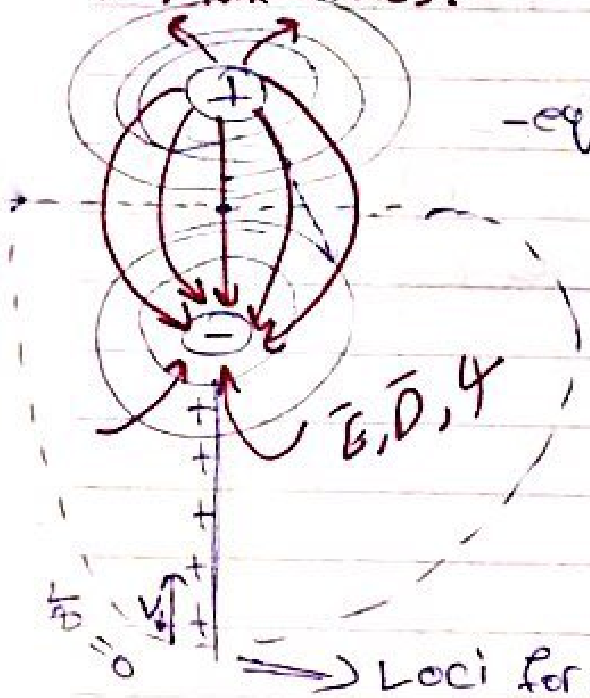
$\bar{r_2} = (0,0,0) - (0,0,3) = \underline{-3\hat{a_z}}$

$r_1 = 2m, r_2 = 3m$

(*) ref is ∞

$V_{\infty 0} = V_0 - V_\infty = V_0 - \text{zero} = V_{origin}$

*Flux lines:



-equi-potential surface (in blue)

⟹ Loci for all equipotential surface.

~~*Energy Density in Electrostatic Fields:~~

$\overline{E} = \overline{E}_1 + \overline{E}_2$, $\overline{E} = -\nabla V$

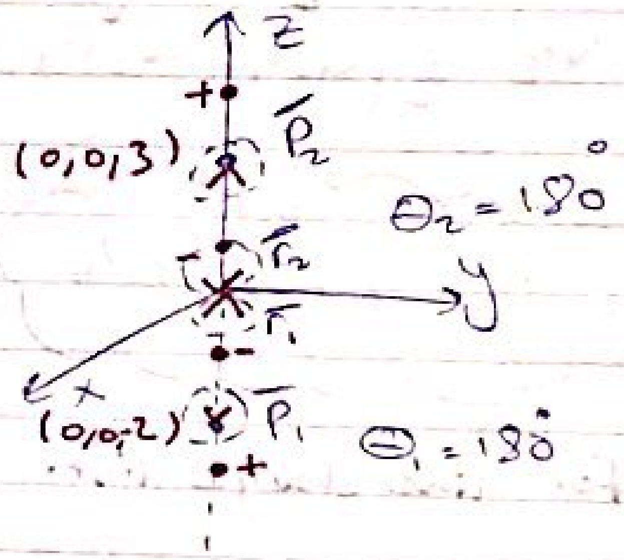
$\overline{E}_1 = \frac{P_1}{4\pi\epsilon_0 r_1^3}(2\cos\theta_1 \hat{a}_r + \sin\theta_1 \hat{a}_\theta)$

$P_1 = 5 nC \cdot m$

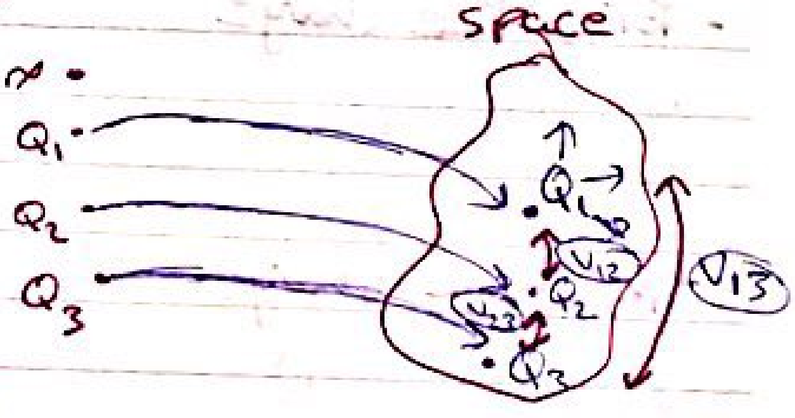
$P_2 = 9 nC \cdot m$

$r_1 = 2m$

$r_2 = 3m$



*Energy density in Electrostatic fields:-



⊛ $W = Q V_{diff}$

$Q_1 \rightarrow Q_2 \rightarrow Q_3$

electrical energy → $W_E = W_1 + W_2 + W_3$

$= 0 + Q_2 V_{12} + Q_3(V_{13} + V_{23})$ — ①

if $Q_3 \to Q_2 \to Q_1$

$$W_E = W_1 + W_2 + W_3$$

$$= Q_1(V_{31} + V_{21}) + Q_2(V_{32}) + 0 \to ②$$

$$2W_E = Q_1(\underbrace{V_{31} + V_{21}}) + Q_2(\underbrace{V_{12} + V_{32}}) + Q_3(\underbrace{V_{13} + V_{23}})$$

$$W_E = \frac{1}{2}\left(Q_1V_1 + Q_2V_2 + Q_3V_3\right)$$

⊛ $\boxed{W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k}$ (J)

- for line Charge dist.

$$W_E = \frac{1}{2}\int \rho_L V dL$$

- for surface charge

$$W_E = \frac{1}{2}\int_S \rho_s V ds$$

- for volume Charge.

$$W_E = \frac{1}{2}\int_V \rho_v V dv$$

* How to relate W_E with $\bar{E}$.

$$W_E = \frac{1}{2}\int_v \boxed{\rho_v}\, v\, dv$$

$$\nabla\cdot\bar{D} = \rho v \quad , \quad \bar{D} = \epsilon.\bar{E}$$

$$W_E = \frac{1}{2}\int_v (\nabla\cdot\bar{D})\, v\, dv$$

- in general: $\boxed{\nabla\cdot(v\bar{D}) = v(\nabla\cdot\bar{D}) + \bar{D}\cdot\nabla v}$

$$W_E = \frac{1}{2}\int_N [\nabla\cdot(v\bar{D}) - \frac{1}{2}\int_v \bar{D}\cdot\nabla v\, dv$$

- Apply divergence theorem in(1) .

$$W_E = \underbrace{\frac{1}{2}\oint_s v\bar{D}\cdot\bar{ds}}_{\propto\frac{1}{r}} - \frac{1}{2}\int_v \bar{D}\cdot\boxed{\nabla v}\, dv$$

$E = -\nabla V$

$N \propto \frac{1}{r}$

$\bar{D} \propto \frac{1}{r^2}$

$vD \propto \frac{1}{r^3}$

$ds \propto r^2$

$$\boxed{W_E = \frac{1}{2}\int_v \bar{D}\cdot\bar{E}\, dv}\ (J)$$

$$W_E = \frac{1}{2}\int \epsilon.\, E^2\, dv \quad (J)$$

$$W_E = \frac{1}{2}\int \frac{D^2}{\epsilon.}\, dv\ (J)$$

$$w_E = \frac{W_E}{\text{Volume}}$$

$$\left.\begin{aligned} w_E &= \frac{1}{2}\cdot\bar{D}\cdot\bar{E} \\ &= \frac{1}{2}\epsilon_0 E^2 \\ &= \frac{1}{2}\frac{D^2}{\epsilon_0} \end{aligned}\right\} \longrightarrow J/m^3$$

$$\boxed{W_E = \int_v \underbrace{w_E}_{J/m^3}\, \underbrace{dv}_{m^3}} \Longrightarrow (J)$$

*example: The point charge -1nc, 4nc and 3nc are located (0,0,0), (0,0,1), and (1,0,0). find the energy in the system

$$W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \text{, or } W_E = \frac{1}{2}\int \epsilon_0 E^2 dv$$

<u>N=3</u>

$$W_E = \frac{1}{2}\left(Q_1 N_1 + Q_2 V_2 + Q_3 V_3\right)$$

$$\Rightarrow \frac{1}{2}\left[Q\left(\frac{Q_2}{4\pi\epsilon_0(1)} + \frac{Q_3}{4\pi\epsilon_0(1)}\right) + Q_2\left(\frac{Q_1}{4\pi\epsilon_0(1)} + \frac{Q_3}{4\pi\epsilon_0 r_2}\right) + Q_3\left(\frac{Q_1}{4\pi\epsilon_0(1)} + \frac{Q_2}{4\pi\epsilon_0 r_2}\right)\right] = 13.37\,nJ$$

chapter 5 :- Electric fields in Materials.

* Types of materials based on its electrial properities

- → conductors (metals), $\sigma >>> 1$ $\sigma = \infty \rightsquigarrow$ supe
- → semi-conductor $\sigma > 1$
- → Dielectrics (Insulators) $0 < \sigma <<< 1$

σ = conuctivity s/m or $(\Omega \cdot m)^{-1}$, V/m

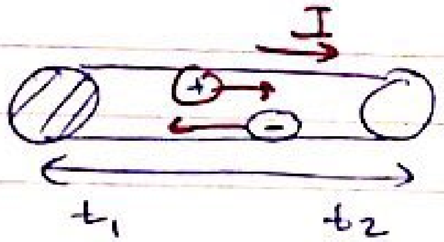
$\sigma = \frac{1}{\rho}$ → resistivity $(\Omega \cdot m)$

σ depends mainly on: $\rightsquigarrow$ tempreture $\rightsquigarrow$ frequency $(\overset{em}{CH}_{10})$

*types of currents:

1) conduction current
2) convection current.
3) displacement current. ⇒ CH9

$$* \; I = \frac{DQ}{Dt}$$



define the current density $(\bar{J})$ in A/m²

$$DI = J\,DS = \int dI = \int J\,ds \Rightarrow I = \int_s \bar{J} \cdot d\bar{s}$$

$*\ I = \frac{\Delta Q}{\Delta t}$, $Q = \int_v \rho_v \, dv$ $\Delta Q = \rho_v \Delta V$

$S\Delta y = \Delta V = \Delta x \Delta y \Delta z$

( ) $\boxed{I = \rho_v S \Delta y}$ A

* How to relate $\bar{J}$ with $\bar{E}$?

$F = Q E_{ext}$ ⇒ the force on one electron

$Q = -e$

$\bar{F} = -e\bar{E} = m\bar{u} = \frac{m\bar{u}}{\tau}$

*τ: time between collasions and its constant at constant temprecare.

$(\tau\uparrow, \uparrow)$ $\tau\uparrow T\downarrow$

$\bar{u} = \left(\frac{-e\tau}{m}\right)\bar{E}$, $\underline{\bar{u} = \mu \bar{E}}$ ↳ mobility

$\bar{u} = v_d = \mu \bar{E}$

↳ drif velocity if $\rho_v = -ne$ → n is $/m^3$

$\bar{J} = \rho_v \bar{u}$

$\bar{J} = -ne\bar{u}$ ⇒ $\bar{J} = -ne\left(\frac{-e^{\tau}}{m}\right)\bar{E} = \left(\frac{n^2 e^{\tau}}{m}\right)\bar{E}$

$\boxed{\bar{J} = \sigma \bar{E}}$ → conducton current density

→ point form of ohm's law.

$I = GV$ ⇒ $I = \frac{V}{R}$

$$I = \int \bar{J} \cdot \bar{ds} \quad , \quad V = -\int_L E \, \bar{dL}$$

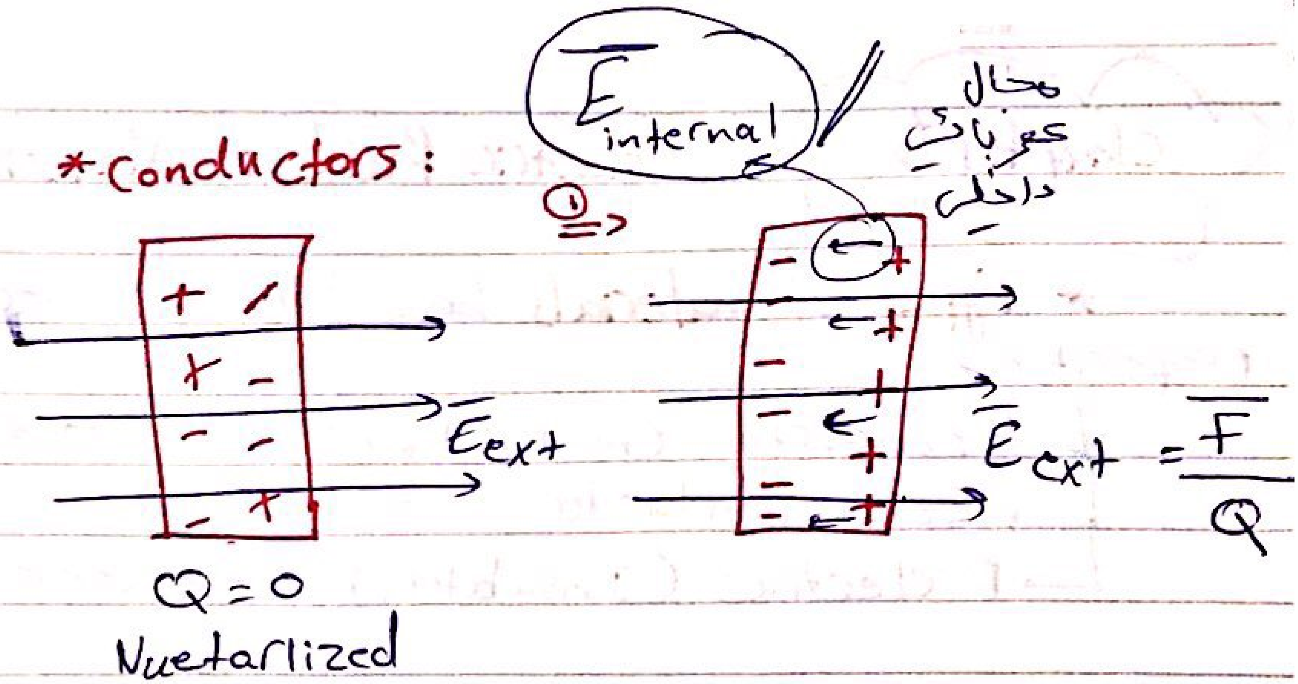
$$I = Js \quad , \quad V = E\ell$$

$$R = \frac{V}{I} = \frac{E\ell}{Js} = \frac{\cancel{E}\ell}{\sigma \cancel{E}_s}$$

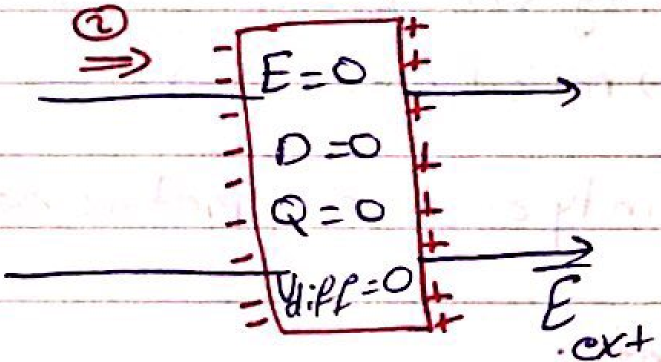
$$R = \frac{\ell}{\sigma s} = \frac{\rho L}{s}$$

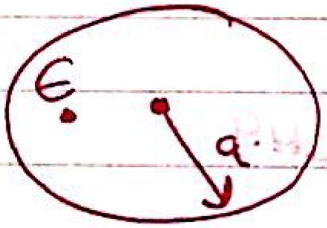
$$\rho = \frac{\sigma}{M}$$

* conductors :



$\bar{E}_{ext} = \frac{\bar{F}}{Q}$

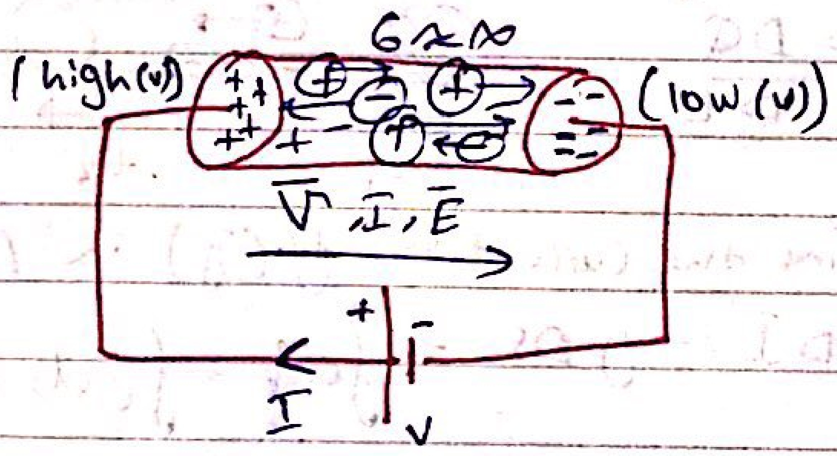




$\epsilon = \epsilon_0$

conductors ($\sigma = \infty$)

free space ($\sigma = 0$)



