Dsp

8/9 Thu

*Chapter 2:

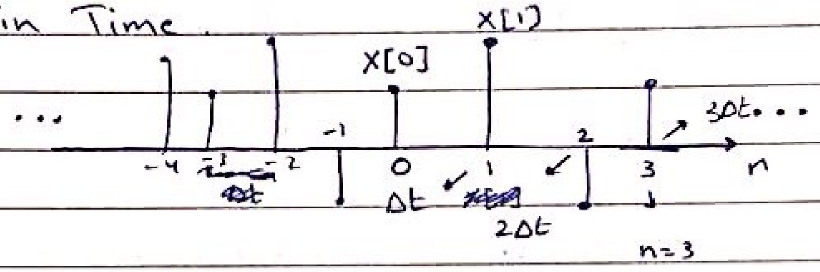
Discrete time Signals Systems:

- Continous in Amplitude (different level)

contazation يعطي level.

[ ] square ≡ Discrete

- Discrete in Time.



*graphical Representation: زمن ثابت بين كل اتنين.

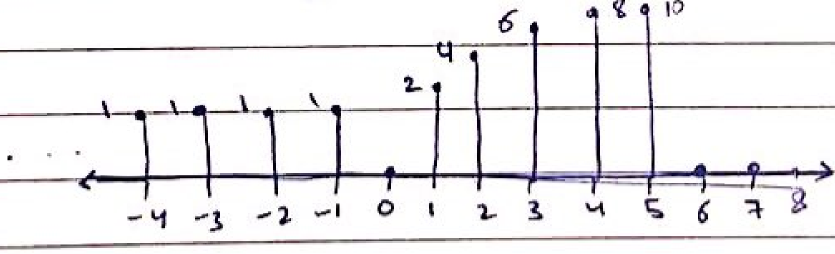
Δt fixed.

x[n] : Discrete Time Signal [sequence]

x[n] :

[1] functional Method :

$$x[n] = \begin{cases} 1 & n<0 \\ 2n & 0<n\le 5 \\ 0 & \text{else where.} \end{cases}$$



$x[n] = A\cos(\omega n + \phi)$ for all n. $\equiv x(t) = A\cos(\omega t + \phi)\big|_{t=nT}$

ω: Discrete time frequency (rad/sample), $\Omega = 2\pi f$ (rad/sec)

[2] Sequential Method :

- Infinite length sequence (Two side sequence).

- Finite length sequence.

$x[n] = \{1, 2, \underset{\uparrow}{3}, 4\}$ finite, $N_1 = -2$, $N_2 = 1$

-2 -1 0 1 → index. هو الترتيب

$x[n] = \{1.1, 2.3, \underset{\uparrow}{5.6}, 7, 5, 9\}$ $N_1 = -2$

$N_2 = 3$.

length $= -2 + 3 + 1 = 6$

The length of seqence. $N_2 - N_1 + 1$

* Some sequences $x[n] = \{1, 2, 5, 6, \ldots\}$ infinite. بتبقى محدودة من جهة

Right - sideal sequence, causel

$x[n] = [1, 2, \underset{\uparrow}{5}, 6, \ldots\}$ R.S.S بتبدأ من مكان ~ وتمشي لجهة اليمين بنسميها R.S.S

-2 -1 0 1 2 ...

!? $x[n] = 0 \rightarrow n < N_1$

$n < -2$

If $N_1 \geq 0$

$x[n] = \{ \ldots, 0, 0, \underset{\uparrow}{1}, 2, 3, 4, \ldots\}$

↳ Right - sideal seqence, Causal seqence.

* left Side seqence:

$[\ldots, 6, 5, 3, 1]$

OR $[\ldots, 7, 5, \underset{\uparrow}{1}, 5, 6]$

-1 0 1 2

Anti - causal

Non Causal

$\{x[n]\} = \{ \dots, 1, 1, 1, \underset{\underset{n=0}{\uparrow}}{0}, 2, 4, 6, 8, 10, 0, 0, 0 \dots \}$

* ما بعرف وين الصفر منشان اهيك دايماً بحط صفر عند قيمة x[0]

$x[n] = \{1, 2, 3, 4\}$ ما اعطاني السهم بفرض اول قيمة هي صفر

$x[0] = 1, \quad x[1] = 2, \quad x[2] = 3, \quad x[3] = 4.$

$x[n] = \{1, 2, \underset{\uparrow}{3}, 4\} \equiv \{0 \dots, 0, 0, 0, \underset{\uparrow}{1}, 2, 3, 4, 0, 0, 0\}$ length = 4

$x[-2] = 1, \quad x[-1] = 2, \quad x[0] = 3, \quad x[1] = 4.$

$\boxed{3}$ Matrix Method 80

* دائماً في Matrix الـ index دائماً موجب.
* و value بتكون Real.
* بستخدمها في applications معينة.
* دائماً أول قيمة بتكون هي x[0]

$$x[n] = \left.\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}\right\} N \text{ length.}$$

↳ colom matrix

$$x[n] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$ 4-length sequence.

Y, **X** bold face, captial latter → vector $= \underline{X}$

A, B, C, D → matrix.

$$\underline{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$ vector X

The length of the Sequence: $x[n]$

$$N_1 \leq n \leq N_2$$

where $N_1 > -\infty$ and $N_2 < \infty$ and $N_1 \leq N_2$

$N_1 = N_2 \rightarrow$ one element.

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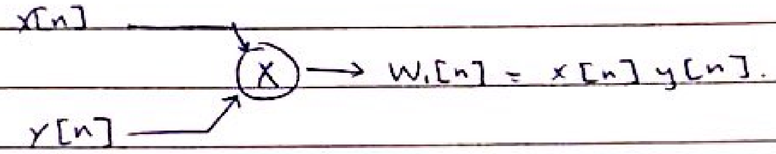
* Elementary operations on sequence. 8Δ

$\boxed{1}$ The product operation

$$w_1[n] = x[n] \cdot y[n]$$



$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ * multiplication element by element.

$x[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\}$

$*$

$y[n] = \{\underset{\uparrow}{2} \quad 2 \quad 1\}$

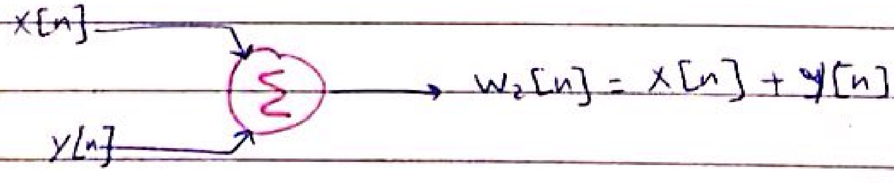
$= 1 \quad \underset{\uparrow}{2} \quad 3 \quad 0$

$\quad 0 \quad \underset{\uparrow}{2} \quad 2 \quad 1$

$w_1[n] = \{0 \quad \underset{\uparrow}{4} \quad 6 \quad 0\}$

$\boxed{2}$ The addition of seqence 8Δ

$$w_2[n] = x[n] + y[n]$$



$x[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\}$ $\qquad$ $y[n] = \{\underset{\uparrow}{2} \quad 2 \quad 1\}$

-	1	$\underset{\uparrow}{2}$	3	0
+	0	$\underset{\uparrow}{2}$	2	1

$w_2[n] = [1 \quad \underset{\uparrow}{4} \quad 5 \quad 1]$

$\boxed{3}$ Multiplication by a Constant 8Δ

$w_3[n] = A\, x[n]$, A +ve
−ve → phase shift 180°

$x[n] \longrightarrow \boxed{A} \longrightarrow w_3[n] = A\, x[n]$

$A = 5 \qquad x[n] = \{1 \; 2 \; 3\}$

$w_3[n] = \{5 \quad \underset{\uparrow}{10} \quad 15\}$

$\boxed{4}$ Time-delay operation (unit)

$w_4[n] = x[n-1]$ you have shift by 1

$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow w_4[n] = x[n-1]$

Ex: $x[n] = [\overset{-1}{1} \quad \underset{\uparrow}{\overset{0}{2}} \quad \overset{1}{3}]$

$w_4[n] = x[n-1]$

$w_4[-2] = x[-2-1] = x[-3] = 0$

$w_4[-1] = x[-1-1] = x[-2] = 0$

$w_4[0] = x[0-1] = x[-1] = 1$

$w_4[1] = x[1-1] = x[0] = 2$

$w_4[2] = x[2-1] = x[1] = 3$

$w_4[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3\}$

$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-2]$

$\equiv \quad x[n] \longrightarrow \boxed{z^{-2}} \longrightarrow x[n-2]$

5 Unit-Advance operation :

$$w_5[n] = x[n+1]$$

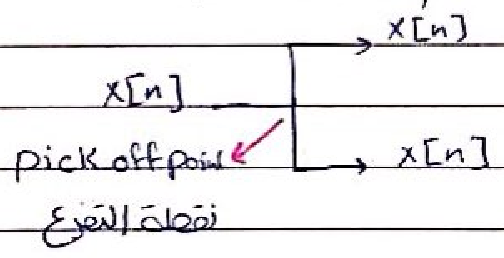
$$x[n] \longrightarrow \boxed{z} \longrightarrow w_5[n] = x[n+1]$$

$$x[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\}$$

$$w_5[-2] = x[-1] = 1$$
$$w_5[-1] = x[0] = 2$$
$$\rightarrow w_5[0] = x[1] = 3$$
$$w_5[1] = x[2] = 0$$

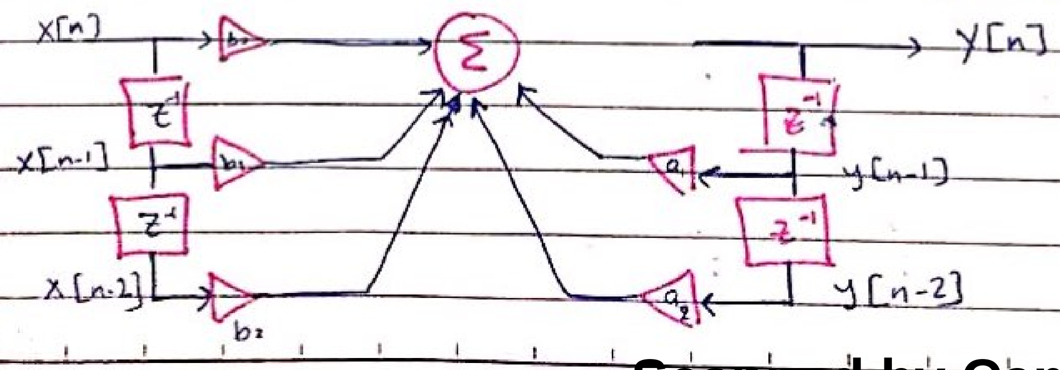
$$w_5[n] = \{1, \ 2 \quad \underset{\uparrow}{3}\}$$

6 The pick-off point



Combination of Elementary operations :

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$



Classification of seqences 8:

1 Base on Symmetry 8:

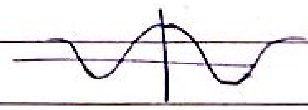
a sequence $x[n]$ is called a conjugate - symmetric sequence if: $x[n] = x^*[-n]$

$x[n] = \{1+j \quad \underset{\uparrow}{5-3j} \quad 2 \quad 5+2j\}$ → complex* sequence

* → Real same, imag convert the sign. [conjeugent].

If $x[n]$ is real then $x[n] = x[-n]$ ⇒ even symmetry.

cos → even.



A sequance $x[n]$ is called conjugate anti symmetric if $x[n] = -x^*[-n]$

If $x[n]$ is real $x[n] = -x[-n]$ ⇒ odd symmetry.

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Classification of sequence.

[I] Based on symmetry:

Conjugate Symmetric sequence. $X_{cs}[n]$

Conjugate Antisymmetric sequence $X_{ca}[n]$

Any sequence $X[n] = X_{cs}[n] + X_{ca}[n]$

where

$$X_{cs}[n] = \frac{X[n] + X^*[-n]}{2}$$

and

$$X_{ca}[n] = \frac{X[n] - X^*[-n]}{2}$$

Ex: $x[n] = \{0 \quad 1+j4 \quad -2+j3 \quad \underset{\uparrow}{4-j2} \quad -5-j6 \quad -j2 \quad 3\}$

sol:

$X^*[n] = \{0 \quad 1-j4 \quad -2-j3 \quad \underset{\uparrow}{4+j2} \quad -5+j6 \quad +j2 \quad 3\}$

$X^*[-n] = \{3 \quad j2 \quad -5+j6 \quad \underset{\uparrow}{4+j2} \quad -2-j3 \quad 1-j4 \quad 0\}$

$X_{cs}[n] = \{\frac{0+3}{2} \quad \frac{1+j4+j2}{2} \quad \dots$

$X_{cs} = \{1.5 \quad 0.5+j3 \quad -3.5+j4.5 \quad \underset{\uparrow}{4} \quad -3.5-j4.5 \quad 0.5-j3 \quad \dots 1.5\}$

$X_{ca}[n] = \{\frac{0-3}{2} \quad \frac{1+j4-j2}{2} \quad \dots$

$X_{ca} = \{-1.5 \quad 0.5+j \quad 1.5-j1.5 \quad -j2 \quad -1.5-j1.5 \quad -0.5+j \quad 1.5\}$

$\tilde{x}_a[n] \cdot \tilde{x}_b[n]$ is also perodic

$$N = \frac{N_a N_b}{GCD(N_a, N_b)}$$

3 Energy and power seq.

A total Energy in a sequence is given by

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Ex: $x[n] = x_{real} + j\, x_{im}[n]$.

$$|x[n]|^2 = x[n] * x^*[n]$$
$$= x_{real}^2[n] + x_{im}^2[n]$$

$x[n] = [1 \;\; 2 \;\; 3]$

$\mathcal{E}_x = 1 + 4 + 9 = 14$ J

$$P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$$

* Finite sequences are energy sequence.
[Finite $\mathcal{E}_x$ and Zero P_x]

Infinite sequences :

1 if E_x is finite → Energy seq [power = 0]

2 if P_x is " → power seq [Energy → ∞]

If $x[n]$ is real

$$x[n] = x_{even}[n] + x_{odd}[n]$$

where $$x_{even}[n] = \frac{x[n] + x[-n]}{2}$$

$$x_{odd}[n] = \frac{x[n] - x[-n]}{2}$$

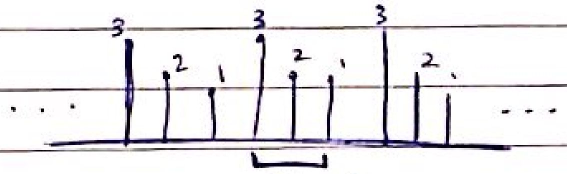
$\boxed{2}$ Based on Periodicity 8

a sequence is periodic if $\tilde{x}[n] = \tilde{x}[n + kN]$ for all n

where K: is any integer.

N: is a positive integer (the period).

N_f: the fundamental period (The least positive integer satisfies the above equation.



$N = 3$

$x[n] = A \cos(\omega_0 n + \phi)$ discreat

$x(t) = A \cos(\Omega_0 t + \phi)$ contious

$\tilde{x}_a[n] + \tilde{x}_b[n]$ is also perodic with a period of

$$N = \frac{N_a N_b}{GCD(N_a, N_b)}$$

قاسم المشترك

Ex: $N_a = 12$ $\quad 3 \times 4 = 3 \times 2 \times 2$

$N_b = 16$ $\quad 4 \times 4 = 2 \times 2 \times 2 \times 2$

$GCD(12, 16) = 4 \longrightarrow N = \frac{16 \times 12}{4} = 48$

Ex: $x_1[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$ $x_2[n] = \begin{cases} 1/\sqrt{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$

sol: $\sum x_1 = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$

$\sum x_2 = \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ diverges ∞

Ex: $x[n] = \begin{cases} 3(-1)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

sol: $P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=0}^{K} (3(-1)^n)^2$

$= \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=0}^{K} 9$ $(-1)^{2n} = 1$

*عديت الـ 9 من 0 لـ K ← K+1

$= \lim_{K \to \infty} \frac{9(K+1)}{2K+1}$

$= 4.5$ Watt.

∴ Power sequence.

other types of classification:

* Bounded sequence:

$|x[n]| \leq B_x < \infty$

* absolutely summable sequence

$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

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Tue

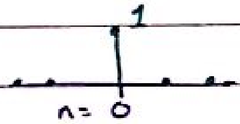
* square summable sequence:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

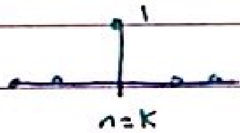
* Elementary sequances and sequance Represntation 8:

1 The unit sample sequence. (Discrete - time - Impulse)
(unit impulse)

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

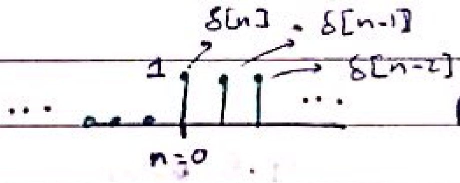


$$\delta[n-K] = \begin{cases} 1 & n=K \\ 0 & n \neq K \end{cases}$$



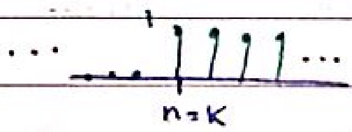
2 The unit step sequence

$$M[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Right sided Sequence. (causal)

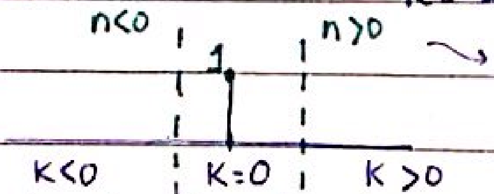
$$M[n-K] = \begin{cases} 1 & n \geq K \\ 0 & n < K \end{cases}$$



* ممكن تكون causal أو non causal حسب قيم الـ K.

$$\delta[n] = M[n] - M[n-1]$$

$$M[n] = \sum_{m=0}^{\infty} \delta[n-m] = \sum_{K=-\infty}^{n} \delta[K]$$



تجمع لحد 1

Five Apple

for $n<0$

$M[n]=0$

for $n \geq 0$

$M[n]=1$

* جمع الاصفار مع الـ 1

$\delta[0]$ بس بتكون قيمتها 1 والباقي اصفار.

Ex: $M[-10]=\sum_{-\infty}^{-10} \delta[K] = \delta[-\infty] + \cdots + \delta[-11] + \delta[-10] = 0$

هي الوحيدة إلي قيمتها 1.

$M[5] = \delta[-\infty] + \cdots + \delta[0] + \delta[1] + \delta[2] + \cdots + \delta[5]$

$= 0 + \cdots + 1 + 0 + 0 + \cdots + 0$

$= 1$

3 Sinusoidal sequence

$x[n] = A\cos(w_0 n + \phi) \qquad -\infty \leq n < \infty$

!? Perodic or not

$\tilde{x}[n] = \tilde{x}[n+N]$

$x[n+N] = A\cos(w_0(n+N)+\phi)$

$= A\cos(\underbrace{w_0 n + \phi} + \underbrace{w_0 N})$

$= A[\cos(w_0 n+\phi)\cos(w_0 N) - \sin(w_0 n+\phi)\sin(w_0 N)]$

$x[n] = A\cos(w_0 n + \phi)$?! will be true if $w_0 N = 2\pi r$

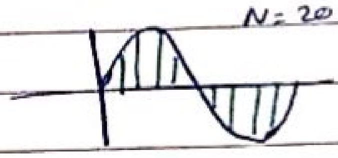
$\cos(w_0 N) = 1$ and $\sin(w_0 N) = 0$

$\boxed{\frac{N}{r} = \frac{2\pi}{w_0}}$ rational number (a ratio of 2 integer)

Ex: [1] $w_0 = 0.1\pi$ rad / Sample.

$$\frac{2\pi}{0.1\pi} = \frac{2}{0.1} = \frac{20}{1}$$ one cycle with 20 sample

$N = 20$



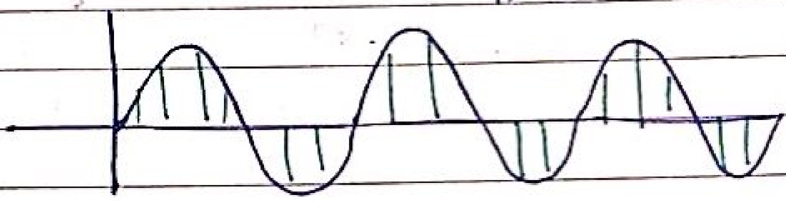
[2] $w_0 = 0.3\pi$ rad / sample.

$$\frac{2\pi}{0.3\pi} = \frac{20}{3}$$ 20 sample, 3 cycle.

$N = 20$

N = fundemntal period.

$N_f = 20.$



$(\Omega_0 = 2\pi f_0$, f_0 = freq Hz , Ω_0 rad/s$)$

$x(t) = A\cos(\Omega_0 t + \phi)$

$t = nT$

$= A\cos(\Omega_0 nT + \phi)$

$\boxed{\Omega_0 T = w_0}$ → T: Sampling period.

$\boxed{T = \frac{1}{f_T}}$

[3] $W_0 = 0.1$ rad/sample.

$$\frac{2\pi}{0.1} = \frac{20\pi}{1}$$ irrational. (not perodic)

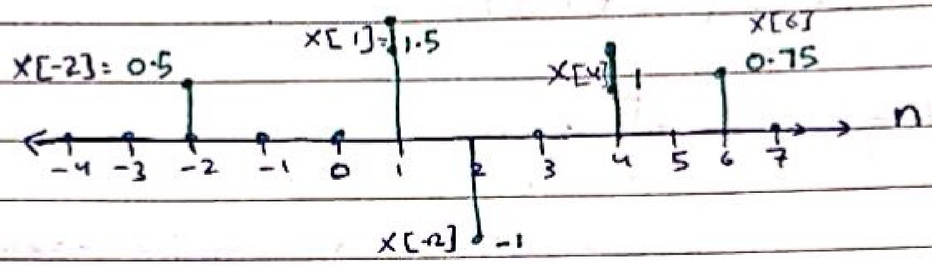
[4] $w_0 = 0.314$ rad / sample

$$\frac{2\pi}{0.314} = \frac{2000\pi}{314}$$ irrational.

