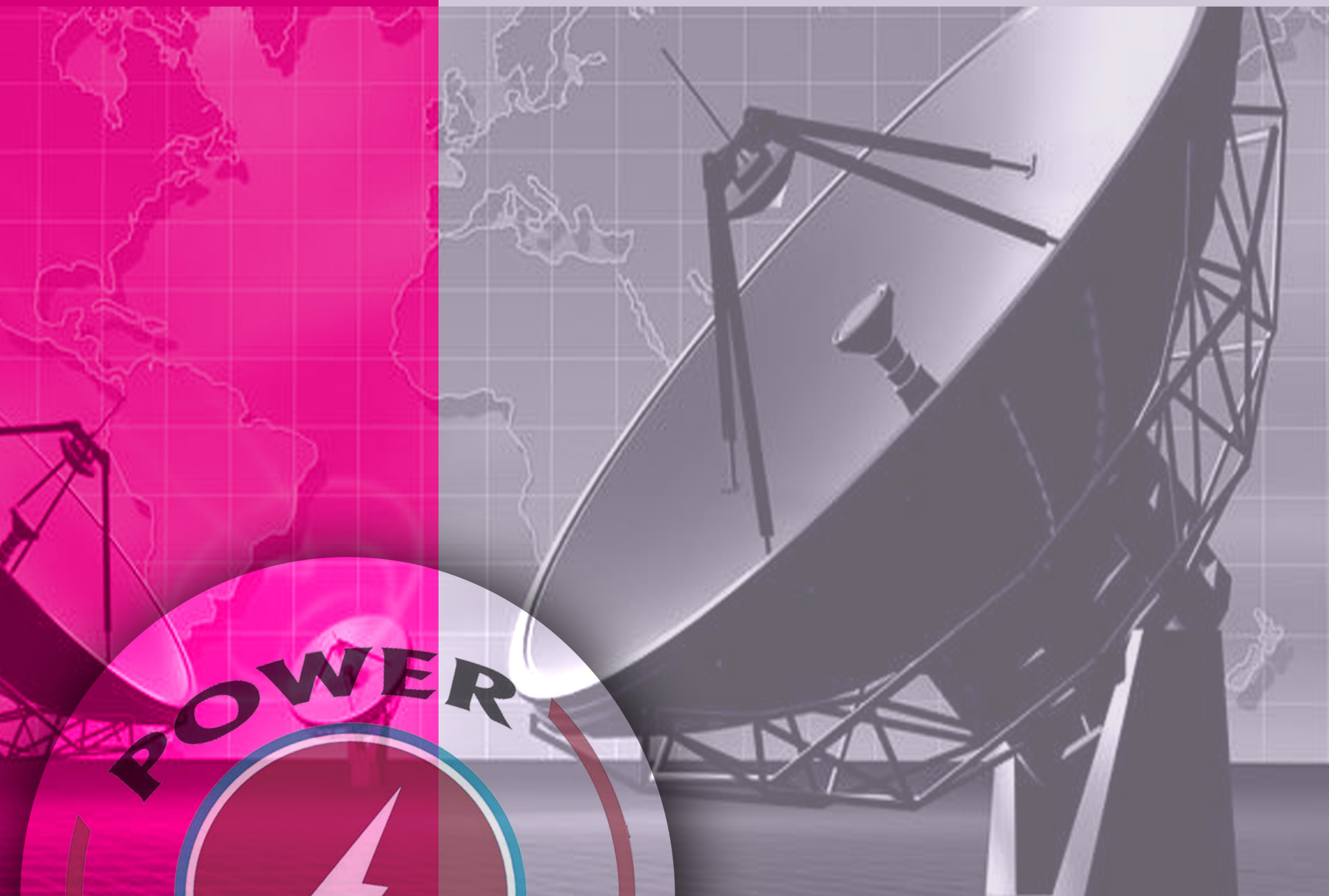
1,2
WEEK

DSP Notebook



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BY : SAUSAN ALMOHTASEB

DSP

Thursday
1/10

* Chapter 2 :

discrete time signals in time domain .

$\underbrace{\text{discrete time signals}}_{\text{sequence}}$

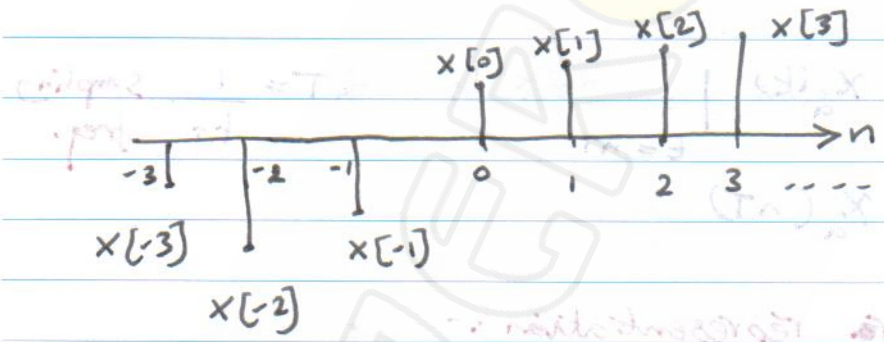
(time) $x(t) \longrightarrow X[k]$ (frequency)

$x[n]$

lower case

integer value

undefined for non-integer values



* set theory :

$x[n] = \{1, \underset{\uparrow}{2}, 5, 6\}$

means $x[0]$

$x[1] = 5$

$x[2] = 6$

$x[-1] = 1$

$x[0] = 2$

* $x[n] = \{ \cdots 1, 2, 3, 5, -1, \cdots \}$
two sided sequence

$x[n] = A\cos(\omega_0 n + \phi) \qquad -\infty < n < \infty$

A and ϕ are known constants

$$x[n] = \begin{cases} \frac{1}{n}, & n > 0 \\ 0, & n \le 0 \end{cases}$$

$x[n] = x_a(t)\Big|_{t = nT} \qquad T = \frac{1}{F_T}$ sampling freq.

$= x_a(nT)$

* Vector representation :-

$$\underline{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \Big\} N$$

vector

$\underline{x} = [x[0] \; x[1] \; \cdots \; x[N-1]]^T$

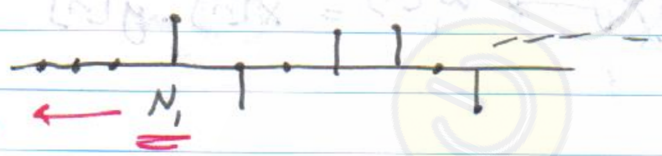
* (capital x @ freq. domain)

* finite sequence or infinite sequence:

$$N_1 \leq n \leq N_2$$

* length of a sequence:

$$L = N_2 - N_1 + 1$$



zeros ←

right sided sequence

if $N_1 \geq 0 \rightarrow$ causal sequence

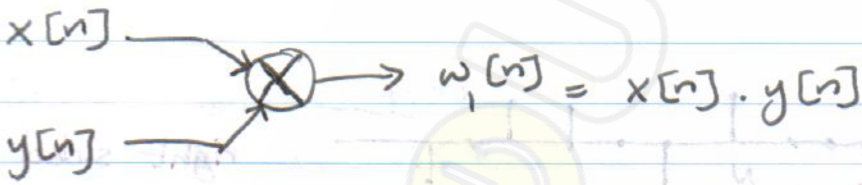
left sided sequence

non causal sequence

* Elementary operations on sequences :-

① **Product** : (mixer) (modulation)

$$w_1[n] = x[n] \cdot y[n]$$



$x[n] = \{1, \underset{\uparrow}{2}, 5, 6\}$

index : ↑

$y[n] = \{-1, 3, \underset{\uparrow}{2}, 1\}$

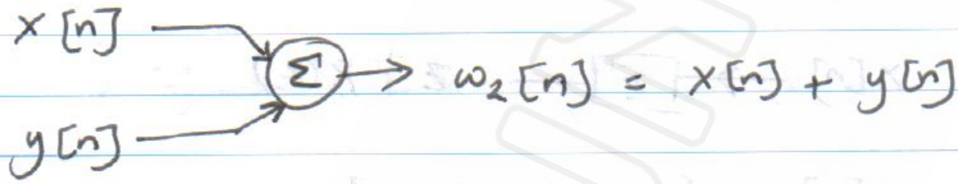
⇓

0	1	2	5	6
-1	3	2	1	0

* $w_1[n] = \{0 \quad 3 \quad 4 \quad 5 \quad 0\}$

② addition:

$$w_2[n] = x[n] + y[n]$$



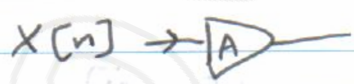
$$w_2[n] = \{-1, 4, \underset{\uparrow}{4}, 6, 6\}$$

③ multiplication by scalar (A):

$$w_3[n] = A \; x[n]$$

$|A| > 1$: Amplification

$|A| < 1$: Attenuation



$$w_3[n] = A\,x[n]$$

(4) Time delay:

$$w_4[n] = x[n-1]$$

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow z^{-1}X(z)$$

$$x[n] = \{1, \underset{\uparrow}{2}, 5, 6\}$$

$\uparrow \rightarrow$ means $x[0]$

$$w_4[-1] = x[-1-1] = x[-2] = 0$$

$$w_4[0] = x[0-1] = x[-1] = +1$$

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-2]$$

$$\downarrow$$

$$x[n-1]$$

$$x[n] \longrightarrow \boxed{z^{-2}} \longrightarrow x[n-2]$$

$$x[n] \longrightarrow \boxed{z^{-N}} \longrightarrow x[n-N]$$

(5) Time advance:

$$w_5[n] = x[n+1]$$

$$x[n] \longrightarrow \boxed{z} \longrightarrow w_5[n] = x[n+1]$$

predictor

lecture 2 4/10 Sunday

⑥ Pick off point :

$$x[n] \rightarrow \begin{cases} x[n] \\ x[n] \end{cases}$$

Ex: $y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + b_1 y[n-1] + b_2 y[n-2]$

difference equation.

$\ddot{y} + a_1 \dot{y} + a_2 y = a_0 x + a_1 x$ ---

* Strength of a sequence :

  * defined by a norm of sequence

→ $\mathcal{L}$-p norm $= \left(\sum_{n=-\infty}^{\infty} |x[n]|^p \right)^{1/p} = \|x\|_p$

→ $\mathcal{L}$-2 norm $= \|x\|_2 = \left(\sum_{-\infty}^{\infty} |x[n]|^2 \right)^{1/2}$

$= \sqrt{\sum |x[n]|^2}$

→ $\mathcal{L}$-1 norm $= \|x\|_1 = \sum_{-\infty}^{\infty} |x[n]|$

→ $\mathcal{L}$-∞ norm $= \|x\|_\infty = |x|_{max}$

* rms $= \|x[n]\|_2 / \sqrt{N}$

N length sequence

* Classifications of sequences :-

① Based on symmetry :

* a sequence is conjugate symmetric if
$$x[n] = x^*[-n]$$

$\Rightarrow x_{cs}[n]$

if the sequence is real: $x[n] = x[-n]$ even symm.