- Resistance (R)

$$\boxed{\begin{array}{l} R = \frac{V}{I} \\ R = \frac{\rho L}{A} = \frac{1}{\sigma A} \\ R = \frac{El}{\sigma E s} \end{array}} = \frac{-\int_{l} \overline{E} \cdot \overline{dl}}{\int \underbrace{\sigma \overline{E} \cdot \overline{ds}}_{J}}$$

$\Downarrow$

*only (uniform)

*Power:-

$$P = VI \quad (\text{watt})$$

$$P = \frac{\text{work}}{\text{time}} \;, \left(\frac{J}{s} = W\right)$$

$$= \frac{-Q\int \overline{E} \cdot \overline{dl}}{\Delta t} \;, Q = \int_v \rho_v \, dv$$

$$= +Q\overline{E}\overline{v}$$

$$P = \int_v \rho_v E v \, dv = \int_v \overline{J} \overline{E} \, dv \;, \overline{J} = \sigma \overline{E}$$

$$\left.\begin{array}{l} P = \int_v \overline{J} \cdot \overline{E} \, dv \\ = \int_v \sigma E^2 \, dv \\ = \int_v \frac{J^2}{\sigma} \, dv \end{array}\right\} \Longrightarrow \begin{array}{l} \overline{J} = \sigma \overline{E} \\ \text{in } (W) \end{array}$$

$w_p \equiv$ Power density (W/m^3)

power density $\equiv w_p = \frac{P}{volume}$

$$w_p = \bar{J}\cdot\bar{E}$$
$$= \sigma E^2$$
$$= \frac{J^2}{\sigma}$$

in uniform:

$$\boxed{P = \int_v \bar{J}\cdot\bar{E}\, dv} \Rightarrow \text{Joule's law.}$$

$$dv = ds\, dl$$

$$P = \int_l \int_s \bar{J}\cdot\bar{E}\, ds\, dl$$

$$P = \underbrace{\int_s \bar{J}\cdot d\bar{s}}_{I} \underbrace{\int_l \bar{E}\cdot d\bar{l}}_{V}$$

$$P = I * V$$

$$P = \sigma V^2 = \frac{V^2}{R}$$

$$= \frac{I^2}{\sigma} = RI^2$$

$$I = \int_s J\cdot d\bar{Z} \quad , \quad V = -\int_l \bar{E}\cdot d\bar{l}$$

***example:** If $\overline{J} = \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)$ A/m³

Find the current passing through:

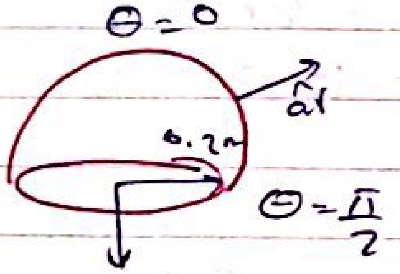
a) A hemi-spherical shell of radius = 20 cm

$0 < \theta < \frac{\pi}{2}$, $0 < \phi < 2\pi$.

b) A spherical shell of r = 10cm.

sol.

a)



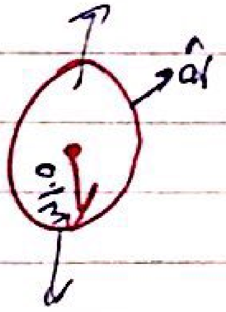
$$I = \int_s \overline{J}\cdot\overline{ds} \quad , \overline{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r$$

$$I = \frac{1}{r}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta\, d\theta\, d\phi \Big|_{r=0.2\text{ m}}$$

$= 10\pi$ A

$= 31.4$ A

b)

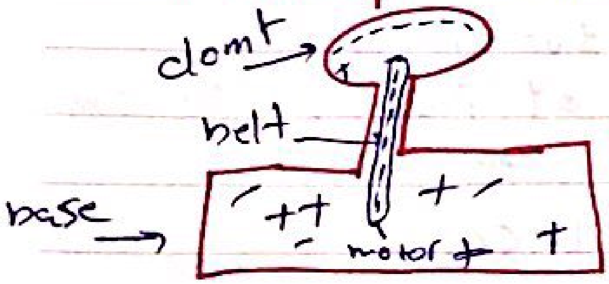


$$\overline{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r$$

$$I = \oint_s \overline{J}\cdot\overline{ds} = \int_v \nabla\cdot\overline{J}\, dv$$

$$\nabla\cdot\overline{J} = 0$$

* example: In Van de Graff generator:



if $P_s = 10^{-7}$ c/m², u = 2m/s

width of the belt = 10 cm

calculat the charge.

$$I = \frac{\Delta Q}{\Delta t} = \frac{P_s \Delta s}{\Delta t}$$

$$P_s = \frac{w_L}{\Delta t} \Rightarrow I = P_s w u$$

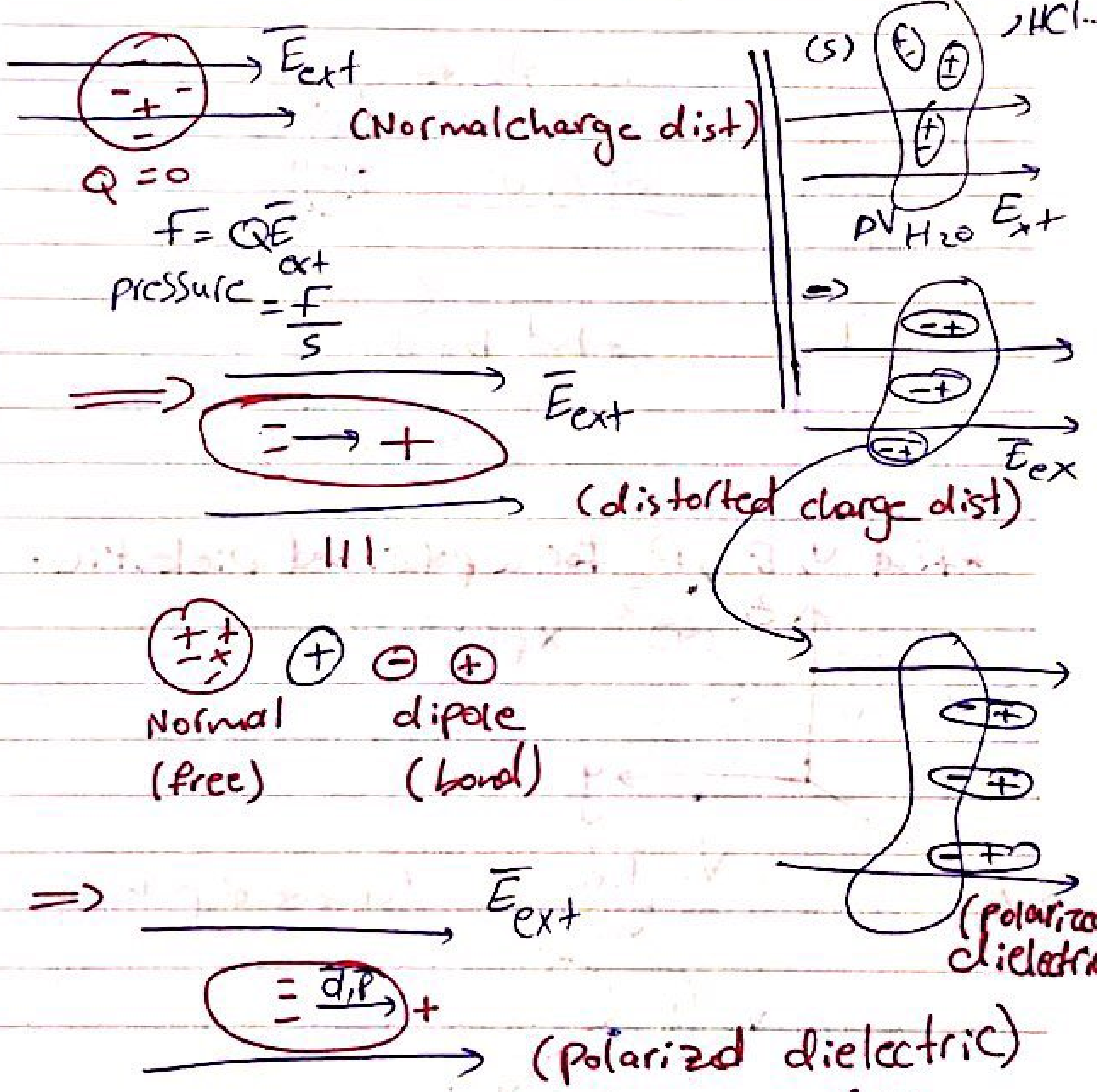
$$Q = I t$$

$$Q = P_s w u t$$

$$Q = 100 nc.$$

*polarization in dielectrics:

- Types of dielectrics:
  - non-polar
    O_2, H_2, N_2, Rarg gase, Xe, Ne, kr.
  - polar (Permerat dipoles) . ex. H_2O, NH_3, HCl..



Q = 0

$F = Q\bar{E}_{ext}$

$\text{pressure} = \frac{F}{S}$

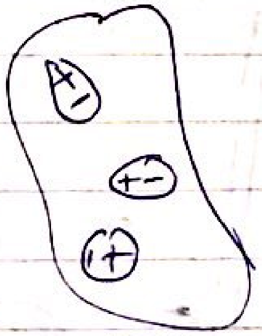
(6)

- define Polarization ($\overline{P}$)

$$\overline{P} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{N} Q_k d_k}{\Delta V} \quad \text{in } (c/m^3)$$

(*) for N-dipoles

$$\overline{P} = \sum_{k=1}^{N} Q_k d_k \quad (c.m)$$

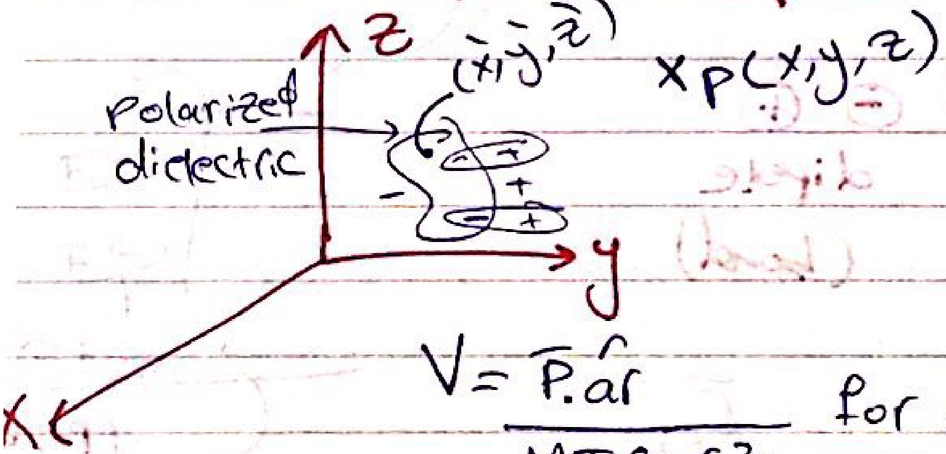


$\overline{P} = 0$

طلعنا قطعة
و صغرنا عدد
ال dipoles

, but by shrinking and taking only one dipole

*find $V, \overline{E}, \overline{D}$ for a polarized Dielectric.



$$V = \frac{\overline{P} \cdot \hat{a}_r}{4\pi\epsilon_0 r^2} \quad \text{for one dipole}$$

$$v = \int_{v} \frac{\overline{P} \cdot \hat{a}_r}{4\pi\epsilon_0 r^2} dv$$

→ for dipole charges.

$$V = \oplus \oint \frac{\bar{P} \cdot \hat{a_n}}{4\pi\epsilon \cdot r} ds \ominus \int \frac{\nabla' \cdot \bar{P}}{4\pi\epsilon \cdot r} dv$$

→ for free charge:

from Ch. 4

$$V = \int \frac{\rho_s ds}{4\pi\epsilon \cdot r} , \quad V = \int_v \frac{\rho_v dv}{4\pi\epsilon \cdot r}$$

- equating terms:

$$\rho_{ps} = \bar{P} \cdot \hat{a_n}$$

↳ polarized (bound) surface chargeddist.

$$\rho_{pv} = -\nabla \cdot \bar{P}$$

↳ polarized (bond) volume charge dist.

- bond charge :- (Q_b)

$$Q_{b+} = \int_s \rho_{\rho_s} ds , \quad Q_{b-} = \int_v \rho_{\rho_v} dv$$

$$Q_{b\,total} = Q_{b+} + Q_{b-} = \boxed{0}$$

↳ must (Neuteralized).

$\overline{E} = -\nabla V$, $\overline{D} = \epsilon_\circ \overline{E}$

* for some dielectrics (Linear Dielectrics)

$$* \boxed{\overline{P} = \chi_e \epsilon_\circ \overline{E}}$$

$\chi_e \equiv$ (electrical susceptebolity).
(constant).

Mat.	χ_e
Air	0
good cond	0
⋮	

$\longrightarrow \overline{P} = 0$

$\longrightarrow \overline{P} = 0$

* polarized in dielectrics:

$\rho_{Ps} = \overline{P} \cdot \hat{an}$

$\rho_{PV} = -\nabla \cdot \overline{P}$

$D = \epsilon_\circ \overline{E}$ in freespace and conductors.

$\rho_{V\,total} = \rho_V + \rho_{PV} = \nabla \cdot \overline{D}$

$\rho_V = \rho_V\,total - \rho_{PV} = \nabla \cdot \overline{D} - -\nabla \cdot \overline{P}$

$\rho V = \nabla \cdot (\overline{D} + \overline{P})$

$\rho_V = \nabla \cdot (\epsilon_\circ \overline{E} + \overline{P}) = \nabla \cdot \overline{D}$

for some dielectrics (Linear dielectric).

$\overline{P} = \chi_e \epsilon_\circ \overline{E}$

$\chi_e \equiv$ electrical susceptebality. (constant)

Material	χ_e
Air	0.
conductors	0

$\boxed{\overline{D} = \epsilon.\overline{E} + \overline{P}}$ → in general

$D|_{\text{dielectric}} > D|_{\text{conductors \& freespace.}}$

$\overline{D} = \epsilon.\overline{E} + \chi_e \epsilon.\overline{E}$

$= (1+\chi_e)\,\epsilon.\overline{E}$

$\overline{D} = \underline{\epsilon_r}\,\epsilon.\overline{E}$

$\overline{D} = \epsilon\overline{E}$

ϵ = permetivity F/m

ϵ_r = relative permitivity (Dielectric constant)

$\epsilon_r = \frac{\epsilon}{\epsilon.}$ or $1+\chi_e$

- bound charge (Q_b)

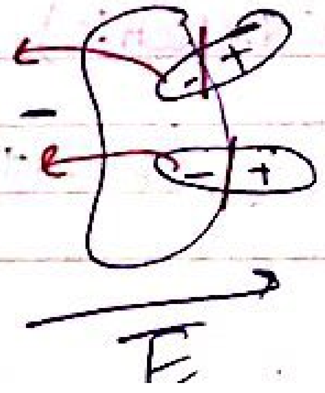
$$Q_{b+} = \int_s \rho_{p_s}\, ds$$

$$Q_- = \int_v \rho_{pv}\, dv$$

$$Q_{b\,total} = Q_{b+} + Q_{p-} = \boxed{0}$$

* Dielectric breakdown.

if $Q_{b\,total} \neq 0$



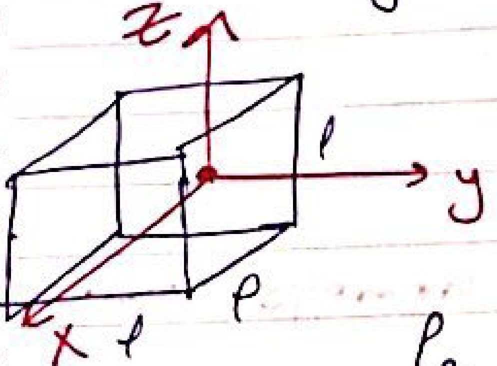
* breakdown depends on:

1) The nature of the material.
2) Temperature T↓
3) Humidity↑
4) The applied E_{ext}↑

5) The time that $\overline{E}_{ext}$ is applied ↑.