[5] $w_0 = \sqrt{3}$ rad / sample

irrational.

* Representation of a general sequence 8Δ



$$x[n] = \underbrace{0.5}_{x[-2]}\,\delta[n+2] + \underbrace{1.5}_{x[1]}\,\delta[n-1] - \delta[n-2] + \delta[n-4] + 0.75\,\delta[n-$$

$$\boxed{x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K].}$$

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* The sampling Process :

$X_a(t) = A\cos(\Omega_0 t + \phi)$

Sampling : $t_n = \underset{\downarrow}{nT} = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$

sampling period → sampling frequency

$X[n] = X_a(t)\Big|_{t=t_n} = A\cos(\underbrace{\Omega_0 T}_{\omega_0} n + \phi)$

$x[n] = A\cos(\omega_0 n + \phi)$

where $\boxed{\omega_0 = \Omega_0 T}$

Ex: $g_1(t) = \cos(6\pi t)$: 3 HZ $\boxed{\Omega_0 = 6\pi}$

$g_2(t) = \cos(14\pi t)$: 7 HZ

$g_3(t) = \cos(26\pi t)$: 13 HZ

The sampling frequency $F_T = 10$ HZ.

$T = 0.1$ sec

t = [0:0.1:10]

vector t = [0, 0.1 0.2 0.3 ...]

sol:

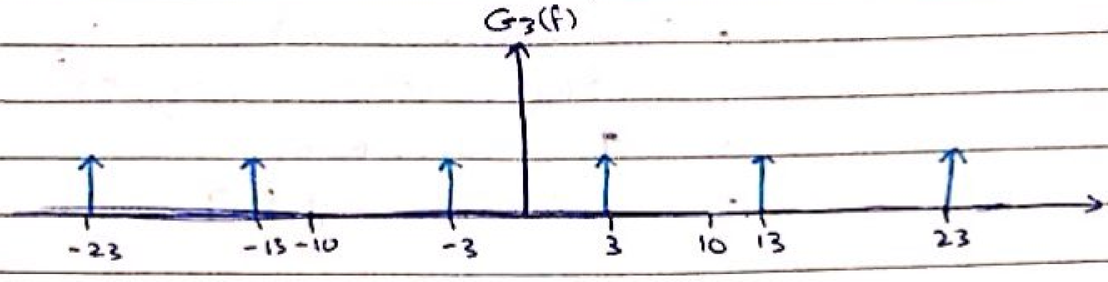
$g_1[n] = \cos(6\pi * 0.1\, n)$

$= \cos(0.6\pi n)$

$g_2[n] = \cos(1.4\pi n)$

$g_3[n] = \cos(2.6\pi n)$

13 HZ:



Discrete Time Systems:-

$$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n] = T[x[n]]$$

$T[\cdot]$: is a Discrete-Time System.
$x[n]$: input of the system
$y[n]$: the output of the system.

1) $y[n] = T[x[n]] = x[n] - x[n-1]$

2) $y[n] = T[x[n]] = x^2[n]$

$$x[n] \xrightarrow{T} y[n].$$

* Classifications of Discrete-Time System:-

[1] linear and nonlinear System:-

A System $T[\cdot]$ is linear if

$$T[a_1 x_1[n] + a_2 x_2[n]] = a_1 T[x_1[n]] + a_2 T[x_2[n]]$$
$$= a_1 y_1[n] + a_2 y_2[n]$$

$g_2[n] = \cos(1.4\pi n) = \cos((2-0.6)\pi n)$

$= \cos(2\pi n - 0.6\pi n)$

$= \cos(2\pi n)\cos(0.6\pi n) + \underbrace{\cancel{\sin(2\pi n)}}_{0}\sin(0.6n)$

$= 1 \cdot \cos(0.6\pi n)$

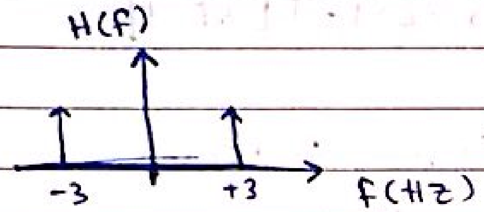
$= \cos(0.6\pi n) = g_1[n]$

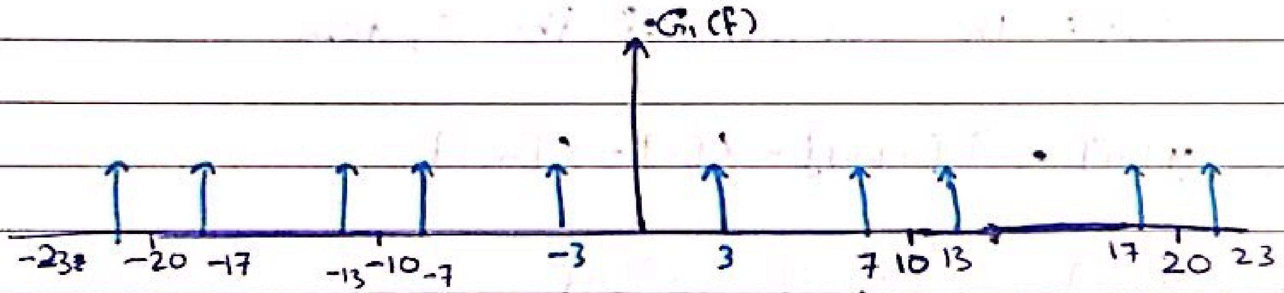
$g_3[n] = \cos((2+0.6)\pi n) = \cos(2\pi n - 0.6\pi n)$

$= \underbrace{\cancel{\cos(2\pi n)}}_{1}\cos(0.6\pi n) - \underbrace{\cancel{\sin(2\pi n)}}_{0}\sin(0.6\pi$

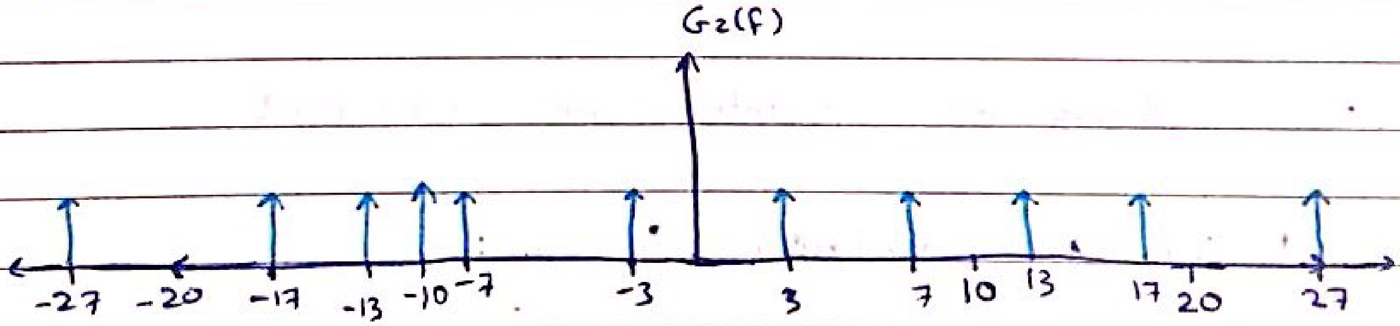
$= \cos(0.6\pi n) = g_1[n]$

3HZ :





7HZ :



$-10-7=-17$

$-20-7=-27$

$10+7=17$

$20+7=27$ بنزل عشرة ، 7HZ او بنزيد

where $y_1[n] = T[x_1[n]]$ and $y_2[n] = T[x_2[n]]$.

Ex1: $y[n] = T[x[n]] = n\,x[n]$

sol: $y_1[n] = T[x_1[n]] = n\,x_1[n]$

$y_2[n] = T[x_2[n]]$ $n\,x_2[n]$

let $x[n] = a_1 x_1[n] + a_2 x_2[n]$

$$y[n] = T[x[n]] = T\left[a_1 x_1[n] + a_2 x_2[n]\right]$$
$$= n\left[a_1 x_1[n] + a_2 x_2[n]\right]$$
$$= \underline{n a_1 x_1[n]} + \underline{n a_2 x_2[n]}$$
$$= a_1 \underbrace{n x_1[n]} + a_2 \underbrace{n x_2[n]}$$
$$= a_1 y_1[n] + a_2 y_2[n]$$

The system is linear.

Ex 2: $y[n] = T[x[n]] = x^2[n]$

sol: $y_1[n] = T[x_1[n]] = x_1^2[n]$.

$y_2[n] = T[x_2[n]] = x_2^2[n]$

let $x[n] = a_1 x_1[n] + a_2 x_2[n]$

$$y[n] = T[x[n]] = T\left[a_1 x_1[n] + a_2 x_2[n]\right]$$
$$= \left(a_1 x_1[n] + a_2 x_2[n]\right)^2$$
$$= a_1^2 x_1^2[n] + 2a_1 a_2 x_1[n] x_2[n] + a_2^2 x_2^2[n]$$
$$\neq a_1 y_1[n] + a_2 y_2[n]$$
$$a_1 x_1^2[n] + a_2 x_2^2[n].$$

The system is non linear.

Ex 3: $y[n] = T[x[n]] = A\,x[n] + B$ check linerity ?!

↳ sol: non linear.

Invariant.

[2] Shift - ~~[illegible]~~. Systems :

$x[n] \xrightarrow{\tau} y[n]$

$x[n-K] \xrightarrow{\tau} y[n-K]$

Shift invariant system.

[1] apply $x[n]$ to get $y[n]$

[2] apply $x_1[n] = x[n-K] \xrightarrow{\tau} y_1[n]$

[3] Shift $y[n]$ in step 1 by $K \rightarrow y[n-K]$
if $y[n-K] = y_1[n] \Rightarrow$ Shift Invariant system

Ex 1 $y[n] = x[n] - x[n-1]$

Sol: 1) $y[n] = x[n] - x[n-1]$

2) $x_1[n] = x[n-K]$.

$y_1[n] = x_1[n] - x_1[n-1]$.

$y_1[n] = x_1[n-k] - x_1[n-K-1]$

3) $y[n-K] = x[n-K] - x[n-K-1]$

$x[n-K] = y_1[n] \rightarrow$ Shift. Invariant System.

Ex 2 $y[n] = n\,x[n]$.

1) $y[n] = n\,x[n]$

2.) $x_1[n] = x[n-K]$

$y_1[n] = n\,x_1[n]$

$y_1[n] = n\,x[n-k]$

3) $y[n-K] = (n-K)\,x[n-k]$

$y[n-K] = nx[n-K] - K\,x[n-K]$

$y[n-K] \neq y_1[n] \Rightarrow$ Shift variant system.

$\boxed{3}$ Causal system :

if the output of the system $y[n]$ depends on input $x[n], x[n-1], x[n-2], \dots \Rightarrow$ Causal system.

$$y[n] = F\left[x[n], x[n-1], x[n-2], \dots\right]$$

Ex: $y[n] = x[n] - x[n-1] \longrightarrow$ Causal system.

Ex: $y[n] = x[-n] \longrightarrow$ non causal system.

$y[-1] = x[1]$ ↳ future value.

$\boxed{4}$ stable System :

The system is BIBO stable if :

$|x[n]| < B_x$ for all value of n.

then

$|y[n]| < B_y$

BIBO: Bounded input bounded output stability

Ex: $y[n] = e^{x[n]}$

$$|y[n]| = |e^{x[n]}| \leq e^{|x[n]|} < e^{B_x}$$

$x[n]$ +ve

$|y[n]| < \underbrace{e^{B_x}}_{B_y} \Rightarrow$ BIBO stable.

Ex: $y[n] = x[n] \cos(w_0 n)$

$|y[n]| = |x[n] \cos(w_0 n)| = |x[n]| \, |\cos(w_0 n)|$

$|y[n]| \leq B_x \quad \cdot (1)$ upper bond for cos = 1

$B_x = B_y \Rightarrow$ BIBO.

Ex: $y[n] = \sum_{k=-\infty}^{N} x[k]$. accumelator.

$|y[n]| = \left| \sum_{k=-\infty}^{n} x[k] \right| \leq \sum_{k=-\infty}^{n} |x[k]| \leq \sum_{k=-\infty}^{\infty} B_x$

" System un stable. "

* Passive and lossless system 8Δ

$$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n]$$

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

$\Rightarrow$ Passive system.

$$\sum |y[n]|^2 = \sum |x[n]|^2$$

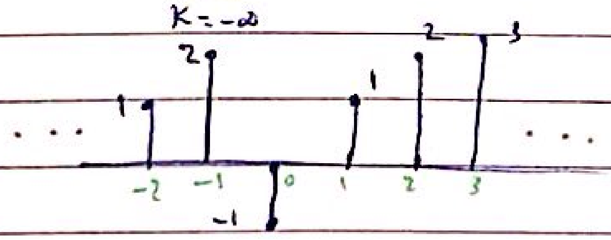
$\Rightarrow$ The system is lossless.

* Static and Dynamic systems 8Δ
(Memoryless).

$y[n] = a\,x[n]$: static
$y[n] = n\,x[n] - b\,x^3[n]$: static
$y[n] = x[n] - 3x[n-1]$: Dynamic.

$$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n]$$

$$x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K]$$



$\delta[n+2] + 2\delta[n+1] - \delta[n] + \delta[n-1] + 2\delta[n-2] + 3\delta[n-3]$

$$y[n] = T[x[n]]$$

$$= T\left[\sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K]\right]$$

For a linear system:

$$= \sum_{K=-\infty}^{\infty} \underbrace{x[K]}_{\text{Constant}}\, T[\delta[n-K]]$$

$$x[n] = \delta[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n] = h[n]$$

impulse input

impulse response h[n] as output.

$$h[n] = T[\delta[n]]$$

If $x[n] = \mu[n]$: unit step

$$s[n] = T[\mu[n]] \quad \text{: The step Response.}$$

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n,K]$$

* The system is shift Invariant:-

$$h[n] = T[\delta[n]]$$

$$h[n-K] = T[\delta[n-K]]$$

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K] \quad \text{: LTI system.}$$

(linear time invariant

* The Convolution Sum :

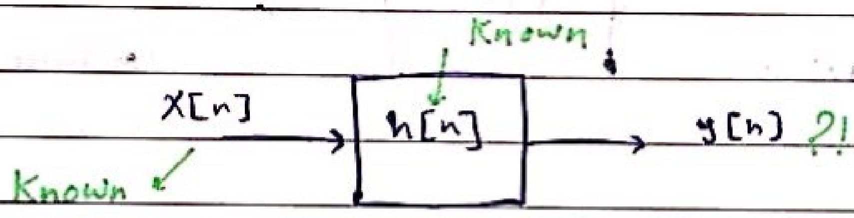
$$y[n] = x[n] \circledast h[n] :$$

let $n - k = m$

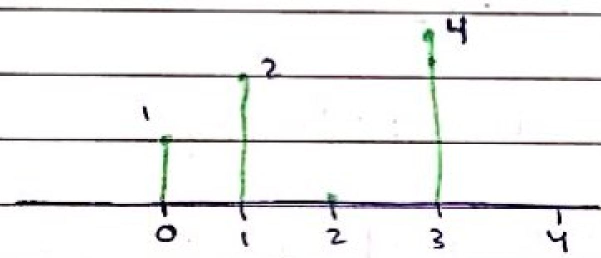
$k = n - m$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

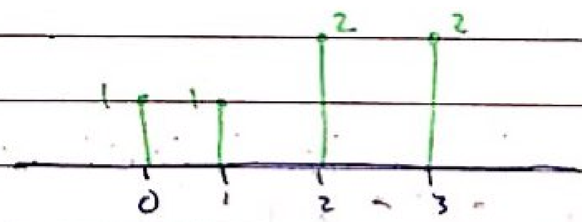
$$= h[n] \circledast x[n]$$



Ex: $x[n]$

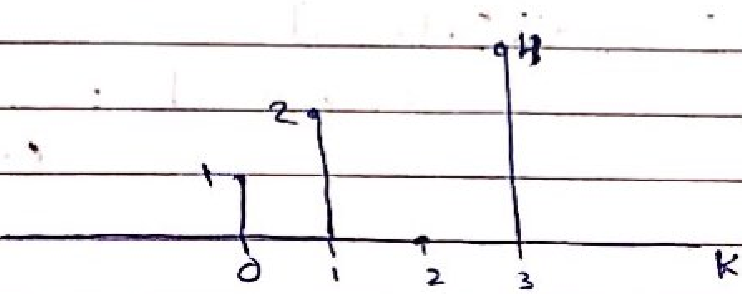


$h[n]$

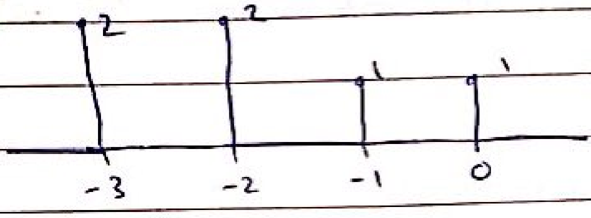


Sol:

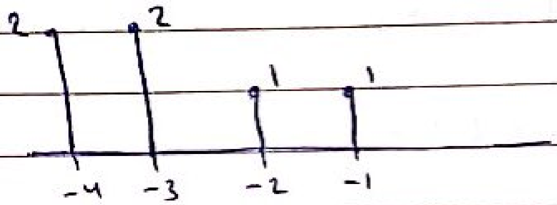
$\boxed{1}$



[2] $h[-k]$



$h[-1-k]$

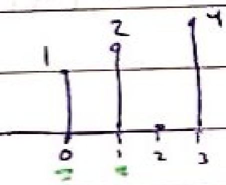


$h[k] \rightarrow h[-1-k]$

$h[0] \rightarrow h[-1]$, $h[1] \rightarrow h[-2]$.

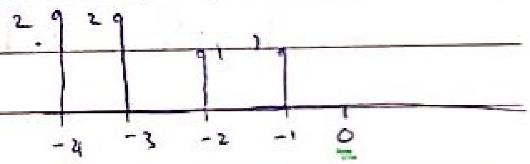
$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]\, h[-1-k].$$

$x[k]$



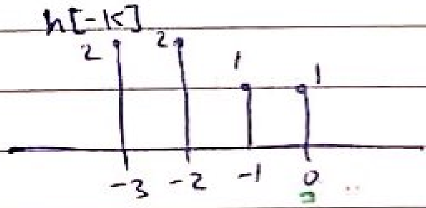
$= 0$

$h[k-1]$



$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k]$$

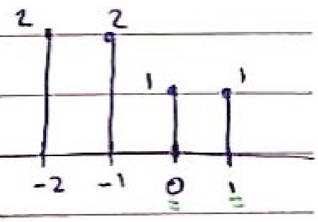
$h[-k]$



$= 1 \cdot 1 \quad = 1$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k].$$

$h[1-k]$

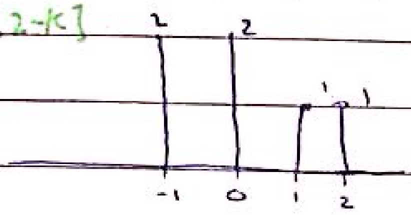


$= 1.1 + 2.1$

$= 3$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\, h[2-k]$$

h[2-k]

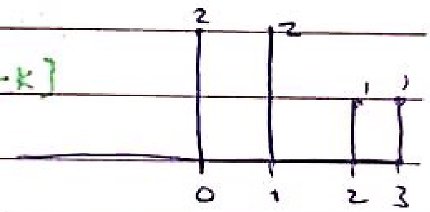


$$= 1\cdot 2 + 2\cdot 1 + 1\cdot 0$$

$$= 4.$$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]\, h[3-k]$$

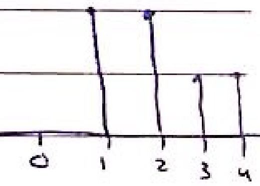
h[3-k]



$$= 1\cdot 4 + 1\cdot 0 + 2\cdot 2 + 2\cdot 1$$

$$= 10$$

h[4-k]

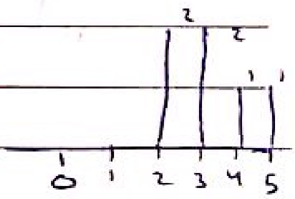


$$y[4] = \sum_{k=-\infty}^{\infty} x[k]\, h[4-k]$$

$$= (1\cdot 4) + 2\cdot 0 + 2\cdot 2$$

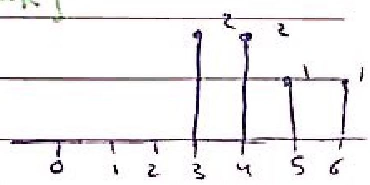
$$= 8$$

h[5-k]



$$y[5] = \sum_{k=-\infty}^{\infty} 2\cdot 4 = 8$$

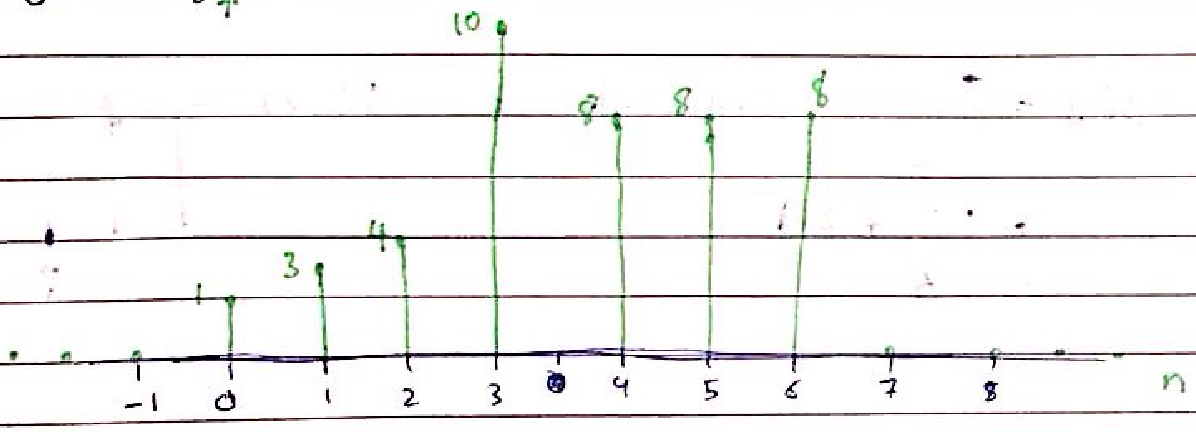
h[6-k]



$$y[6] = 2\cdot 4 = 8$$

$$y[7] = 0$$

$$y[n] = \{\underset{\uparrow}{1}, 3, 4, 10, 8, 8, 8\}$$



* Tabular Method for Convolution Sum

Ex: $x[n] = \{ \underset{\uparrow}{1} \;\; 2 \;\; 0 \;\; 4 \}$ $\quad h[n] = \{ \underset{\uparrow}{1} \;\; 1 \;\; 2 \;\; 2 \}$

n	0	1	2	3	4	5	6	7
x[n]	1	2	0	4				
h[n]	1	1	2	2				
	1	2	0	4				
	-	1	2	0	4			
	-	-	2	4	0	8		
	-	-	-	2	4	0	8	
+	1	3	4	10	8	8	8	

بنبلش بأول رقم ① من h[n] نضرب بكل الـ x[n] ونعيد بنفس الطريقة الثاني من h[n] بـ x[n] وبنزيح ← one digit
وهكذا وبعدين بنجمع .