$\Rightarrow x_{even}[n]$

* a sequence is conjugate antisymmetric if
$$x[n] = -x^*[-n]$$

$\Rightarrow x_{ca}[n]$

if the sequence is real
$$x[n] = -x[-n] \quad \text{odd symm.}$$

$\Rightarrow x_{odd}[n]$

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

$$x_{cs}[n] = \frac{x[n] + x^*[-n]}{2}$$

$$x_{ca}[n] = \frac{x[n] - x^*[-n]}{2}$$

* **Ex:** $g[n] = \{0,\ 1+j4,\ -2+j3,\ 4-2j,\ -5-j6,\ -j2,\ 3\}$

find $g_{cs}[n]$ and $g_{ca}[n]$?

→ $g^*[n] = \{0,\ 1-j4,\ -2-j3,\ 4+2j,\ -5+j6,\ j2,\ 3\}$

→ $g^*[-n] = \{3,\ j2,\ -5+j6,\ 4+j2,\ -2-j3,\ 1-j4,\ 0\}$

→ $g_{cs}[n] = \{1.5,\ 0.5+j3,\ -3.5+j4.5,\ 4,\ -3.5-j4.5,\ 0.5-j3,\ 1.5\}$

→ $g_{ca}[n] =$

* **Ex:** $X[n] = \{1, 2, 3\}$

$$X_{even}[n] = \frac{x[n] + x[-n]}{2}$$

$$X_{odd}[n] = \frac{x[n] \cancel{+} - x[-n]}{2}$$

$$X[n] = \{0, 0, 1, 2, 3\}$$

$$X[-n] = \{3, 2, 1, 0, 0\}$$

$$X_{even}[n] = \{1.5, 1, 1, 1, 1.5\}$$

$$X_{odd}[n] = \{-1.5, -1, 0, 1, 1.5\}$$

② ~~periodic~~

6/10
Tuesday

② Periodic and aperiodic sequences :-

$$\tilde{x}[n] = x[n + kN] \quad \text{for all } n$$

k is any integer

N is ~~the~~ +ve integer (period)

N_f the smallest N satisfies the above equation
↓
(the fundamental period)

* $\tilde{x}_a[n]$ & $\tilde{x}_b[n]$ are periodic sequences with periods N_a and N_b, respectively

* $\tilde{x}_a[n] + \tilde{x}_b[n]$ will be periodic with $N = LCM(N_a, N_b)$
(least common multiple)

* $LCM(N_a, N_b) = \dfrac{N_a N_b}{GCD(N_a, N_b)}$

greatest common divider

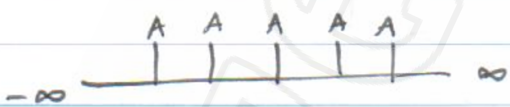
$N_a = 21 \longrightarrow 7 * 3$

$N_b = 12 \longrightarrow 2 * 2 * 3$

$N = 7 * 4 * 3 = 84$

$\Rightarrow \dfrac{21 * 12}{3} = \dfrac{7 * 3 * 2 * 2 * \cancel{3}}{\cancel{3}} = 84$

for $\tilde{x}_a[n]$, $\tilde{x}_b[n]$ the same equations apply.



$N_b = 1$

③ Energy and Power sequences :-

* The energy in a sequence :

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

* finite sequences :-

$$x[n] = \{1, 2, 3\}$$

$$\mathcal{E}_x = 1^2 + 2^2 + 3^2 = 14 \text{ J}$$

* for infinite sequences :-

Ex1 : $x[n] = \begin{cases} 1/n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ J}$$

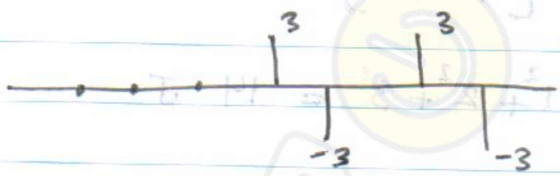
Ex2 : $x[n] = \begin{cases} 1/\sqrt{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$

$$\mathcal{E}_x = \sum_{n=1}^{\infty} 1/n \Rightarrow \infty \text{ diverge}$$

* Average power in a signal:

→ $P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |x[n]|^2$

* $x[n] = \begin{cases} 3(-1)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$



→ $P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=0}^{k} \left(3(-1)^n\right)^2$

$= \lim_{k\to\infty} \frac{9}{2k+1} \sum_{n=0}^{k} (-1)^{2n}$

$= \lim_{k\to\infty} \frac{9(k+1)}{2k+1} = \frac{\text{coeff. of highest order}}{\text{coef of highest order}}$

$= \lim \frac{9k+9}{2k+1} = 4.5 \ W$

→ $P_x = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$: periodical sequence

* Other Classifications :-

→ A sequence is bounded if:

$$|x[n]| \leq B_x < \infty$$

→ A sequence is absolutely - summable if:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

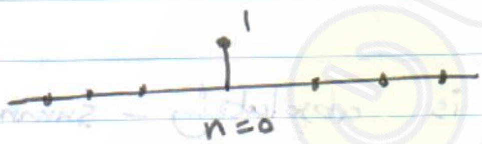
→ A sequence is squre - summable if :

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

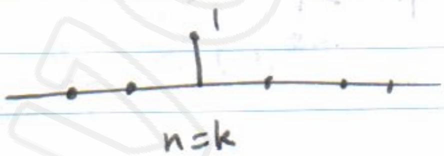
* Typical sequences:

① unit sample sequence (Discrete time impulse) (Unit impulse)

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$



* $x[n] = \{1 \quad \underset{\uparrow \text{index}}{2} \quad 3\}$

$+ \; 3\,\delta[n] : \{3\}$

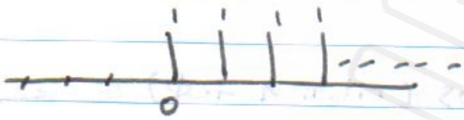
$\Rightarrow \{1 \quad 5 \quad 3\}$

if without index : $\{4 \quad 2 \quad 3\}$

Thursday
8/10

② Unit step sequence:

$$\mu[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$\mu[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$$

③ Sinosoidal sequences:

$$x[n] = A\cos(\omega_0 n + \phi) \text{ ; for all } n$$

* $x[n] = x[n+N]$ periodicity.

$$= A\cos(\omega_0(n+N) + \phi)$$

$$= A\cos((\omega_0 n + \phi) + \omega_0 N)$$

$$= A\left[\cos(\omega_0 n + \phi)\underbrace{\cancel{\cos(\omega_0 N)}}_{1} - \sin(\omega_0 n + \phi)\underbrace{\cancel{\sin(\omega_0 N)}}_{0}\right]$$

$(\omega_0 N = 2\pi r) \rightarrow$

$\Rightarrow A\cos(\omega_0 n + \phi)$

$$\boxed{\frac{2\pi}{\omega_0} = \frac{N}{r}}$$ rational number

* irrational numbers : $\pi, \sqrt{3}, \sqrt{5}$

$w_0 = 0.1$

$x[n] = A\cos(0.1n + \phi) \Rightarrow \frac{2\pi}{0.1} = \frac{N}{r}$ irrational

$\therefore$ not periodic

$w_0 = 0.1\pi$

$x[n] = A\cos(0.1\pi n + \phi)$

$\Downarrow$

$\frac{2\pi}{0.1\pi} = \frac{20}{1}$ periodic

* if $\begin{cases} \frac{12}{4} = \frac{3}{1} & N_f = 3 \quad \text{simplify} \\ \frac{25}{15} = \frac{5}{3} \end{cases}$

* $0 < w_1 < \pi$

$w_2 = w_1 + 2\pi k$

$x_1[n] = A\cos(w_1 n + \phi)$

$x_2[n] = A\cos(w_2 n + \phi)$

$= A\cos((w_1 + 2\pi k)n + \phi)$

$= A\cos(w_1 n + \phi) = x_1[n]$

* $0 < \omega_1 < \pi$

$\pi < \omega_2 < 2\pi$

$\omega_2 = 2\pi - \omega_1$

* $0 \rightarrow \pi \longrightarrow 2\pi$

low freq. $\longrightarrow$ high frequency $\rightarrow$ low

$\begin{cases} 0.1\pi & \text{low} \\ 0.9\pi & \text{high} \\ 1.9\pi & \text{low} \end{cases}$

* low : around (0) , high : around (π)

* Sampling Proces :-

$x_a(t) = A\cos(\Omega_0 t + \phi)$ $\Rightarrow$ $A\cos(\Omega_0 nT + \phi)$

$A\cos(\underbrace{\Omega_0 T}_{w_0} n + \phi)$

Ω_0 : rad/sec

w_0 : rad/samples

$\boxed{w_0 = \Omega_0 T}$

→ $x[n] = x_a(t)\Big|_{t=nT}$; where $T = \dfrac{1}{F_T}$

F_T : Sampling frequency.

$\Omega_T : 2\pi F_T$

Ex :

$g_1(t) \Rightarrow 3$ Hz

$g_2(t) \Rightarrow 7$ Hz

$g_3(t) \Rightarrow 13$ Hz

$F_T = 10 \Rightarrow T = 0.1$ sec.