*** Example:** A dielectric cube of length (ℓ) is centered at the origin has $\overline{P} = a\overline{r}$ where (a) is constant and $\overline{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

Find: All bound charge dist. and the total bound charge



$$\rho_{ps_{front}} = \overline{P}\cdot\hat{a}_n \,,\quad \hat{a}_n = \hat{a}_x$$

$$= ax\Big|_{x=-\frac{\ell}{2}} \qquad = \frac{a\ell}{2}\ C/m^2$$

$$\rho_{ps_{back}} = -ax\Big|_{x=\frac{\ell}{2}} = \frac{a\ell}{2}\ C/m^2$$

$$= \overline{P}\cdot(-\hat{a}_x)$$

$$\Longrightarrow \underline{\underline{All}}\ \rho_{ps} = \frac{a\ell}{2}\ C/m^2$$

$$\rho_{pv} = -\nabla\cdot\overline{P}$$

$$= -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right)$$

$$= -(a + a + a)$$

$$\rho_{pv} = -3a\ C/m^3$$

$$Q_{bt} = 6\iint \rho_{ps}\, ds = 6\int_{\frac{\ell}{2}}^{\frac{\ell}{2}}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{a\ell}{2}\, dy\, dz$$

$$Q_{bt} = 3a\ell^3\ C$$

$$Q_{b-} = \int_N \rho_{rv} \, dv$$

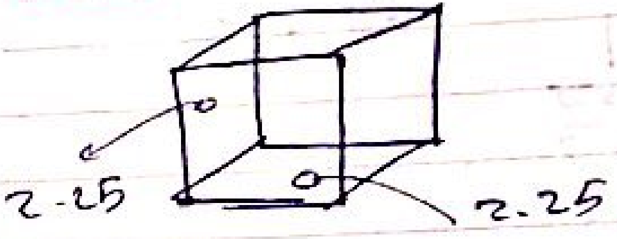
$$= \iiint_{-\frac{\ell}{2}}^{\frac{\ell}{2}} -3a \, dx\,dy\,dz$$

$$Q_{b-} = -3a\ell^3_c$$

$$\boxed{Q_{b\,total} = 0}$$

$\boxed{5.7}$ → linear, Isotropic, homogeneous.

ϵ_r

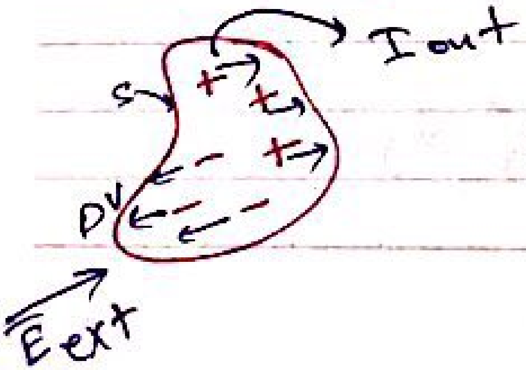


$$\epsilon_r = 2.25 * r$$

$$\nabla \cdot \overline{D} = \nabla \cdot \epsilon \overline{E} = \nabla \cdot \epsilon . \epsilon_r \overline{E}$$
$$= \epsilon \cdot \nabla \cdot \epsilon_r \overline{E}$$

(5.8) → conductivity equation & relaxation time.

$\boxed{KCl}$: $I_{out} = I_{in}$



$$I_{out} = \oint_s \overline{J} \cdot \overline{ds} \quad , I_{in} = -\frac{dQ_{in}}{dt}$$

$$\oint_s \bar{J}\cdot\bar{ds} = -\frac{dQ_{in}}{dt} , \quad Q_{in} = \int_v P_v\,dv$$

$$\oint_s \bar{J}\cdot\bar{ds} = -\frac{d}{dt} = \int_v P_v\,dv$$

$$\oint_s \bar{J}\cdot\bar{ds} = -\int_v \frac{\partial P_v}{\partial t}\,dv \xrightarrow{} P_v(x,y,z;t)$$

$$\int_v \nabla\cdot\bar{J}\,dv = -\int \frac{\partial P_v}{dt}\,dv$$

$$\boxed{\nabla\cdot\bar{J} = -\frac{\partial P_v}{dt}} \rightarrow \text{continuity of Eq.}$$

start from $\nabla\cdot\bar{J} = \frac{-\partial P_r}{\partial t}$

with: $\nabla\cdot\bar{D} = P_v$, $\bar{D} = \epsilon\bar{E}$, $\bar{J} = \sigma\bar{E}$

$\nabla\cdot\epsilon\bar{E} = P_v$ if ϵ is homogeneous

$$\boxed{\nabla\cdot\bar{E} = \frac{P_v}{\epsilon}}$$

$$\nabla\cdot\sigma\bar{E} = \frac{-\partial P_v}{\partial t}$$

$$\sigma\frac{P_v}{\epsilon} = -\frac{\partial P_v}{\partial t}$$

$$\int_{P_{v0}} \frac{\partial P_v}{P_v} = \int_{t_0}^{t} -\frac{\sigma}{\epsilon}\,dt$$

I.c.

$P_v(t=t_0) = P_{v0}$

$$\ln \rho_v - \ln \rho_{v_0} = \frac{-\sigma}{\epsilon}(t-t_0)$$

$$\ln\left(\frac{\rho_r}{\rho_{v_0}}\right) = \frac{-\sigma}{\epsilon}(t-t_0)$$

$$\cancel{\ln\left(\frac{\rho_f}{\rho_{v_0}}\right)}$$

$$\frac{\rho_v}{\rho_{v_0}} = e^{-\frac{\sigma}{\epsilon}(t-t_0)}$$

$$\rho_v = \rho_{v_0} e^{-\frac{\sigma}{\epsilon}(t-t_0)}$$

$$\rho_v = \rho_{v_0} e^{-\frac{\sigma}{\epsilon}(t-t_0)}$$

$$\tau_r = \frac{\epsilon}{\sigma}$$

$$t = \tau_r \longrightarrow e^{-1} = 36.8$$

* for copper $\epsilon_r = 1$, $\sigma = 5.7 \times 10^{8}$

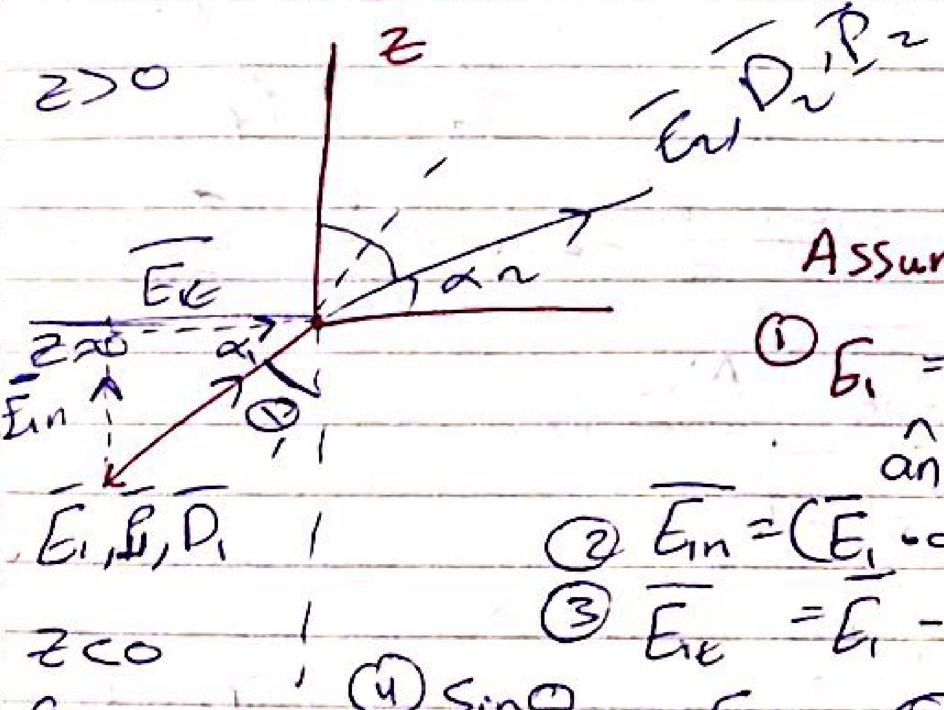
$$\tau_r = \frac{\epsilon \cdot \epsilon_r}{\sigma} = 1.53 \times 10^{-17}$$

for fused

$$\tau_r = \frac{\epsilon \cdot \epsilon_r}{\sigma} = 51.2$$

* Boundary condition.

*① Dielectric to Dielectric.



Assume $\bar{E}_1$ is given

① $\bar{E}_1 = \bar{E}_{1n} + \bar{E}_{1t}$

$\hat{a}_n = +\hat{a}_z$

② $\bar{E}_{1n} = (\bar{E}_1 \cdot \hat{a}_n)\hat{a}_n$

③ $\bar{E}_{1t} = \bar{E}_1 - \bar{E}_{1n}$

④ $\sin\theta_1 = \frac{E_{1t}}{E_1}$ ⑤ $\cos\theta = \frac{E_{1n}}{E_1}$

⑥ $\tan\theta_1 = \frac{E_{1t}}{E_{1n}}$, $\alpha_1 = 90^\circ - \theta_1$

to find $\bar{E}_2$

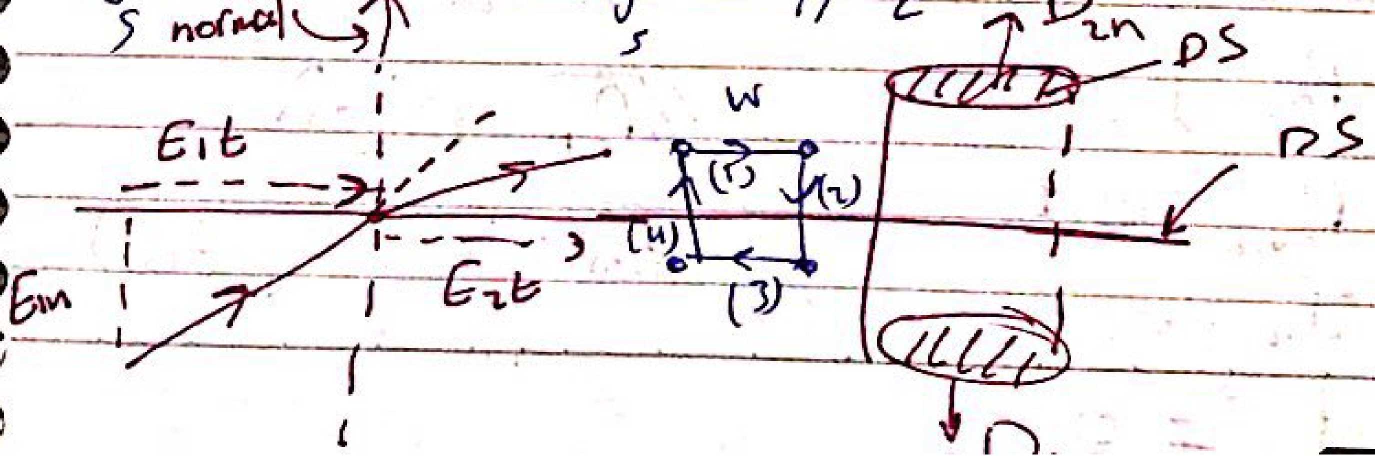
$\bar{E}_2 = \bar{E}_{2n} + \bar{E}_{2t}$

⊛ Apply 1st maxwell eq. & 2nd

on the ref. (z=0)

$$\oint_S D \cdot \bar{ds} = Q_{enc} = \int_S \rho_s \, ds \quad // \quad \oint_L E \cdot dl = 0$$

normal



$\oint_{\ell} E\cdot dl = 0$, on $z=0$

$$\oint_{\ell} \bar{E}\cdot \bar{dl} = \int_{\ell} \bar{E}\cdot \bar{dl}_1 + \int_{\ell_2} \bar{E}\cdot \bar{dl}_2 + \int_{\ell_3} \bar{E}\cdot \bar{dl}_3 + \int_{\ell_4} \bar{E}\cdot dl_4 = 0$$

$$E_{2t} W - E_{2n}\frac{h}{2} - E_{1n}\frac{h}{2} - E_{12} W \frac{h}{2} - E_{2n}\frac{h}{2} = 0$$

$\boxed{\bar{E}_{2t} = \bar{E}_{1t}}$ Always

* Applying $\oint \bar{D}\cdot \bar{ds} = \int \rho_s ds$

$D_{2n}\,\Delta s - D_{1n}\,\Delta s = \rho_s\,\Delta s$

$\boxed{D_{2n} - D_{1n} = \rho_s}$ if $\rho_s = 0 \Rightarrow \bar{D}_{1n} = \bar{D}_{2n}$

$\epsilon_0 \epsilon_{r_1} \bar{E}_{1n} = \epsilon_0 \epsilon_{r_2} \bar{E}_{2n}$

$$\boxed{\bar{E}_{2n} = \frac{\epsilon_{r_1}}{\epsilon_{r_2}} \bar{E}_{1n}}$$

line ① $E_2 = E_{2n} + E_{2t}$

② $\bar{D}_2 = \epsilon_0 \epsilon_{r_2} E_2$

③ $\bar{P}_2 = (\epsilon_{r_2} - 1)\,\epsilon_0 \bar{E}_2$

④ $\rho_{ps_2} = \bar{P}_2\, \hat{an}$

⑤ $\rho_{pv_2} = \nabla\cdot \bar{P}_2$

⑥ $\bar{J}_2 = \sigma_2 \bar{E}_2$

⑦ if $\rho_s \neq 0$

$D_{2n} - D_{1n} = \rho_s$

$D_{2n} = D_{1n} + \rho_s$, $D_{1n} = \epsilon_0 \epsilon_{r_1} E_{1n}$

$\bar{D}_{2n} = D_{2n}\, \hat{an}$

$\bar{E}_{2n} = \bar{D}_{2n}\,(\epsilon_0 \epsilon_{r_2})$

if $\rho_s = 0$ (relation between θ_1 & θ_2)

$E_{1t} = E_{2t}$

$$\boxed{E_1 \sin\theta_1 = E_2 \sin\theta_2} — ①$$

$D_{1n} = D_{2n}$

$\epsilon_0 \epsilon_{r_1} E_{1n} = \epsilon_0 \epsilon_{r_2} E_{2n}$

$\epsilon_{r_1} E_1 \cos\theta_1 = \epsilon_{r_2} E_2 \cos\theta_2$

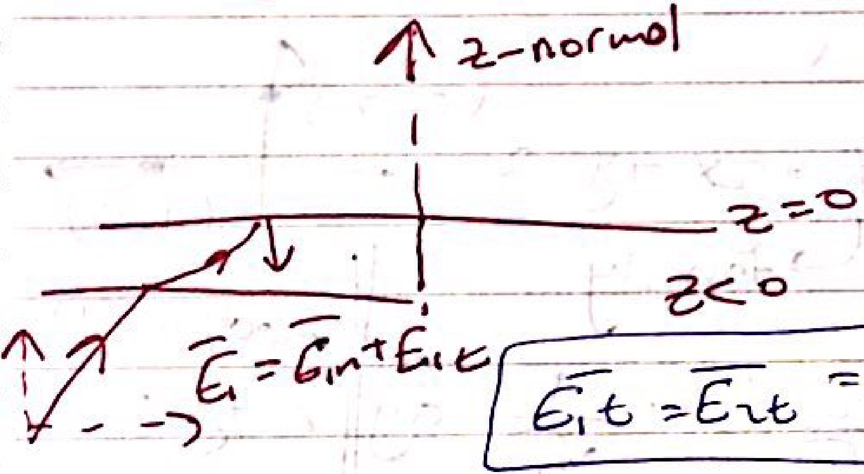
dividing ① by ②

$$\boxed{\frac{\tan\theta_1}{\epsilon_{r_1}} = \frac{\tan\theta_2}{\epsilon_{r_2}}}$$

(*) ② Conductor - to - Dielectric

$\overline{E}_2 = 0 \begin{cases} E_{2t} = 0 \\ E_{2n} = 0 \end{cases}$

$\overline{D}_2 = 0 \begin{cases} D_{2n} = 0 \\ \overline{D}_{2t} = 0 \end{cases}$



$\overline{E}_1 = \overline{E}_{1n} + \overline{E}_{1t}$

$\boxed{\overline{E}_{1t} = \overline{E}_{2t} = 0}$, $\cancelto{0}{\overline{D}_{2n}} - D_{1n} = \rho_s$

$$\boxed{D_{1n} = -\rho_s}$$

$E_n = \dfrac{D_{1n}}{\epsilon_0 \epsilon_{r_1}} \Longleftarrow \boxed{\overline{D}_{1n} = -\rho_s (\hat{a}_n)}$

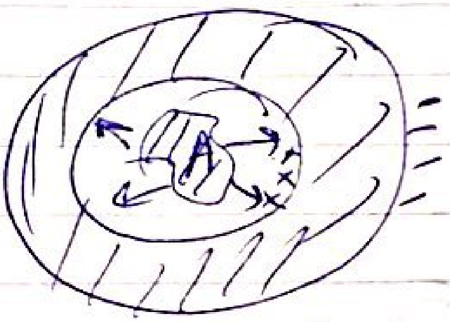
(*) c) freespace - to conctor :-

$\overline{E}_{1t} = \overline{E}_{2t} = 0$

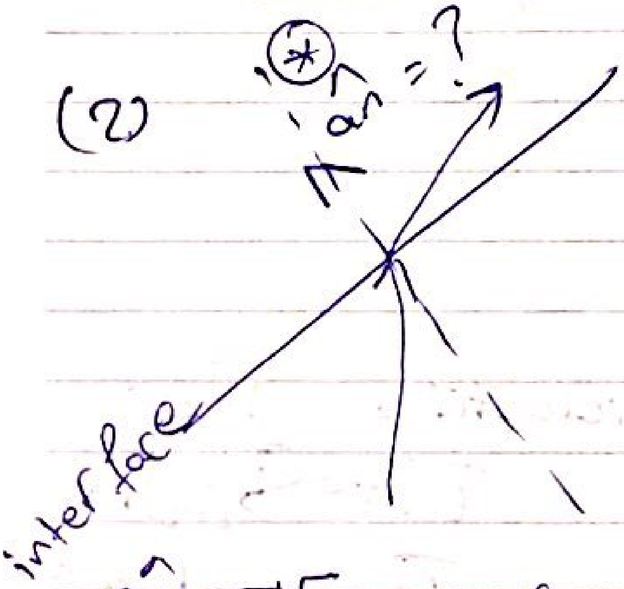
$\overline{D}_{1n} = -\rho_s \, \hat{a}_n$, $\overline{E}_n = \dfrac{\overline{D}_1}{\epsilon_0 (1)}$