$y[n] = \{ \underset{\uparrow}{1} \;\; 3 \;\; 4 \;\; 10 \;\; 8 \;\; 8 \;\; 8 \}$

Ex: $x[n] = \{ 1 \;\; \underset{\uparrow}{2} \;\; 0 \;\; 4 \}$ $\quad h[n] = \{ 1 \;\; 1 \;\; \underset{\uparrow}{2} \;\; 2 \}$ → check at home graphically ?!

* بس حل السؤال ← decimal point

n	0	1	2	3	4	5	6	7
x[n]	1	2.	0	4				
	1	1	2.	2				
	1	2	0	4				
	-	1	2	0	4			
	-	-	2	4	0	8		
	-	-	-	2	4	0	8	
+	1	3	4	10	8	8	8	

$y[n] = \{1 \;\; 3 \;\; 4 \;\; \underset{\uparrow}{10.} \;\; 8 \;\; 8 \;\; 8\}$

* عندي ٣ أرقام بعد الـ decimal point

* Stability Condition in terms of Impulse Response $h[n]$

$x[n] = \delta[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n] = h[n]$ Impulse response

$x[n] = u[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n] = S[n]$ step response.

BIBO stability

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\, |x[n-k]| \leq \sum_{k=-\infty}^{\infty} |h[k]|\, B_x$$

$$= B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_y$$

the condition should be

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad \text{absolutly summable}$$

Ex: Consider a causal LTI system with $h[n] = \alpha^2 u[n]$ Find the condition on α such the system is BIBO stable ?!!

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \;\; ?!!$$

Sol: $\sum_{n=-\infty}^{\infty} \alpha^n u[n] = \sum_{n=0}^{\infty} |\alpha^n| = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|}$, $|\alpha|<1$

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad |r|<1}$$ geometric series.

for $|\alpha| \geq 1$ the series will diverge.

$$\boxed{\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}}$$ always true

If $|r|<1$ and $N \to \infty$ $\Rightarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

proof:

$$S_N = \sum_{n=0}^{N} r^n \quad , \quad S_{N-1} = \sum_{n=0}^{N-1} r^n$$

$$S_{N-1} = 1 + r + r^2 + r^3 + \cdots + r^{N-1}$$
$$S_N = 1 + r + r^2 + \cdots + r^{N-1} + r^N$$

$$S_N = S_{N-1} + r^N$$
$$S_N = 1 + r[\underbrace{1 + r + r^2 + \cdots + r^{N-1}}_{S_{N-1}}]$$

$$S_N = 1 + r S_{N-1} = S_{N-1} + r^n$$

$$S_{N-1}[1-r] = 1 - r^N$$

$$S_{N-1} = \frac{1-r^N}{1-r} = \sum_{n=0}^{N-1} r^n$$

Causality Condition interms of the impulse Response.

$h[n]$ to exist must be LTI

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n_0 - k]$$

$$y[n_0] = T\left[\, x[n_0],\ x[n_0 - 1],\ \ldots \right]$$

$$y[n_0] = \sum_{k=0}^{\infty} h[k]\, x[n_0 - k] + \sum_{k=-\infty}^{-1} h[k]\, x[n_0 - k]$$

$$y[n_0] = \left[h[0]\, x[n_0] + h[1]\, x[n_0 - 1] + h[2]\, x[n_0 - 2] + \ldots \right]$$
$$+ \left[h[-1]\, x[n_0 + 1] + h[-2]\, x[n_0 + 2] + \ldots \right]$$

$$\boxed{h[n] = 0 \qquad n < 0}$$

Causality of an LTI system.

$$\delta[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow h[n]$$

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Tue

* The Discrete Time Fourier Transform (DTFT)

$$X(e^{j\omega}) := \sum_{n=-\infty}^{\infty} \underbrace{x[n]}_{C_n}\, e^{-j\omega n}$$

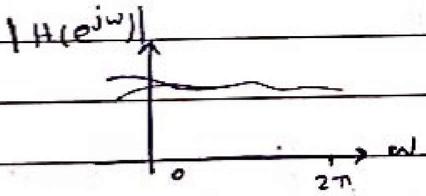
① $X(e^{j\omega})$ is a continuous function in ω.

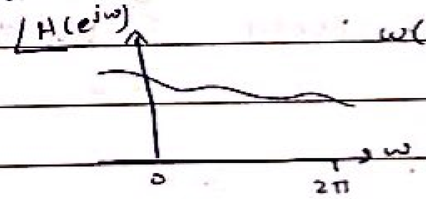
Ex: $x[n] = \{1 \quad 2 \quad \underset{\uparrow}{5} \quad 3 \quad 1.2\}$

n: $-2 \quad -1 \quad n=0 \quad 1 \quad 2$

$$X(e^{j\omega}) = 1\cdot e^{j2\omega} + 2\cdot e^{j\omega} + 5\cdot e^{j0} + 3e^{-j\omega} + 1.2e^{-j2\omega}$$

$$\underbrace{X(e^{j\omega})}_{\text{complex function in } \omega} = \underbrace{e^{j2\omega}}_{\cos 2\omega + j\sin 2\omega} + 2e^{j\omega} + 5 + 3e^{-j\omega} + 1.2e^{-j2\omega}$$





$\omega(0, 2\pi)$

② $X(e^{j\omega})$ is periodic with a period of 2π.

$$X(e^{j\omega}) = X\left(e^{j(\omega+2\pi k)}\right)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi k)n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \cdot \underbrace{e^{-j2\pi k n}}_{=1} \rightarrow \cos(\cancel{2\pi n k})^{1} - j\sin(\cancel{2\pi n k})^{0}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{jw}) \Rightarrow \text{Fourier transform pair}$$

↳ function in frequency domain.

* The inverse Discret-Time Fourier transform : IDTFT

* Fourier series Expansion 84

a_n, b_n (~~Sem~~ sines and asines)

C_n (complex exponential).

$$\boxed{C_n = x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw}$$

$x(t)$ periodic with a period T :

$$\cancel{X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt}$$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \cos w n t + \sum_{n=-\infty}^{\infty} b_n \sin w t$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{-jw_0 n t}, \quad C_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{jw_0 n t}\, dW.$$

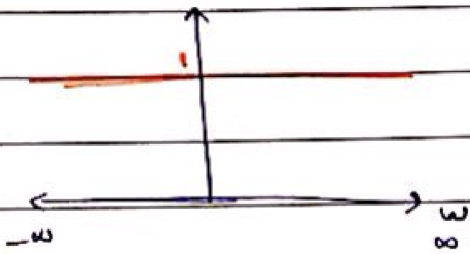
$$\boxed{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) \cdot e^{jwn}\, dw}$$

$$F[x[n]] = X(e^{jw})$$

$$F^{-1}[X(e^{jw})] = x[n]$$

Ex: $x[n] = \delta[n]$.

$$\underset{\text{(delta)}}{\Delta(e^{jw})} = \sum_{n=-\infty}^{\infty} \delta[n] . e^{-jwn} = e^{-j0w} = 1 \quad \text{for all } w.$$



continous

peroidc

Ex: $x[n] = \alpha^n u[n]$ $\qquad |\alpha| < 1$

$$X(e^{jw}) = \sum_{n=0}^{\infty} \alpha^n e^{-jwn} = \sum_{n=0}^{\infty} (\alpha e^{-jw})^n$$

$$\overbrace{\underbrace{|\alpha|}_{<1}\,\underbrace{|e^{-jw}|}_{\hookrightarrow 1\angle -w}}^{<1}$$

$$= \frac{1}{1-\alpha e^{-jw}}$$

$$X(e^{-j(w+2\pi)}) = \frac{1}{1-\alpha e^{-j(w+2\pi)}} = \frac{1}{1-\alpha e^{-jw}}$$

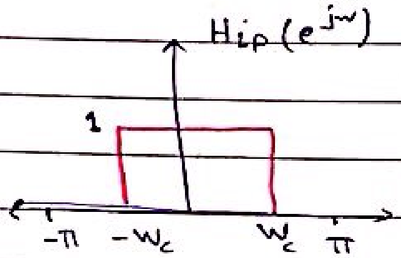
* Discrete-Time Fourier Transform (DTFT)

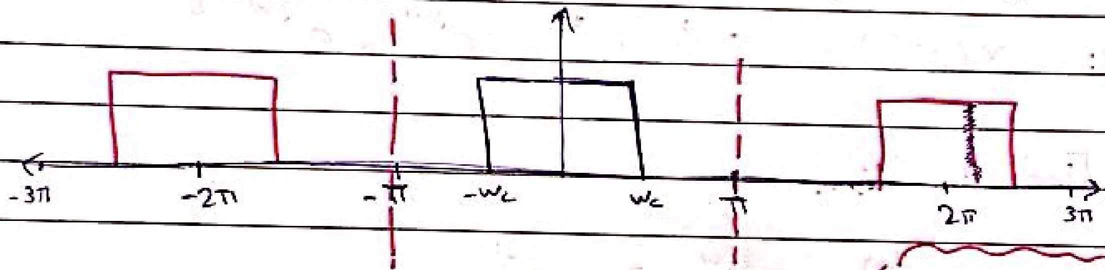
$$x[n] \xleftrightarrow{DTFT} X(e^{jw})$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) \cdot e^{jwn}\, dw$$

Ex:





$$H_{LP}(e^{jw}) = \begin{cases} 1 & 0 \le |w| \le w_c \\ 0 & w_c < |w| \le \pi \end{cases}$$

maximum frequency π

repeated every $n\pi$

Ideal low pass Filter

$\Rightarrow h_{LP}[n]$?!

Sol: $h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{jw}) \cdot e^{jwn}\, dw$

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-w_c}^{w_c} 1 \cdot e^{jwn}\, dw = \frac{1}{2\pi} \cdot \frac{e^{jwn}}{jn} \Bigg|_{-w_c}^{w_c} = \frac{e^{jw_c n} - e^{-jw_c n}}{2\pi jn}$$

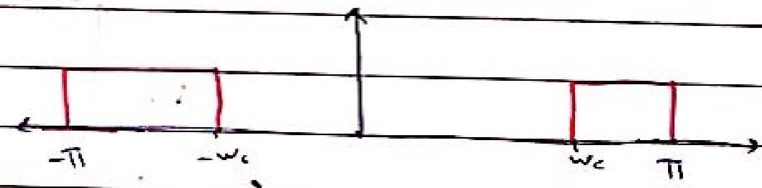
$$= \frac{\sin w_c n}{\pi n} \qquad -\infty < n < \infty \;,\; n \neq 0$$

$$h_{lp}[0] = \frac{1}{2\pi}\int_{-w_c}^{w_c} 1.1 \, \partial w = \frac{w_c}{\pi} \qquad \boxed{n=0}$$

$$h_{lp}[n] = \begin{cases} \dfrac{w_c}{\pi} & , n=0 \\[2ex] \dfrac{\sin w_c n}{\pi n} & , n \neq 0 \end{cases}$$

$$h_{lp}[n] = \frac{\sin w_c n}{\pi n}$$

HPF:



Ex: $X(e^{jw}) = (\cos^2(w)) = \left(\dfrac{e^{jw} + e^{-jw}}{2}\right)^2$

Find $x[n]$?!!

$$X(e^{jw}) = \frac{e^{2jw} + 2 + e^{-j2w}}{4}$$

$$= \frac{1}{4}e^{j2w} + \frac{1}{2} + \frac{1}{4}e^{-j2w}$$

$$x[n] = \left\{\frac{1}{4}, 0, \underset{\uparrow}{\frac{1}{2}}, 0, \frac{1}{4}\right\}$$

$$X(e^{jw}) = \sum x[n]e^{-jwn}$$

$$\frac{1}{2\pi} = \ldots + x[-2]e^{j2w} + x[-1]e^{jw} + x[0] + x[1]e^{-jw} + x[2]e^{-j2w}$$

* DTFT Theorems 81

$$g[n] \xleftrightarrow{DTFT} G(e^{jw})$$

$$h[n] \xleftrightarrow{DTFT} H(e^{jw})$$

[1] linearity theorem 84

$$a_1 g[n] + a_2 h[n] \xleftrightarrow{DTFT} a_1 G(e^{jw}) + a_2 H(e^{jw})$$

[2] Time-Reversal Theorem 84

$$g[-n] \xleftrightarrow{DTFT} G(e^{-jw})$$

$$F\{g[-n]\} = \sum_{n=-\infty}^{\infty} g[-n]\, e^{-jwn} \qquad \text{let } \boxed{m = -n}$$

$$= \sum_{m=-\infty}^{\infty} g[m]\, e^{jwm} = X(e^{-jw})$$

$$g[n] = \{1 \quad \underset{\uparrow}{2} \quad 3\}$$

$$G(e^{jw}) = e^{jw} + 2 + 3e^{-jw} \Rightarrow G(e^{-jw}) = e^{-jw} + 2 + 3e^{jw}$$

$$g[-n] = [3 \quad \underset{\uparrow}{2} \quad 1]$$

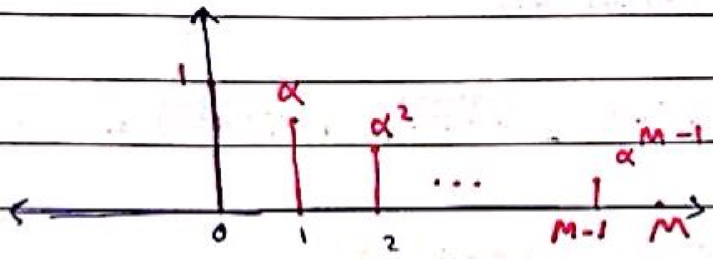
$$G(e^{-jw}) = 3e^{jw} + 2 + e^{-jw}$$

[3] Time-shifting Theorem 81

$$g[n - n_0] \xleftrightarrow{DTFT} e^{-jwn_0} G(e^{jw})$$

Ex: $y[n] = \begin{cases} \alpha^n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \qquad |\alpha| < 1$

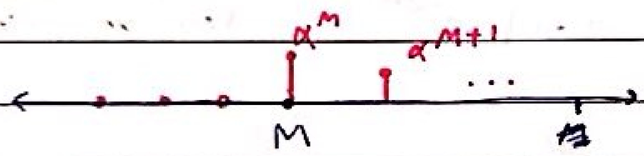
Sol:



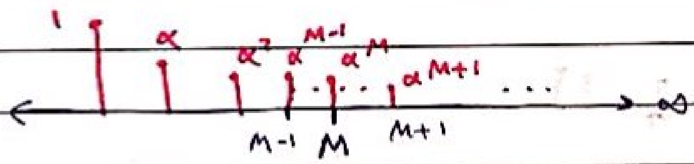
$$y[n] = \alpha^n \mu[n] - \alpha^{n\cancel{M}} \mu[n-M]$$

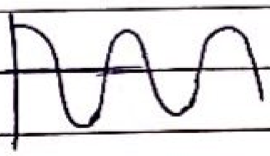
$$\alpha^{n-M} \mu[n-M] \quad X$$

$\alpha^n \mu[n-m]$



$\alpha^n [\mu[n]$





$\Rightarrow \cos(\omega t)\,[u(t) - u(t-10)]$

$$y[n] = \alpha^n \mu[n] - \alpha^M \cdot \alpha^{-M} \alpha^n \mu[n-M]$$

$$= \alpha^n \mu[n] - \alpha^M \underbrace{\alpha^{n-M} \mu[n-M]}$$

$$Y(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} - \alpha^M \cdot \frac{e^{-jwM}}{1-\alpha e^{jw}}$$

$$= \frac{1-(\alpha e^{-jw})^M}{1-\alpha e^{jw}}$$

OR $\sum_{i=0}^{M-1} r^n = \frac{1-r^M}{1-r}$, $Y(e^{jw}) = \sum_{n=0}^{M-1} \alpha^n e^{-jwn} = \frac{1-(\alpha e^{-jw})^M}{1-\alpha e^{-jw}}$

[4] Frequency - Shifting Theorem

$$e^{jw_0 n} g[n] \xleftrightarrow{DTFT} G(e^{j(w-w_0)})$$

Ex: $y[n] = (-1)^n \alpha^n \mu[n] \qquad |\alpha|<1$

Find $Y(e^{jw})$?!

Sol. $y[n] = e^{j\pi n} \alpha^n \mu[n]$ or $(-\alpha)^n \mu[n]$. $\boxed{e^{j\pi n} = (-1)^n}$

$g[n] = \alpha^n \mu[n]$

$$G(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}$$

$$Y(e^{jw}) = G(e^{j(w-\pi)})$$

$$Y(e^{jw}) = \frac{1}{1-\alpha e^{-j(w-\pi)}} = \frac{1}{1-\alpha e^{j\pi} e^{-jw}}$$ $\boxed{e^{j\pi} = -1}$

$$= \frac{1}{1+\alpha e^{-jw}}$$

[5] Differentiation in Freqency Theorem :-

$$n g[n] \xleftrightarrow{DTFT} j \frac{dG(e^{jw})}{dw}$$

$$\frac{d}{dw}\left[G(e^{jw}) = \sum_{n=-\infty}^{\infty} g[n] e^{-jwn} \right]$$

$$j \frac{d G(e^{jw})}{dw} = j \sum_{n=-\infty}^{\infty} g[n] (-jn) e^{-jwn}$$ $\boxed{-j \cdot j = 1}$

$$j \frac{dG(e^{jw})}{dw} = \sum_{n=-\infty}^{\infty} n g[n] e^{-jwn}$$

Ex: $y[n] = (n+1)\alpha^n \mu[n]$ $\quad |\alpha| < 1$

let $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$

$$X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}$$

$$y[n] = n\,x[n] + x[n]$$

$$j\frac{d\,X(e^{jw})}{dw} = \frac{\cancel{j} \times \alpha \cancel{j} e^{-jw}}{(1-\alpha e^{-jw})^2}$$

$$= \frac{\alpha e^{-jw}}{(1-\alpha e^{-jw})^2}$$

$$Y(e^{jw}) = \frac{\alpha e^{-jw}}{(1-\alpha e^{-jw})^2} + \frac{1}{1-\alpha e^{-jw}}$$

$$Y(e^{jw}) = \frac{\cancel{\alpha e^{-jw}} + 1 - \cancel{\alpha e^{-jw}}}{(1-\alpha e^{-jw})^2} = \frac{1}{(1-\alpha e^{-jw})^2}$$

$\boxed{6}$ The Convolution Theorem 84

$$g[n] \circledast h[n] \xleftrightarrow{DTFT} G(e^{jw})H(e^{jw})$$

Ex: $g[n] = \{1 \;\; 2 \;\; 0 \;\; 4\}$
$h[n] = \{1 \;\; 1 \;\; 2 \;\; 2\}$

Sol: $G(e^{jw}) = 1 + 2e^{-jw} + 0 + 4e^{-j3w}$
$H(e^{jw}) = 1 + e^{-jw} + 2e^{-j2w} + 2e^{-j3w}$

$$G(e^{jw}) \cdot H(e^{jw}) = (1 + 2e^{-jw} + 0 + 4e^{-j3w})(1 + e^{-jw} + 2e^{-j2w} + 2e^{-j3w})$$

$$= 1 + 2e^{-jw} + 0 + 4e^{-j3w}$$
$$+ 1 \cdot e^{-jw} + 2e^{-j2w} + 0 + 4e^{-j4w}$$
$$+ 2e^{-j2w} + 4e^{-j3w} + 0 + 8e^{-j5w}$$
$$+ 2e^{-j3w} + 4e^{-j4w} + 0 + 8e^{-j6w}$$

+

$$= 1 + 3e^{-jw} + 4e^{-j2w} + 10e^{-j3w} + 8e^{-j4w} + 8e^{-j5w} + 8e^{-j6w}$$

$$y[n] = \{1 \quad 3 \quad 4 \quad 10 \quad 8 \quad 8 \quad 8\}$$

$\boxed{7}$ Modulation Theorem

$$g[n]\,h[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\,H(e^{j(w-\theta)})\,d\theta$$

⇒ Gibbs phenomenon

$\boxed{8}$ Parsevals Relation :-

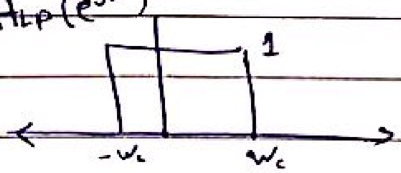
$$\sum_{n=-\infty}^{\infty} g[n]\,h^*[n] \overset{\text{equal}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{jw})\,H^*(e^{jw})\,dw$$

If $g[n] = h[n]$

$$\sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{jw})|^2\,dw$$

$$S_{gg}(e^{jw}) = |G(e^{jw})|^2$$ energy density Spectrum

Ex:



$$h_{LP}[n] = \begin{cases} \dfrac{w_c}{\pi} & n = 0 \\ \dfrac{\sin w_c n}{\pi n} & n \neq 0 \end{cases}$$

$$\sum |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{LP}(e^{+jw})|^2 \, dw$$

$$= \frac{1}{2\pi} \int_{-w_c}^{w_c} 1^2 \, dw = \frac{w_c}{\pi} < \infty$$

Finite-length Discrete Transforms :-

N-length sequence

$$x[n] = \{ \underset{\uparrow}{x[0]} \quad x[1] \quad \dots \quad x[N-1] \}$$

* Orthogonal Transforms :-

if we let $x[n]$ denote a length N-sequence

$$X[K] = \sum_{n=0}^{N-1} x[n] \, \psi^*[k,n] \qquad 0 \leq K \leq N-1$$

"Analysis equation"

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[K] \, \psi[K,n] \qquad 0 \leq n \leq N-1$$

"Synthysis equation"

orthogonal transformation.