$g_1(t) = \cos(6\pi t) \rightarrow 2\pi f t = 2\pi \times 3\, t$

$g_2(t) = \cos(14\pi t)$

$g_3(t) = \cos(26\pi t)$

$g_1[n] = \cos(0.6\pi n) \rightarrow$

* $\boxed{\omega_0 = \Omega_0 T}$
$= 6\pi \times 0.1$
$= 0.6\pi$

$$
\begin{aligned}
g_2[n] &= \cos(1.4\pi n) \\
&= \cos((2-0.6)\pi n) \\
&= \cos(2\pi n - 0.6\pi n) \\
&= \cos 2\pi n \cos 0.6\pi n + \sin 2\pi n \sin 0.6\pi n \\
&= \cos(0.6\pi n) \\
&= g_1[n]
\end{aligned}
$$

$$
\begin{aligned}
g_3[n] &= \cos(2.6\pi n) \\
&= \cos((2+0.6)\pi n) \\
&= \cos(0.6\pi n) \\
&= g_1[n]
\end{aligned}
$$

* $\boxed{\pm(\Omega_0 t + \phi) + k\,\Omega_T t \quad k = 0, \pm 1, \pm 2 \ldots}$

Sunday
11/10

* Discrete time systems :-

$$x[n] \rightarrow \boxed{T} \rightarrow y[n]$$

$$x[n] \xrightarrow{T} y[n]$$

$$T[x[n]] = y[n]$$

* Classifications of discrete time (DT) systems :-

$\boxed{1}$ static or Dynamic DT system :-

→ (memoryless) : static

→ (system with memory) : Dynamic

a) $y[n] = a\,x[n]$ static

b) $y[n] = n\,x[n] + b\,x^2[n]$ static

c) $y[n] = x[n] + 3x[n-1]$ dynamic

$\boxed{2}$ time invarient and time varient systems :
(shift invarient)

$$x[n] \xrightarrow{T} y[n]$$

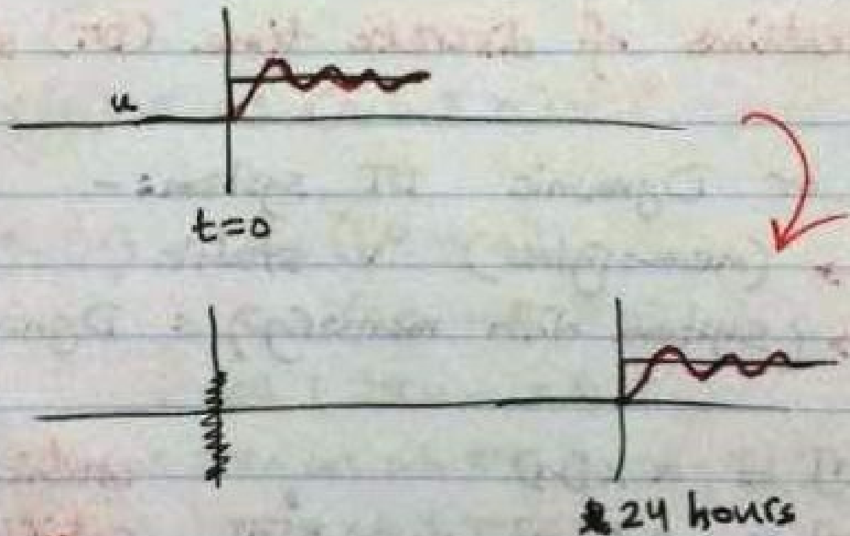
$$x[n-k] \xrightarrow{T} y[n-k]$$

1) apply $x[n] \xrightarrow{T} y[n]$
2) apply $x_1[n] = x[n-k]$

$$x_1[n] \xrightarrow{T} y_1[n]$$

3) Shift $y[n]$ by $k \rightarrow y[n-k]$
4) If $y[n-k] = y_1[n] \rightarrow$ time invarient

Ex:



shift invarient sys.

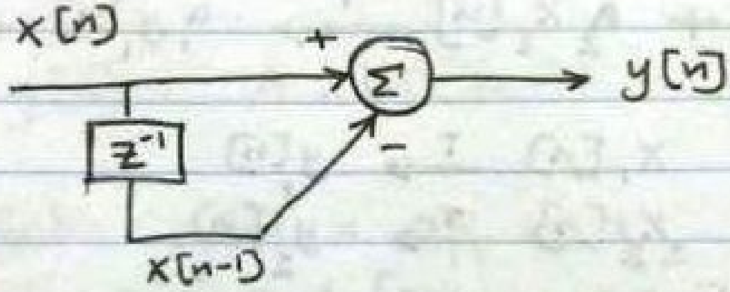
* if the system changed after 24 hours
→ varient

Ex:

1) $y[n] = x[n] - x[n-1]$ — differentiator
2) $y[n] = n\,x[n]$ — multiplier
3) $y[n] = x[-n]$ — folder
4) $y[n] = x[n]\cos(\omega_0 n)$ — modulator

sol:

①



$$y[n] = x[n] - x[n-1]$$ invarient

$$x_1[n] = x[n-k]$$

$$y_1[n] = x_1[n] - x_1[n-1]$$

$$y_1[n] = x[n-k] - x[n-k-1]$$

$$y[n-k] = x[n-k] - x(n-k-1]$$

② $y[n] = n\,x[n]$

$$x_1[n] = x[n-k]$$ varient

$$y_1[n] = n\,x_1[n]$$

$$y_1[n] = n\,x[n-k]$$

$$y[n-k] = (n-k)\,x[n-k]$$

$$= n\,x[n-k] - k\,x[n-k]$$

do 3, 4!

3 linear and non linear systems:

$$T\left[a_1 x_1[n] + a_2 x_2[n]\right] = ?$$

$$a_1 T\left[x_1[n]\right] + a_2 T\left[x_2[n]\right]$$

$$a_1 x_1[n] + a_2 x_2[n] \xrightarrow{T} a_1 y_1[n] + a_2 y_2[n]$$

where:

$$x_1[n] \xrightarrow{T} y_1[n]$$
$$x_2[n] \xrightarrow{T} y_2[n]$$

* Ex:

a) $y[n] = n\,x[n]$
b) $y[n] = x[n^2]$
c) $y[n] = x^2[n]$
d) $y[n] = A\,x[n] + B$

sol: a) $y[n] = n\,x[n]$

let: $x[n] = a_1 x_1[n] + a_2 x_2[n]$

$$y[n] = n\left[a_1 x_1[n] + a_2 x_2[n]\right]$$
$$y[n] = a_1 n\,x_1[n] + a_2 n\,x_2[n]$$

a_1 * $y_1[n] = n\,x_1[n]$ * a_1
a_2 * $y_2[n] = n\,x_2[n]$ * a_2

$$a_1 y_1[n] = a_1 n\, x_1[n]$$

$\Rightarrow$ $$a_2 y_2[n] = a_2 n\, x_2[n] \quad +$$

$$= a_1 n x_1[n] + a_2 n x_2[n]$$

$\therefore$ linear system

d) $$y[n] = A\,x[n] + B$$

let: $$x[n] = a_1 x_1 + a_2 x_2$$

$$y[n] = A\,[a_1 x_1 + a_2 x_2] + B$$

$$= a_1 A x_1 + a_2 A x_2 + B$$

$$(y_1[n] = A\,x_1[n] + B) * a_1$$

$$(y_2[n] = A\,x_2[n] + B) * a_2$$

$\Rightarrow$ $$= a_1 A x_1 + a_2 A x_2 + (a_1 + a_2)\,B$$

$\therefore$ non linear sys.

* do ⓑ, ⓒ

↙ linear   ↙ non linear

(b) $x[n] = a_1 x_1[n] + a_2 x_2[n]$

$y[n] = x[n^2]$

$= a_1 x_1[n^2] + a_2 x_2[n^2]$

$(y_1[n] = x_1[n^2]) * a_1$

$(y_2[n] = x_2[n^2]) * a_2$

$\vdots$

linear

Tuesday
13/10

* Causal and non causal systems :-

$$y[n] = T\,[x[n],\ x[n-1], \ldots]$$

Ex :

1) $y[n] = x[n] - x[n-1]$ causal
2) $y[n] = x[2n]$ non causal
3) $y[n] = x[-n]$ non causal
4) $y[n] = x[n] + 3x[n+4]$ non causal

2) $y[1] = x[2] \rightarrow$ non causal
3) $y[-1] = x[1] \rightarrow$ non causal

* Stability (stable and unstable) :

BIBO stability criterion.
(bounded input bounded output)

→ If you apply $|x[n]| \leq B_x < \infty$

$\Rightarrow |y[n]| \leq B_y < \infty$

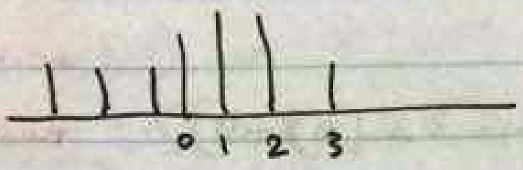
→ The system is BIBO stable.
otherwise, unstable!

Ex:

1) $y[n] = e^{x[n]}$

2) $y[n] = \sum_{k=0}^{n} x[k]$ (accumulator)

3) $y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$ (M-length moving avg)



$$y[2] = \frac{1}{3} \sum_{k=0}^{2} x[2-k]$$

$$= \frac{1}{3} \left[x[2] + x[1] + x[0] \right]$$

sol: 1

$$|y[n]| = |e^{x[n]}| \leq e^{|x[n]|} \leq \underbrace{e^{B_x}}_{B_y}$$

عادي تكون + ، 0 ، −

دائمًا موجبة

B_x is finite

$\therefore B_y$ finite

$$y[n] \leq B_y \leq \infty$$

$\therefore$ BIBO stable system

(2) $y[n] = \left| \sum_{k=0}^{n} x[k] \right| \leq \sum_{k=0}^{n} |x[k]| \leq \sum_{k=0}^{n} B_x$

$|y[n]| \leq (n+1) B_x$

$\therefore$ unstable system

(3) $|y[n]| = \left| \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \right|$

$= \frac{1}{M} \left| \sum_{k=0}^{M-1} x[n-k] \right| \leq \frac{1}{M} \sum_{k=0}^{M-1} |x[n-k]|$

$\leq \frac{1}{M} \sum_{k=0}^{M-1} B_x$

$\leq \cancel{\frac{1}{M}} \cancel{M} B_x$

By

$B_x = B_y \rightarrow$ BIBO stable system.

Passive and lossless systems:-

$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leqslant \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Ex: $y[n] = \alpha\, x[n-N]$, where N is a positive integer

$$\Rightarrow \sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{-\infty}^{\infty} |\alpha\, x[n-N]|^2$$

$$= |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n-N]|^2$$

let $m = n-N$

$$= |\alpha|^2 \sum_{m=-\infty}^{\infty} |x[m]|^2$$

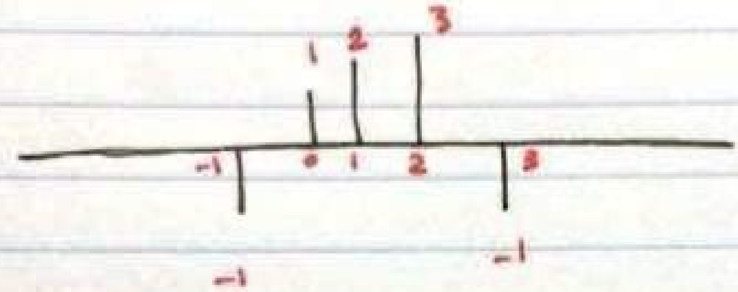
$m = -\infty - N$
$= -\infty$

if: $|\alpha|^2 = 1 \longrightarrow$ lossless

$0 < |\alpha|^2 < 1 \longrightarrow$ passive

$\alpha > 1 \longrightarrow$ active

* LTI systems :
↳ (linear time invarient)



$x[n] = \{-1, \underset{\uparrow}{1}, 2, 3, -1\}$

$\Rightarrow x[n] = -1\,\delta[n+1] + 1\,\delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2] - 1\,\delta[n-3]$

↳ representation using unit sample

$x[n] = x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1]$
$\cancel{+3\delta} + x[2]\,\delta[n-2) - x[3]\,\delta[n-3)$

$\Rightarrow x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$ general representation