*Application :-

Sheilding

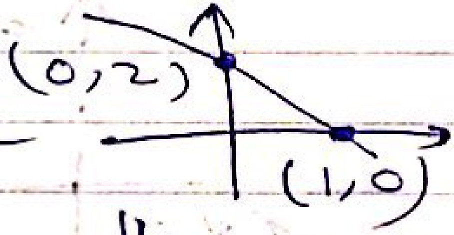


(2)



$\hat{a_n} = \frac{\nabla f}{|\nabla f|}$, $f = 2x + y - 2$

↳ family



$= \frac{(2,1,0)}{\sqrt{5}} = \frac{2}{\sqrt{5}}\hat{a_x} + \frac{1}{\sqrt{5}}\hat{a_y}$

medium (1)
$2x+y-2 \leq 0$
(2)
$2x+y-2 \geq 0$

$\bar{E}_{1n} = (\bar{E}_1 \cdot \hat{a_n}) \cdot \hat{a_n}$

*example: Two homogeneous isotropic dielectrics meet on plane $z=0$. for $z>0$, $\epsilon_{r_1}=4$ has $\bar{E} = 5a\hat{x} - 2a\hat{y} \textcircled{+} 3a\hat{z}$ kV/m

$\rightarrow$ مهما يعرف الاتجاه

Find:

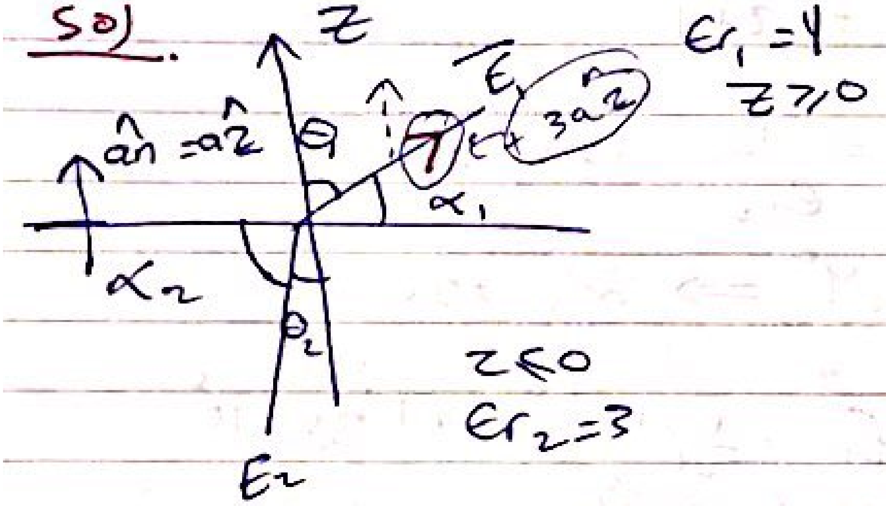
a) ϵ_2 for $z \leq 0$ if $\epsilon_{r_2}=3$

b) $\theta_1, \theta_2, \alpha_1, \alpha_2$

c) energy & density in both regions.

d) the energy in a cube of side (2m) centered at (3,4,−5).

sol.



$\hat{a}n = a\hat{z}$, $\bar{E}_1 = \bar{E}_{1n} + \bar{E}_{1t}$

$$E_{1n} = (\bar{E}_1 \cdot \hat{a}n)\hat{a}n$$

$$\bar{E}_{1n} = 3a\hat{z} \text{ kV/m}$$

$$\bar{E}_{1t} = \bar{E}_1 - \bar{E}_{1n} = 5a\hat{x} - 2a\hat{y} \text{ kV/m}$$

$$= \bar{E}_{2t}$$

$$\boxed{\bar{E}_{1t} = \bar{E}_{2t}}$$

$\Rightarrow D_{1n} - D_{2n}$, $\rho_s = 0$

$$D_{1n} = D_{2n}$$

$$\overline{E}_{2n} = \frac{\epsilon_{r1}}{\epsilon_{r2}} E_{1n}$$

$\overline{E}_{2n} = 4\, a\hat{z}$ kV/m

$\overline{E}_2 = \overline{E}_{2t} + \overline{E}_{2n}$

$= 5\, a\hat{x} + -2\, a\hat{y} + 4\, a\hat{z}$

b) $\theta_1 = \sin^{-1}\left(\frac{E_{1t}}{E_1}\right) = \sin^{-1}\left(\frac{\sqrt{29}}{\sqrt{38}}\right) = 60.9^\circ$

$\alpha_1 = 90^\circ - \theta_1 = 29.1$

$\theta_2 = \frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}$ if $\rho_s = 0$

$\theta_2 = 53.4^\circ \Longrightarrow \alpha_2 = 36.6^\circ$

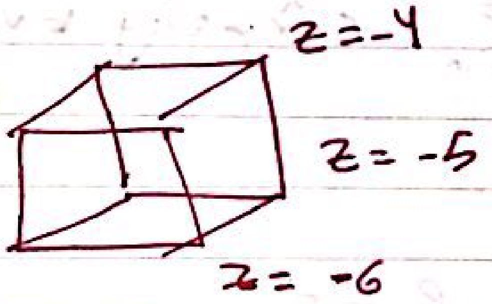
c) $w_{E_1} = \frac{1}{2}\epsilon_1 E_1^2 = \frac{1}{2}\left(\frac{10^{-9}}{36\pi}\right)(4)\, 38 * 10^{6}$

$= 672\, \mu J/m^3$

$w_{E_2} = \frac{1}{2}\epsilon_0 \epsilon_{r2} E_2^2 = \frac{1}{2} * \left(\frac{10^{-9}}{36\pi}\right)(3)(45) * 10^{6}$

$= 597\, \mu J/m^3$

d)



$W_E = \int_v w_{E_2}\, dv = \int_{-6}^{-4}\int_{3}^{5}\int_{2}^{4} 597\times10^{-6}\, dx\, dy\, dz$

$= 598\times10^{-6} * 8$

$= 4.776$ mJ

b

in region **2**

$\Rightarrow$ resolve (d) if the center is (0,0,0)

$$W_E = \int_{V_1} w_{E_1} dv + \int_{V_2} w_{E_2} dV$$

$$= \int_0^1\int_{-1}^1\int_{-1}^1 672 \times 10^{-6} dxdydz + \int_{-1}^0\int_{-1}^1\int_{-1}^1 597 \times 10^{-6} dxdydz$$

chapter 6 : electro Static Boundary value problems (B.V.P)

Coloumb's law
Gauss's law } $\rightarrow$ charge is known

potential

$\overline{E} = -\nabla V \longrightarrow$ potential is given

If the charge or potential is known for part of the surface

$\Longrightarrow$ use B.V.P

$$\nabla \cdot \overline{D} = \rho_v$$

$$\overline{E} = -\nabla V \leftarrow \nabla \times \overline{E} = 0$$

$$\overline{D} = \epsilon E , \quad \epsilon = \epsilon_0 \epsilon_r$$

$$\nabla \cdot \epsilon (-\nabla V) = \rho_v$$

Poisson's eq - for non-homo.

ϵ_r is not constant

if ϵ_r is homo

$$\nabla \cdot \nabla V = \frac{-\rho_v}{\epsilon} \longrightarrow \boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon}}$$

→ poisson's eq. for hom media

for surface free region → $\rho_v = 0$

$\nabla \cdot -\epsilon \nabla V = 0$ → laplace's eq for non-hom media

$\nabla^2 V = 0$

$$\nabla = \frac{\partial \hat{a}_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial \hat{a}_z}{\partial z}$$

*procedure to solve laplace's or poisson's:

$\boxed{1}$ choose a suitable coordinates (Based on the structure).

$\boxed{2}$ select the proper partial eq. to solve (based on ϵ_r and ρ_v).

$\boxed{3}$ solve the selected eq. using:

a) Direct integral if (V) is function of on variable.

b) separation of variables if (V) is function of (2) or (3) variables.

$\boxed{4}$ Apply two boundary (Initial) conditions to find two unknowns.

⟶ to find the unique solution of (V)

$\boxed{5}$ Find $\bar{E} = -\nabla V$

mainly:

$\boxed{6}$ $R = \frac{V}{I} = \frac{-\int_{p} E . dL}{\int_{s} \sigma \bar{E} . ds}$

$$C = \frac{Q}{V} = \frac{\oint_{s} \epsilon \bar{E} \cdot \bar{ds}}{-\int_{p} E . dL}.$$

in cart. $$\nabla^2 V = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

(scalar)

in cyl. (given)

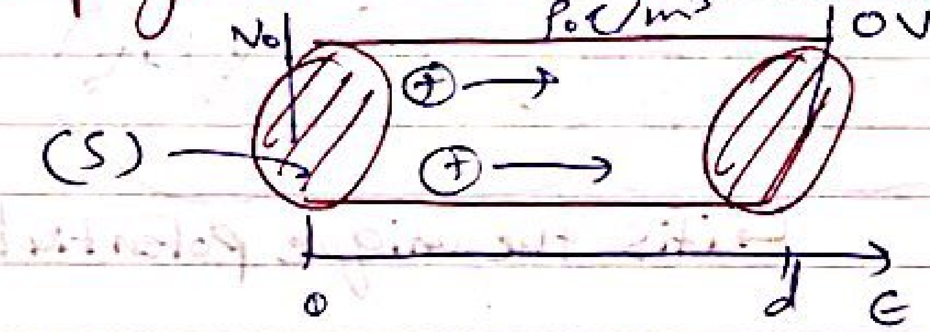
$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial v}{\partial \rho}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2}$$

in sph. (given)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) \oplus \frac{1}{r^2 \sin\theta}\frac{\partial (\sin\theta)}{\partial \theta}\frac{dv}{d\theta}$$

$$\oplus \frac{1}{r^2 \sin\theta}\frac{\partial^2 v}{\partial \phi^2}$$

* example: Electrohydrodynomos (EHD) pumping has ρ_0 C/m³ exists between electrods



if $\rho_0 = 25$ mc/m³ and $V_0 = 22$ kV
Find the pressure of the pump.

- solve the poissons eq. in cyl. coordinates

$$\nabla^2 V = \frac{-\rho_v}{\epsilon}, \quad \rho_v = \rho_0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial v}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial z^2} = \frac{-\rho_v}{\epsilon}$$

$$\int \frac{d^2 v}{dz^2} = \int \frac{-\rho_v}{\epsilon}$$

$$\int \frac{dV}{dz} = \int \left(-\frac{\rho_v}{\epsilon} z + A\right)$$

$$V = \frac{-\rho_v}{\epsilon}\frac{z^2}{2} + Az + B$$

B.C.s are , $V(z=0) = V_0$

$V(z=d) = 0$

$\xrightarrow{} \boxed{B = V_0}$

$$0 = \frac{-\rho_v}{\epsilon}\frac{d^2}{2} + Ad + V_0$$

$$\boxed{A = \frac{1}{d}\left[\frac{\rho_v}{\epsilon}\frac{d^2}{2} - V_0\right]}$$

$$\boxed{V = \frac{-\rho_v}{\epsilon}\frac{z^2}{2} + \frac{z}{d}\left(\frac{\rho_v}{\epsilon}\frac{d^2}{2} - V_0\right) + V_0}$$

↳ it is the unique potential.

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial z}\,\hat{a}_z \quad V/m$$

$$\overline{E} = -\left(\frac{-\rho_v}{\epsilon} + \frac{1}{d}\left(\frac{\rho_v d^2}{2\epsilon} - V_0\right)\right)\hat{a}_z$$

$$\overline{E} = \left(\frac{\rho_v z}{\epsilon} - \frac{\rho_v d}{2\epsilon} + \frac{V_0}{d}\right)\hat{a}_z$$

$$\overline{F} = Q\overline{E}, \quad Q = \int_v \rho_v \, dv$$

$$\overline{F} = \int_v \rho_v \overline{E}\, dv, \quad dv = ds\, dz$$

B.V.P

3) Solve laplace eq. and apply the B.C.S to find the unique (V)

4) $\overline{E} = -\nabla V$

5) $Q = \oint_s \overline{E} \cdot \overline{ds} \rightarrow C = \frac{Q}{\boxed{V_0}}$ → Point ②

$I = \int_s \sigma \overline{E} \cdot \overline{ds} \rightarrow R = \frac{V_0}{I}$

Gauss

3) Find $\overline{E}$ using Gauss.

$\oint_s \epsilon \overline{E} \cdot \overline{ds} = Q$

4) $V = -\int_{\ell} \overline{E} \cdot \overline{dl}$

5) $C = \frac{Q}{V}$

$R = \frac{V}{\int_s \sigma \overline{E} \cdot \overline{ds}}$

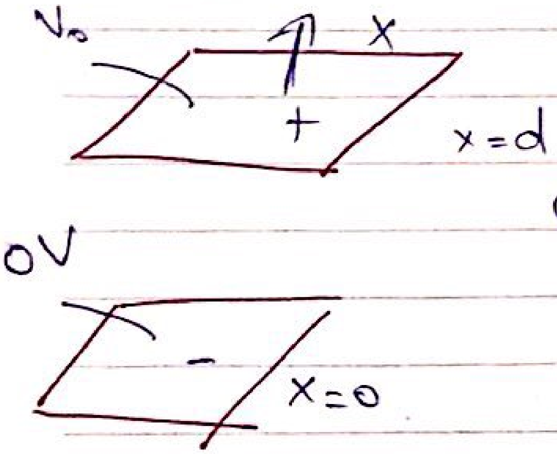
* types of capacitors:

① parallel plate capacitor

~~using~~

1) parallel plate capacitor

(Gauss)



Gauss

$$\bar{E} = \frac{Q}{\epsilon_s}(-\overset{n}{a}_x) \quad v/m$$

$$V = -\int_{\ell} \bar{E}\cdot d\bar{l} \quad = \frac{\epsilon s}{d} \quad (F)$$

$$C = \frac{Q}{V} \quad , \quad I = \int_s \sigma \bar{E}\cdot d\bar{s}$$

higher ⊕ شحنة
potential

using (B.v.P)

- Assume a V₀ potential diff.
- cart

$$\nabla^2 V = 0 \longrightarrow \int \frac{d^2 v}{dx^2} = \int 0$$

$$\int \frac{dv}{dx} = \int A$$

$$V = A x B$$

B.C.s $v(x=0) = 0 \longrightarrow \boxed{B = 0}$

$v(x=d) = v_o \implies A = \frac{v_o}{d}$

$$\boxed{V = \frac{v_o}{d} x} \quad (v)$$

$\overline{E} = -\nabla V = \frac{-dv}{dx} \hat{a}x$

$\overline{E} = \frac{-v_o}{d} \hat{a}x = \frac{v_o}{d} (-\hat{a}x) \quad v/m$

$Q = \oint \epsilon \overline{E} \cdot \overline{ds} = \int_z \int_y (\epsilon) \frac{+v_o}{d} (-\hat{a}x) \cdot dydz (-\hat{a}x)$

$Q = \frac{\epsilon V_o}{d} s \, (C)$

$C = \frac{Q}{V_o} = \frac{\frac{\epsilon V_o}{d}}{V_o} = \frac{\epsilon s}{d} (F)$

Energy stored in the Capacitor:

$W_E = \frac{1}{2} \int_v \epsilon E^2 dv$

$W_E = \frac{1}{2} \epsilon \frac{Q^2}{\epsilon^2 S^2} sd$

$= \frac{Q^2}{2C}, \quad C = \frac{Q}{V}, \quad Q = CV$

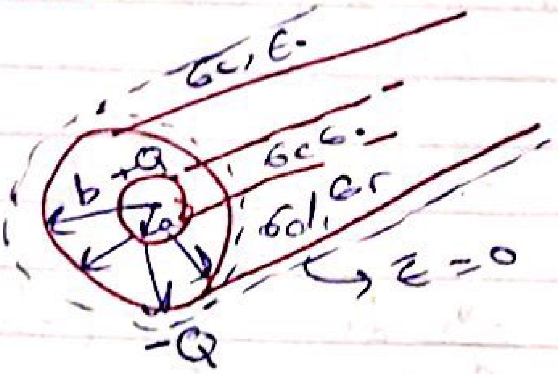
$= \frac{C^2 V^2}{2C} = \frac{1}{2} C V^2 \rightarrow$ for uniform

$\rightarrow C = \frac{\epsilon s}{d}$

2) cylindrical capacitor (coaxial cable).

C ? , R_d ?