$\psi[K,n]$ called the basis sequences.

* The Discrete - Fourier Transform (DFT) :-

$$\psi[k,n] = e^{j \frac{2\pi k n}{N}}$$

$$X[K] = \sum_{n=0}^{N-1} x[n] \, e^{-j \frac{2\pi k n}{N}} \qquad 0 \leq K \leq N-1$$

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[k] \, e^{j \frac{2\pi k n}{N}} \qquad 0 \leq n \leq N-1$$

let $W_N = e^{-j \frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j \sin\left(\frac{2\pi}{N}\right)$

$$X[K] = \sum_{n=0}^{N-1} x[n]\, W_N^{Kn} \qquad 0 \leq K \leq N-1$$

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[K]\, W_N^{-Kn} \qquad 0 \leq n \leq N-1$$

$$x[n] = \{ x[0],\ x[1] \cdots\cdots x[N-1] \}$$

$$X[0] = \sum_{n=0}^{N-1} x[n]\, W_N^{0} = \sum_{n=0}^{N-1} x[n] = x[0] + x[1] + \cdots + x[N-1]$$

$$X[1] = \sum_{n=0}^{N-1} x[n]\, W_N^{1n} = x[0]\cdot 1 + x[1]\, W_N^{1} + x[2]\, W_N^{2} + \cdots + x[N-1] \times W_N^{N-1}$$

$$X[2] = \sum_{n=0}^{N-1} x[n]\, W_N^{2n} = x[0]\cdot 1 + x[1]\, W_N^{2} + x[2]\, W_N^{4} + \cdots + x[N-1] \times W_N^{2(N-1)}$$

$$\vdots$$

$$X[N-1] = \sum_{n=0}^{N-1} x[n]\, W_N^{(N-1)n} = x[0]\cdot 1 + x[1]\, W_N^{(N-1)} + \cdots + x[N-1]\, W_N^{(N-1)(N-1)}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{1} & W_N^{2} & \cdots & & W_N^{N-1} \\ 1 & W_N^{2} & W_N^{4} & W_N^{6} & \cdots & W_N^{2(N-1)} \\ 1 & W_N^{3} & W_N^{6} & W_N^{9} & & W_N^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & & W_N^{(N-1)(N-1)} \end{bmatrix}}_{D_N} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\boxed{\underline{X} = D_N \underline{x}}$$

$$X[K] = \{ X[0] \quad X[1] \quad \dots \quad X[N-1] \}$$

$$x[0] = \frac{1}{N} \sum_{K=0}^{N-1} X[K] W_N^{0} = \frac{1}{N} \left[\sum_{K=0}^{N-1} X[K] \right.$$

$$= \frac{1}{N} \left[X[0] + X[1] + \dots X[N-1] \right]$$

$$x[1] = \frac{1}{N} \sum_{K=0}^{N-1} X[K] W_N^{-K} = \frac{1}{N} \left[1 X[0] + X[1] W_N^{-1} + X[2] W_N^{-2} + \dots + X[N-1] W_N^{-(N-1)} \right.$$

$$x[2] = \frac{1}{N} \sum_{K=0}^{N-1} X[K] W_N^{-K} = \frac{1}{N} \left[X[0] + X[1] W_N^{-2} + X[2] W_N^{-4} + \dots + X[N-1] W_N^{-2(N-1)} \right.$$

$$\vdots$$

$$x[N-1] = \frac{1}{N} \left[X[0] \cdot 1 + X[1] W_N^{-(N-1)} + X[2] W_N^{-2(N-1)} + \dots + X[N-1] W_N^{-(N-1)(N-1)} \right]$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & & \dots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & & \dots & W_N^{-2(N-1)} \\ \vdots & & & & & \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & & \dots & W_N^{-(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\boxed{\underline{x} = \frac{1}{N} D_N^{*} \underline{X}}$$

$$\underline{x} = D_N^{-1} \underline{X}$$

$$D_N^{-1} = \frac{1}{N} D_N^{*}$$

23/10
SUN

D_N : DFT Matrix (dftmtx(N))

$$y[n] = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} \qquad h[n] = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$e^{\frac{-j2\pi(3)}{4}}$

نعوضها.

$$D_2 = \begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$W_2^1 = e^{\frac{-j2\pi(1)}{2}} = e^{-j\pi} = -1$$

$$G[K] = \begin{bmatrix} G(0) \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1+2j \\ -5 \\ 1-2j \end{bmatrix}$$

$X(e^{jw}) = 1 + 2e^{-jw} + 0.e^{-j2w} + 4e^{-j3w}$.

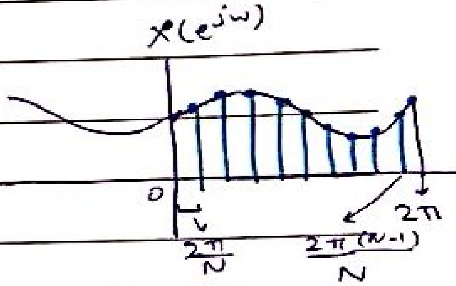
"Continous function in w"

$$H[K] = D_4 \cdot h[n] = \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix} \begin{array}{l} w=0 \\ w=\pi/2 \\ w=\pi \\ w=\frac{3\pi}{2} \end{array}$$

* The Relationship between the Fourier Transform and the DFT

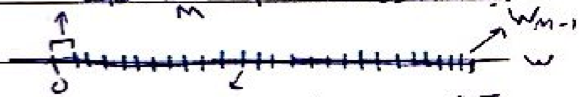
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-jwn}$$



$$X(e^{jw})\Big|_{w_k = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}\cdot n}. \qquad 0 \le K \le N-1$$

بين كل نقطتين $\frac{2\pi}{M}$ = period



$w_k = \frac{2\pi k}{M}$

$$X(e^{jw}) = \sum_{n=0}^{N-1} x[n]\, e^{-jwn}$$

بدي اقسمها إلى نقاط صغيرة كثير.

$[M >> N]$

والنقاط أعوضها بـ $X(e^{jw})$.

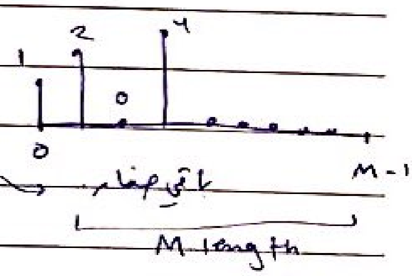
$$W_{M-1} = \frac{2\pi(M-1)}{M}$$

$$X(e^{jw})\Big|_{w_k = \frac{2\pi k}{M}} = \sum_{n=0}^{N-1} x[n]\, e^{-jw_k n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{M}}$$

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}}$$

$$X_e[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le M-1 \end{cases}$$



$$X_e[K] = \sum_{n=0}^{M-1} x_e[n]\, e^{-j\frac{2\pi nk}{M}} \qquad 0 \le K \le M-1$$

$$X_e[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi n k}{M}} \quad 0 \le k \le M-1$$

$$= X(e^{j\omega})\Big|_{\omega_k = \frac{2\pi k}{M}}$$

Zero padding

in matlab
```
fft(x)
fft(x,M)
fft(x,256)   2^N
```

* بدي ال resolution الي بكون شي ال M .

```
D8 = dftmtx(8);
```

$$d_8 * \begin{bmatrix} 1 \\ 3 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = ?!$$

$g[n]$, $h[n]$

$\boxed{1}$ $G(e^{jw})$, $H(e^{jw})$

$$\underbrace{g[n] \circledast h[n]}_{2N-1 \text{ point}} \xleftrightarrow{DTFT} G(e^{jw})\, H(e^{jw})$$

$\boxed{2}$ $G[K]$, $H[K]$

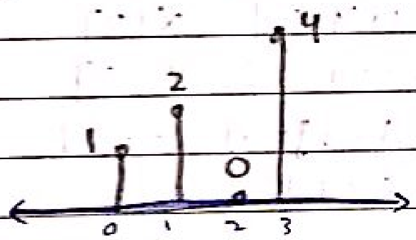
N length N-length.

$$\underbrace{\underset{N}{G[K]} \cdot \underset{N}{H[K]}}_{N\text{-length}} \xleftrightarrow{DFT} g[n] \bigcirc h[n]$$

What is the operation ?!

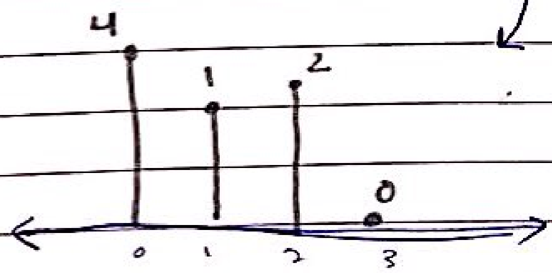
$$\underset{N\text{-length}}{Y[K]} \xleftrightarrow{DFT} \underset{N\text{-length.}}{y[n]}$$

Circular shift of a sequence :-



4-length.

one circular shift.



$r = \langle m \rangle_N$ (r equal m modulo N

$r = m + \ell N$

where L is an integer chosen to make

$r = m + \ell n$ a number between 0 and N-1.

Ex: 1) $\langle 25 \rangle_7 = 4$

$25 - 3\times 7 = 4.$

$\ell = -3$

2) $\langle -16 \rangle_7 = -16 + 3\times 7 = 5$ $\left[-3 + \frac{5}{7} = \frac{-16}{7} \right]$

3) $\langle 56 \rangle_7 = +56 - 7\times 8 = 0.$

~~Xc[n]~~

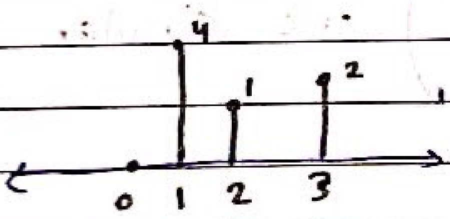
$X_c[n] = X[\langle n - n_0 \rangle_N]$ ← circular shift of a seq.

$X_c[0] = X[\langle -2 \rangle_4] = X[-2 + 1\times 4 = 2] = X[2]$

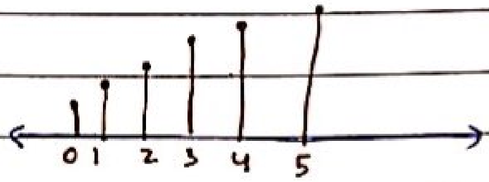
$X_c[1] = X[\langle -1 \rangle_4] = X[-1 + 1\times 4 = 3] = X[3]$

$X_c[2] = X[\langle 0 \rangle_4] = X[0 + 0\times 4 = 0] = X[0]$

$X_c[3] = X[\langle 1 \rangle_4] = X[1 - 0\times 4] = X[1]$

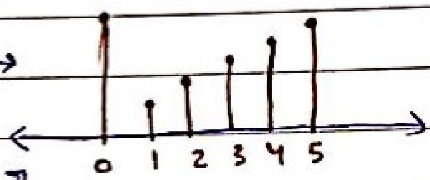


Ex:



$x[\langle n-1\rangle_6] =$

$5+1=6$



$x[\langle n+5\rangle_6] =$

Circular Convolution 8∆

$$\underset{2N-1}{y[n]} = \underset{N}{x[n]} \circledast \underset{N}{h[n]} \qquad 0 \le n \le 2N-2$$

$$y_L[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m] \quad : \quad \text{linear convolution}$$

$$y_c[n] = \sum_{m=0}^{N-1} x[m]\, h[\langle n-m\rangle_N]$$

$$y_c[N] = x[n] \; Ⓝ \; h[n]$$

$y_c[n] \equiv$ Circular convolution.

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Thu

Circular Convolution:

$$y_L[n] = \sum_{m=0}^{N-1} g[m]\, h[n-m] \qquad 0 \le n \le 2N-2.$$

$$y_c[n] = \sum_{m=0}^{N-1} g[m]\, h[\langle n-m \rangle_N] \qquad 0 \le n \le N-1$$

$$\underset{\text{N-length}}{g[n]} \xleftrightarrow{DFT} \underset{\text{N-length.}}{G[K]}$$

$$\underset{\text{N-length.}}{h[n]} \xleftrightarrow{DFT} \underset{\text{N length.}}{H[K]}$$

$$\underset{\text{N-length}}{x[n]} \rightarrow \boxed{\underset{\text{N-length}}{h[n]}} \rightarrow \underset{\text{2N-1 length.}}{y[n] = x[n] \circledast h[n].}$$

$h[-2] = h[N-2]$

$$\begin{bmatrix} y_c[0] \\ y_c[1] \\ \vdots \\ y_c[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & & h[2] \\ h[2] & h[1] & h[0] & & \vdots \\ \vdots & h[2] & \vdots & & \vdots \\ & \vdots & & & \\ h[N-1] & h[N-2] & h[N-3] & & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ \vdots \\ g[N-1] \end{bmatrix}$$

Circulant Matrix.

Tabular Method:

Ex: $g[n] = \{1 \quad 2 \quad 0 \quad 4\}$
$h[n] = \{1 \quad 1 \quad 2 \quad 2\}$

n	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$ ...
$g[n]$	1	2	0	4			
$h[n]$	1	1	2	2			
	1	2	0	4			
	4	1	2	0	$\cancel{4}$		
	0	8	2	4	$\cancel{0}$	$\cancel{8}$	
	4	0	8	2	$\cancel{4}$	$\cancel{0}$	$\cancel{8}$
$y_c[n]$	9	11	12	10			

$\therefore \langle 4\rangle_4 = 0$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ 12 \\ 10 \end{bmatrix}$$

$$G[K] = D_4 \, g[n]$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1+2j \\ -5 \\ 1-2j \end{bmatrix}$$

$H[K] = D_4 \cdot h[n]$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & +1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix}$$

$Y_c[K] = G[K] \cdot H[K]$

$$= \begin{bmatrix} 7 \\ 1+2j \\ -5 \\ 1-2j \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix} = \begin{bmatrix} 42 \\ -3-j \\ 0 \\ -3+j \end{bmatrix}$$

$y_c[n] = \frac{1}{4} D_4^* \cdot Y[K]$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & +1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} \begin{bmatrix} 42 \\ -3-j \\ 0 \\ -3+j \end{bmatrix} = \begin{bmatrix} 42 - 3 \cancel{-j} \cancel{-3+j} \\ 42 - 3j + 1 + 3j + 1 \\ 42 + 3 + j + 3 - j \\ 42 + j3 - 1 - 3j - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 11 \\ 12 \\ 10 \end{bmatrix}$$

*

n	0	1	2	3	4	5	6	$\langle 7\rangle_7$	$\langle 8\rangle_7$	
$g_e[n]$	1	2	0	4	0	0	0			
$h_e[n]$	1	1	2	2	0	0	0			
	1	2	0	4						
	-	1	2	0	4					
	-	-	2	4	0	8				
	-	-	-	2	4	0	8			
	-	-	-	-	0	0	0	0		
	-	-	-	-	-	0	0	0	0	
	-	-	-	-	-	-	0	0	0	0
	1	3	4	10	8	8	8			

* Discrete - Fourier Transform Theorems :-

$$g[n] \xleftrightarrow{DFT} G[K]$$
$$h[n] \xleftrightarrow{DFT} H[K]$$

1 linearty :-

$$\alpha g[n] + \beta h[n] \xleftrightarrow{DFT} \alpha G[K] + \beta H[K].$$

2 circular Time - shifting Theorem :-

$$g[\langle n-n_o \rangle_N] \xleftrightarrow{DFT} W_N^{Kn_o} G[K].$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

3 Circular Freqency Shifting theorem :-

$$W^{-Kn} g[n] \xleftrightarrow{DFT} G[\langle K-K_o \rangle_N]$$

4 Duality theorem : -

$$G[n] \xleftrightarrow{} N g[\langle -K \rangle_N]$$

5 Circular Convolution :-

$$g[n] \; \text{Ⓝ} \; h[n] \xleftrightarrow{DFT} G[K] H[K].$$

* Using matlab: $\gg g = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$; $h = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$

$\gg y_1 = $ conv (g, h)

$\gg g_e = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $he = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

DFT
$\gg$ GE = fft (ge) ;
$\gg$ HE = fft (he) ;
$\gg Y_2 = GE .* HE$

⑥ Modulation Theorem :-

↳ Multiplication in time domain.

$$g[n]\,h[n] \xleftrightarrow{DFT} \frac{1}{N} \sum_{L=0}^{N-1} G[L]\,H[\langle K-L \rangle_N]$$

⑦ Parseval's Relation :-

$$\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |G[K]|^2$$

$$\sum_{n=0}^{N-1} g[n]\,h^*[n] = \frac{1}{N} \sum_{K=0}^{N-1} G[K]\,H^*[K]$$

$\boxed{Ex.}$ $x[n] = \{-3 \;\; 5 \;\; 45 \;\; -15 \;\; -9 \;\; -19 \;\; -8 \;\; 21 \;\; -10 \;\; 23\}$

10-point seqence $X[K]$.

$\boxed{a}$ $X[0]$ $\boxed{b}$ $X[5]$ $\boxed{c}$ $\sum_{K=0}^{9} X[K]$ $\boxed{d}$ $\sum_{K=0}^{9} |X[K]|^2$

Do not evaluate $X[K]$.

Sol:

$$X[K] = \sum_{n=0}^{9} x[n]\, W_{10}^{nK} \quad , \quad W_{10} = e^{\frac{-j2\pi}{10}}$$

$\boxed{a}$ $X[0] = \sum_{n=0}^{9} x[n]\, W_{10}^{0} = \sum_{n=0}^{9} x[n] = -3+5+45+\dots+23$

$\boxed{b}$ $X[5] = \sum_{n=0}^{9} x[n]\, \underbrace{e^{\frac{-j\pi 2(5)n}{10}}}_{W_{10}^{5n}} = \sum_{n=0}^{9} x[n]\,(-1)^n$

$= -3 - 5 + 45 + 15 + \dots$

((odd (-1)))

$\boxed{c}$ $x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[K] \; W_N^{-nK}$

$N \cdot x[0] = \sum_{K=0}^{9} X[K].$

$\sum_{K=0}^{9} X[K] = N \cdot x[0] = 10 \cdot -3 = -30.$

$\boxed{d}$ $N \cdot \sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |G[K]|^2$

* complex. $(2+j)(2-j) = 4+1 = 5.$

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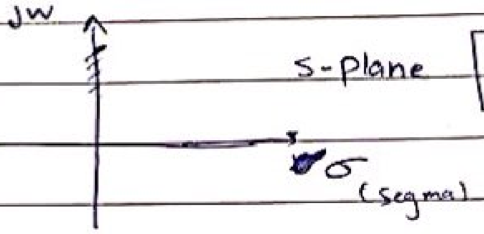
Tue

The Z-transform :-

$H(s)$
$h(t)$

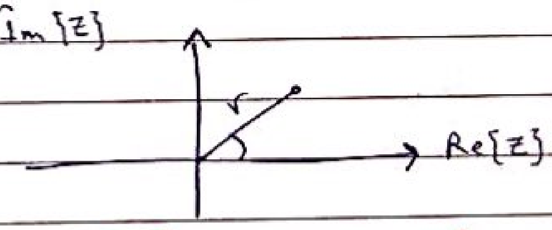


$\boxed{s = \sigma + jw}$

Definition 8.1

* $x[n] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

Z is a complex variable.



Z plane
$Z = re^{jw}$.

Z بمثلها بـ نقطة وزاوية
أو Re و Im

* The region of convergence (ROC)

$X(Z) = Z\{x[n]\}$

$x[n] \xleftrightarrow{Z} X(Z)$

$x[n] = Z^{-1}\{X(Z)\}$.

$\boxed{\text{Ex:}}$ $x_1[n] = \{\underset{\uparrow}{1}\ 2\ 5\ 7\ 0\}$

$x_2[n] = \{1\ 2\ \underset{\uparrow}{5}\ 7\ 0\}$

$x_3[n] = \{0\ 0\ 1\ 2\ 5\ 7\ 0\}$

$x_4[n] = \delta[n]$

$x_5[n] = \delta[n-K]$, $K > 0$

$x_6[n] = \delta[n+K]$ $k > 0$

Find the Z transform for the above sequence ?!

sol:

$\boxed{1}$ $X_1(Z) = 1 + 2Z^{-1} + 5Z^{-2} + 7Z^{-3} + Z^{-5}$

[illegible] ال ROC   ROC : entire Z plane except $Z = 0$.

$\boxed{2}$ $X_2(Z) = 1Z^{+2} + 2Z^{1} + 5 + 7Z^{-1} + 0 + Z^{-3}$.

ROC: entire Z plane except $Z = 0$ and $Z = \infty$

لل negative ((القوى)).   لل positive

$\boxed{3}$ $X_3(Z) = Z^{-2} + 2Z^{-3} + 5Z^{-4} + 7Z^{-5} + Z^{-7}$

ROC: entire Z plane except $Z = 0$.

$\boxed{4}$ $X_4(Z) = 1$ $\delta[n] \xleftrightarrow{Z} 1$

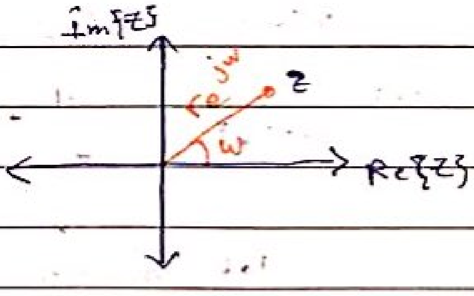
$\boxed{5}$ $X_5(Z) = Z^{-K}$, $K > 0$

ROC: the entire Z plane except. $Z = 0$.