Sunday
18/10

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$x[n] \rightarrow \boxed{T} \rightarrow y[n]$$

$$y[n] = T\left[\sum_{-\infty}^{\infty} x[k]\,\delta[n-k]\right]$$

* for linear system:

$$= \sum_{k=-\infty}^{\infty} x(k]\,T[\delta[n-k]]$$

$$x[n] = \delta[n] \;—\boxed{T}—\; h[n]$$

impulse response

$$x[n] = \mu[n] \;—\boxed{T}—\; s(n]$$

step response

$$\delta[n-k] \;—\boxed{T}—\; h[n,k]$$

* for time-invarient system:

$$\delta[n-k] \xrightarrow{T} h[n-k]$$

$$y[n] = \sum_{-\infty}^{\infty} x[k]\,h(n-k]$$

convolution sum

$$y[n] = x[n] \circledast h[n]$$

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$\Rightarrow$ let: $m = n-k$

$k = n-m$

$$y[n] = \sum_{m=-\infty}^{\infty} x[n-m] \cdot h[m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

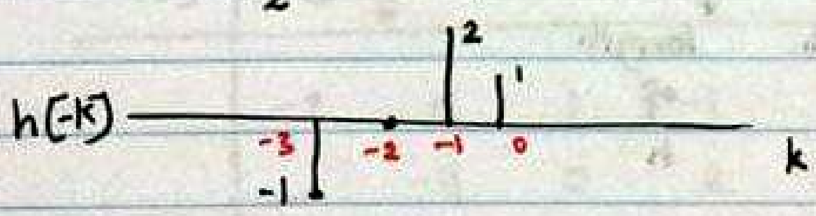
$$y[n] = h[n] \circledast x[n]$$

Ex:- ① $x[n] = \{-2, 0, 1, -1, 3\}$

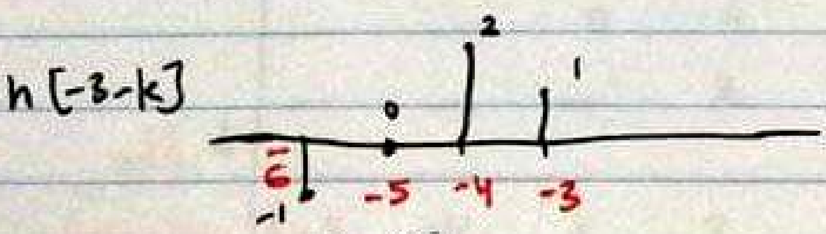
$h[n] = \{1, 2, 0, -1\}$

find $y[n]$?

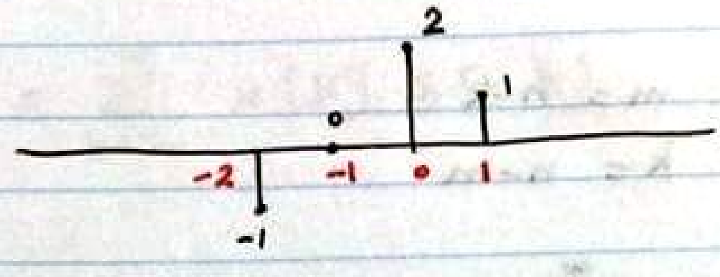
Sol:



(folding)



$h[1-k]$



$y[2]: \longrightarrow$ shift by 2 $\Rightarrow y = 1 \times 1 = 1$

$y[3]: \rightarrow$ shift $\Rightarrow y = -1 \times 1 + 2 \times 1 + -1 \times -2$
$= 3$

$y[4]: \rightarrow$ shift $\Rightarrow y = 3 \times 1 + 2 \times -1 = 1$

$y[5] = 5$

$y[6] = -1 \times -1 = 1$

$y[7] = 3$

$y[8] = 0$

$\Rightarrow y[n] = \{-2, \bar{4}, 1, 3, 1, 5, 1, -3\}$

* the length of the output sequence:

$$\underbrace{N_1}_{\substack{\text{length} \\ \text{of} \\ x}} + \underbrace{N_2}_{\substack{\text{length} \\ \text{of} \\ h}} - 1$$

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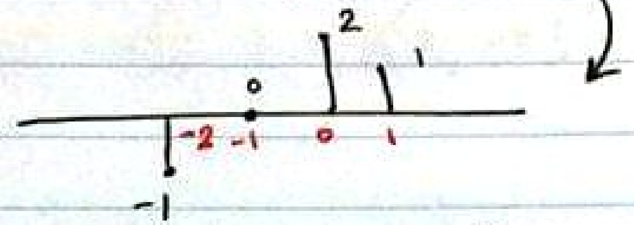
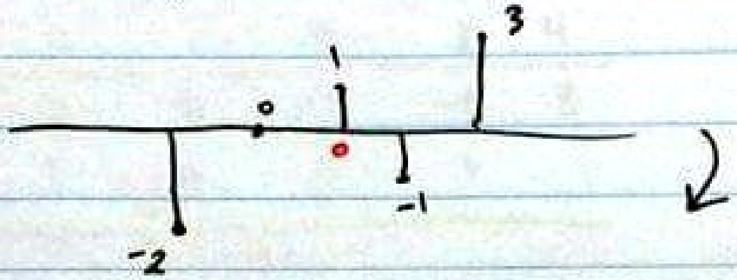
* EX 2 :

* if : $h[n] = \{1, \underset{\uparrow}{2}, 0, -1\}$

h[-k]



x[k]



$x[k] = \{-2, 0, \underset{\uparrow}{1}, -1, 3\}$

* do it

* Tabular Method for Convolution:

①

n	0	1	2	3	4	5	6	7
x[n]	-2	0	1	-1	3			
h[n]	1	2	0	-1				
	-2	0	1	-1	3			
	—	-4	0	2	-2	6		
	—	—	0	0	0	0		
	—	—	—	2	0	-1	1	-3
add ↓								
y[n] =	{-2	-4	1	3	1	5	1	-3}

Tuesday
20/10

* $x[n] = \{-2, 0, 1, -1, 3\}$

$h[n] = \{1, 2, 0, -1\}$

$y[n] = \{-2, -4, 1, 3, 1, 5, 1, -3\}$

n	0	1	2	3	4			
	-2	0	+1	-1	3			
	1	2	0	1				
		index						
⇒	-2	-4	1	3	1	5	1	-3

* Causal linear time invarient of system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$y[n_0] = T\,[x[n_0],\ x[n_0-1],\ x[n_0-2], \ldots]$$

$$\Rightarrow y[n_0] = \sum_{k=-\infty}^{\infty} h[k]\, x[n_0-k]$$

$$= \sum_{k=0}^{\infty} h[k]\, x[n_0-k] + \sum_{k=-\infty}^{-1} h[k]\, x[n_0-k]$$

↳

$$\Rightarrow y[n_0] = \left[h[0]\,x[n_0] + h[1]\,x[n_0-1] + h[2]\,x[n_0-2] + \cdots\right] + \left[h[-1]\,x[n_0+1] + h[-2]\,x[n_0+2] + \cdots\right]$$

$$\boxed{h[n] = 0 \qquad n < 0}$$

* Stable linear time invarient system:

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,\underline{|x[n-k]|}$$

$$\leq \left(\sum_{k=-\infty}^{\infty} |h[k]|\,\underline{B_x}\right)$$

$$= \left(\underbrace{B_x}_{\text{constant}} \underbrace{\sum_{-\infty}^{\infty} |h[k]|}_{S_h}\right)$$

$$\Rightarrow S_h = \sum_{-\infty}^{\infty} |h[k]| < \infty \quad (\text{finite})$$

* the impulse response $h[n]$ must be absolutely summable (BIBO stable)

* Discrete time fourier transform (DTFT)

* discrete fourier transform (DFT) (fft)

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$x[n] = \{2, -1, \underset{\uparrow}{3}, 5, 6, 4\}$$

-2 -1 0 1 2 3 ← index (n)

$$X(e^{jw}) = 2e^{j2w} - 1e^{jw} + 3 + 5e^{-jw} + 6e^{-j2w} + 4e^{-j3w}$$

* $X(e^{jw})$ is periodic with a period of 2π

$$X(e^{j(w+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(w+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} \cdot \underbrace{e^{-j2\pi kn}}_{1}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} = X(e^{jw})$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) \cdot e^{jwn} \cdot dw$$

* $\boxed{x[n] \xleftrightarrow{F} X(e^{jw})}$

$\Rightarrow X(e^{jw}) = F[x[n]]$, $\Rightarrow x[n] = F^{-1}[X(e^{jw})]$

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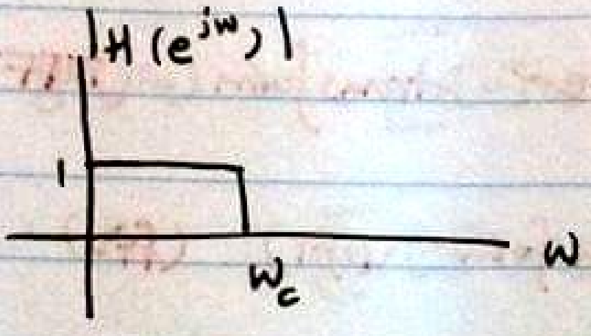
8

*



(LPF)

→ ideal LPF ⇒ $h[n]$

1^{st} exam

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3RD WEEK

DSP Notebook

DR . DEYA' ABULNADIE
BY : SAUSAN ALMOHTASEB

Thursday
22/10

* Discrete-time fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad -\infty < \omega < \infty$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$$

$\Rightarrow$ $X(e^{j\omega})$: 1) continuous in ω
2) periodic with a period of 2π
3) Complex quantity.