$G = \frac{1}{R_d}$



$\boxed{\text{using Gauss}}$

$\oint \epsilon \bar{E} \cdot \bar{ds} = Q$

$\bar{E} = E_r \hat{a}_\rho \cdot \bar{ds} = \rho d\phi dz \hat{a}_\rho$

$$\int_0^{\ell} \int_0^{2\pi} \epsilon E_r \hat{a}_\rho \cdot \rho d\phi dz$$

$\epsilon E_r \rho 2\pi \ell = Q$

$\boxed{\bar{E} = \frac{Q}{2\pi \epsilon \rho \ell} \hat{a}_\rho}$ N/m

$\underline{\text{ch.4}}$

$\bar{E} = \frac{\rho_L}{2\pi \epsilon \rho} \hat{a}_\rho$

$\rho_L = \frac{Q}{\ell}$

$V = -\int_{\ell} \bar{E} \cdot \bar{dl}$

$$= -\int_{\rho=b}^{\rho=a} \frac{Q}{2\pi \epsilon \rho \ell} \hat{a}_\rho \cdot d\rho \hat{a}_\rho$$

$$= \frac{-Q}{2\pi \epsilon \ell} \ln\left(\frac{a}{b}\right)$$

$\therefore$

$$V = \frac{Q}{2\pi \epsilon \ell} \ln\left(\frac{b}{a}\right) \quad (V)$$

Scanned with CamScanner

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\epsilon l}\ln\left(\frac{b}{a}\right)}$$

$$\boxed{C = \frac{2\pi\epsilon l}{\ln\left(\frac{b}{a}\right)}}\ (F)$$

*capacitance per unit length.

$$C/l = \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)}\ (F/m)$$

$$I = \int_s \sigma d\,\bar{E}.\bar{ds} = \int_0^{l}\int_0^{2\pi} \sigma d \frac{Q}{2\pi\epsilon\rho l}\hat{a\rho}.\rho d\phi dz\,\hat{a\rho}$$

$$= \frac{\sigma d\, Q}{2\pi\epsilon\rho l}\hat{a\rho}.\rho d\phi dz\,\hat{a\rho}$$

$$\boxed{I = \frac{\sigma d\, Q}{\epsilon}}\ A$$

$$Rd = \frac{V}{I} = \frac{\frac{Q}{2\pi\epsilon l}\ln\left(\frac{b}{a}\right)}{\frac{\sigma d\, Q}{\epsilon}}$$

$$Rd = \frac{\ln\left(\frac{b}{a}\right)}{2\pi\sigma d\, l}\ (\Omega)$$

$$\boxed{R_d C = \frac{\epsilon}{\sigma d}} \quad or \quad \boxed{\frac{C}{G} = \frac{\epsilon}{\sigma d}}$$

$$G/l = \frac{2\pi\sigma d}{\ln\left(\frac{b}{a}\right)}\ s/m \Rightarrow G = \frac{2\pi\sigma d\, l}{\ln\left(\frac{b}{a}\right)}\ s \quad \Big|\Big| \quad \rho = \Omega.m$$

using B.V.P

$$\nabla^2 V = 0$$

$$\frac{1}{\rho}\int\int\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = \int\int 0$$

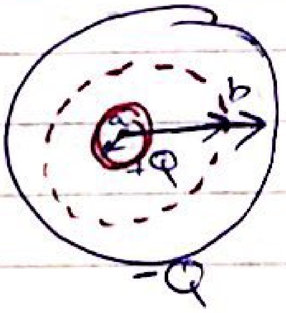
$$\boxed{\begin{array}{l} V(\rho=b)=0 \\ V(r=a)=V_0 \end{array}}$$

$$\int\frac{\partial V}{\partial\rho} = \int\frac{A}{\rho} \longrightarrow \boxed{V = A\ln\rho + B}$$

3) spherical capacitor

C=? , Rd ?

using Gauss :



$$\oint_s \epsilon\overline{E}\cdot\overline{ds} = Q$$

$$\overline{E} = E_r\hat{a_r}$$

$$\overline{ds} = r^2\sin\theta\, d\theta\, d\phi\, \hat{a_r}$$

$$\int_0^{2\pi}\int_0^{\pi}\epsilon E_r\hat{a_r}\, r^2\sin\theta\, d\theta\, d\phi\, \hat{a_r} = Q$$

$$\epsilon E_r/r^2\,(2)(2\pi) = Q$$

$$\boxed{E = \frac{Q}{4\pi\epsilon r^2}\hat{a_r}} \text{ N/m}$$

$$V = -\int_{\rho} \bar{E}\cdot d\bar{l}$$

$$= -\int_{r=b}^{r=a} \frac{Q}{4\pi\epsilon r^2}\,\hat{a_r}\cdot dr\,\hat{a_r}$$

$$= \frac{+Q}{4\pi\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right) \quad (v)$$

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)}$$

$$C = \frac{4\pi\epsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \quad (f)$$

$$I = \int_{s} \sigma_d \bar{E}\cdot d\bar{s} = \int_{0}^{2\pi}\int_{0}^{\pi} \frac{\sigma_d Q}{4\pi\epsilon r^2}\,\hat{a_r}\cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{a_r}$$

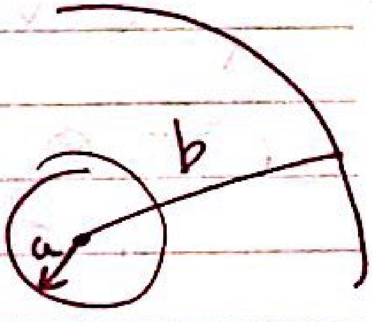
$$I = \frac{\sigma Q}{\epsilon} \quad (A)$$

$$R_d = \frac{V}{I} = \frac{\frac{1}{a}-\frac{1}{b}}{4\pi\sigma_d} \quad (\Omega)$$

$$G = \frac{4\pi\sigma_d}{\frac{1}{a}-\frac{1}{b}} \quad (s)$$

4) Isolated Capacitor

when $b \longrightarrow \infty$



$C = 4\pi \epsilon_a (f)$

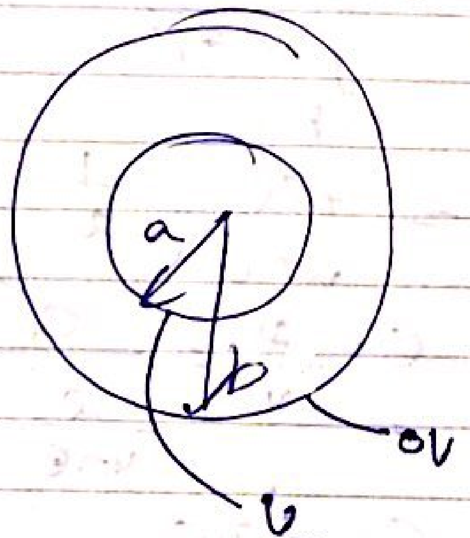
$R = \frac{1}{4\pi \epsilon_0 a} (\Omega)$

Sph Cap. using B.V.P

Sph.

$\nabla^2 V = 0$

$\frac{1}{r^2} \int \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) = \int 0$



$\cancel{r^2} \int \frac{\partial v}{\partial r} = \int \frac{A}{r^2}$

Ex. $V = \frac{-A}{r} + B$

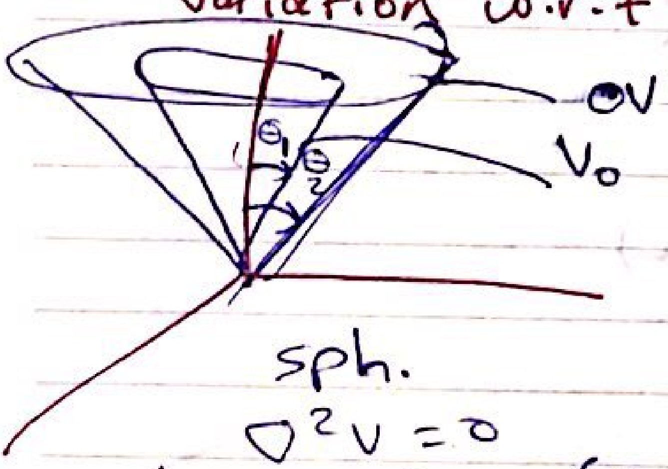
$\boxed{\begin{matrix} \bar{E} \\ \bar{D} \\ V \\ C \\ R \end{matrix}}$

B.C.s

$V(r=a) = V_o$

$V(r=b) = 0$

Variation w.r.t θ.



sph.

$\nabla^2 V = 0$

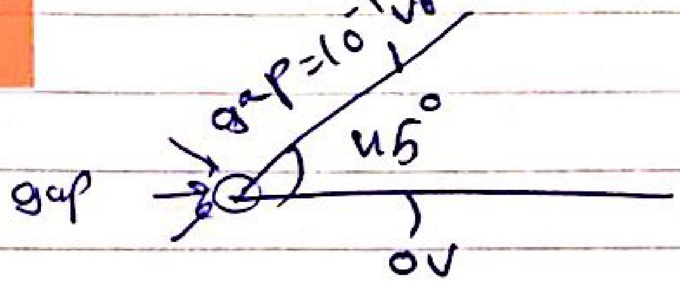
$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial v}{\partial \theta} \right) = 0$

$\cancel{\sin\theta} \int \frac{\partial v}{d\theta} = \int A \cos\theta$

$V(\Theta=\Theta_2)=V_0$

$V(\Theta=\Theta_1)=0$

variation w.r.t ϕ

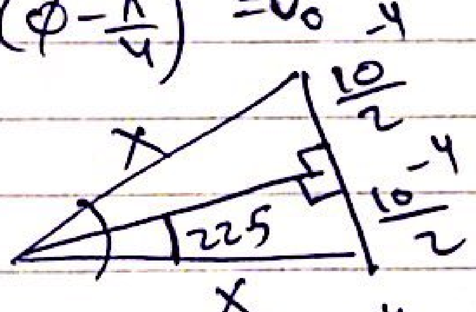


$$\frac{1}{\rho_2}\frac{d^2V}{d\phi^2}=0$$

$$\boxed{V=A\phi+R}$$

$V(\Theta=0)=0$

$V(\phi-\frac{\pi}{4})=V_0$



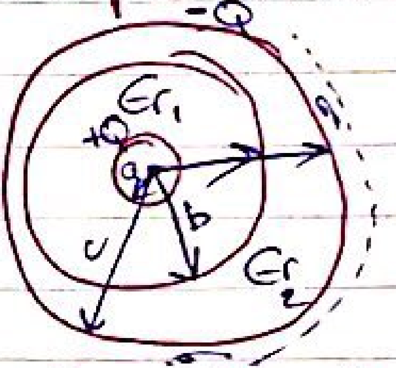
$$\sin(22.5)=\frac{\frac{10^{-4}}{2}}{X}$$

$$Q=\oint_s \varepsilon\bar{E}\cdot d\bar{s}=\int_0^{\ell}\int_x^{x+1}\varepsilon E_\phi\,\hat{az}\,d\rho\,dz\,\cancel{\hat{az}}$$

*example: find the capacitance of the following

a)

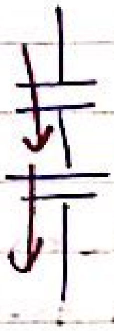
cyl



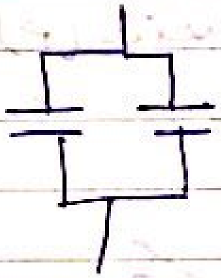
→ cylindrical cross section.

a
b
c
→ given

ϵ_{r_1}
ϵ_{r_2}
→ homogeneous من ابداية
→ non- ما وس



$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$ => في الحالة هوقع فتح ال capaciter ①
+
capaciter ②
so series.



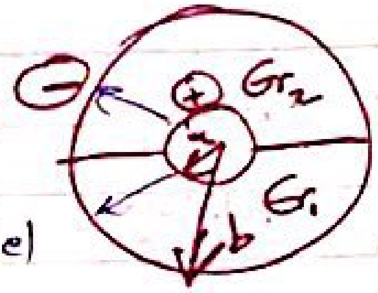
$C_{eq} = C_1 + C_2$

$$C = \frac{2\pi\epsilon\ell}{\ln\left(\frac{b}{a}\right)}$$

$$\rightarrow C_1 = \frac{2\pi\epsilon_0\epsilon_{r_1}\ell}{\ln\left(\frac{b}{a}\right)} \quad (F)$$

$$\rightarrow C_2 = \frac{2\pi\epsilon_0\epsilon_{r_2}\ell}{\ln\left(\frac{c}{b}\right)}$$

b) cyl.



parallel

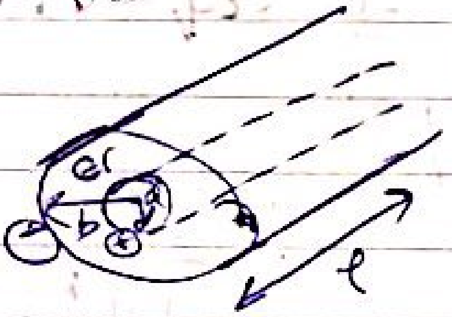
$$C_1 = \frac{\pi \epsilon_0 \epsilon_{r1} \ell}{\ln\left(\frac{b}{a}\right)} \ (F)$$

$$C_2 = \frac{\pi \epsilon_0 \epsilon_{r2} \ell}{\ln\left(\frac{b}{a}\right)} \ (F)$$

*example: A coaxial cable has a = 1cm b = 2.5 cm, if the dielectric between the two conductors has $\epsilon_r = \frac{10+\rho}{\rho}$, where ($\rho$) is in cm.

find C/ℓ ?

$\Longrightarrow$ Gauss

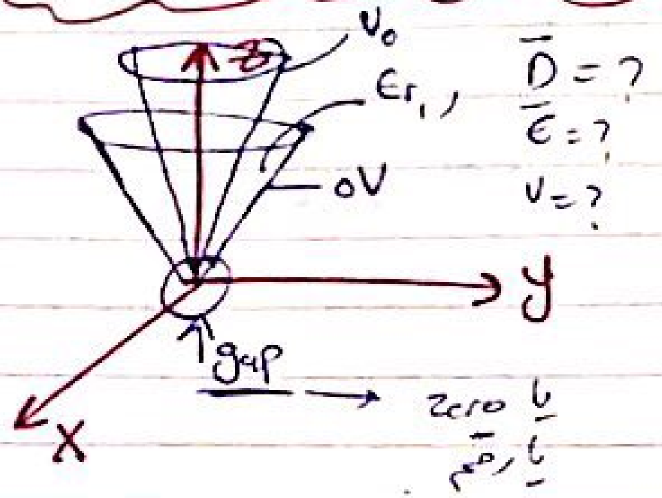


$$\oint_s \epsilon \overline{E} \cdot \overline{ds} = Q$$

$$\int_0^{\ell} \int_0^{2\pi} \epsilon_0 \epsilon_r E_\rho \, \rho \, d\phi \, dz = Q$$

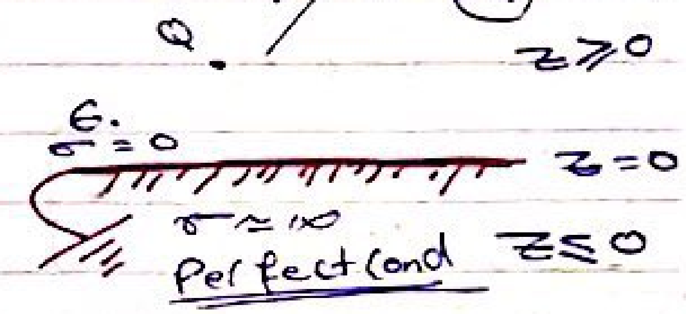
$$\boxed{\overline{E} = \frac{Q}{2\pi \epsilon_0 \epsilon_r \rho \ell} \hat{a}_\rho \ V/m}$$

- variation w.r.t θ or ϕ :-



$\bar{D} = ?$
$\bar{E} = ?$
$V = ?$

gap → zero يا
رقم يا



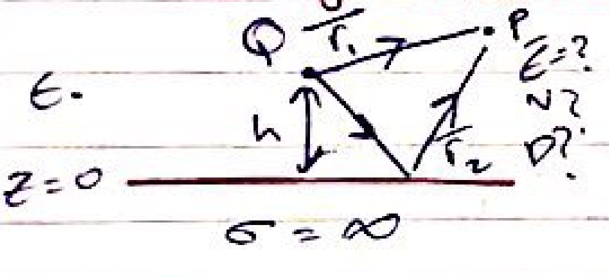
$\bar{E} = ?$
V ?
D ?
$z \geq 0$
$z \leq 0$

*** Method of Images:**

(electric feild لإيجاد ≗ طريقة)

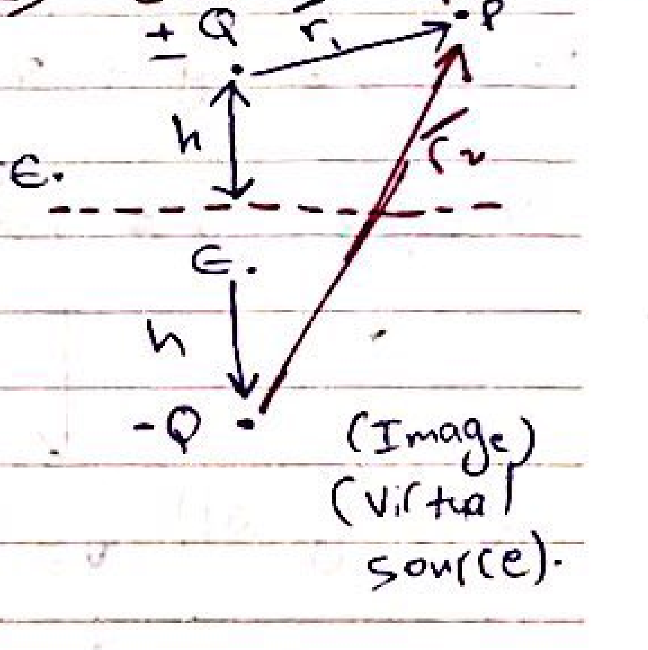
A) Point charge above perfect ground conducting plane

① original problem: ⟹ ② equivilant problem



$\bar{E}$=?
V?
D?