$\boxed{6}$ $X_6(Z) = Z^{K}$, $K > 0$

ROC: the entire Z plane except. $Z = \infty$.

$$X(Z) = \sum_{n=-\infty}^{\infty} X[n] \, Z^{-n}$$



$$X(Z) = \sum_{n=-\infty}^{\infty} X[n] \, (re^{j\omega})^{-n}$$

If $r = 1$

$$X(Z)\Big|_{r=1} = \sum_{n=-\infty}^{\infty} x[n] \, (e^{-j\omega n}) = X(e^{j\omega})$$

Ex: Right sided sequence.

$$X[n] = a^n u[n].$$

Sol:-

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}.$$

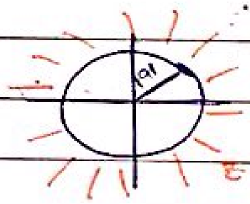
$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$X(z) = \frac{1}{1 - a z^{-1}} \qquad |a z^{-1}| < 1$$

$$\boxed{\text{ROC: } |a| < |z|}$$

* اذا كانت $|a| > |z|$ ← diverge والـ $X(z)$ ⇐ not ~~exsist~~ (مش موجود).
exsist.



(a=2)
* اذا كانت الـ a اكبر من دائرة الوحدة فبالتالي الـ $X(z)$ مش موجود.
* " " " a اقل " " " " موجود.
(a=0.5)
والـ Z transform موجود.

Ex left sided sequence :-

$$X[n] = -a^n u[-n-1].$$

Sol:-

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n}.$$

let $m = -n$

$$X(z) = -\sum_{m=1}^{\infty} a^{-m} z^{m}$$

بدي ابلش الـ m مساوي 0.
$-1 = z^0 a^0$ فبطلع واحد

$$X(z) = 1 - \sum_{m=0}^{\infty} (a^{-1} z)^m$$

*Properties of the Z-tranform :-

[1] linearity :-

$$x_1[n] \xleftrightarrow{z} X_1(z)$$
$$x_2[n] \xleftrightarrow{z} X_2(z)$$

then

$$x[n] = a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{z} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

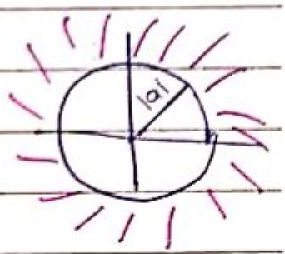
ROC: $R_1 \cap R_2$.

Ex: $x[n] = a^n \mu[n] + b^n \mu[-n-1]$

Find $X(z)$ and the ROC.

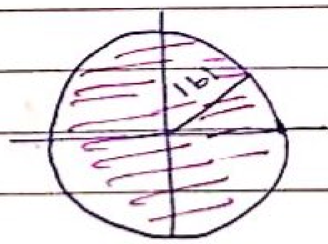
$x_1[n] = a^n \mu[n]$

$X_1(z) = \dfrac{1}{1 - az^{-1}}$ ROC: $|z| > |a|$



$x_2[n] = -b \mu[-n-1]$.

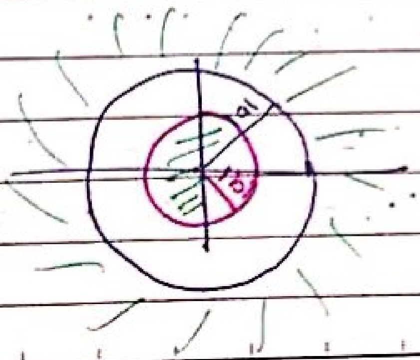
$X_2(z) = \dfrac{1}{1 - bz^{-1}}$ ROC: $|z| < |a|$



$X(z) =$

Case ① $|b| < |a|$

$X(z)$ does not exist.



$X(z) = 1 - \dfrac{1}{1-a^{-1}z}$ $\quad |a^{-1}z| < 1$

$\boxed{|z| < |a|}$

$X(z) = \dfrac{1 - a^{-1}z - 1}{1 - a^{-1}z}$

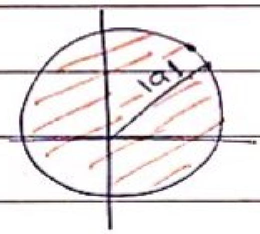
$X(z) = \dfrac{-a^{-1}z \,/\, (-a^{-1}z)}{1 - a^{-1}z \,/\, (-a^{-1}z)}$

$X(z) = \dfrac{1}{1 - a^{-1}z}$ $\quad \boxed{\text{ROC}: |a| > |z|}$

نفس الـ Right بس بتفرق عند الـ ROC.

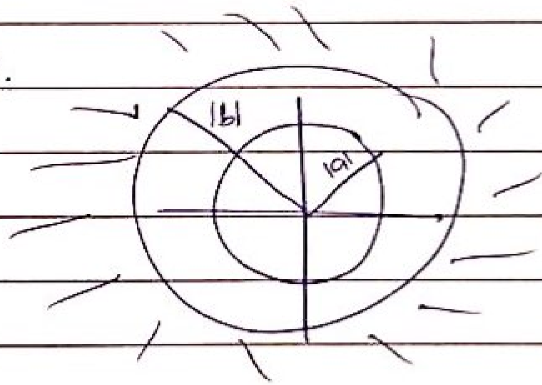
LSS → Region goes down.

RSS → " " up.



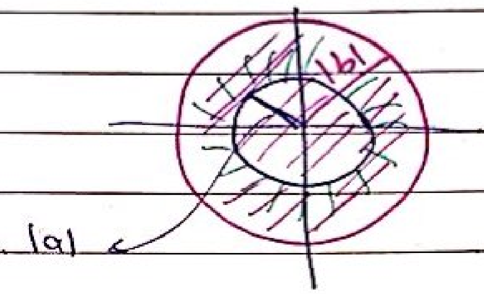
$x[n] = a^n u[n] + b^n u[n].$

Region بعد الـ b.



Case 2: $|b| > |a|$

$$X(z) = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}} \qquad ROC: |a| < |z| < |b|$$



$\boxed{Ex\ 2:1}$ Determine the Z transform of the signals:

a) $x[n] = \cos(w_0 n)\, u[n]$

b) $x[n] = \sin(w_0 n)\, u[n]$

sol: a) $x[n] = \cos(w_0 n) u[n]$

$$= \left[\frac{1}{2} e^{jw_0 n} + \frac{1}{2} e^{-jw_0 n}\right] \cdot u[n]$$

$$= \frac{1}{2}\left(e^{jw_0}\right)^n u[n] + \frac{1}{2}\left(e^{-jw_0}\right)^n u[n]$$

$$\left|e^{\pm jw_0}\right| = 1$$

$$= \frac{1/2}{1-e^{jw_0}z^{-1}} + \frac{1/2}{1-e^{-jw_0}z^{-1}}$$

ROC: $|z| > 1$

$$= \frac{1/2\left(1-e^{-jw_0}z^{-1} + 1-e^{jw_0}z^{-1}\right)}{(1-e^{jw_0}z^{-1})(1-e^{-jw_0}z^{-1})}$$

$$= \frac{\frac{1}{2}\left[2-\overbrace{\left(\left(e^{jw_0}+e^{-jw_0}\right)/2\right)}^{\cos(w_0)}\cdot 2z^{-1}\right.}{1-2z^{-1}\underbrace{\left(\frac{e^{jw_0}+e^{-jw_0}}{2}\right)}_{\cos(w_0)}+z^{-2}}$$

$$= \frac{1-\cos(w_0)\,z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}}.$$ ROC : $|z|>1$

$$\cos(w_0 n)\,u[n] \xleftrightarrow{z} \frac{1-\cos(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}}$$ ROC : $|z|>1$

$X(z) = \frac{N(z)}{D(z)}.$

* اذا كان في complex $x(n)$ ⟵ . Sinusiodal $x(z)$

* \" \" السبب complex ⟵ . (e) exponental.

sol:

b) $x[n] = \sin w_0 n \; u[n]$

$$X(z) = \frac{z^{-1}\sin(w_0)}{1-2z^{-1}\cos(w_0)+z^{-2}}$$ ROC: $|z|>1$

$$\sin(w_0 n)\,u[n] \xleftrightarrow{z} \frac{z^{-1}\sin(w_0)}{1-2\cos(w_0)z^{-1}+z^{-2}}$$

$\boxed{2}$ time Shifting :-

$x[n] \xleftrightarrow{z} X(z)$

then

$x[n-k] \xleftrightarrow{z} z^{-k}X(z).$

ROC is the Same as ROC of $X(z)$ except for $z=0$ if $k>0$ and $z=\infty$ if $k<0$.

Ex: $x[n] = \{\underset{\uparrow}{1} \; 2 \; 5 \; 7 \; 0 \; 1\}$

Sol: $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$.

$x_1[n] = x[n-2] = \{\underset{\uparrow}{0}\ 0\ 1\ 2\ 5\ 7\ 0\ 1\}$.

$\therefore X_1(z) = z^{-2} + 2z^{-3} + 5z^{-4} + \cancel{z}\,7z^{-5} + 1z^{-6}$.

or $z^{-2}[1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}]$

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[3] Scaling in the Z-domain 84

$x[n] \xleftrightarrow{Z} X(z)$ ROC: $r_1 < |z| < r_2$

then.

$a^n x[n] \xleftrightarrow{Z} X(a^{-1}z)$ ROC: $|a| r_1 < |z| < |a| r_2$

$X\left(\frac{z}{a}\right)$

where a any constant, real or complex.

Ex: 1) $x[n] = a^n \cos(w_0 n)\, u[n]$. $|z| > 1$

2) $x[n] = a^n \sin(w_0 n)\, u[n]$. $|z| > |a|$.

Sol: $x[n] = \cos w_0 n \; u[n]$.

$$X(z) = \frac{1 - \cos w_0 z^{-1}}{1 - 2\cos(w_0) z^{-1} + z^{-2}}$$

For $x[n] = a^n \cos(w_0 n)\, u[n]$.

$$X(z) = \frac{1 - \cos w_0 \left(\frac{z}{a}\right)^{-1}}{1 - 2\cos(w_0)\left(\frac{z}{a}\right)^{-1} + \left(\frac{z}{a}\right)^{-2}}$$

$$= \frac{1 - a\cos w_0 (z^{-1})}{1 - 2a\cos(w_0) z^{-1} + a^2 z^{-2}} \quad |z| > |a|.$$

OR

$$x[n] = a^n \left[\frac{e^{jw_0 n} + e^{-jw_0 n}}{2}\right] u[n].$$

$$= \left[\frac{1}{2}(ae^{jw_0})^n + \frac{1}{2}(ae^{-jw_0})^n\right] u[n].$$

special case: if $a=1$

$$n\,u[n] \xleftrightarrow{Z} \frac{z^{-1}}{(1-z^{-1})^2} \qquad |z|>1.$$

Z-transform of the unit ramp signal.

$\boxed{6}$ Convolution of two sequences

$$x_1[n] \xleftrightarrow{z} X_1(z). \qquad \text{ROC 1}$$
$$x_2[n] \xleftrightarrow{z} X_2(z) \qquad \text{ROC 2}$$

$$x[n] = x_1[n] \circledast x_2[n] \xleftrightarrow{z} X(z) = X_1(z)\,X_2(z)$$

ROC: ROC1 $\cap$ ROC 2.

$\boxed{\text{Ex:}}$ $x_1[n] = \{1\ 2\ 0\ 4\}$

$x_2[n] = \{1\ 1\ 2\ 2\}$

sol: $X_1(z) = 1 + 2z^{-1} + 4z^{-3}$

$X_2(z) = 1 + z^{-1} + 2z^{-2} + 2z^{-3}.$

$$X(z) = X_1(z)\,X_2(z) = 1 + 2z^{-1} + 0z^{-2} + 4z^{-3}$$
$$+ 1z^{-1} + 2z^{-2} + 0z^{-3} + 4z^{-4}$$
$$+ 2z^{-2} + 4z^{-3} + 0 + 8z^{-5}$$
$$+ 2z^{-3} + 4z^{-4} + 0 + 8z^{-6}$$

$$X(z) = 1 + 3z^{-1} + 4z^{-2} + 10z^{-3} + 8z^{-4} + 8z^{-5} + 8z^{-6}$$

$x[n] = \{1\ 3\ 4\ 10\ 8\ 8\ 8\}$

[4] Time Reversal :

$$x[n] \xleftrightarrow{z} X(z) \qquad \text{ROC: } r_1 < |z| < r_2$$

then

$$x[-n] \xleftrightarrow{z} X(z^{-1}) \qquad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Ex:

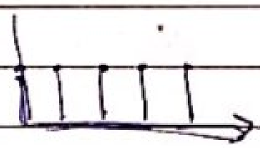
$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}} \qquad \text{ROC: } |z| > 1$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-1}$$

$$X(z) = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}} \qquad |z^{-1}| < 1$$

$$|z| > 1$$

* $X(e^{jw})$ for $u[n]$ $\neq \frac{1}{1-e^{-jw}}$



[5] Differentiation in the Z domain

$$x[n] \xleftrightarrow{z} X(z) \qquad \text{ROC}$$

$$n\,x[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz} \qquad \text{the same ROC}$$

Ex: $x[n] = n\,\underbrace{a^n u[n]}$

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}} \qquad \text{ROC: } |z| > a \quad \text{RSS}$$

$$-z\frac{dX(z)}{dz} = \frac{-(-a)(-1)z^{-2}\cdot(-z)}{(1-az^{-1})^2} = \frac{a z^{-1}}{(1-az^{-1})^2} \qquad |z| > a$$

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$ $\quad x[n] \circledast h[n] = y[n].$

$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$ $\quad X(z)\, H(z) = Y(z).$

$$H(z) = \frac{Y(z)}{X(z)}$$ $\Rightarrow$ the transfer function.

⑦ The Initial Value Theorem 8Δ

If $x[n]$ is causal [i.e $x[n] = 0$ for $n < 0$]

then $x[0] = \lim_{z \to \infty} X(z)$

$$\lim_{z \to \infty} X(z) = x[0] + x[1] \underbrace{z^{-1}}_{\frac{1}{\infty} = 0} + x[2]\, z^{-2} + \cdots$$

$$\lim_{z \to \infty} X(z) = x[0]$$

* Final value theorem — مش مطلوبة !!

$$\lim_{z \to 1} (1 - z^{-1})\, X(z) = x[\infty]$$

15/11

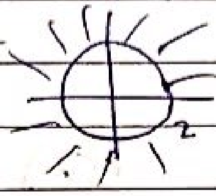
Tue.

The inverse Z-transform:-

Inspection Method: (the table).

$X(z) = \frac{1}{1-2z^{-1}}$ ROC: $|z| > 2$

left side sequence.



$x[n] = (2)^n u[n]$

$X(z) = \frac{A_1}{1-2z^{-1}}$ ROC: $|z| < 2$

$x[n] = A_1 (2)^n u[-n-1]$ left sided sequence.

* partial function Expansion:

$Y(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-M}}$ ROC!

~~ROC~~

* case 1: $N > M$ (proper function).

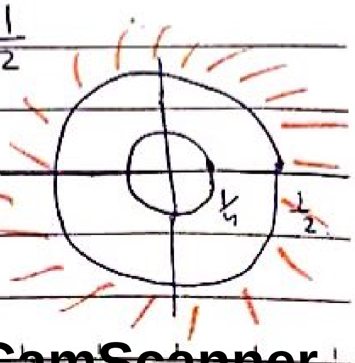
$X(z) = \frac{A_1}{1-d_1 z^{-1}} + \frac{A_2}{1-d_2 z^{-2}} + \cdots + \frac{A_N}{1-d_N z^{-1}}$. $\boxed{A_1 = \left.\frac{N(z)(1-d z^{-1})}{D(z)}\right|_{z=d_1}}$

$A_K = \left. Y(z)(1-d_K z^{-1}) \right|_{z=dK}$

Ex: $X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$ ROC: $|z| > \frac{1}{2}$

OR $|z| < \frac{1}{4}$

OR $\frac{1}{4} < |z| < \frac{1}{2}$



$X(z) = \frac{A_1}{(1-\frac{1}{4}z^{-1})} + \frac{A_2}{(1-\frac{1}{2}z^{-1})}$

$$A_1 = \frac{1}{1-\frac{1}{2}\times 4} = -1$$

$$A_2 = \frac{1}{1-\frac{1}{4}*2} = 2$$

$$X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

if ROC $|z| > \frac{1}{2}$ both RSS

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n].$$

if ROC $|z| < \frac{1}{4}$ both LSS

$$x[n] = -2\left(\frac{1}{2}\right)^n u[-1-n] + \left(\frac{1}{4}\right)^n u[-n-1].$$

if ROC $\frac{1}{4} < |z| < \frac{1}{2}$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[-1-n].$$

* Case 2: $N \leq M$ (improper function)

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

Ex: $X(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$

- بما إنو نفس ال order ح يطلع من القسمة ثابت (Constant)

$$\begin{array}{r|l} & \quad 2 \\ \hline \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ \ominus & z^{-2} - 3z^{-1} + 2 \\ \hline & \quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{5z^{-1} - 1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$= 2 + \frac{5z^{-1} - 1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

$$= 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

$$A_1 = \frac{5 \times 2 - 1}{1 - 2} = -9$$

$$A_2 = \frac{5 \times 1 - 1}{1 - 2} = 8$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

$$x[n] = 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] + 8\,u[n]$$

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Thu

EX 1/8 a linear Time-Invariant system is characterized by

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

specify the ROC of H(z) and determine h[n] for the following conditions:-

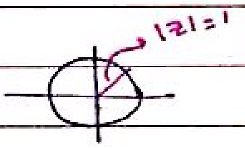
[1] the system is stable.
[2] the system is causal.

→ BIBO stability : $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$|H(z)| = \left|\sum_{n=-\infty}^{\infty} h[n]\, z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} |h[n]|\,|z^{-n}|$$

if $|z| = 1$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]|$$

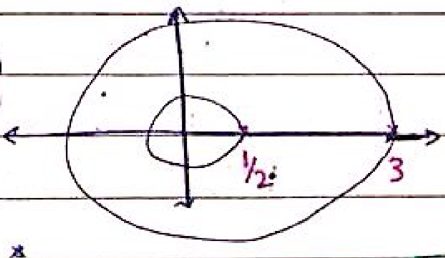


the system is stable if ROC includes the unit circle.

the system is causal : $h[n] = 0 \quad n < 0$
Right sided sequence.

Sol: $H(z) = \frac{3 - 4z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 3z^{-1})}$ → Poles of system.

properties of ROC:
① No poels ⇒ ROC داخل ال
② Rings or disck.
③ is always conected. ROC



* ممنوع يكون عندي اصفار (اقطاب) اللي فيها ال $H(z) = \infty$
ما بين ال Disck وال ROC من 0 ← $\frac{1}{2}$ او من 3 ← ∞.

① عشان يكون stable لازم تكون الـ ROC بين الـ $\frac{1}{2}$ و 3 .
وبتحتوي على دائرة الوحدة .

[1] stable: ROC:

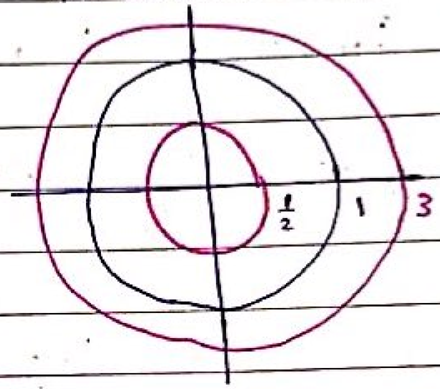
$\frac{1}{2} < |z| < 3$ contain the unit circle to ensure stability.

* احتمالات الـ ROC :

① داخل الـ $\frac{1}{2}$

② خارج الـ 3 .

③ او بين الـ $\frac{1}{2}$ و 3 .



((بعد ما بتحتوي 2poles دائرة الوحدة)) .

Poles means $H(z) = \infty$.

* اذا كان عندي pole = 1 ⟵ system unstable.

$$H(z) = \frac{A_1}{1+\frac{1}{2}z^{-1}} + \frac{A_2}{1-3z^{-1}}$$

$$A_1 = \frac{3-4(2)}{(1-3(2))}$$

* نعوض $1/2 = z$
$z^{-1} = 2$.

$$A_1 = \frac{-5}{-5} = 1$$

$$A_2 = \frac{3-4(1/3)}{(1-\frac{1}{2}(\frac{1}{3}))} = \frac{5/3}{5/6} = 2.$$

* نعوض $3 = z$
$z^{-1} = 1/3$.

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-3z^{-1}}$$

① ROC: $\frac{1}{2} < |z| < 3$

$h[n] = (\frac{1}{2})^n u[n] - 2(3)^n u[-n-1]$. ⇒ stable system

b Causal if ROC $|z| > 3$.

$h[n] = \left(\frac{1}{2}\right)^n u[n] + 2(3^n)u[n] \Rightarrow$ unstable.

Ex2: $$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Determine the causal signal $x[n]$ for which the z-transform is given above.

$\Rightarrow$ roots : complex.