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j X_{im}(e^{j\omega})$$

$$X(e^{j\omega}) = |X(e^{j\omega})| \cdot e^{j\theta(e^{j\omega})}$$

* Geometric series:

$$S_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \cdots + r^N$$

$$S_{N+1} = \sum_{n=0}^{N+1} r^n = 1 + r + r^2 + \cdots + r^N + r^{N+1}$$

$$S_{N+1} = 1 + r\,[\underbrace{1 + r + r^2 + \cdots + r^N}_{S_N}]$$

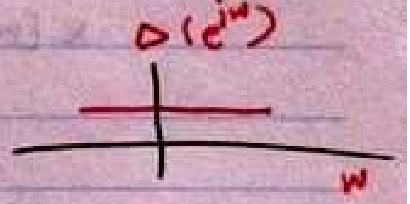
$$S_{N+1} = 1 + r S_N$$
$$= S_N + r^{N+1}$$

$$S_N(1-r) = 1 - r^{N+1} \Rightarrow \boxed{S_N = \frac{1-r^{N+1}}{1-r}}$$

⇒ if $|r|<1$ and $N \to \infty$

$$S_\infty = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

* **Ex:** $x[n] = \delta[n]$



$$\Delta(e^{jw}) = X(e^{jw}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-jwn}$$

$$= e^{jw(0)} = 1 \qquad -\infty < w < \infty$$

* **Ex:** $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$

$$X(e^{jw}) = \sum_{n=\cancel{-\infty}\ 0}^{\infty} \alpha^n \underbrace{\cancel{\mu[n]}}_{1} e^{-jwn}$$

$$= \sum_{0}^{\infty} \alpha^n e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-jw})^n$$

$$X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} * \frac{1-\alpha e^{jw}}{1-\alpha e^{jw}}$$ ← complex conjugate

$$X(e^{jw}) = \frac{1-\alpha e^{jw}}{1-\alpha e^{jw} - \alpha e^{-jw} + \alpha^2}$$

$$= \frac{1-\alpha[\cos w + j \sin w]}{1 - 2\alpha \cos w + \alpha^2}$$

$$\Rightarrow \frac{[1-\alpha\cos w]-j\alpha\sin w}{1-2\alpha\cos w+\alpha^2}$$

$$X(e^{jw}) = \frac{1-\alpha\cos w}{1-2\alpha\cos w+\alpha^2} = \frac{-\alpha\sin w}{1-2\alpha\cos w+\alpha^2}$$

$$\rightarrow |X(e^{jw})|^2 = X(e^{jw})\cdot X^*(e^{jw})$$

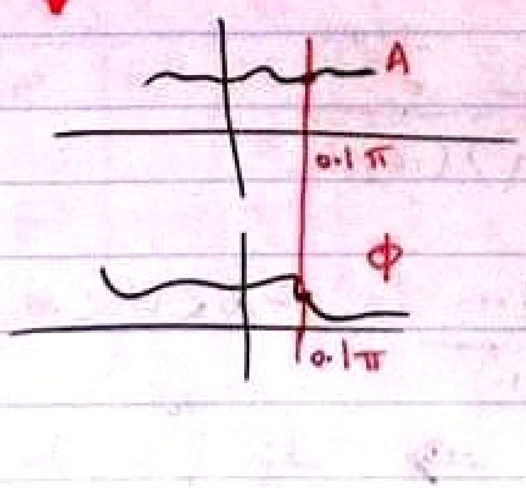
$$= \frac{1}{1-2\alpha\cos w+\alpha^2}$$

$$\rightarrow |X(e^{jw})| = \frac{1}{\sqrt{1-2\alpha\cos w+\alpha^2}}$$

$$\rightarrow \theta(e^{jw}) = \tan^{-1}\left(-\frac{\alpha\sin w}{1-\alpha\cos w}\right)$$

$\rightarrow X(e^{j0.1\pi})$ < mag, phase $= A\angle\phi$

$\downarrow$

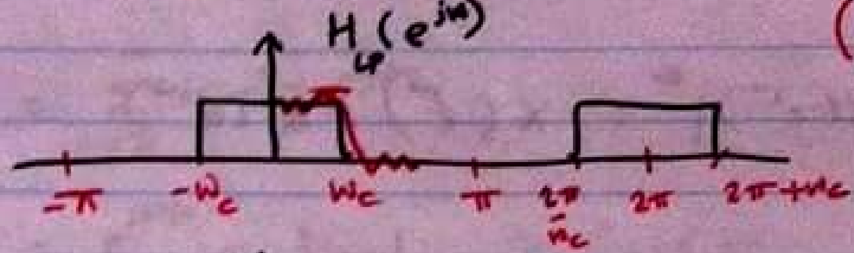


Ex: $H_{lp}(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$

find $h_{lp}[n]$?

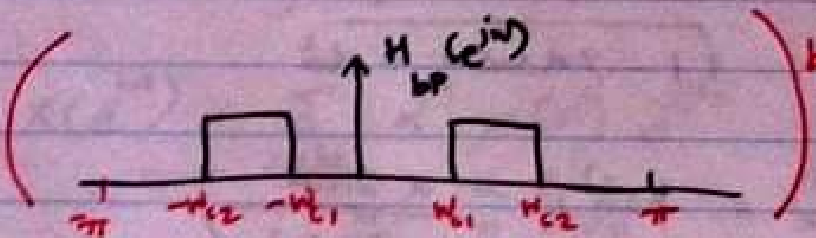
(low pass)

(low pass)



$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n} \cdot d\omega$$

band pass



$$h_{lp}[n] = \frac{1}{2\pi} \left. \frac{e^{j\omega n}}{jn} \right|_{-\omega_c}^{\omega_c}$$

$$= \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{(\pi n)(2j)}$$

$$\boxed{h_{lp}[n]} = \frac{\sin \omega_c n}{\pi n} \qquad -\infty < n < \infty$$

$$* \ \boxed{h_{lp}[0]} = \frac{\omega_c}{\pi} \Rightarrow \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 . 1 . \, d\omega = \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

$$h_{lp}[n] = \begin{cases} \frac{\sin \omega_c n}{\pi n} & n \neq 0 \\ \omega_c / \pi & n = 0 \end{cases}$$

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Sunday
25/10

* Discrete time fourier transform theorems :-

$$g[n] \xleftrightarrow{F} G(e^{jw})$$
$$h[n] \xleftrightarrow{F} H(e^{jw})$$

① linearity theorem:

$$\alpha g[n] + \beta h[n] \xleftrightarrow{F} \alpha G(e^{jw}) + \beta H(e^{jw})$$

② time - reversal theorem:

$$g[-n] \xleftrightarrow{F} G(e^{-jw})$$

* $\sum_{n=-\infty}^{\infty} g[-n]\, e^{-jwn}$

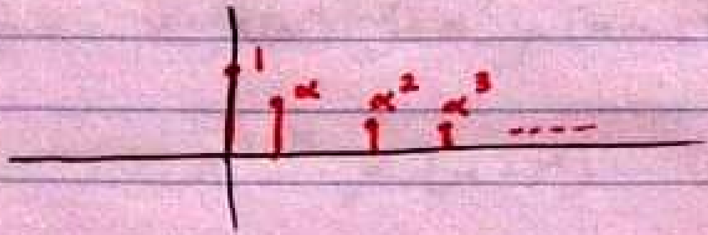
let $(m = -n) \rightarrow \sum_{m=-\infty}^{\infty} g[m]\, e^{-(-jwm)} = G(e^{-jw})$

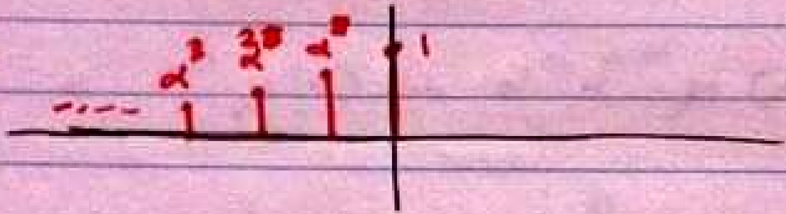
→ $x[n] = \alpha^{n} \mu[n] \quad ; \ |\alpha| < 1$

$g[n] = x[-n] = \alpha^{-n} \mu[-n]$

$\alpha^n u[n]$

$\Rightarrow$



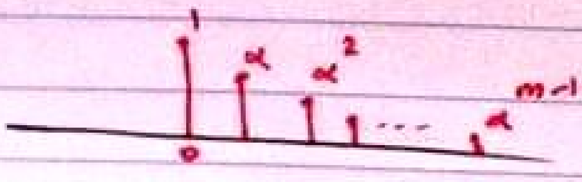


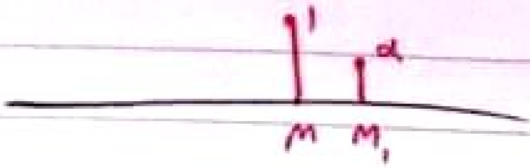
$$G(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

③ time-shifting theorem :-

$$g[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} G(e^{j\omega})$$

ex: $y[n] = \begin{cases} \alpha^n & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$ $\quad |\alpha| < 1$





$$y[n] = \alpha^n \mu[n] - \alpha^n \mu[n-M]$$

$$= \alpha^{n-M} \mu[n-M]$$

$$= \alpha^n \mu[n] - \alpha^n \alpha^{-M} \alpha^M \mu[n-M]$$

$$= \alpha^n \mu[n] - \alpha^M \alpha^{n-M} \mu[n-M]$$

$$= \frac{1}{1-\alpha e^{-jw}} - \frac{\alpha^M}{1-\alpha e^{-jw}}$$

$$Y(e^{jw}) = \frac{1-\alpha^m e^{-jwM}}{1-\alpha e^{-jw}}$$

$$\sum_{n=0}^{M} r^n = \frac{1-r^M}{1-r}$$

$$= \sum_{n=0}^{M-1} \alpha^n e^{-jwn} = \sum_{n=0}^{M-1} (\alpha e^{-jw})^n$$

$$= \frac{1-(\alpha e^{-jw})^M}{1-\alpha e^{-jw}}$$

* **Ex:** $d_0\, v[n] + d_1\, v[n-1] = P_0\, \delta[n] + P_1\, \delta[n-1]$

find $V(e^{jw})$?

$\boxed{\delta(n) \xrightarrow{F} 1}$

$$\Rightarrow V(e^{jw}) + d_1\, e^{-jw(1)}\, V(e^{jw}) = P_0(1) + P_1\, e^{-jw1} \cdot 1$$

$$V(e^{jw}) = \frac{P_0 + P_1 e^{-jw}}{d_0 + d_1 e^{-jw}}$$

④ frequency shifting theorem:-

$$e^{jw_0 n}\, g[n] \longleftrightarrow G(e^{j(w-w_0)})$$

ex: $y[n] = (-1)^n \alpha^n \mu[n]$, $|\alpha| < 1$

$y[n] = e^{j\pi n} \alpha^n \mu[n]$

$$Y(e^{jw}) = \frac{1}{1-\alpha e^{-j(w-\pi)}} = \frac{1}{1-\alpha e^{j\pi} \cdot e^{-jw}}$$

$$= \frac{1}{1+\alpha e^{-jw}}$$

⑤ differentiation in frequency :-

$$n\, g[n] \xleftrightarrow{F} j \frac{d\, G(e^{jw})}{dw}$$

$$\frac{d}{dw}\left(G(e^{jw}) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-jwn} \right)$$

$$j \frac{d\, G(e^{jw})}{dw} = \sum_{n=-\infty}^{\infty} -jn\, g[n]\, e^{-jwn}$$

$$= \sum_{-\infty}^{\infty} n\, g[n]\, e^{-jwn}$$

Ex: $y[n] = (n+1)\, \alpha^{n} \mu[n] \qquad |\alpha| < 1$

let: $x[n] = \alpha^{n} \mu[n] \qquad |\alpha| < 1$

$y[n] = n\, x[n] + x[n]$

$$\Rightarrow X(e^{jw}) = \frac{1}{1-\alpha e^{jw}}$$

$$j \frac{d\, X(e^{jw})}{dw} = j \frac{-1(-j\alpha e^{-jw})}{(1-\alpha e^{-jw})^{2}} = \frac{\alpha e^{-jw}}{(1-\alpha e^{-jw})^{2}}$$

$$\Rightarrow Y(e^{jw}) = \frac{1}{1-\alpha e^{jw}} + \frac{\alpha e^{-jw}}{(1-\alpha e^{-jw})^{2}} = \frac{1-\alpha e^{-jw}+\alpha e^{-jw}}{(1-\alpha e^{-jw})^{2}}$$