$\equiv$



(Image)
(Virtual source).

original || equivilent

$$\bar{E} = \begin{cases} \quad , z \geq 0 \\ \quad , z = 0 \\ 0 \quad , z \leq 0 \end{cases} \qquad \bar{E} = \begin{cases} \text{sol.} \quad , z \geq 0 \\ \underline{\text{sol}} . \quad , z = 0 \\ \text{sol} \quad , z \leq 0 \end{cases}$$

Sol. for the eq. problem

$V = V_+ + V_-$

$$= \frac{Q}{4\pi\epsilon_0 \cdot r_1} + \frac{-Q}{4\pi\epsilon_0 \cdot r_2}$$

$$= \boxed{\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) v} \quad , z > 0$$

$\bar{r}_1 = (x, y, z) - (0, 0, h) = (x, y, z-h)$

$\bar{r}_2 = (x, y, z) - (0, 0, -h) = (x, y, z+h)$

$r_1 = \sqrt{x^2 + y^2 + (z-h)^2}$

$r_2 = \sqrt{x^2 + y^2 + (z+h)^2}$

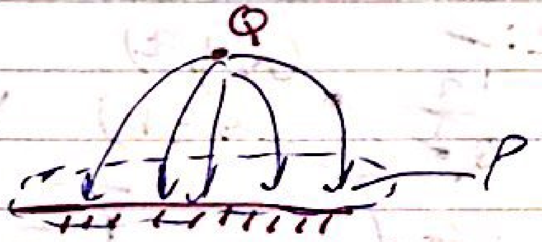
$$V(\text{equivilent}) = \begin{cases} \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) & , \underline{z > 0 , z < 0} \\ 0 & , z = 0 \end{cases}$$

$$V(\text{original}) \quad V = \begin{cases} \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) & , z > 0 \\ 0 & , z \leq 0 \end{cases}$$

$$\overline{E} = \overline{E}_+ + \overline{E}_-$$

$$= \frac{Q\overline{r_1}}{4\pi\epsilon_0 r_1^3} \overset{-}{\cancel{+}} \frac{Q\overline{r_2}}{4\pi\epsilon_0 r_2^3}$$

$$= \begin{cases} \frac{Q}{4\pi\epsilon_0}\left(\frac{\overline{r_1}}{r_1^3} - \frac{\overline{r_2}}{r_2^3}\right), & z>0 \\ \frac{Q}{4\pi\epsilon_0} \frac{(-2h\hat{a_z})}{(x^2+y^2+h^2)^{\frac{3}{2}}}, & z=0 \quad \leftarrow \boxed{E_n} \\ 0, & z<0 \end{cases}$$



$$\overline{D} = \epsilon_0 \overline{E}$$

$$\rho_s = \overline{D} \cdot \hat{a_n}$$

$$= \frac{-Qh}{2\pi\epsilon_0 (x^2+y^2+h^2)^{\frac{3}{2}}} \ C/m^2$$

$Q \equiv$ induced charge

$$Q_i = \iint_s \rho_s \, ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_s \, dx\,dy$$

convert to cyl (z) is common

$$ds = \rho \, d\rho \, d\phi$$

$$x^2+y^2 = \rho^2$$

$$Q_i = \int_0^{2\pi}\int_0^{\infty} \frac{-hQ}{2\pi\epsilon_0(\rho^2+h^2)^{\frac{3}{2}}} \rho \, d\rho \, d\phi$$

let $\rho^2 + h^2 = u \rightarrow \rho \, d\rho = \frac{du}{2}$

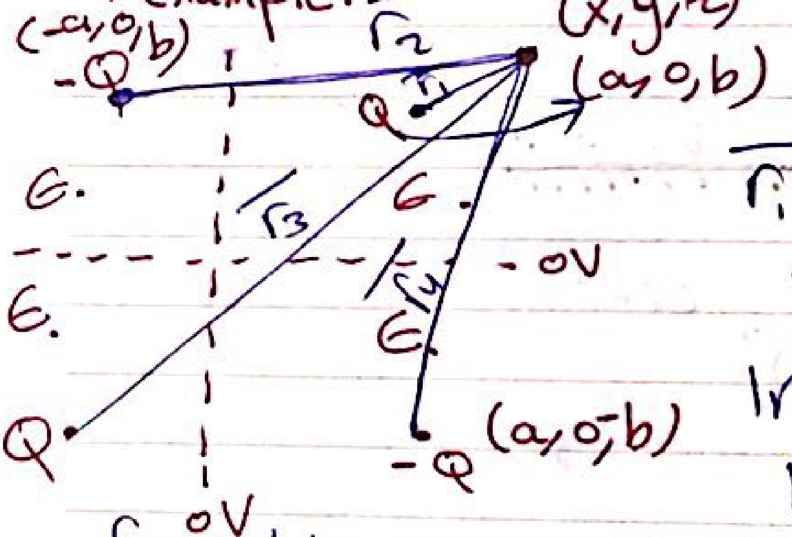
$$Q_i = \frac{-hQ}{(2)} \int \frac{dd}{u^{\frac{3}{2}}}$$

$$Q_i = \frac{+hQ}{2} \frac{u^{-\frac{1}{2}}}{\frac{1}{2}} \Big|$$

$$Q_i = hQ * \frac{1}{\sqrt{\rho^2+h^2}} \Big|_0^\infty = hQ\left(0 - \frac{1}{h}\right)$$

$$= -Q \; \cancel{\times}$$

*example:



$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} \ominus \frac{1}{r_2} + \frac{1}{r_3} \ominus \frac{1}{r_4}\right)$$

$\overline{r_1} = \langle x-a, y, z-b \rangle, |r_1| = \sqrt{(x-a)^2+y^2+(z-b)^2}$

$|r_2| = \sqrt{(x+a)^2+y^2(z-b^2}$

$|r_3| = \sqrt{(x+a)^2+y^2+(z+b)}$

$|r_4| = \sqrt{(x-a)^2+y^2+(z+b)^2}$

(must be $\underline{0}$ to preserve oV conditions).

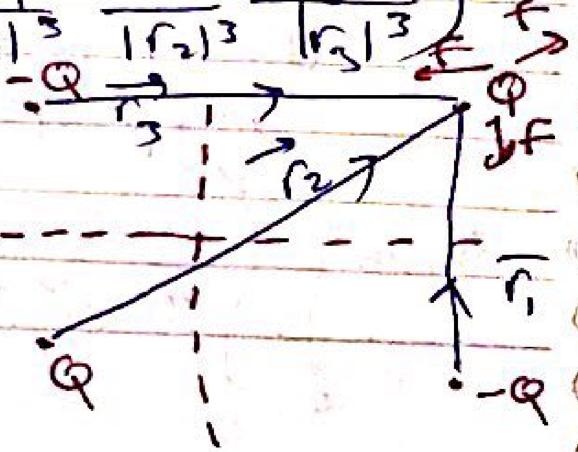
$N = \frac{360}{\emptyset} * -1 = 3$ images

b) $\overline{F} = \overline{E}Q \rightarrow \overline{E} = \frac{Q}{4\pi\epsilon_0}\left(\frac{-\overline{r_1}}{|r_1|^3} \; \frac{-\overline{r_2}}{|r_2|^3} + \frac{\overline{r_3}}{|r_3|^3}\right)$

$\overline{r_1} = \langle 0, 0, 2b \rangle, |r_1| = 2b$

$\overline{r_2} = \langle 2a, 0, 0 \rangle, |r_2| = 2a$

$\overline{r_3} = \langle 2a, 0, b \rangle, |r_3| = 2\sqrt{a^2+b^2}$



*chapter 7 :

Duality :

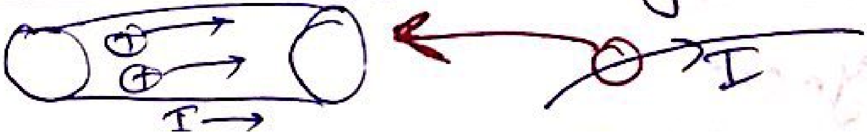
Electro static	Megreto Static
$E \equiv V/m$	$H \rightarrow$ Magnatic field intensity
$D \equiv c/m^2$	$H \equiv A/m$
	$B \rightarrow$ Magnatic flux density
$\epsilon \equiv F/m$	$B \equiv Wb/m^2$ or (T)
$D = \epsilon . \overline{E}$	$1 G = 10^{-15} T$
$\overline{E} = -\nabla V$	$\mu \equiv$ Permebility (H/m)
$W_E = \frac{1}{2}\int_v \overline{E} . \overline{D} dv$	$B = \mu H$
	$\overline{H} = -\overline{V}$ V/m
	$W_m = \frac{1}{2}\int_v \overline{H} . \overline{B} dv$

$$\psi = \oint_s \overline{D} . \overline{ds} \longrightarrow \psi = \oint_s \overline{B} . \overline{ds}$$

Gauss law

→ sources for magnetostatic :-

1) permenant magnet (Neodumium Nd).
2) Charge moving with constant velocity.
   (uniform) ⇒ $\underline{\underline{0}}$ accelaration
3) DC current flowing in the wire.

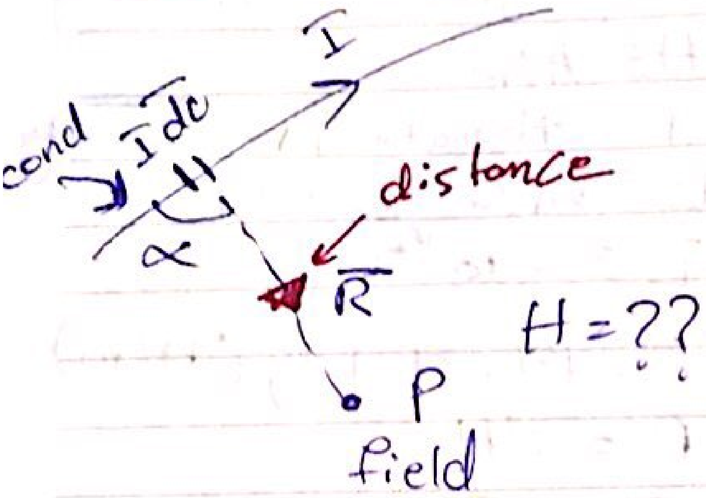


* Major laws:

$\boxed{1}$ → Biat-Savart's law:

$\boxed{2}$ → Amper's

$\boxed{1}$ Biat-savart's law: by exp



H = ??

$$dH \propto \frac{I dL \cdot \sin\alpha}{R^2} \rightarrow \int dH = \int \frac{K I dL \sin\alpha}{R^2}$$

$$K = \frac{1}{4\pi}$$

$$H = \int_L \frac{I dL \sin\alpha}{4\pi R^2} \equiv A/m$$

magnitude only.

→ to find the direction:-

$$I dL \sin\alpha = |I \overline{dL} \times a_R|$$

$$\overline{H} = \int \frac{I dL \times a_R}{4\pi R^2}, \quad E = \int \frac{\rho_v \, dv}{4\pi \epsilon R^2}$$

Any 1-D segment,

$$= \int_L \frac{I\bar{dl} \times \bar{R}}{4\pi R^3} , \quad \bar{dl} = dl\,\hat{a}_L$$

* note that:

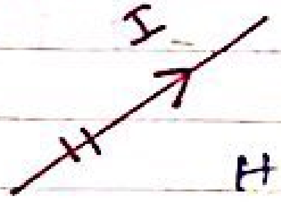
$\bar{A} \times \bar{B} = |A||B| \sin\theta$

$\bar{A} \times \bar{B} = - \bar{B} \times \bar{A}$

$$\bar{H} = \int_L \frac{I \hat{a}_L \times R}{4\pi R^3} \, d\bar{l}$$

* current distribution:-

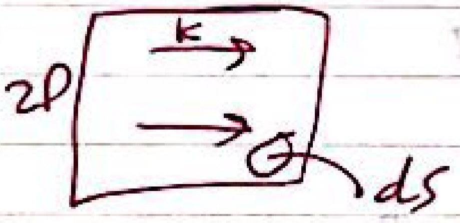
[1] line current.



$\bar{I} dl \equiv A.m$

$$H = \int_L \frac{I \bar{dl} \times \bar{R}}{4\pi R^3}$$

[2] surface current:

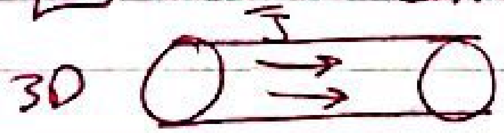


$\bar{K} ds \equiv A.m$

$\hookrightarrow m^2$

$$\bar{H} = \int_s \frac{\bar{K} ds \times \bar{R}}{4\pi R^3} \equiv A/m$$

[3] volume current distribution:



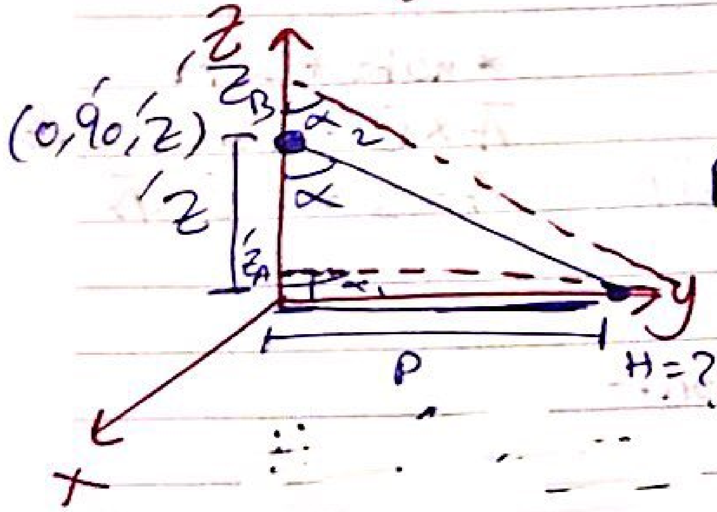
$\bar{J}\, dv \equiv A.m$

$\hookrightarrow m^3$

$\hookrightarrow A/m^2$

$$\bar{H} = \int_v \frac{J\, dv \times \bar{R}}{4\pi R^3}$$

*example: find $\overline{H}$ due to striaght finite wire along- z axis carry I.



$P(\rho,90°,0)$

$P(0,y,0)$

$\overline{dl} = dz'\,\hat{a}_z$

$\overline{R} = \rho\hat{a}_\rho - z'\hat{a}_z$

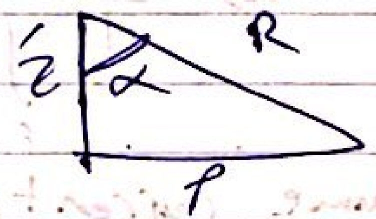
$$H = \int \frac{I\,\overline{dl} \times \overline{R}}{4\pi R^3}$$

$$\overline{H} = \frac{I}{4\pi} \int_{z'_A}^{z'_B} \frac{\hat{a}_z \times (\rho\hat{a}_\rho - \overset{\text{zero}}{\cancel{z'\hat{a}_z}})}{(\rho^2+z^2)^{\frac{3}{2}}}\, dz'$$

$$\begin{tikzcd} & \hat{a}_\rho \arrow[dr] & \\ \hat{a}_z \arrow[ur] \arrow[rr, dash] & & \hat{a}_\phi \end{tikzcd}$$

$$H = \frac{I}{4\pi} \int_{z'_A}^{z'_B} \frac{\rho}{(\rho^2+z^2)^{\frac{3}{2}}}\, dz'\,\hat{a}_\phi$$

$\sin\alpha = \frac{\rho}{R}$



$\cos\alpha = \frac{z'}{R}$

$\tan\alpha = \frac{\rho}{z'} \longrightarrow z'\ \rho\cot\alpha$

$R^2 = \rho^2 + z^2 = \rho^2 + \rho^2\cot^2\alpha$

$= \rho^2(1 + \cot^2\alpha)$

$R^2 = \rho^2\csc^2\alpha$

$R^3 = \rho^3\csc^3\alpha$

$$H = \frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho}{\rho^2 \csc^2\alpha} (-\rho \csc^2\alpha \, d\alpha) \, \hat{a\phi}$$

$$H = \frac{I}{4\pi\rho} \int_{\alpha_1}^{\alpha_2} -\sin\alpha \, d\alpha \, \hat{a\phi}$$

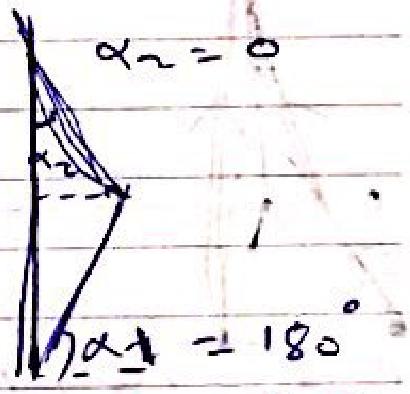
$$H = \frac{I}{4\pi\rho} (\cos\alpha_2 - \cos\alpha_1) \, \hat{a\phi}$$

→ for any finite straigt wire.