$z^2 - z + 0.5 = 0$

$(1 - z^{-1} + 0.5z^{-2}) * z^2$

$$\text{poles, roots} = \frac{+1 \pm \sqrt{1-(4)(1)(0.5)}}{2} = \frac{1}{2} \pm j\frac{1}{2}.$$

* عشان عندي معامل z^{-2} قيمته $= \frac{1}{2}$ ← بطلع عندي الجواب $a^2 \cos w_0 n\, u[n]$ او $a^2 \sin w_0 n\, u[n]$

* اذا كان معامل $z^{-1} = 1$ → $\cos w_0 n\, u[n]$ / $\sin w_0 n\, u[n]$

$$a^n \cos(w_0 n)\, u[n] \xleftrightarrow{z} \frac{1 - a z^{-1}\cos w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-2}}.$$

$$a^2 \sin(w_0 n)\, u[n] \xleftrightarrow{z} \frac{a z^{-1} \sin w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-2}}$$

$a^2 = 0.5$ $\boxed{a = \frac{1}{\sqrt{2}}}$ باخذ القيمة الموجبة

$+1 = 2\frac{1}{\sqrt{2}}\cos w_0 \Rightarrow \cos w_0 = \frac{1}{\sqrt{2}}$, $\boxed{w_0 = \frac{\pi}{4}}$

$$\sin w_0 = \frac{1}{\sqrt{2}} \Rightarrow \boxed{w_0 = \frac{\pi}{4}}$$

$$a \cdot \cos w_0 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$X(z) = \underbrace{\frac{1 - \frac{1}{2} z^{-1}}{1 - z^{-1} + 0.5 z^{-1}}}_{\cos} + \underbrace{\frac{3\left(\frac{1}{2} z^{-1}\right)}{1 - z^{-1} + 0.5 z^{-1}}}_{\sin}$$

$$1 + z^{-1} = \left(1 - \frac{1}{2} z^{-1}\right) + \left(3\left(\frac{1}{2} z^{-1}\right)\right)$$ لازم يصير عندي :

$$x(z) = \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4} n\right) u[n] + 3\left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi}{4} n\right) u[n].$$

$$= \sqrt{10} \left(\frac{1}{\sqrt{2}}\right) \cos\left(\frac{\pi}{4} n - 71.565^\circ\right) u[n].$$

19/11
Sun

$x[n] \longrightarrow \boxed{H(e^{jw})} \longrightarrow y[n]$

$h[n]$

$$x[n] = \sum_{i=1}^{L} \underbrace{A_i \cos(w_i n + \theta_i)}_{\sum A_i \angle \theta_i} \qquad -\infty < n < \infty$$

$$y[n] = \sum_{i=1}^{L} A_i \, |H(e^{jw_i})| \; \cos(w_i n + \theta_i + \Psi(e^{jw_i}))$$

$\boxed{\text{Ex:}}$ a system with $\underbrace{h[n] = (\frac{1}{2})^n u[n]}_{\text{impulse response}}$.

① Find $H(e^{jw})$

② Find the out put of the system if

$x[n] = 10 - 5\sin \frac{\pi}{2} n + 20 \cos \pi n \qquad -\infty < n < \infty$

Sol: ① $H(e^{jw}) = \dfrac{1}{1 - \frac{1}{2} e^{-jw}}$

a) $w = 0$

$$H(e^{j0}) = \frac{1}{1 - \frac{1}{2} e^{-j0}} = 2 \angle 0^\circ$$

b) $w = \pi/2$.

$$H(e^{j\frac{\pi}{2}}) = \frac{1}{1 - \frac{1}{2} e^{-j\frac{\pi}{2}}} = \frac{1}{1 - \frac{1}{2}(\underbrace{\cancel{\cos \frac{\pi}{2}}}_{0} - j \sin \frac{\pi}{2})}$$

$$= \frac{2}{\sqrt{5}} \angle -26.6^\circ$$

c) $\omega = \pi$

$$H(e^{j\pi}) = \frac{1}{1 - \frac{1}{2} e^{-j\pi}} = \frac{1}{1 - \frac{1}{2}[\cos \pi - j \underbrace{\sin \pi}_{0}]}$$

$$= \frac{1}{1 + 1/2} = \frac{2}{3} \angle 0^\circ$$

$$\Rightarrow y[n] = 20 - \frac{5 \times 2}{\sqrt{5}} \sin\left(\frac{\pi}{2} n - 26.6^\circ\right) + 20\left(\frac{2}{3}\right) \cos \pi n$$

$$= 20 - \frac{10}{\sqrt{5}} \sin\left(\frac{\pi}{2} n - 26.6^\circ\right) + \frac{40}{3} \cos \pi n$$

$$-\infty < n < \infty$$

29-11
TUE

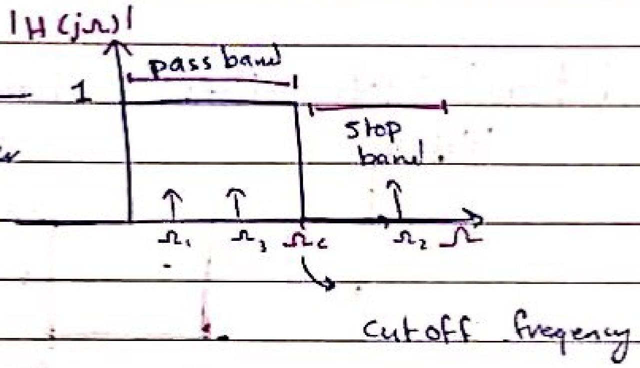
* Types of filters:-

[1] Classification based on Spectrum (frequency)

- Ideal filters:-

[A] LPF, low pass filter.

passive system ← 1
if > 1, active filter
↳ op-amp.



$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

$x(t) = A_1 \cos(\Omega_1 t) + A_2 \cos(\Omega_2 t) + A_3 \cos(\Omega_3 t)$.

I check each Ω_i

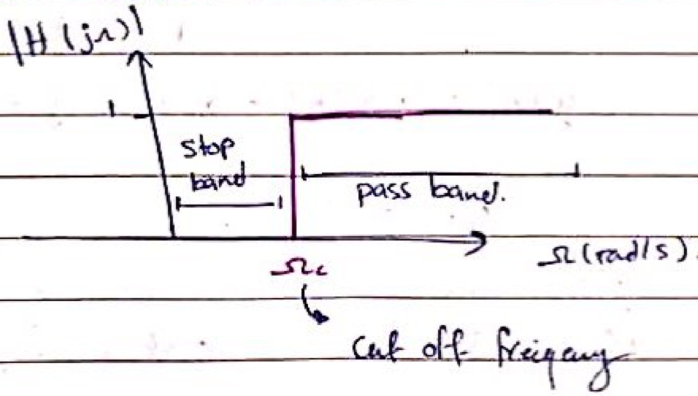
if $\Omega_i < \Omega_c \rightarrow y(t) = x(t)$.

if $\Omega_i > \Omega_c \rightarrow y(t) = 0$

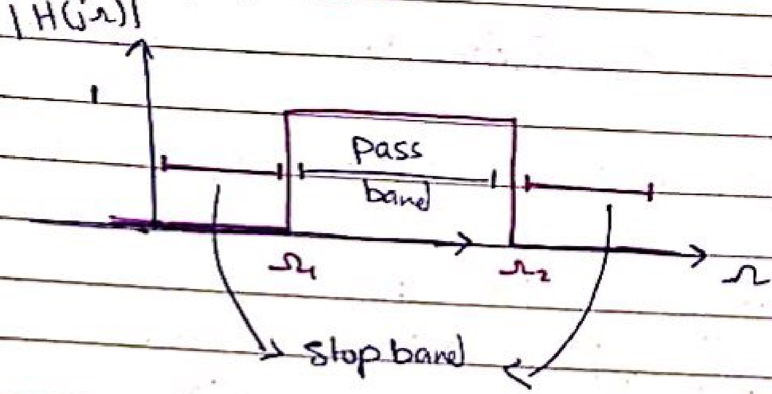
if $\Omega_1, \Omega_2, \Omega_3$ as shown above ⇒

$y(t) = A_1 \cos(\Omega_1 t) + A_3 \cos(\Omega_3 t)$

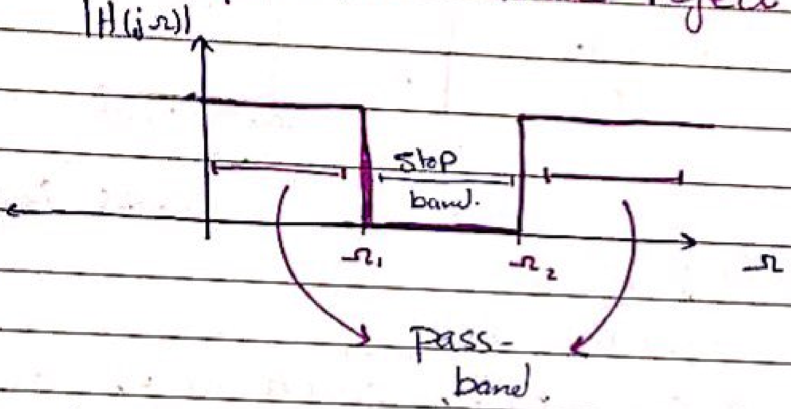
[B] HPF, High pass filter:-



[C] Band-pass filter (BPF) :-

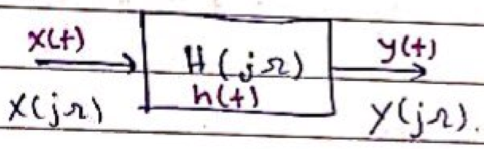


[D] Band-stop filter : (band-reject) (notch)



[2] classification based on Impulse Response :-

Filter.



[a] Finite impulse response filter (FIR).

$$H(j\Omega) = \frac{N(j\Omega)}{1}$$ ⇝ denamotor = 1

$$y[n] = a_0 x[n] + a_1 x[n-1] + \cdots + a_M x[n-M]$$

$$x[n] = \delta[n].$$

$y[0] = a_0$ when I put an impulse
$y[1] = a_1$ input, I got a finite.
$\vdots$
$y[M] = a_M$
$y[M+1] = 0.$

$$Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + \dots + a_M z^{-M} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + \dots a_M z^{-M}}{1}$$

$$H(z) = \frac{N(z)}{1}$$

$\boxed{b}$ infinite impulse response filter (IIR):-

$$y[n] = b_1 y[n-1] + b_2 y[n-2] + \dots + b_N y[n-N] + a_0 x[n] + a_1 x[n-1] + \dots + a_M x[n-M].$$

function of previous output and current and previous input [ARMA auto regresive MA]

Auto Regresive system: previous outputs + current input

$$H(z) = \frac{a_0 + a_1 z^{-1} + \dots + a_M z^{-M}}{b_0' + b_1' z^{-1} + \dots + b_N' z^{-N}} = \frac{N(z)}{D(z)}.$$

* MA: moving average. $\frac{a_0 + \dots a_M}{M+1}.$

here $b_0' = 0$

$b_i' = -b_i$

if $x[n] = \delta[n]$.

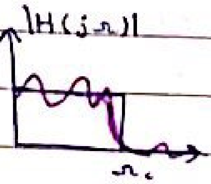
$X(z) = 1$

$$y[n]\Big|_{x[n]=\delta[n]} = h[n] = Z^{-1}\{H(z) \cdot 1\}.$$

Thu
1/12

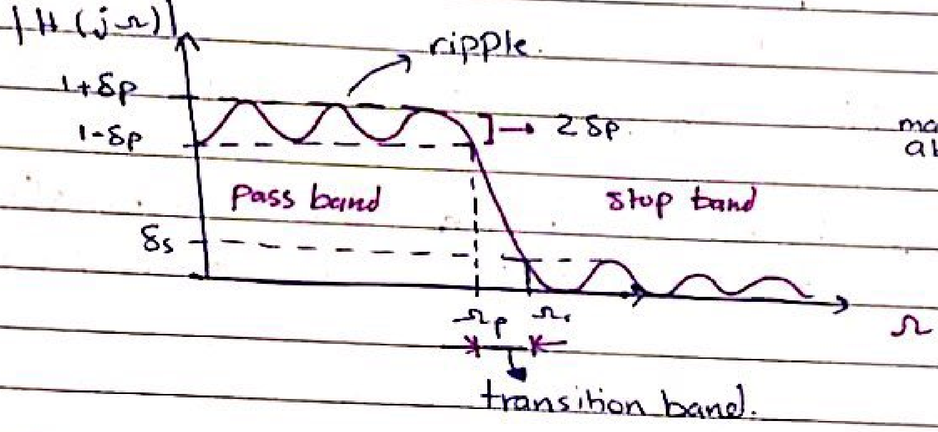
* IIR Analog filter design :-

Analog lowpass filter Design.

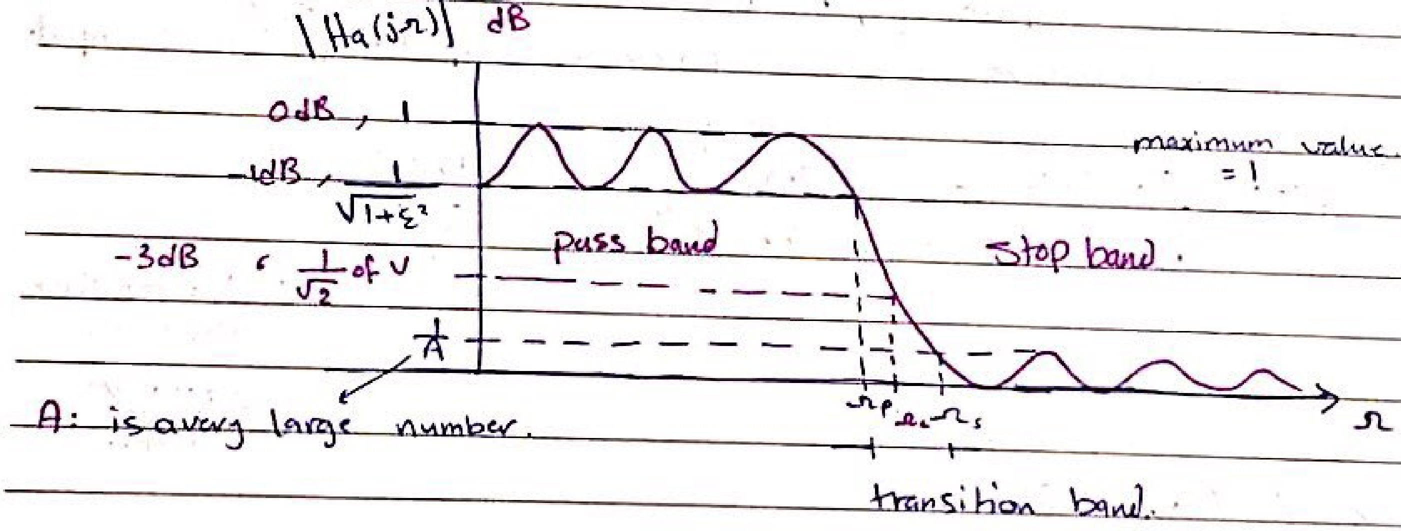


ideal low-pass filter.

Specification :-



maximum above 1



* transition ratio (selectivity parameter) :-

$$\boxed{K = \frac{\Omega p}{\Omega s}}$$

* Discrimination ~~parameter~~ parameter :-

$$\boxed{K_1 = \frac{\epsilon}{\sqrt{A^2-1}} \approx \frac{\varepsilon}{A}}$$

$K_1 \ll 1$

I Butterworth approximation :-

The magnitude squared response of an analog low-pass Butterworth filter of N^{th} order is given by :-

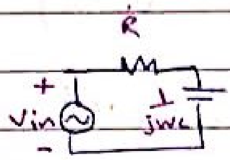
$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

N : order of the filter.

$\Omega_c \rightarrow$ 3dB frequency.

* 1st order Butterworth analog low pass filter :-

$$H(s) = \frac{1}{s+1}$$



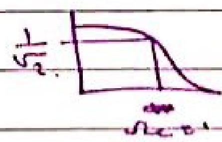
$$= \frac{\frac{1}{j\omega c}}{\frac{1}{j\omega c} + R}$$

$$H(j\Omega) = \frac{1}{j\Omega + 1}$$

$$\frac{(1/Rc)/1}{(1/Rc)/1 + Rj\omega c} = \frac{1/Rc}{1/Rc + j\omega}$$

$$H(j1) = \frac{1}{1+j} \Rightarrow \frac{1}{\sqrt{2}} = |H(j1)|$$

$$\boxed{\frac{1}{Rc} = \omega_c}$$



$$\boxed{\Omega = \frac{\Omega}{\Omega_c}}$$

$$H(j\Omega) = \frac{1}{j\left(\frac{\Omega}{\Omega_c}\right)+1} = \frac{\Omega_c}{\Omega_c + j\Omega}.$$

$$|H(j\Omega)|^2 = H(j\Omega) \cdot H^*(j\Omega)$$

$$= \frac{\Omega_c}{\Omega_c + j\Omega} \cdot \frac{\Omega_c}{\Omega_c - j\Omega}$$

$$= \frac{\Omega_c^2}{\Omega_c^2 + \Omega^2} = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^2}$$

* 2nd order butterworth analog low pass filter:-

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$H_a(j\Omega) = \frac{1}{(j\Omega)^2 + \sqrt{2}(j\Omega) + 1}$$

$$H_a(j\Omega) = \frac{1}{\left(j\frac{\Omega}{\Omega_c}\right)^2 + \sqrt{2}\, j\left(\frac{\Omega}{\Omega_c}\right) + 1}$$

$$= \frac{\Omega_c^2}{-\Omega^2 + j\sqrt{2}\,\Omega\Omega_c + \Omega_c^2}$$

$$= \frac{\Omega_c^2}{(\Omega_c^2 - \Omega^2) + j\sqrt{2}\,\Omega\Omega_c}$$

$$|H_a(j\Omega)|^2 = \frac{\Omega_c^4}{(\Omega_c^2 - \Omega^2)^2 + 2\Omega_c^2\Omega^2}$$

$$= \frac{\Omega_c^4}{\Omega_c^4 + \Omega^4} = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^4}$$

4/12
SUN

[1] Butterworth approximation:-

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Analog low pass filter design

$\Omega_p \rightarrow R_p$: ripple , attenuation in the ~~pas~~ pass-band

$\Omega_s \rightarrow R_s$: Minimum attenuation in the Stop band

* First order:-

$$H_a(s) = \frac{1}{s+1}$$

num = [Ω_c] ;
den = [1 Ω_c]

$$H_a(s) \quad \frac{\Omega_c}{s+\Omega_c}$$

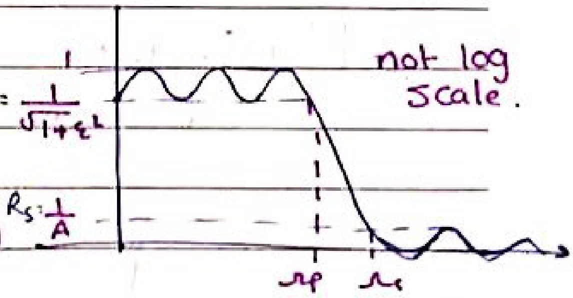
* Second order:-

$$H_a(s) = \frac{\Omega_c^2}{s^2+\sqrt{2}\,\Omega_c s+\Omega_c^2}$$

num : [Ω_c^2]
den = [1 $\sqrt{2}\,\Omega_c$ Ω_c^2]

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$|H_a(j\Omega_p)|^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = \frac{1}{1+\varepsilon^2} \quad \ldots (1)$$



$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \frac{1}{A^2} \quad \ldots (2)$$

Five Apple

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \varepsilon^2$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = A^2-1$$

$$\log_{10} : \left[\left(\frac{\Omega_s}{\Omega_p}\right)^{N} = \sqrt{\frac{A^2-1}{\varepsilon^2}}\right]$$

$$N \cdot \log\left(\frac{\Omega_s}{\Omega_p}\right) = \log_{10}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)$$

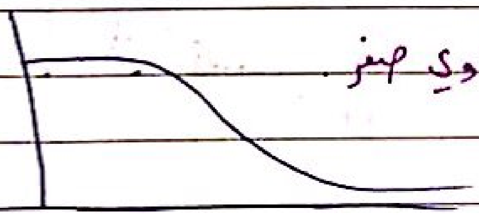
$$\boxed{N = \frac{\log(1/K_1)}{\log(1/K)}}$$

$$K_1 = \frac{\varepsilon}{\sqrt{A^2-1}}$$

$$K = \frac{\Omega_p}{\Omega_s}$$

* Maximally flat filter :-

↳ $2N-1$ derivatives of $|H_a(j\Omega)|^2\Big|_{\Omega=0} = 0$



كل الـ مشتقات بتساوي صفر

Ex: Find the order of a maximally flat filter with the following specsifiation:-

1 dB att at $f_p = 1\,KHZ$.

40 dB att at $f_s = 5\,KHZ$.

Sol:- بالـ dB

$\boxed{\Omega = 2\pi f}$

$$10 \log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -1\ dB$$

$$\boxed{\varepsilon^2 = 0.25895}$$

$$10 \log_{10}\left(\frac{1}{A^2}\right) = -40.$$

$$\boxed{A^2 = 10{,}000}$$

$$K_1 = \frac{1}{196.5} \Rightarrow \frac{1}{K_1} = 196.5$$

$$\frac{1}{K} = 5$$

$$N = \frac{\log(196.5)}{\log(5)} = 3.28 \rightarrow \boxed{N = 4}$$

قرب لأكبر

2 Chebyshev approximation

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N\left(\frac{\Omega}{\Omega_p}\right)}$$

$\Rightarrow T_N(\Omega)$ is the chebyshev polynomial of order N.

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}(\Omega)) & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}(\Omega)) & |\Omega| \ge 1 \end{cases}$$

نفس ال formula تحقق نفس الشرط ← * Recurrence relation :-

$$T_r(\Omega) = 2\Omega\, T_{r-1}(\Omega) - T_{r-2}(\Omega)$$

$T_0(\Omega) = 1$

$T_1(\Omega) = \Omega$

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TUE.