Sunday
1/11

6) Modulation theorem:

$$g[n]\,h[n] \xleftrightarrow{F} \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta}) \cdot H(e^{j(\omega-\theta)}) \cdot d\theta$$

7) Convolution theorem:

$$g[n] \circledast h[n] \xleftrightarrow{F} G(e^{j\omega})\,H(e^{j\omega})$$

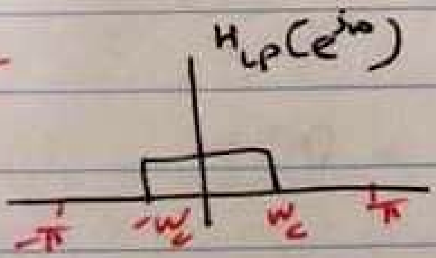
8) Parseval's theorem:

$$\sum_{-\infty}^{\infty} g[n]\,h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega}) \cdot H^*(e^{j\omega}) \cdot d\omega$$

$$\sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2\, d\omega$$

$|G(e^{j\omega})|^2 = S_{gg}$ ⇒ energy density spectrum

*



* $h_{LP}[n] = \begin{cases} \dfrac{\sin \omega_c n}{\pi n}, & n \neq 0 \\ \dfrac{\omega_c}{\pi}, & n = 0 \end{cases}$

⇒ $\sum_{-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1^2 \cdot d\omega$ $\qquad -\infty < n < \infty$

$$= \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

* LTI systems in frequency domain:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] \circledast h[n]$$

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\rightarrow \quad H(e^{j\omega}) = \frac{Y(n)}{X(n)} \Bigg|_{x(n) = e^{j\omega n}}$$

$$* \quad y[n] = \sum_{k=-\infty}^{\infty} h(k)\, x[n-k]$$

$$y(n) = \sum_{-\infty}^{\infty} h[k]\, e^{j\omega(n-k)}$$

$$y(n) = e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$

$$-\infty < n < \infty$$

$$x[n] = \sum_{i=1}^{L} A_i \cos(\omega_i n + \phi_i)$$

$\hookrightarrow$

$$x[n] = \sum_{i=1}^{L} A_i \cos(\omega_i n + \phi_i) \qquad -\infty < n < \infty$$

$$x[n] \longrightarrow \boxed{\begin{matrix} h[n] \\ H(e^{j\omega}) \end{matrix}} \longrightarrow y[n]$$

$$y(n) = \sum_{i=1}^{L} A_i \left| H(e^{j\omega_i}) \right| \cos\left(\omega_i n + \phi_i + \theta(e^{j\omega_i})\right)$$

$$\rightarrow \sum_{i=1}^{L} A_i \angle \phi_i \quad \begin{cases} \text{phasor} \\ \text{frequency domain} \end{cases}$$

Ex: $y(n) = 0.9\, y[n-1] + 0.1\, x(n)$

find: 1) $H(e^{j\omega})$

2) the output of the system for an input: ~~x[n] =~~

$$x[n] = 5 + 12 \sin\left(\frac{\pi}{2} n\right) - 20 \cos(\pi n + 45^\circ)$$

$$\rightarrow Y(e^{j\omega}) = 0.9\, e^{-j\omega}\, Y(e^{j\omega}) + 0.1\, X(e^{j\omega})$$

$$\rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{0.1}{1 - 0.9\, e^{-j\omega}}$$

$\hookrightarrow$ a)

a) $w=0$

$$H(e^{j0}) = \frac{0.1}{1-0.9} = 1\angle 0^\circ$$

b) $w = \pi/2$

$$H(e^{j\pi/2}) = \frac{0.1}{1-0.9\,e^{-\pi/2 j}}$$

$$= \frac{0.1}{1-0.9(\cancel{\cos(\frac{\pi}{2})}^{0} - j\cancel{\sin(\frac{\pi}{2})}^{1})}$$

$$= \frac{0.1}{1+0.9j} = 0.074 \angle \overline{42}$$

c) $w = \pi$

$$H(e^{j\pi}) = \frac{0.1}{1-0.9\,e^{-j\pi}} = 0.053 \angle 0^\circ$$

$$y[n] = 5*1 + 12 * 0.074 \sin\left(\frac{\pi n}{2} - 42^\circ\right)$$

$$-20(0.053)\cos(\pi n + 45^\circ)$$

$$\rightarrow y[n] = 5 + 0.888 \sin\left(\frac{\pi}{2}n - 42^\circ\right) - 1.06\cos(\pi n + 45^\circ)$$

$$-\infty < n < \infty$$

Thursday
5/11

No lecture!

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Sunday
8/11

[1] finite-length Discrete-time:

$$x[n] \qquad 0 \leq n < N-1$$

* Orthogonal transforms:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, \psi^*[k,n] \qquad 0 \leq k \leq N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, \psi[k,n]$$

* $\psi[k,n]$: basis sequence

$$\rightarrow \frac{1}{N} \sum_{n=0}^{N-1} \psi[k,n]\, \psi^*[l,n] = \begin{cases} 1 & l=k \\ 0 & l \neq k \end{cases}$$

$\Rightarrow$ orthonormal

* Discrete Fourier Transform (DFT):

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}} \qquad 0 \leq k \leq N-1$$

$$\psi[k,n] = e^{\frac{j2\pi kn}{N}} \qquad \begin{matrix} 0 \leq n \leq N-1 \\ 0 \leq k \leq N-1 \end{matrix}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{\frac{j2\pi kn}{N}} \qquad 0 \leq n < N-1$$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

$$X[0] = \sum_{n=0}^{N-1} x[n]\, e^{-j0}$$

$$= x[0] + x[1] + \cdots + x[N-1]$$

$$X[1] = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi(1)n}{N}}$$

$$= x[0] + x[1]e^{\frac{-j2\pi}{N}} + x[2]e^{\frac{-j2\pi \ast 2}{N}} + \cdots + x[N-1]\, e^{\frac{-j2\pi(N-1)}{N}}$$

$$X[2] = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi(2)n}{N}}$$

$$= x[0] + x[1]\, e^{\frac{-j2\pi(2)}{N}} + x[2]\, e^{\frac{-j2\pi(2)(2)}{N}} + \cdots + x[N-1]\, e^{\frac{-j2\pi(2)(N-1)}{N}}$$

$\vdots$

$$X[N-1] = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi(N-1)n}{N}}$$

$$= x[0] + x[1]\, e^{\frac{-j2\pi(N-1)}{N}} + \cdots + x[N-1]\, e^{\frac{-j2\pi(N-1)(N-1)}{N}}$$

* let: $W_n = e^{\frac{-j2\pi}{N}}$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_n^{nk} \qquad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-nk} \qquad 0 \le n \le N-1$$

→ $X[0] = x[0] + x[1] + \cdots + x[N-1]$

→ $X[1] = x[0] + x[1] W_N^{1} + x[2] W_N^{2} + \cdots + x[N-1] W_N^{N-1}$

→ $X[2] = x[0] + x[1] W_N^{2} + x[2] W_N^{4} \cdots + x[N-1] W_N^{2(N-1)}$

$\vdots$

→ $X[N-1] = x[0] + x[1] W_N^{N-1} + x[2] W_N^{2(N-1)} + \cdots + x[N-1] W_N^{(N-1)(N-1)}$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^{1} & W_N^{2} \cdots & W_N^{N-1} \\ 1 & W_N^{2} & W_N^{4} \cdots & W_N^{(N-1)} \\ \vdots & & & \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{D_n} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$N \times N$

@ matlab: >> dftmtx(N)

$$\underline{X} = D_N\, \underline{x}$$

$$D_N^{-1} \underline{X} = D_N^{-1} D_N\, \underline{x}$$

$$D^{-1} \underline{X} = \underline{x}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}$$

$$\underline{x} = \frac{1}{N} D_N^{*}\, \underline{X}$$

$$\underline{x} = D_N^{-1} \underline{X} \longrightarrow D_N^{-1} = \frac{1}{N} D_N^{*}$$

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Tuesday

$$\Rightarrow \quad \underline{X} = D_N \, \underline{x}$$

$$\underline{x} = \frac{1}{N} D_N^* \, \underline{X}$$

$$x(n) = \{1 \quad 2 \quad 0 \quad 1\}$$

$$\rightarrow \quad D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\rightarrow \quad X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2+1 \\ 1-2j+1 \\ 1-2-1 \\ 1+2j-j \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\rightarrow \quad x(n) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4+(1-j)-2+(1+j) \\ 4+j(1-j)+2-j(1+j) \\ 4-(1-j)-2-(1+j) \\ 4-j(1-j)+2+j(1+j) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ DTFT

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

$0 \le \omega \le 2\pi$, $\omega_k = \frac{2\pi k}{M}$ $\rightarrow$ $0 \le k \le M-1$

cont. function $\rightarrow$

$$X(e^{j\omega k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega k n} \qquad 0 \le k \le M-1$$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi k n}{M}} \qquad 0 \le k \le M-1$$

DFT $\rightsquigarrow$ $$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi k n}{N}} \qquad 0 \le k \le N-1$$

* $$X_e[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le M-1 \end{cases}$$ zero padding

extended value

$$X_e[k] = \sum_{n=0}^{M-1} x_e[n]\, e^{\frac{-j2\pi nk}{M}} \qquad 0 \le k \le M-1$$

$$X_e(k) = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi nk}{M}} \qquad 0 \le k \le M-1$$

* Computation of the DTFT using the DFT:

matlab >> $\underline{X} = fft(\underline{x})$ → DFT

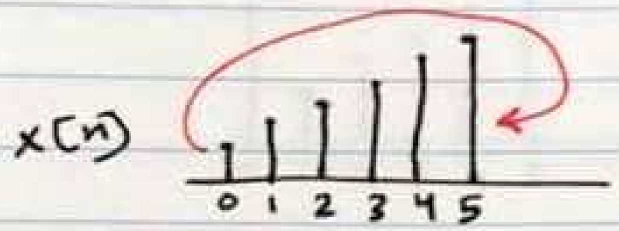
fft (x, M);

length = N → resolution

length of a sequence → M>N

* Circular shift of a sequence :

$x[n-n_0]$ $\quad\quad$ $0 \leq n \leq N-1$



→ $r = \langle m \rangle_N$ $\quad$ m modulus N

→ $r = m + lN$ $\quad$ * where l is an integer chosen that r is between (0) and $N-1$