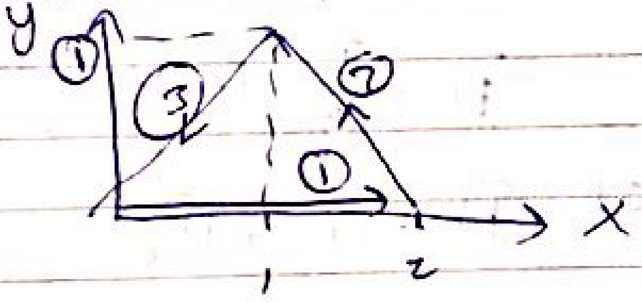
$$\hat{a\phi} = \hat{aL} \times \hat{a\rho}$$

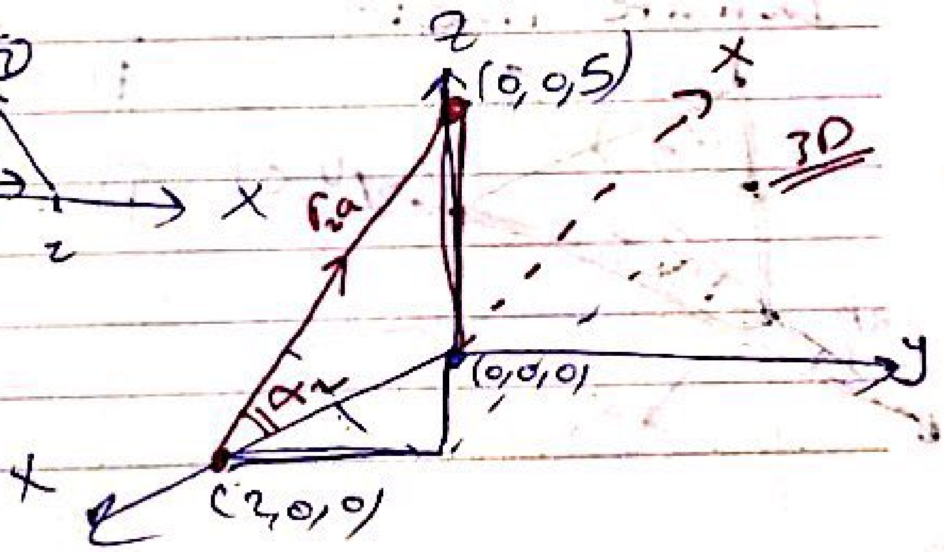
for an infinite line:-

$$\boxed{\overline{H} = \frac{I}{2\pi\rho}} \Rightarrow \text{always}$$



* example: Find I at (0,0,5) due to side ① of the loop.





$$H = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1) . \hat{a}_\phi$$

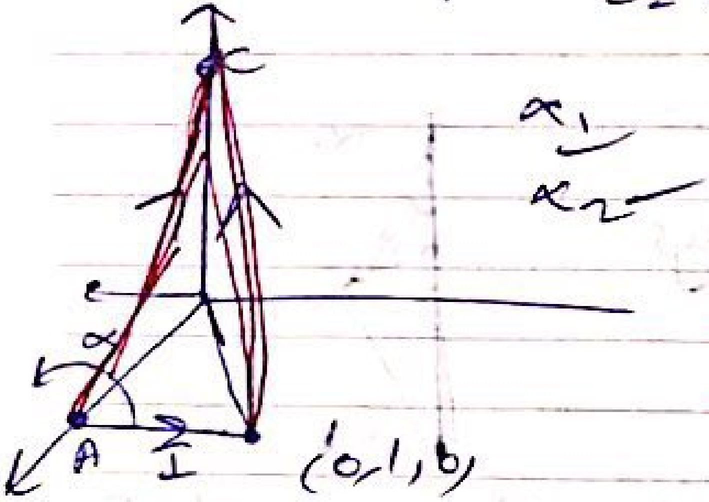
$$\begin{cases} \rho = 5 \\ \alpha_1 = 20^\circ \\ \alpha_2 \longrightarrow \cos\alpha_2 = \frac{2}{\sqrt{29}} \\ \hat{a}_\phi = \bar{dL} \times \hat{a}_\rho \end{cases}$$

$= \hat{a}_x \times \hat{a}_z$

$= -\hat{a}_y$

$H = -59.1\, \hat{a}_y$ mA/m

$\theta_1 + \alpha_2 = 120^\circ$



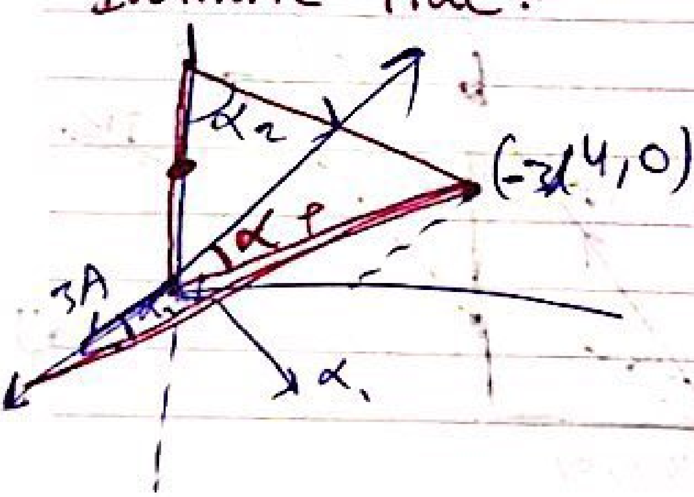
$\overline{BA} \cdot BC = |BA||BC|\cos\theta_2$

$\overline{AB} \cdot \overline{AC} = |AB||AC|\cos\theta_1$

$\cos\alpha_1 = -\cos\theta_1$

* example: Find $\bar{H}$ at (−3, 4, 0)

Infinite line:-



$$\bar{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1) \, \hat{a}_\phi$$

$$\bar{H} = \bar{H}_1 + \bar{H}_2$$

for line 1: -

$\rho^2 = (-3,4,0) \longrightarrow \rho = 5$

$\alpha_1 = 90$

$\alpha_2 = 0$

$a\phi = a_l \times a\rho$

$= -\hat{a_z} \times \frac{(-3,4,0)}{5}$

$\frac{3}{5}\hat{a_y} + \frac{4}{5}\hat{a_x}$

$\overline{H}_1 = \frac{I}{4\pi\rho}\hat{a\phi} = 32.2\hat{a_x} + 28.65\hat{a_y}$ mA/m

for line 2:

$\rho = 4$

$\alpha_2 = 0$

$\cos\alpha_1 = \frac{-3}{5}$

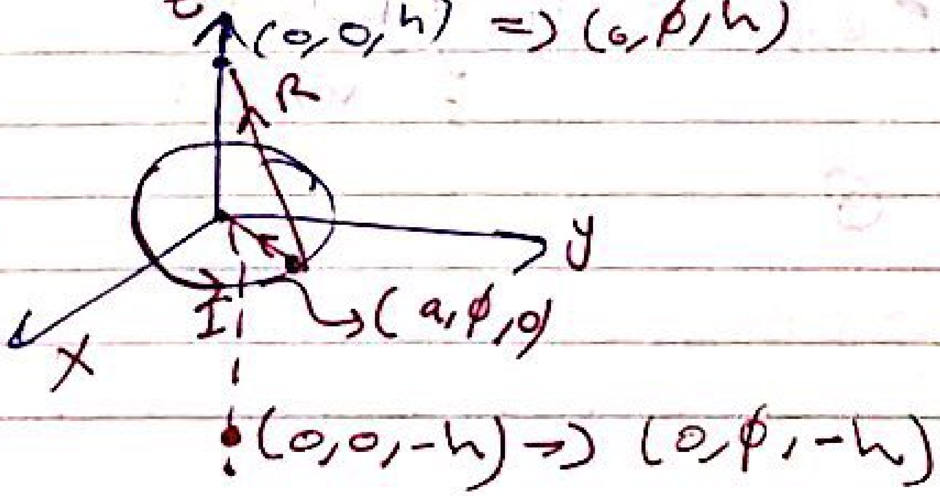
$\hat{a\phi} = \hat{a_l} \times \hat{a\rho}$

$= \hat{a_x} \times \hat{a_y} = \hat{a_z}$

$\overline{H}_2 = \frac{I}{4\pi\rho}\left(\frac{3}{5}\right)a\phi$

$H_2 = 23.88\,\hat{a_z}$ mA/m.

* example: find $\overline{H}$ at (0,0,h)



$$H = \int_L \frac{I\,\bar{dl} \times \bar{R}}{4\pi R^3}$$

$$\bar{dl} = \rho\, d\phi\, \hat{a}_\phi = a\, d\phi$$

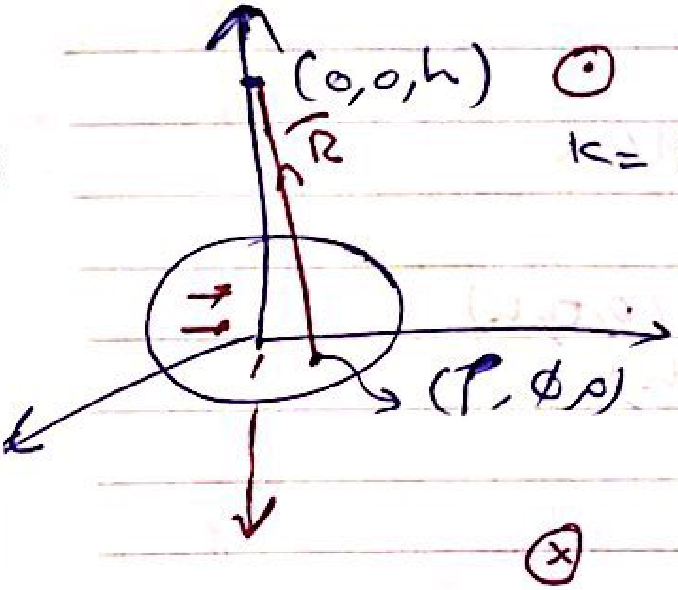
$$\bar{R} = -a\,\hat{a}_\rho + h\,\hat{a}_z$$

$$H = \frac{Ia}{4\pi}\int_0^{2\pi} \frac{\hat{a}_\phi \times (-\hat{a}_\rho + h\hat{a}_z)\,d\phi}{(a^2+h^2)^{\frac{3}{2}}}$$

$$H = \frac{Ia^2}{4\pi}\int_0^{2\pi} \frac{(a\hat{z} + \overset{zero}{\cancel{h\hat{a}_\rho}})\,d\phi}{(a^2+h^2)^{\frac{3}{2}}}$$

$$\boxed{H = \frac{I a^2}{2(m^2+h^2)^{\frac{3}{2}}}\hat{a}_z \quad A/m}$$

$\bar{H}$ for an infinite sheet.



$K = K_x\hat{a}_x + K_y\hat{a}_y$

$$\bar{H} = \int_s \frac{K\,ds \times \bar{R}}{4\pi R^3}$$

~~$ds = \rho \, d\rho \times \bar{R}$~~
~~$4\pi R^3$~~

$ds = \rho \, d\rho \, d\phi$

$\bar{R} = -\rho \hat{a_\rho} + h \hat{a_z}$

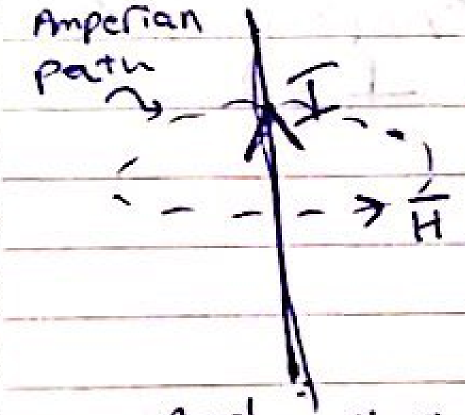
$R = \sqrt{\rho^2 + h^2}$

$$H = \frac{\bar{k}_y}{4\pi} \int_0^{2\pi} \int_0^{\infty} \frac{\hat{a_y} \times (-\rho \hat{a_\rho} + h \hat{a_z})}{(\rho^2 + h^2)^{\frac{3}{2}}} \rho \, d\rho \, d\phi$$

$\boxed{2}$ ** Amper's law:-

$$\oint_L \bar{H} \cdot dL = I_{enclosed}$$

$\circledast$ → 3rd maxwell's eq. in integral form.



$\boxed{*}$ Duality:

① $\oint_s D \, ds = Q_{enc}$ ↷

$\oint_s \bar{B} \cdot ds = zero$ zero

② $\oint_s \bar{E} \cdot dL = zero$ ↷

$\oint \bar{H} \cdot dL = I_{encosed}$

$\circledast$ 4th maxwell's eq. in integral form.

* Apply steche's theorem:-

$$\oint_L \bar{H} \cdot dL = \int_s \nabla \times H \, ds = I_{enclosed} = \int_s \bar{J} \cdot ds$$

$\circledast$ $\nabla \times \bar{H} = \bar{J}$

→ 3rd maxwell's eq in diff form.

* Application on Amperes law:-

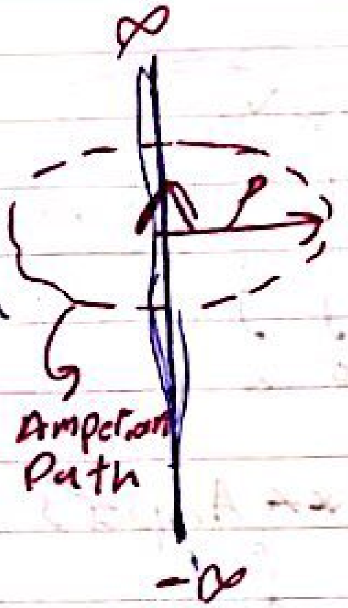
$\boxed{1}$ $\bar{H}$ due to an inf line of Current

$$\oint_L \bar{H}\cdot\bar{dl} = I_{enc} = I$$

R.H.R → $\bar{H} = H_\rho \hat{a}_\phi$

$\bar{dl} = \rho d\phi \hat{a}_\rho$

↳ Amperian path



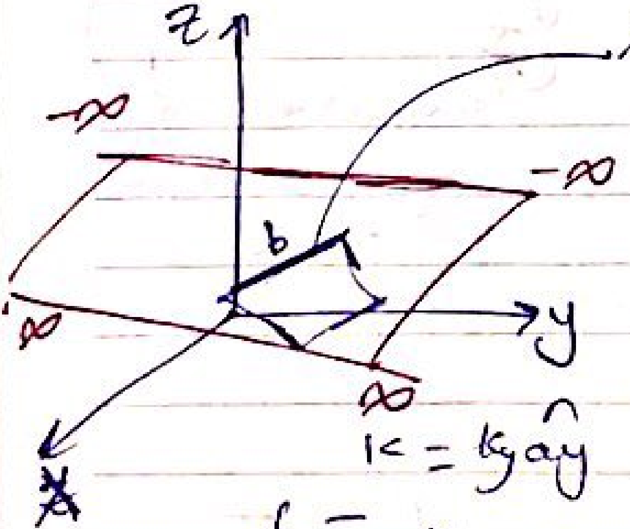
$$\int_0^{2\pi} H_\phi \rho d\phi = I$$

$\Rightarrow 2\pi\rho H_\rho = I$

$$\bar{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

$\boxed{2}$ $\bar{H}$ for an inf sheet of Current.

XZ plane (Amperian path ⊥ y).



$K = K_y \hat{a}_y$

$$\oint_L \bar{H}\cdot dl = I_{enc}$$

R.H.R → $H = \begin{cases} H.(\hat{a}_x), & z>0 \\ H.(-\hat{a}_x), & z<0 \end{cases}$

$$\oint_{L} \overline{H}.\overline{dL} = \int_{L_1} H.dL_1 + \int_{L_2} H dL_2 + \int_{L_3} dL_3 . H + \int_{L_4} H.dL_4$$

$$\int_{0}^{b} H.\hat{ax}.dx\,\hat{ax} + \int_{b}^{0} H.(-\hat{ax}).dx\,\hat{ax}$$

$$H.\cancel{b} + H.\cancel{b} + K_y.\cancel{b}$$

$$H_{\circ} = \frac{K_y}{2} \text{ A/m}$$

$$H = \begin{cases} Ky(\hat{ax}), & z>0 \\ \frac{ky}{2}(-\hat{ax}), & z<0 \end{cases} \quad ?$$

(*) In general

$$\overline{H} = \frac{1}{2} K \times \hat{an}$$

in this example.