* Type I Chebyshev Approximations:

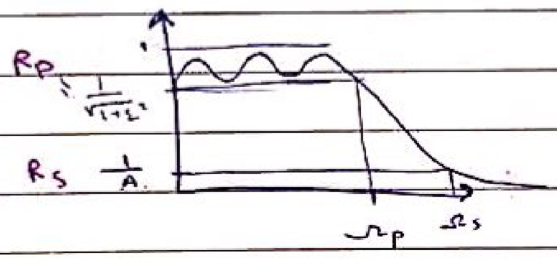
$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}(\Omega)) & |\Omega|<1 \\ \cosh(N\cosh^{-1}(\Omega)) & |\Omega|>1 \end{cases}$$

* Recursive formula:-

$$T_r(\Omega) = 2\Omega\, T_{r-1}(\Omega) - T_{r-2}(\Omega) \qquad r \geq 2$$

$$T_0(\Omega) = 1 \quad , T_1(\Omega) = \Omega$$



$$|H_a(j\Omega_p)|^2 = \frac{1}{1+\varepsilon^2} = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_p}{\Omega_p}\right)}$$

* No information is given

$$|H_a(j\Omega_s)|^2 = \frac{1}{A^2} = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} \qquad \Omega_s > \Omega_p$$

$|\Omega| > 1$

$$\frac{1}{A^2} = \frac{1}{1+\varepsilon^2\left[\left(\cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)\right)\right]^2}$$

$$\frac{\sqrt{A^2-1}}{\varepsilon} = \cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)$$

$$N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right) = \cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)$$

$$N = \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\boxed{N = \frac{\cosh^{-1}\left(\frac{1}{K_1}\right)}{\cosh^{-1}\left(1/K\right)}}$$

calculator. $\cos^{-1} \Rightarrow hyp \rightarrow 5$

Ex: $\frac{1}{K_1} = 196.5$, $\frac{1}{K} = 5$

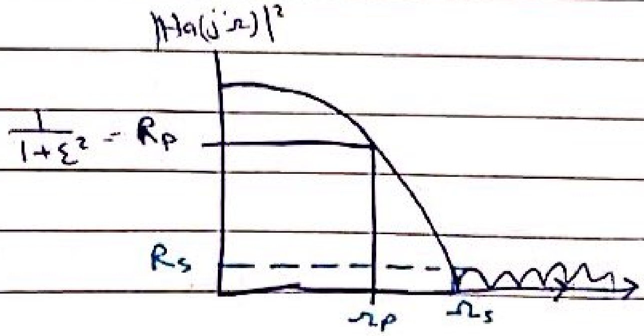
$$N = \frac{\cosh^{-1}(196.5)}{\cosh^{-1}(5)} = 2.6$$

$$\boxed{N = 3}$$

* order ↓ → less complexity

* Type 2: Chebyshev Approximation

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\frac{T_N\left(\frac{\Omega_s}{\Omega_p}\right)}{T_N\left(\frac{\Omega_s}{\Omega}\right)}\right]^2}$$

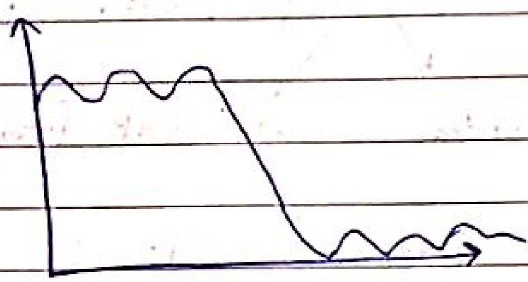


$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\varepsilon^2\left[\frac{T_N\left(\frac{\Omega_s}{\Omega_p}\right)}{T_N(1)}\right]^2}$$

$$\boxed{T_N(1) = 1}$$

$$\boxed{N = \frac{\cosh^{-1}\left(\frac{1}{K_1}\right)}{\cosh^{-1}(1/K)}}$$

3 Elliptic Approximations (Caver Filter)



$$H_a(j\Omega) = \frac{1}{1+\varepsilon^2 R_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

$R_N(\Omega)$ is a rational polynomial of order N that satisfies $R_N\left(\frac{1}{\Omega}\right) = \frac{1}{R_N'(\Omega)}$

$$\boxed{N \simeq \frac{2\log_{10}\left(\frac{4}{K_1}\right)}{\log_{10}(1/\rho)}}$$

$$K' = \sqrt{1-K^2}$$

$$\rho_0 = \frac{1-\sqrt{K'}}{2(1+\sqrt{K'})}$$

$$\rho = \rho_0 + 2(\rho_0)^5 + 15(\rho_0)^9 + 150(\rho_0)^{13}$$

((مش محفوظ))

" الدكتور بيعطينا اياها "

* Analog Filter Design Using Matlab 3Δ.

[1] Butterworth :-

analog region.

[N, Wn] = buttord (Wp, Ws, Rp, Rs , 'S') ;

rad/sec   +ve dB

$f_p = 1 KHZ$, $f_s = 5 KHZ$, $R_p = 1 dB$, $R_s = 40 dB$.

[N, Wn] = buttord (2*pi*1000 , 2*pi*5000 , 1 , 40 , 'S')

order W_c (3dB freqency).

[num , den] = butter (N, Wn, 'S') ;

* $H_a(s)$, num = C , den = [1 a_1 a_2 a_3]

$$H_a(s) = \frac{C \quad (num)}{S^3 + a_1 S^2 + a_2 S^1 + a_3 \quad (den)}.$$

[2] Type1 Chebyshev :-

[N, Wn] = Cheb1ord (Wp , Ws , Rp , Rs , 'S') ;

rad/s   +ve dB

$\boxed{W_n = W_p}$

[num, den] = Cheby1 (N, Rp, Wn, 'S') ;

[3] Type 2 Chebyshev :-

[N, Wn] = Cheb2ord (Wp, Ws , Rp , Rs , 'S') ;

$\boxed{W_n = W_s}$

[num, den] = cheby2 (N, Rs, Wn, 'S') ;

4 Elliptic Filter :-

[N, Wn] = ellipord (Wp, Ws, Rp, Rs, 'S');

$\boxed{Wn = [Wp, Ws]}$

[num, den] = ellip (N, Rp, Rs, Wn, 'S').

* Design of Analog Highpass, Bandpass and Band stop Filters:-

$$S = F(\hat{S})$$

low pass filter ↙ ↘ desired filter.

$$H_D(\hat{S}) = H_{LP}(S)\Big|_{S=F(\hat{S})}$$

$$H_{LP}(S) = H_D(\hat{S})\Big|_{\hat{S}=F^{-1}(S)}$$

* Analog Highpass Filter Design:-

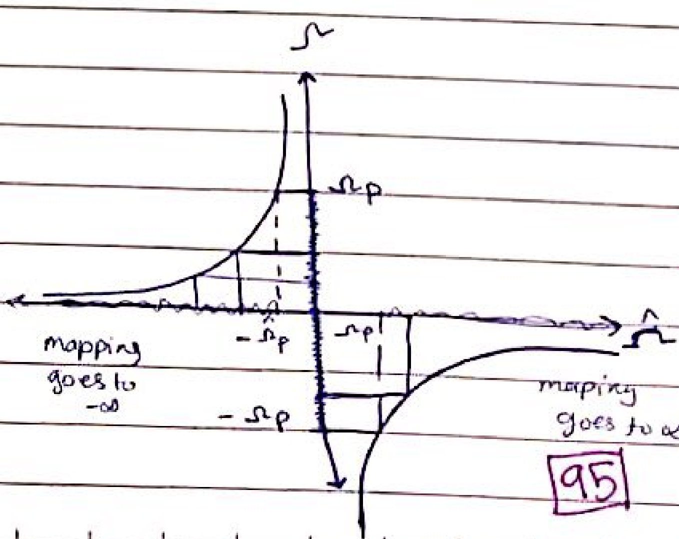
$$S = \frac{\Omega_p \hat{\Omega}_p}{\hat{S}}$$

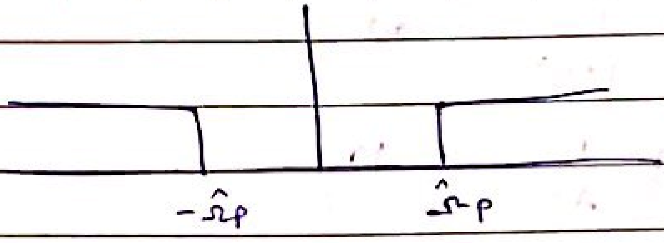
on the imaginary axis:-

$$j\Omega = \frac{\Omega_p \hat{\Omega}_p}{j\hat{\Omega}}$$

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

$$\boxed{\Omega = \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}}$$

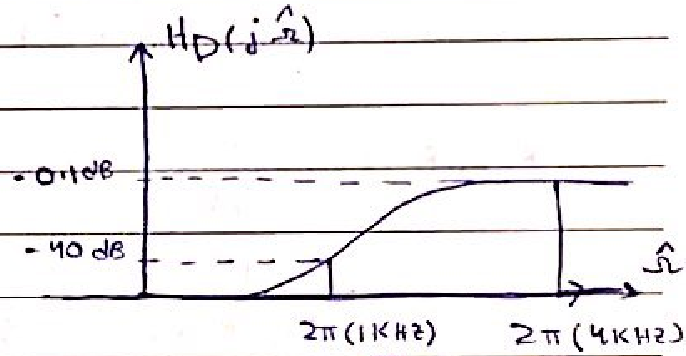


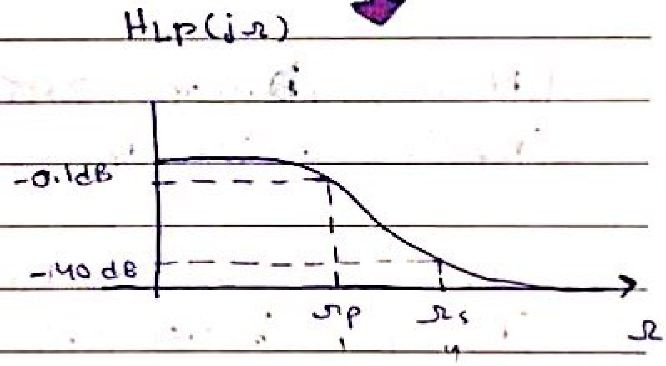


Ex: Design of an Analog Butterworth High pass filter :-

$\hat{\Omega}_p = 4\,KHZ$, $\hat{\Omega}_s = 1\,KHZ$

$R_p = 0.1\,dB$. $R_s = 40\,dB$.





⇒ let $\Omega_p = 1$ rad/sec

$$\Omega_s = \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s} = \frac{1 * (2\pi)(4K)}{(1K)(2\pi)} = 4 \text{ rad/sec}$$

* The prototype low pass filter specification is:

$\Omega_p = 1$ with $R_p = 0.1\,dB$

$\Omega_s = 4$ with $R_s = 40\,dB$

$$N = \frac{\log\left(\frac{1}{K_1}\right)}{\log(1/K)}, \quad K_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}}, \quad K = \frac{\Omega_p}{\Omega_s}$$

$$R_p \Rightarrow 10 \log \frac{1}{1+\varepsilon^2} = -0.1$$

$$R_s \Rightarrow 10 \log \frac{1}{A^2} = -40.$$

$$H_{LP}(s) = \frac{Num}{Den.}$$

$$H_D(\hat{s}) = H_{LP}(s)\Big|_{s=F(\hat{s})} \rightarrow s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$$

$$* \text{ LPF} \Rightarrow \frac{1}{s+1}$$

$$\frac{1}{\frac{\Omega_p \hat{\Omega}_p}{\hat{s}} + 1} = \frac{\hat{s}}{\hat{s} + \Omega_p \hat{\Omega}_p}$$

~~[N,Wn] = buttord(2*pi*1000, 2*pi*4000, 0.1, 40, 'S');~~
~~[B,A] = butter(N,~~

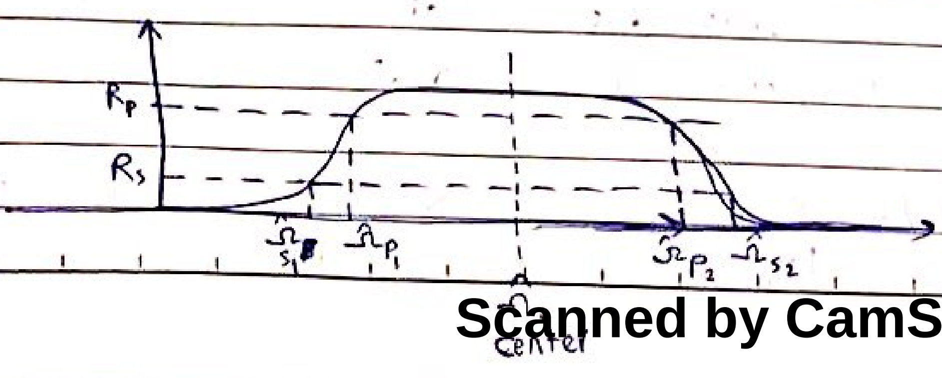
```
[N,Wn] = buttord(1, 4, 0.1, 40, 'S');
[B,A] = butter(N, Wn, 'S');
[num, den] = lp2hp(B, A, 2*pi*4000);
```

$$\underbrace{2*pi*4000}_{\hat{\Omega}_p}$$

* Analog Bandpass Filter design :-

$$S = \Omega_p \frac{\hat{S}^2 + \hat{\Omega}_o^2}{\hat{S}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$$

without hat (^) → LPF.
with (^) → bandpass.



$\hat{\Omega}_0 \rightarrow$ the center frequency.

$$\boxed{BW = \hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}}$$

$$\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} = \hat{\Omega}_{s_1} \hat{\Omega}_{s_2} = \hat{\Omega}_0^2$$

$$\Omega_0 = \frac{1}{\sqrt{LC}},$$

$$\Omega = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\, BW}$$

$$\Omega = \Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\, BW}$$

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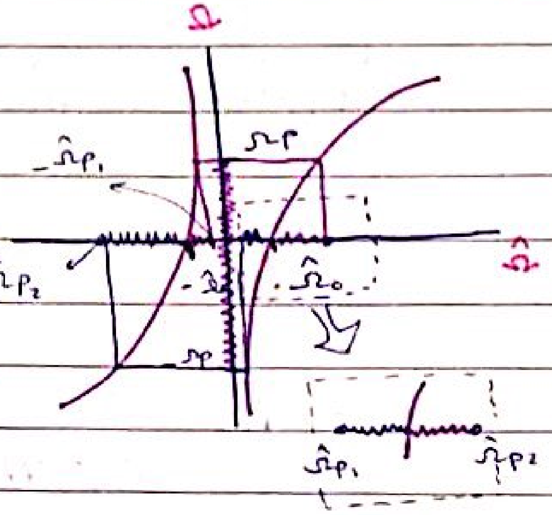
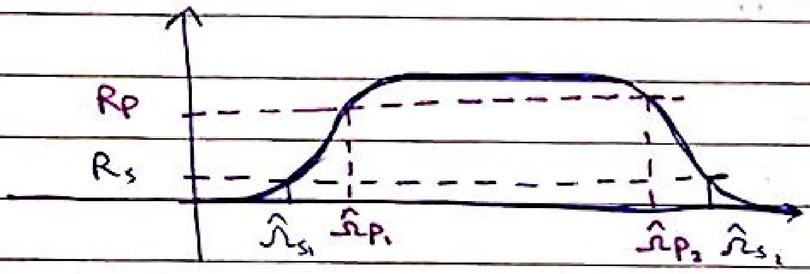
TUE

Analog Bandpass Filter Design 8Δ

$$\Omega = \Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\, BW}$$



$$\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} = \hat{\Omega}_{s_1} \hat{\Omega}_{s_2} = \hat{\Omega}_0^2 \quad \dots (1)$$



$$\boxed{BW = \hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}}$$ fixed.

* لازم يتحقق الشرط ① . اذا ما تحقق ~~بغير~~ بغير قيم Ω_s ما بغير قيم Ω_p عشان احافظ ع نفس BW.

* $\hat{\Omega}_{s_1} \hat{\Omega}_{s_2} > \hat{\Omega}_{p_1} \hat{\Omega}_{p_2}$

* لازم اقل قيم Ω_s مين اختار ؟!

① اذا اخترت Ω_{s_1} رح تزيد قيم transition band.

② اذا اخترت Ω_{s_2} رح تقل عندي قيم transition band وهو الحل الأفضل . لانو بصير عندي Sharp

$$\hat{\Omega}_{s_2} = \frac{\hat{\Omega}_{p_1} \hat{\Omega}_{p_2}}{\hat{\Omega}_{s_1}}$$

* if $\hat{\Omega}_{s_1} \hat{\Omega}_{s_2} < \hat{\Omega}_{p_1} \hat{\Omega}_{p_2}$

* بدي ازيد قيم Ω_s ← $\Omega_{s_1}\uparrow$ ← تقل ال transition band ← الحل الافضل

← $\Omega_{s_2}\uparrow$ ← تزيد ال transition band

$$\hat{\Omega}_{s_1} = \frac{\hat{\Omega}_{p_1} \; \hat{\Omega}_{p_2}}{\hat{\Omega}_{s_2}}$$

$$\boxed{\Omega_s = \Omega_p \, \frac{\hat{\Omega}_o^2 - \hat{\Omega}_{s_1}^2}{\hat{\Omega}_{s_1} \, BW}}$$ $\Rightarrow$ نعوض في Ω_{s_1} الجديدة

$\boxed{\text{Ex}}$ Design an Analog Elliptic Band pass filter with the following specification :-

$\hat{\Omega}_{s_1} = 3 KHZ$ $\quad\quad$ $R_s = 22 dB$

$\hat{\Omega}_{s_2} = 8 KHZ$

$\hat{\Omega}_{p_1} = 4 KHZ$ $\quad\quad$ $R_p = 1 dB.$

$\hat{\Omega}_{p_2} = 7 KHZ$

Sol: $\quad BW = \hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}$

$= 7 KHZ - 4 KHZ$

$\boxed{BW = 3 KHZ}$

$\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} = 7 \cdot 4 = 28 \times 10^6 = \hat{\Omega}_o^2$

$\hat{\Omega}_o = \sqrt{28} = 5.2915 KHZ.$

$\hat{\Omega}_{s_1} \hat{\Omega}_{s_2} = 3 \times 8 = 24 \times 10^6 \neq 28 \times 10^6$

$\hat{\Omega}_{s_2} \hat{\Omega}_{s_1} < \hat{\Omega}_{p_1} \hat{\Omega}_{p_2}$

$$\hat{\Omega}_{s_1} = \frac{\hat{\Omega}_{p_1} \; \hat{\Omega}_{p_2}}{\hat{\Omega}_{s_2}}$$

$$\hat{\Omega}_{s_1} = \frac{28 \times 10^6}{8 \times 10^3} = 3.5 \times 10^3 \; HZ.$$

The prototype LPF :-

let $\Omega_p = 1$ rad/s

$$\Omega_s = (1) \cdot \frac{28 - (3.5)^2}{(3.5)(3)} = 1.5 \; rad/s$$

Specification :-

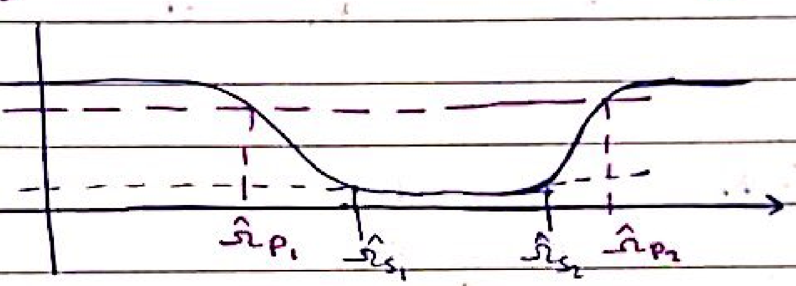
$\Omega_p = 1$ $R_p = 1\,dB$

$\Omega_s = 1.5$ $R_s = 22\,dB$

```
[N, Wn] = ellipord ( 1, 1.5, 1, 22, 's');
[B, A] = ellip ( N, 1, 22, Wn, 's');
[num, den] = Lp2bp ( B, A, 2*pi*5.2915*1000, 2*pi*3000);
```

$\underbrace{2*pi*5.2915\times1000}_{\hat{\Omega}_0}$, $\underbrace{2*pi*3000}_{BW}$

* Analog band stop filter Design 8Δ



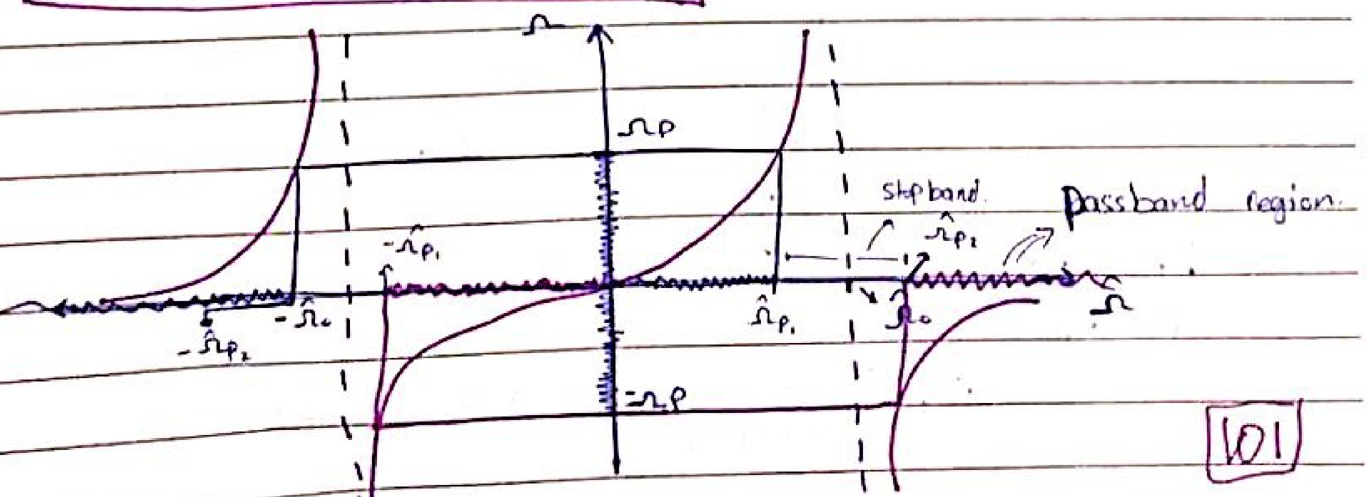
Fixed $\boxed{BW = \hat{\Omega}_{s_2} - \hat{\Omega}_{s_1}}$

$$\boxed{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2} = \underbrace{\hat{\Omega}_{s_1}\hat{\Omega}_{s_2}}_{\text{Fixed}} = \hat{\Omega}_0^2}$$

$S = F(\hat{S})$

$$S = \Omega_s \frac{\hat{S}(\hat{\Omega}_{s_2} - \hat{\Omega}_{s_1})}{\hat{S}^2 + \hat{\Omega}_0^2}$$

$$\boxed{\Omega = \Omega_s \frac{\hat{\Omega}(\hat{\Omega}_{s_2} - \hat{\Omega}_{s_1})}{\hat{\Omega}^2 + \hat{\Omega}_0^2}}$$



if $\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} > \hat{\Omega}_{s_1} \hat{\Omega}_{s_2}$

$$\hat{\Omega}_{p_2} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_1}}$$

else if $\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} < \hat{\Omega}_{s_1} \hat{\Omega}_{s_2}$

$$\hat{\Omega}_{p_1} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_2}}$$

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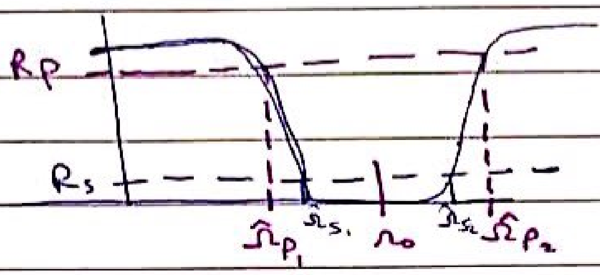
Thu

* Analog Band Stop Filter Design:-

$$S = \Omega_s \frac{\hat{S}(\hat{\Omega}_{s_2} - \hat{\Omega}_{s_1})}{\hat{S}^2 + \hat{\Omega}_o^2}$$



$$\Omega = \Omega_s \frac{\hat{\Omega}\, BW}{\Omega_o^2 - \hat{\Omega}^2}$$

$$\boxed{BW = \hat{\Omega}_{s_2} - \hat{\Omega}_{s_1}}$$

$$\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} = \underbrace{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}_{\text{fixed}} = \hat{\Omega}_o^2$$

if $\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} < \hat{\Omega}_{s_1} \hat{\Omega}_{s_2}$

$$\hat{\Omega}_{p_1} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_2}}$$

else if $\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} > \hat{\Omega}_{s_1} \hat{\Omega}_{s_2}$

$$\hat{\Omega}_{p_2} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_1}}$$

$$\boxed{\Omega_p = \Omega_s \frac{\hat{\Omega}_{p_1} BW}{\Omega_o^2 - \hat{\Omega}_{p_1}^2}}$$

* مع اقرب الصا Ω_{p_1} عشان يطلع عندي ناتج الطرح +ve

let $\Omega_s = 1$

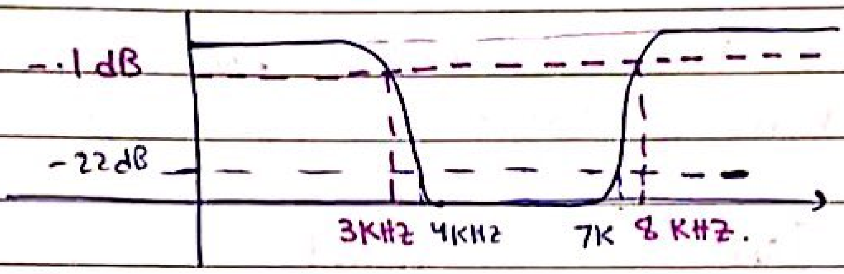
$$\Omega_p = \Omega_s \frac{\hat{\Omega}_{p_1} BW}{\Omega_o^2 - \hat{\Omega}_{p_1}^2}$$

Prototype low pass:

Ω_p with R_p

Ω_s with R_s.