$r = \langle 25 \rangle_7$

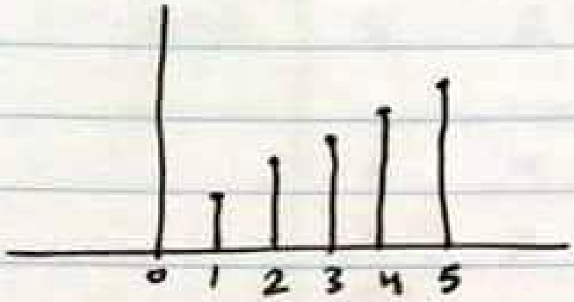
$r = 25 + (-3)(7)$
$= 25 - 21 = 4$

$r = \langle -16 \rangle_7$
$= -16 + (3)(7) = 5$

$r = 5$

$x[\langle n - n_0 \rangle_N]$ circular shift of a sequence

* $x[\langle n-1\rangle_6] = y[n] \Rightarrow x[\langle n+5\rangle_6]$



$y[0] = x[\langle -1\rangle_6] = x[5]$

$y[1] = x[\langle 1-1\rangle_6] = x[\langle 0\rangle_6] = x[0]$

$y[2] = x[\langle 2-1\rangle_6] = x[\langle 1\rangle_6] = x[1]$

* Circular convolution :-

→ $y_L[n] = x[n] \circledast h[n]$; $N_1 + N_2 - 1$ points

$$= \sum_{l=-\infty}^{\infty} x(l)\, h(n-l)$$

→ $y_c[n] = x[n] \; Ⓝ \; h[n]$; N points

$$= \sum_{l=0}^{N-1} x(l)\, h[\langle n-l\rangle_N]$$

* ex :

n	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$
$x[n]$	1	2	0	1			
$h[n]$	2	2	1	1			
	2	4	0	2			
	2	2	4	0	~~2~~		
	0	1	1	2	~~0~~	~~1~~	
	2	0	1	1	~~2~~	~~0~~	~~1~~
$y(n)$	{6	7	6	5}			

$$\Rightarrow \begin{bmatrix} y_c[0] \\ y_c[1] \\ \vdots \\ y_c[N-1] \end{bmatrix} = \begin{bmatrix} x[0] & x[N-1] & \cdots & x[1] \\ x[1] & x[0] & & x[2] \\ \vdots & x[1] & & \vdots \\ & \vdots & & x[N-1] \\ x[N-1] & x[N-2] & & x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-1] \end{bmatrix}$$

circulant matrix

$$= \begin{bmatrix} h[0] & h[N-1] & h[1] \\ \vdots & h[0] & h[2] \\ \vdots & \vdots & \vdots \\ & & h[N-1] \\ h[N-1] & h[N-1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$y_c[n] = x[n] \textcircled{N} h[n] = h(n) \textcircled{N} x[N]$$

* Discrete Fourier Transform Theorems :-

$$g[n] \xleftrightarrow{DFT} G[k]$$
$$h[n] \xleftrightarrow{DFT} H[k]$$

① linearity theorem:

$$\alpha g[n] + \beta h[n] \xleftrightarrow{DFT} \alpha G[k] + \beta H[k]$$

② Circular Time Shifting Theorem:

$$g[\langle n-n_0\rangle_N] \xleftrightarrow{DFT} W_N^{n_0 k} G[k]$$

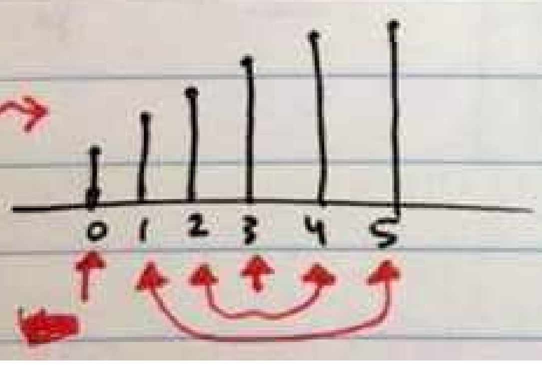
③ circular frequency shifting theorem:

$$W_N^{-k.n} g[N] \xleftrightarrow{DFT} G[\langle k-k_0\rangle_N]$$

④ Duality theorem:

$$G[n] \xleftrightarrow{DFT} N g[\langle -k\rangle_N]$$

* $x[\langle -n\rangle_6]$



⑤ Circular convolution theorem:

$$g[n] \; \textcircled{N} \; h[n] \xleftrightarrow{DFT} G[k]\,H[k]$$

* ex:

$\underline{x}[n] = \{1 \quad 2 \quad 0 \quad 1\}$

$\underline{h}[n] = \{2 \quad 2 \quad 1 \quad 1\}$

→ $$x[k] = D_4 \, \underline{X} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

→ $$H[k] = D_4 \, \underline{h} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

⇨ $$\cancel{y_c[k]} = Y_c[k] = G(k) \, . \, H[k]$$

$$= \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \begin{bmatrix} 24 \\ -2j \\ 0 \\ -2j \end{bmatrix}$$

⇨ $$y_c[n] = \frac{1}{4} D_4^* \, \underline{Y_c} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

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⑥ Modulation Theorem :

$$g[n]\,h[n] \xleftrightarrow{DFT} \frac{1}{N}\sum_{l=0}^{N-1} G[l]\,H[\langle k-l\rangle_N]$$

⑦ Parseval's theorem :

$$\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |G[k]|^2$$

$$\sum_{n=0}^{N-1} g[n]\,h^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} G[k]\,H^*[k]$$

ex: $x[n] = \{-3 \;\; 5 \;\; 45 \;\; -15 \;\; -9 \;\; -19 \;\; -8 \;\; 21 \;\; -10 \;\; 23\}$

10 - length sequence

* find:

a) $X[0]$

b) $X[5]$

c) $\sum_{k=0}^{9} X[k]$ ~~donot~~ * don't evaluate $X[k]$

d) $\sum_{k=0}^{9} |X[k]|^2$

2

$$\Rightarrow \quad X[k] = \sum_{n=0}^{9} x[n]\, W_{10}^{nk}$$

a) $X[0] = \sum_{n=0}^{9} x[n] =$ sub - - - - - - -

b) $X[5] = \sum_{n=0}^{9} x[n]\, e^{-j\frac{2\pi(5)}{10}n}$

$$= \sum_{n=0}^{9} x[n]\, e^{-j\pi n}$$

$$= \sum_{n=0}^{9} x[n]\, (-1)^n$$

c) $x[n] = \frac{1}{10} \sum_{k=0}^{9} X[k]\, W_{10}^{-nk}$

@ $n=0$: $x[0] = \frac{1}{10} \sum_{k=0}^{9} X[k] \Rightarrow \sum_{k=0}^{9} X[k] = (10)(-3) = -30$

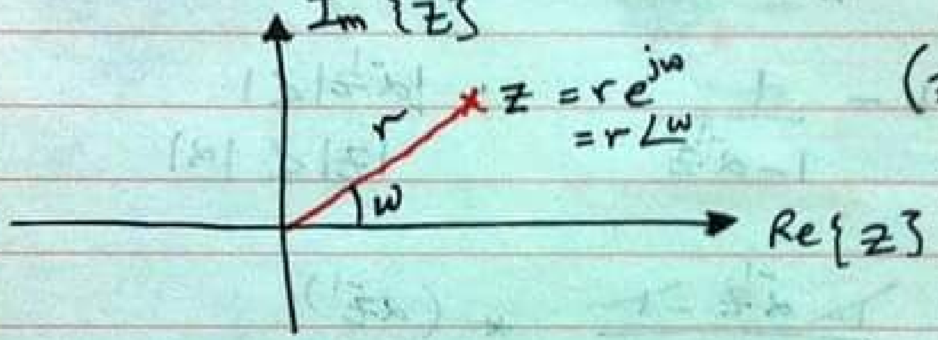
d) $\sum_{k=0}^{9} |X[k]|^2 = 10 \sum_{n=0}^{9} |x[n]|^2$

* Z - transform :

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{F.T}$$

$$x[n] = \alpha^{n} u[n] \qquad |\alpha|<1$$

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$



(z-plane)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

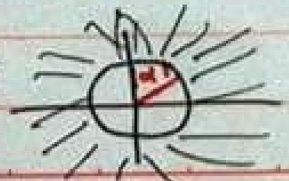
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\,(r e^{j\omega})^{-n}$$: Region of convergens (ROC)

* ex.1 : $x[n] = \alpha^{n} u[n]$: (Right sided sequence)

$$X(z) = \sum_{n=\cancel{-\infty}\,0}^{\infty} \alpha^{n}\, \underbrace{\cancel{u[n]}}_{1}\, z^{-n}$$

$$= \sum_{0}^{\infty} \alpha^{n} z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^{n} = \frac{1}{1-\alpha z^{-1}}$$

Right sided ⇐ ∴ outside



ROC → $\boxed{\begin{array}{c} |\alpha z^{-1}| < 1 \\ |\alpha| < |z| \end{array}}$

* **ex.2:** $x[n] = -\alpha^n u[-n-1]$

(left sided sequence)

$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

$$X(z) = -\sum_{n=1}^{\infty} \alpha^{-n} z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} (\alpha^{-1} z)^n$$

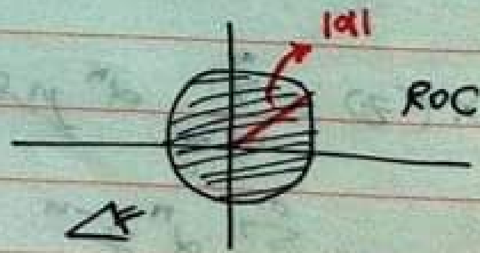
$$= 1 - \frac{1}{1-\alpha^{-1}z} \quad ; \quad |\alpha^{-1} z| < 1$$

$$|z| < |\alpha|$$

$$= \frac{\cancel{1} - \alpha^{-1}z \cancel{-1}}{1-\alpha^{-1}z} * \frac{(\alpha z^{-1})}{\alpha(z^{-1})}$$

$$= \frac{-1}{\alpha z^{-1} - 1}$$

$$= \frac{1}{1-\alpha z^{-1}} \quad ; \quad |\alpha| > |z|$$



left sided

∴ inside

5

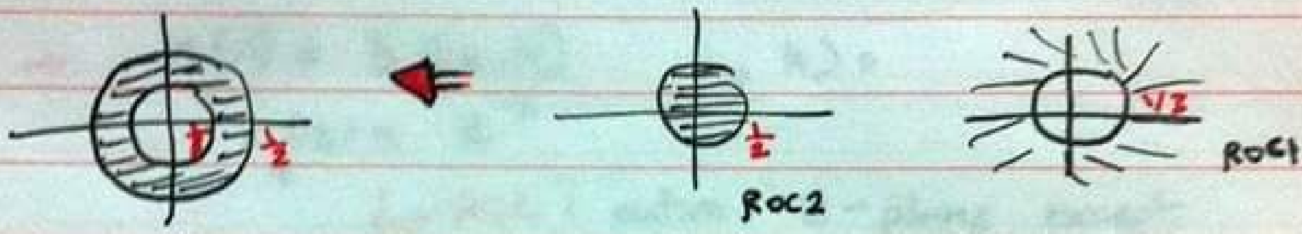
ex.3: Two-sided sequence:

$$x[n] = \left(\frac{-1}{3}\right)^n \mu[n] - \left(\frac{1}{2}\right)^n \mu[-n-1]$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$$