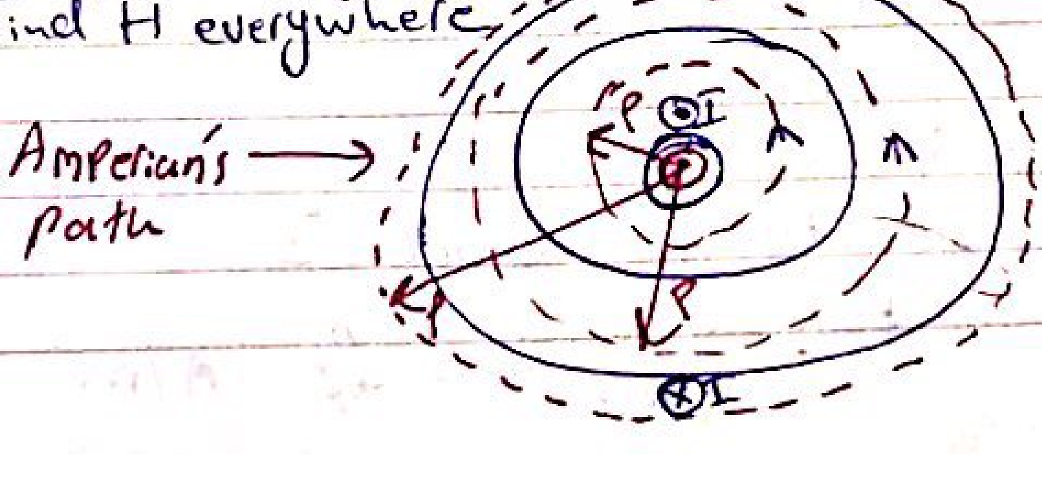
$$= \frac{1}{2} k\, ax$$

$$\frac{1}{2} k\hat{ay} \times -\underline{\hat{az}}$$

$$= \frac{1}{2} k\, -\hat{ax}$$

$\boxed{3}$ for an infanelty large coaxial cable:-

- find $\overline{H}$ everywhere



by R.H.R

$\overline{H} = H_\phi \hat{a\phi}$

$\overline{dl} = \rho d\phi \hat{a\phi}$

$\oint_L \overline{H} \cdot dl = I_{enclosed}$

$\longrightarrow$ for $0<\rho<a$

$\int_0^{2\pi} H_\phi \hat{a\phi} \cdot \rho d\phi \hat{a\phi} = I_{enclosed}$

$2\pi \rho \overset{?}{H_\phi} = I_{enclosed} = \int_S \overline{J} \cdot ds$

at $\rho = a \longrightarrow I_{enclosed} = \int_S \overset{?}{J} \cdot ds$

$I = \int_0^{2\pi} \int_0^{a} J_z \hat{az} \cdot \rho d\rho d\phi \hat{az}$

$I = J_z \pi a^2 \longrightarrow J = \frac{I}{\pi a^2} \hat{az} \quad A/m^2$

$2\pi \rho H_\phi = \int_0^{2\pi} \int_0^{0<\rho<a} \frac{I}{\pi a^2} \hat{az} \cdot \rho d\rho d\phi \hat{az}$

$\overline{H} = \frac{I\rho}{2\pi a^2} \hat{a\phi} \; (A/m) \longrightarrow$ for $0<\rho\leq a$

$\longrightarrow$ for $a<\rho<b$

$2\pi \rho H_\phi = I_{enclosed} = I$

$H = \frac{I}{2\pi\rho} \hat{a\phi} \quad A/m$

→ for $b<\rho<c$

$$2\pi\rho H_\phi = I_{enc} = \int \bar{J}\,ds = \int_0^a \bar{J}\,d\bar{s} + \int_a^b J\,ds + \int_{b}^{c} \bar{J}.ds$$

at $\underline{\rho=c}$: $I_{enc} = I$

$$I = \int_s \bar{J}.\bar{ds} = \int_0^{2\pi}\int_b^c J_z\,(-\hat{a}_z).\,\rho d\rho d\phi\,\hat{a}_z$$

$$\boxed{I + \int_0^{2\pi}\int_0^{\rho} J_z\,\rho d\rho d\phi}$$

$$\bar{J} = \frac{I}{\pi(a^2-b^2)}(-\hat{a}_z) \rightarrow \text{wb.n}$$

$$2\pi\rho H_\phi = I - \int_0^{2\pi}\int_b^{\rho} \frac{I}{\pi(a^2-b^2)}\,\rho d\rho d\phi$$

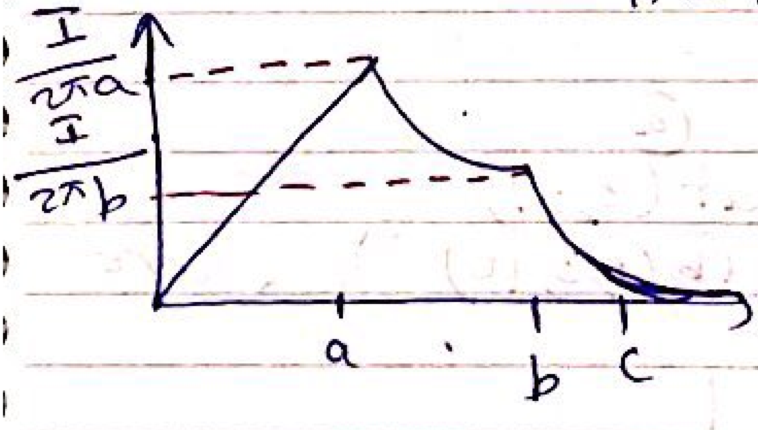
$$2\pi\rho H\phi = I - \frac{I}{\cancel{\pi}(a^2-b^2)} \times \cancel{\pi}(\rho^2-b^2)$$

$$\bar{H} = \frac{I}{2\pi\rho}\left(1-\frac{\rho^2-b^2}{c^2-b^2}\right)\hat{a}_\phi \quad A/m$$

↳ $b \leq \rho \leq c$

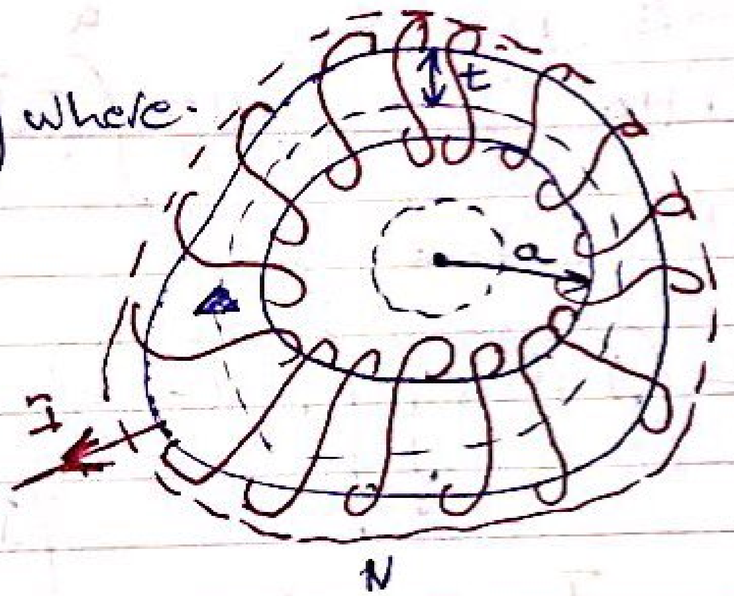
→ for $\rho > c$ ⇒ $2\pi\rho H_\phi = I_{enc} = I - I = 0$

$\bar{H} = \text{zero},\ \rho \geq c$



4 Toroid:-

- find $\overline{H}$ everywhere.



$H = H_\phi \hat{a}_\phi$

for $0 < \rho < a$

$\oint H . dL = I_{enc}$
$= zero$

$H = zero$

for $\rho > a + 2t$

$2\pi\rho H_\phi = I_{enc} = I - I = zero$

$H = zero$

→ for $a < \rho < a + 2t$

$2\pi\rho H_\phi = I_{enc} = \underset{\uparrow}{N}I$

عدد اللفات

$$\boxed{\overline{H} = \frac{NI}{\underset{L}{2\pi\rho}} \hat{a}_\phi \text{ A/m}}$$

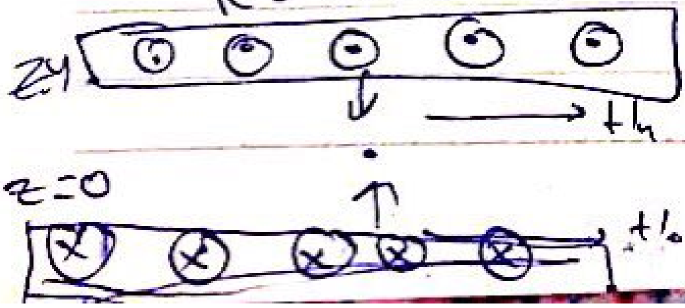
$HL = NI$

$\oint_L H . dL = NI$

* example: Find $\overline{H}$ at ⓐ (0,0,1)

ⓑ (0,0,10) → zero

$K = 10(\hat{a}_x)$ A/m



$K = 10(-\hat{a}_x)$ A/m

$\overline{H} = \frac{1}{2} k \hat{a_n}$

at $(0,0,1)$ $\overline{H_1} = \overline{H_o} + \overline{H_4}$:-

$= \frac{1}{2}\left(10(-\hat{a_x}) \times \hat{a_z}\right) + \frac{1}{2}\left(10(\hat{a_x}) \times -\hat{a_z}\right)$

$= 5\hat{a_y} + 5\hat{a_y}$

$= 10\hat{a_y}$ (A/m)

* Magnatic flux density ($\overline{B}$):-

$\overline{B} = \mu_o \overline{H}$ → consitutuc relation.

$\mu_o \equiv$ free space permctibility

$= 4\pi * 10^{-7}$ H/m

→ unit: $\frac{H}{m} \cdot \frac{A}{m} = \frac{HA}{m^2}$

or $\frac{Wb}{m^2}$, W_b = weber

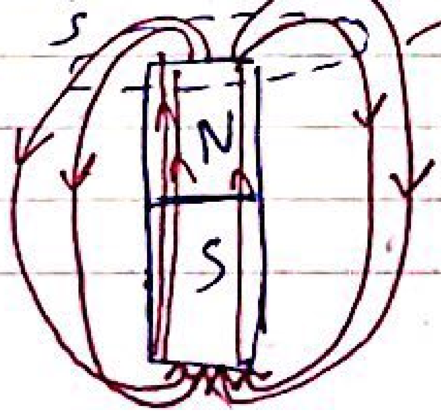
or T

or $G = 10^{-5}$ T (old unit)

* Magnatic flux (Ψ_m): - (scalar)

$\Psi = \int \overline{B} \cdot \overline{ds} \neq$ zero

$\Psi = \oint_s \overline{B} \cdot \overline{ds} = 0$ → magnatic Gauss law



→ Magnatic flux lines

$\Psi_e = \int \overline{D} \cdot \overline{ds}$

$\Psi_e = \oint_s \overline{D} \cdot \overline{ds} = Q_{enc}$

Gauss law.

$$\oint_s \bar{B}\cdot \bar{ds} = \text{zero}$$

↳ 4th maxwell's eq. in integral form

Apply divergence thm.

$$\oint_s \bar{B}\cdot\bar{ds} = \int_v \bar{\nabla}\cdot\bar{B}\,dv = \text{zero}$$

$$\bar{\nabla}\cdot\bar{B} = \text{zero}$$

↳ 4th maxwell's eq. in diff form

**** Magnatic potential :-**

↳ Scalar : V_m (A)

↳ vector : $\bar{A}$ (w_b/m)

we know ;

4th maxwel eq. ← $\nabla\cdot\bar{B} = 0$

$\nabla\times\bar{H} = J$

↳ 3rd Maxwell's eq.

so

$$\bar{B} = \bar{\nabla}\times\bar{A} \;,\; \bar{H} = \frac{B}{\mu_0}$$

$$\downarrow \quad \downarrow$$

$$\frac{w_b}{m^2} \quad \frac{1}{m} \cdot \boxed{\frac{w_b}{m}}$$

(magnatic fluxdensity

-∇V

$* \;\bar{H} = -\nabla V_m \equiv \bar{E} =$

if J = zero → since free region

$\frac{1}{m}$ $\qquad$ A

$dl \nearrow I$

What is A :-

$$B = \int_l \frac{\mu_0 I \bar{dl} \times \bar{R}}{4\pi A}$$

$$\frac{\bar{R}}{R^3} = \frac{\hat{a_R}}{R^2} = \boxed{-\nabla_x\left(\frac{1}{R}\right)}$$

$$B = \frac{-\mu_0 I}{4\pi} \int_L dL \times \underbrace{\nabla\left(\frac{1}{R}\right)}_{\text{Identity:}}$$

* Apply the vector Identity:

$$\nabla \times (f\bar{A}) = f\,\nabla \times \bar{A} + \nabla f \times \bar{A}$$

$$f \equiv \frac{1}{R} = \bar{A} \equiv dL$$

$$B = \frac{\mu_0 I}{4\pi} \int_L \nabla \times \frac{dL}{R}$$

$$\bar{B} = \nabla \times \int_L \frac{\mu_0 I\,dL}{4\pi R}$$

$$\boxed{\bar{B} = \nabla \times A}$$

$$\text{so} \longrightarrow \bar{A} = \int \frac{\mu_0 I\,dL}{4\pi R} \rightarrow \text{line}$$

$$\bar{A} = \int_S \frac{\mu_0 \bar{K}\,ds}{4\pi R} \rightarrow \text{surface}$$

$$\bar{A} = \int_V \frac{\mu_0 I\,dV}{4\pi R} \rightarrow \text{volume}$$

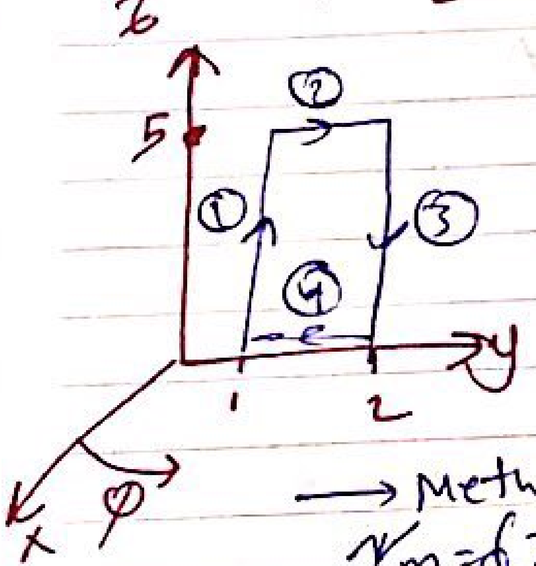
$$\psi_m = \int_S \bar{B}\cdot\bar{ds} = \int_S \nabla \times A \cdot ds\,,\quad B = \nabla \times \bar{A}$$

$$\psi_m = \oint_s \bar{A}\cdot\bar{dL}$$

stoke's

*example: Given $\bar{A} = \frac{\rho^2}{4}\hat{a}_z$ wb/m

Calculate the magnatic flux crossing the surface $\phi = \frac{\pi}{2}$, $1 \leq \rho \leq 2$ m, $0 \leq z \leq 5$ m.



$\overline{dL_1} = dz\,\hat{a}_z$
$\overline{dL_2} = d\rho\,\hat{a}_\rho$
$\overline{dL_3} = dz\,\hat{a}_z$
$\overline{dL_4} = d\rho\,\hat{a}_\rho$

→ Method※1

$$\psi_m = \oint \bar{A}\cdot\overline{dL}$$

zero.

$$\psi_m = \int_{L_1} \bar{A}\cdot dL_1 + \cancel{\int_{L_2} \bar{A}\cdot dL_2} + \int_{L_3} \bar{A}\cdot dL_3 + \cancel{\int_{L_4} \bar{A}\cdot dL_4}$$

$$\psi_m = \int_0^5 \frac{-\rho^2}{4}\Big|_{\rho=1} dz + \int_5^0 \frac{-\rho^2}{4} dz\Big|_{\rho=2}$$

$$\boxed{\psi_m = \frac{15}{4} W_b}$$

→ Method※2:

$$\psi_m = \int_s \bar{B}\cdot\overline{ds} \longrightarrow ds = d\rho\, dz\,\hat{a}_\phi$$

$$\bar{B} = \nabla\times\bar{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{\rho}\left((0-0)\hat{a}_\rho - \left(-\frac{\rho}{2}\right)\rho\,\hat{a}_\phi + (0)\,\hat{a}_z\right)$$

$$= \bar{B} = \frac{\rho}{2}\hat{a}_\phi \ T \Rightarrow \psi_m = \int_0^5\int^2 \frac{\rho}{2}\, d\rho\, dz = \frac{15}{4} \ w_b$$