Ex: 1



Elliptic filter :-

$BW = \hat{\Omega}_{s_2} - \hat{\Omega}_{s_1} = 7 - 4 = 3KHZ.$

$\hat{\Omega}_0^2 = \hat{\Omega}_{s_1} \hat{\Omega}_{s_2} = 7 \cdot 4 = 28 \times 10^6$

$\hat{\Omega}_0 = \sqrt{28 \times 10^6} = 5.2915 \, KHZ.$

$\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} = 8 \cdot 3 = 24 \times 10^6$

$$\hat{\Omega}_{p_1} = \frac{28 \times 10^6}{8 \times 10^3} = 3.5 \, KHZ.$$

$\left.\begin{array}{l}\hat{\Omega}_{p_1} = 3.5KHZ \\ \hat{\Omega}_{p_2} = 8\,KHZ\end{array}\right\} R_p = 0.1\,dB$ $\qquad$ $\left.\begin{array}{l}\hat{\Omega}_{s_1} = 4KHZ. \\ \hat{\Omega}_{s_2} = 7KHZ\end{array}\right\} R_s = 22dB$

low ~~pass~~ prototype Specification:-

let $\Omega_s = 1$

$$\Omega_p = (1)\frac{(3.5)(3)}{28 - (3.5)^2} = \frac{2}{3} = 0.67 \quad ✓ \text{ ok}$$

OR $\Omega_p = 1$

$$1 = \Omega_s \frac{(3.5)(3)}{28 - (3.5)^2}$$

$$\Omega_s = \frac{28 - (3.5)^2}{(3.5)(3)} = \frac{3}{2} = 1.5 \quad ✓ \text{ ok.}$$

* اكمليه صح .

order $[N, W_n]$ = ellipord (1, 1.5, 0.1, 22, 's');
$H_{LP}(s)$ $[B, A]$ = ellip (N, 0.1, 22, wn, 's');
$H_D(\hat{s})$ [num, den] = lp2bs (B, A, 2*pi*5291.5, 2*pi*3000);

* order of low pass $N = 3$, $N \doteq \frac{\log_{10}(1/k_1)}{\log_{10}(1/\rho)}$
* order of bs, bp $= 2N = 6$

den => $[a_N \quad a_{N-1} \quad \dots \quad a_1 \; a_0]$.
↳ $a_N S^N + a_{N-1} S^{N-1} + \dots + a_1 S + a_0$.

$$\frac{num}{den} = \frac{b_M S^M + b_{M-1} S^{M-1} + \dots + b_1 S + b_0}{a_N S^N + a_{N-1} S^{N-1} + \dots + a_1 S + a_0} \qquad \boxed{N > M}$$

* check Lp2Lp ?!

* $S = F(\hat{s})$

$$H_D(\hat{s}) = H_{LP}(s)\Big|_{s = F(\hat{s})}$$

$$H_{LP}(\hat{s}) = H_D(\hat{s})\Big|_{\hat{s} = F^{-1}(\hat{s})}$$

$$H_D(\hat{s}) \xrightarrow{!?} H_D(z)$$

$H_D(z)$ [numT, denT] = bilinear (   );

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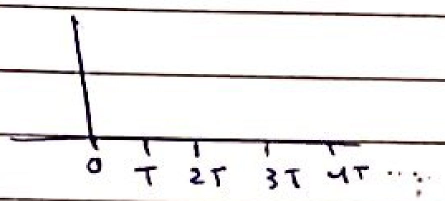
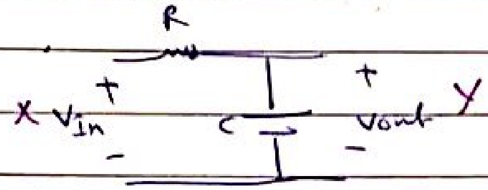
SUN

* Discreatizing of a Continous Time Filter:-

$$\frac{Y(s)}{X(s)} = H(s) = \frac{a}{s+a}$$ (simple lowpass Filter in Continous-time)





1) Backward Differance Method.
2) Forward "
3) Bilinear Transformation

$$\frac{dy}{dt} = -ay + ax$$

$$\int_0^t \frac{dy(t)}{dt}\,dt = -a\int_0^t y(t)\,dt + a\int_0^t x(t)\,dt$$

Subsitute $t = KT$

$$\int_0^{KT} \frac{dy(t)}{dt}\,dt = -a\int_0^{KT} y(t)\,dt + a\int_0^{KT} x(t)\,dt \quad \cdots \text{(A)}$$

$$y(KT) - y(0) = -a\int_0^{KT} y(t)\,dt + a\int_0^{KT} x(t)\,dt. \quad \cdots \text{(B)}$$

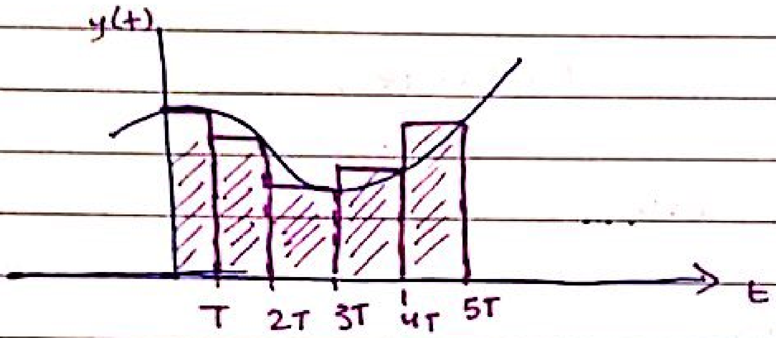
Subsitute $t = (K-1)T$ in (A)

$$\int_0^{(K-1)T} \frac{dy(t)}{dt} = -a\int_0^{(K-1)T} y(t)\,dt + a\int_0^{(K-1)T} x(t)\,dt$$

$$y((K-1)T) - y(0) = -a\int_0^{(K-1)T} y(t)\,dt + a\int_0^{(K-1)T} x(t)\,dt \quad \ldots (C)$$

$$\boxed{y(KT) - y((K-1)T) = -a\int_{(K-1)T}^{KT} y(t)\,dt + a\int_{(K-1)T}^{KT} x(t)\,dt \quad \ldots (D)}$$

Method 1: Backward difference :-



$$\int_{(K-1)T}^{KT} y(t)\,dt = T\cdot y(KT)$$

$$\int_{(K-1)T}^{KT} x(t)\,dt = T x(KT)$$

$$y(KT) - y((K-1)T) = -aT\cdot y(KT) + aT\cdot x(KT)$$

$$y(K) - y(K-1) = -aTy(K) + aTx(T)$$

$$Y(z) - z^{-1}Y(z) = -aTY(z) + aT\,X(z)$$

$$Y(z)\left[1 - z^{-1} + aT\right] = aT\,X(z)$$

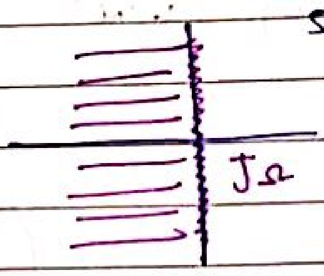
$$H(z) = \frac{Y(z)}{X(z)} = \frac{(aT)/T}{(1-z^{-1}+aT)/T}$$

$$H(z) = \frac{a}{\frac{1-z^{-1}}{T} + a}$$

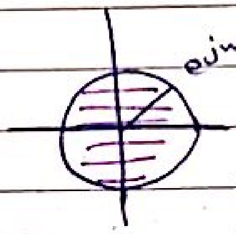
$$H(s) = \frac{a}{s+a}$$

$$\boxed{s = \frac{1-z^{-1}}{T}}$$

$Re\{s\} < 0$



$$* \; Re\left\{\frac{1-z^{-1}}{T}\right\} < 0$$



$z = \sigma + jw$

$$\left(Re\left\{\frac{z-1}{Tz}\right\} < 0 \right) T$$

$z = re^{jw}$

let $z = \sigma + jw$

T is constant > 0.

let $z = \sigma + jw$.

$$Re\left\{\frac{\sigma + jw - 1}{\sigma + jw}\right\} < 0$$

$$Re\left\{\frac{(\sigma-1) + jw}{\sigma + jw} \times \frac{\sigma - jw}{\sigma - jw}\right\} < 0$$

$$Re\left\{\frac{(\sigma-1)\sigma + jw\sigma - jw(\sigma-1) + w}{\sigma^2 + w^2}\right\} < 0$$

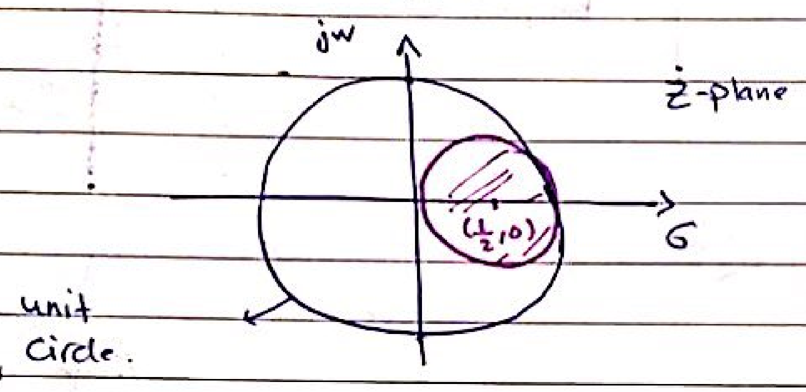
$$Re\left\{\frac{\sigma^2 - \sigma + jw\sigma - jw\sigma + jw + w^2}{\sigma^2 + w^2}\right\}$$

$$Re\left\{\frac{\sigma^2 - \sigma + w^2 + jw}{\sigma^2 + w^2}\right\} < 0$$

$$\boxed{\sigma^2 - \sigma + w^2 < 0}$$ هذا الشرط إلي لازم يتحقق

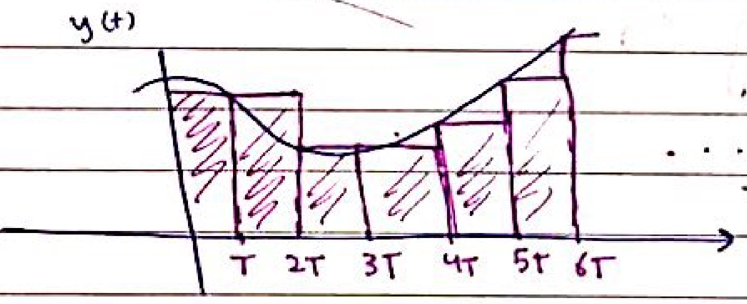
$\sigma^2 - \sigma + \left(\frac{1}{2}\right)^2 + w^2 < \left(\frac{1}{2}\right)^2$ ضفنا $\left(\frac{1}{2}\right)^2$

$\left(\sigma - \frac{1}{2}\right)^2 + w^2 < \left(\frac{1}{2}\right)^2 \Rightarrow$ معادلة دائرة



*** Method 2: Forward Difference Method :-**

take the first one and go forward.



$$\int_{(K-1)T}^{KT} y(t)\,dt = T \cdot y((K-1)T)$$

$$\int_{(K-1)T}^{KT} x(t)\,dt = T \cdot x((K-1)T)\,.$$

$$y(KT) - y((K-1))T = -aT \cdot y((K-1)T) + aT \cdot x((K-1)T)$$

$$y(K) - y(K-1) = -aT\,y(K-1) + aT\,x(K-1)$$

$$Y(z) - z^{-1}Y(z) = -aT\,z^{-1}\,Y(z) + aT\,z^{-1}\,X(z)$$

$$Y(z) * \left[1 - z^{-1} + a z^{-1} T\right] = a T z^{-1} X(z)$$

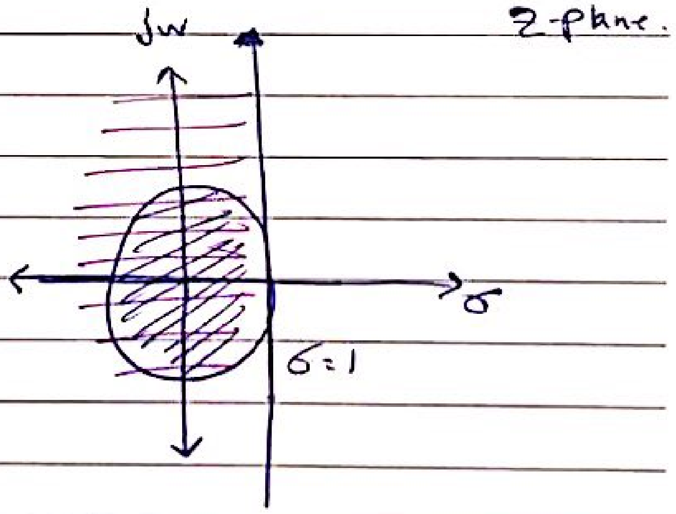
$$H(z) = \frac{Y(z)}{X(z)} = \frac{a T z^{-1}}{1 - z^{-1} + a T z^{-1}} = \frac{a}{\frac{1 - z^{-1}}{T z^{-1}} + a}$$

$$\boxed{S = \frac{1 - z^{-1}}{T z^{-1}}}$$

$$Re\left\{\frac{1 - z^{-1}}{T z^{-1}}\right\} < 0$$

$$\left(Re\left\{\frac{z - 1}{T}\right\} < 0\right) T$$

$$Re\{z - 1\} < 0$$



$$z = \sigma + jw$$

$$Re\{\sigma + jw - 1\} < 0$$

$$\sigma - 1 < 0$$

$$\boxed{\sigma < 1}$$

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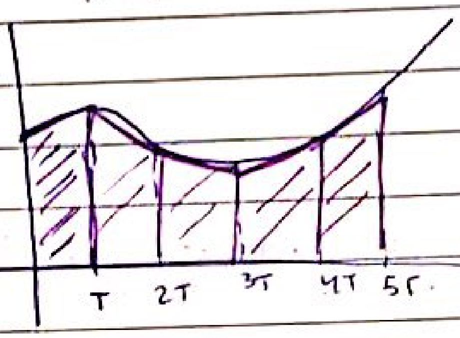
Thu

* Bilinear Transformation Method:

$$y(KT) - y((K-1)T) = -a\int_{(K-1)T}^{KT} y(t)\,dt + a\int_{(K-1)T}^{KT} x(t)\,dt.$$

$$\int_{(K-1)T}^{KT} y(t)\,dt \simeq \frac{1}{2}\left[y(KT) + y((K-1)T)\right]T$$



$$\int_{(K-1)T}^{KT} x(t)\,dt \simeq \frac{1}{2}\left[x(KT) + x((K-1)T)\right]T$$

$$y(KT) - y((K-1)T) = -a\frac{1}{2}T\left[y(KT) + y((K-1)T)\right] + a\frac{1}{2}\left[x(KT) + x((K-1)T)\right]T$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{a}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

$$\boxed{S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$ Bilinear Transformation.

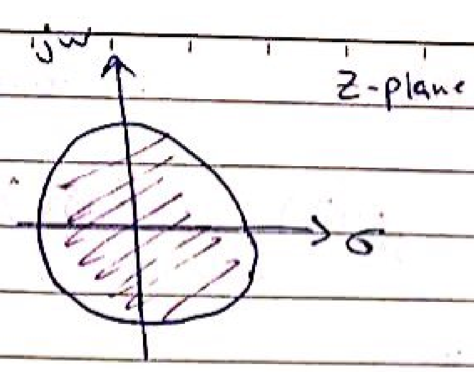
$$Re\left\{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\} < 0 \Rightarrow Re\left\{\left(\frac{z-1}{z+1}\right)\right\} < 0$$

$z = \sigma + jw$

$$Re\left\{\frac{\sigma + jw - 1}{\sigma + jw + 1}\right\} < 0 \Rightarrow Re\left\{\frac{(\sigma - 1) + jw}{(\sigma+1) + jw}\right\} < 0$$

$$\frac{(\sigma-1)+jw}{(\sigma+1)+jw} \times \frac{(\sigma+1)-jw}{(\sigma+1)-jw} \Rightarrow Re\left\{\frac{\sigma^2 - 1 + w^2 + j2w}{(\sigma+1)^2 + w^2}\right\} < 0$$

$$\sigma^2 - 1 + w^2 < 0$$
$$\sigma^2 + w^2 < 1$$



$$\sigma^2 - 1 + w^2 < 0$$
$$\sigma^2 + w^2 < 1$$

* Frequency Prewarping :-

$$j\Omega = \frac{2}{T}\left(\frac{1-e^{-jw}}{1+e^{-jw}}\right)$$

$S = j\Omega$ (imajnery axis)

$Z = e^{jw}$ (unit circle)

take $T = 2$

$$j\Omega = \frac{1-e^{-jw}}{1+e^{-jw}}$$

$$= j\,\frac{\cancel{e^{\frac{-jw}{2}}}}{\cancel{e^{\frac{-jw}{2}}}}\;\frac{\left[e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\right]/2j}{\left[e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\right]/2}$$

$$\boxed{\Omega = \tan\left(\frac{w}{2}\right)}$$

tan (in radians not in degrees)

Ex: Design of a Band pass IIR Filter.

$w_{p_1} = 0.45\pi$ } $R_p = 1dB$
$w_{p_2} = 0.65\pi$

$w_{s_1} = 0.3\pi$ } $R_s = 40dB$
$w_{s_2} = 0.75\pi$

* $w = \Omega T_s$ T_s: sampling period.

Sol: $\hat{\Omega}_{p_1} = \tan\left(\frac{0.45\pi}{2}\right) = 0.854$ } R_p $\boxed{\pi = 180}$

$\hat{\Omega}_{p_2} = \tan\left(\frac{0.65\pi}{2}\right) = 1.632$

$\hat{\Omega}_{s_1} = \tan\left(\frac{0.3\pi}{2}\right) = 0.51$ } R_s

$\hat{\Omega}_{s_2} = \tan\left(\frac{0.75\pi}{2}\right) = 2.4142.$

$BW = \hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}$

$\hat{\Omega}_{p_1} * \hat{\Omega}_{p_2} = \hat{\Omega}_0^2$

⋮   تكملة الحل

$H_{LP}(S) \rightarrow H_D(\hat{S}) \rightarrow H_D(Z).$

⇒ matlab

[BT, AT] = bilinear (num, den, 0.5).
$\downarrow \frac{1}{2}$

[N, wn] = (...)

$H_{LP}(S)$ [B, A] = (...)

~~$H_{LP}(S)$~~ [num, den] = (..)

$H_D(\hat{S})$ ← [BT, AT] = bilinear (num, den, 0.5)

$\frac{2}{T}\left(\frac{1-e^{-jw}}{1+e^{-jw}}\right)$ بياخد ~~..~~ محل كل S وبعوض فيه

* freq z: بيعطي 100 نقطة بين ال $0-\pi$ ويرسم ال magnitude وال phase.

$0 \quad \pi$

* log10, في ماتلاب → يعني log

log ⇒ ln. في ماتلاب يعني