ROC1 $|z| > \frac{1}{3}$

ROC2 $|z| < \frac{1}{2}$



ROC: $\frac{1}{3} < |z| < \frac{1}{2}$ (intersection)

ex4:

$$x_1[n] = \{\underset{\uparrow}{1}\ 2\ 5\ 7\ 0\ 1\}$$

→ $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

↳ ROC1: entire z plane except ($z = 0$)

$$x_2[n] = \{1\ 2\ \underset{\uparrow}{5}\ 7\ 0\ 1\}$$

→ $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$

↳ ROC: entire z plane except ($z = 0$) and ($z = \infty$)

→ $x_3[n] = \delta[n]$

$X_3(z) = 1$

$$X_3(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^{-0} = 1$$

↳ ROC: entire z-plane

→ $x_4[n] = \delta[n-k]$, $k>0$

$X_4(z) = z^{-k}$

↳ ROC: entire z-plane except $(z=0)$

→ $x_5[n] = \delta[n+k]$, $k>0$

$X_5(z) = z^{k}$

↳ ROC: entire z-plane except $(z=\infty)$

* The Inverse Z- Transform :

* inspection method:-

* Partial fraction expansion :

$$\rightarrow X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$\rightarrow X(z) = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$$\rightarrow X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

* where : $A_k = (1 - d_k z^{-1})\, X(z) \Big|_{z = d_k}$

* **Ex:** $$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} \qquad |z| > \frac{1}{2}$$

$$X(z) = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$\underline{\underline{A_1}} = \frac{1}{1-\frac{1}{2}(4)} = \frac{1}{1-2} = -1$$

$$\underline{\underline{A_2}} = \frac{1}{1-\frac{1}{4}(2)} = \frac{1}{1-0.5} = \frac{1}{0.5} = 2$$

$$\Rightarrow X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$\Rightarrow x[n] = -\left(\frac{1}{4}\right)^n \mu[n] + 2\left(\frac{1}{2}\right)^n \mu[n]$$

* **Ex:** $$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} \qquad |z| > 1$$

$$\begin{array}{r|l} & \quad 2 \\ \hline \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & -z^{-2} \pm 3z^{-1} \mp 2 \\ \hline & 5z^{-1}-1 \end{array}$$

* long division

$\Rightarrow$

$$\Rightarrow X(z) = \left(2 + \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}\right)(1-\tfrac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}}$$

$$= 2 + \frac{A_1}{(1-\frac{1}{2}z^{-1})} + \frac{A_2}{(1-z^{-1})}$$

$$\underline{A_1} = \frac{-1+(5)(2)}{1-2} = -9$$

$A_2 = ??$

$$X(z) = \left(2 + \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}\right)(1-z^{-1})\Big|_{z=1}$$

$$\underline{A_2} = \frac{-1+(5)(1)}{1-\frac{1}{2}} = \frac{4}{0.5} = 8$$

$$\Rightarrow X(z) = 2 + \frac{-9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$

$$\Rightarrow x[n] = 2\delta[n] + -9\left(\frac{1}{2}\right)^n u(n) + 8u[n]$$

10

* Z-transform properties:

$$x[n] \overset{z}{\longleftrightarrow} X(z) \qquad (ROC: R_x)$$

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z) \qquad (ROC: R_{x_1})$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z) \qquad (ROC: R_{x_2})$$

① linearity:

$$a x_1[n] + b x_2[n] \overset{z}{\longleftrightarrow} a X_1(z) + b X_2(z)$$

↳ ROC: $R_{x_1} \cap R_{x_2}$

② time shifting:

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0} X(z)$$

↳ ROC: R_x { except for possible addition or deletion of $(z=0)$ or $(z=\infty)$ }

2 ex: $X(z) = \dfrac{1}{z - \frac{1}{4}}$ $\quad |z| > \frac{1}{4}$

$$X(z) = \frac{1}{z - \frac{1}{4}} \begin{matrix} * z^{-1} \\ * z^{-1} \end{matrix}$$

$$= \frac{z^{-1}}{1 - \frac{1}{4} z^{-1}}$$

$$* \quad -\tfrac{1}{4} z^{-1} + 1 \overline{\big) \, \cancel{z^{-1}}} \; -4$$
$$\underline{\cancel{-z^{-1}} + 4}$$
$$4$$

→ $X(z) = -4 + \dfrac{4}{1 - \frac{1}{4} z^{-1}}$

→ $x[n] = -4\,\delta[n] + 4\left(\frac{1}{4}\right)^n \mu[n]$

* alternative solution:

$$X(z) = z^{-1}\left(\frac{1}{1 - \frac{1}{4} z^{-1}}\right)$$

$$= \left(\frac{1}{4}\right)^{n-1} \mu[n-1]$$

③ multiplication by an exponential sequence:

$$z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right)$$

↳ ROC: $|z_0| R_x$

$$e^{j w_0 n} x[n] \xleftrightarrow{z} X\left(e^{j(w-w_0)}\right)$$

$$X\left(\frac{e^{jw}}{j e^{jw_0}}\right)$$

* Ex: $x[n] = r^n \cos w_0 n \, \mu[n]$

* $\mu[n] \xleftrightarrow{z} \dfrac{1}{1-z^{-1}} \qquad |z|>1$

→ $x[n] = r^n \left[\dfrac{e^{jw_0 n} + e^{-jw_0 n}}{2}\right] \mu[n]$

$$= \frac{1}{2}(r e^{jw_0})^n \mu[n] + \frac{1}{2}(r e^{-jw_0})^n \mu[n]$$

$$= \frac{1/2}{1-\left(\frac{z}{re^{jw_0}}\right)^{-1}} + \frac{1/2}{1-\left(\frac{z}{re^{-jw_0}}\right)^{-1}}$$

ROC1: $|z| > 1 * r$
$|z| > r$

ROC2: $|z| > 1 * r$
$|z| > r$

$\therefore |z| > r$ ⟶

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$$X(z) = \frac{1/2}{1 - r e^{j\omega_0} z^{-1}} + \frac{1/2}{1 - r e^{-j\omega_0} z^{-1}} \qquad |z| > r$$

$$= \frac{1/2 - \frac{1}{2} r e^{-j\omega_0} z^{-1} + \frac{1}{2} - \frac{1}{2} r e^{j\omega_0} z^{-1}}{1 - \underbrace{r e^{j\omega_0} z^{-1} - r e^{-j\omega_0} z^{-1}}_{-2r z^{-1} \left(\underbrace{\frac{e^{j\omega_0} + e^{-j\omega_0}}{2}}_{\cos\omega_0} \right)} + r^2 z^{-2}}$$

$$X(z) = \frac{1 - r\cos\omega_0 . z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}} \qquad |z| > r$$

$$r^n \sin\omega_0 n\, \mu[n] \xleftrightarrow{z} \frac{(r\sin\omega_0)\, z^{-1}}{1 - (2r\cos\omega_0) z^{-1} + r^2 z^{-2}} \qquad |z| > r$$

$$\frac{1}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}} - z^{-1}(\,) + (\,) z^{-1}$$

$$1 - 5z^{-1} + 25 z^2$$

DATE 22/11 Sunday

* **Ex:** A causal LTI system is described by the difference equation

$$y[n] = 0.2\,y[n-1] + 0.08\,y[n-1] + 2x[n]$$

a) determine $H(z)$ of the system
b) determine the impulse response $h[n]$ of the sys.
c) determine the step response $s[n]$ of sys.

ⓐ

$$Y(z) = 0.2\,z^{-1}Y(z) + 0.8z^{-2}Y(z) + 2X(z)$$

$$(1 - 0.2z^{-1} - 0.08z^{-2})\,Y(z) = 2X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - 0.2z^{-1} - 0.08z^{-2}}$$

$$H(z) = \frac{2}{(1+0.2z^{-1})(1-0.4z^{-1})} \qquad |z| > 0.4$$

ⓑ

$$h[n] \xleftrightarrow{z} H(z)$$

$$x[n] = \delta[n]$$

$$\delta[n] \xleftrightarrow{z} 1$$

$$H(z) = \frac{A_1}{1+0.2z^{-1}} + \frac{A_2}{1-0.4z^{-1}}$$

$$A_1 = \frac{2}{1-0.4\left(\frac{-1}{0.2}\right)} = \frac{2}{1+2} = \frac{2}{3}$$

$$A_2 = \frac{2}{1+(0.2)\left(\frac{1}{0.4}\right)} = \frac{2}{1.5} = \frac{4}{3}$$

$$H(z) = \frac{2/3}{1+0.2z^{-1}} + \frac{4/3}{1-0.4z^{-1}}$$

$$h(n) = \frac{2}{3}(\bar{0}.2)^n \mu(n) + \frac{4}{3}(0.4)^n \mu(n)$$

$$* \quad Y(z) = H(z).\,X(z)$$
$$= H(z).\,1 \quad \delta$$

(c) $X(z) = \dfrac{1}{1-z^{-1}}$ $\qquad x(n) = \mu(n)$

$$S(z) = H(z).\,Z\{\mu(n)\}$$
$$S(z) = \frac{2}{(1-0.2z^{-1}-0.08z^{-2})} \cdot \frac{1}{(1-z^{-1})}$$

$$S(z) = \frac{2}{(1+0.2z^{-1})(1+0.4z^{-1})(1-z^{-1})}$$

$$= \frac{A_1}{1+0.2z^{-1}} + \frac{A_2}{1-0.4z^{-1}} + \frac{A_3}{1-z^{-1}}$$

$\rightarrow$

$$A_1 = \frac{2}{\left(1-0.4\left(\frac{-1}{0.2}\right)\right)\left(1-\left(\frac{-1}{0.2}\right)\right)} = \frac{2}{(1+2)(6)} = \frac{2}{18}$$

$$= 1/9 = 0.111$$

$$A_2 = \frac{2}{\left(1+0.2\left(\frac{1}{0.4}\right)\right)\left(1-\frac{1}{0.4}\right)} = \frac{2}{(1.5)(\bar{1}.5)} = \bar{0}.8889$$

$$A_3 = \frac{2}{(1+0.2)(1-0.4)} = \frac{2}{(1.2)(0.6)} = 2.7778$$

$$S(z) = \frac{0.1111}{1+0.2z^{-1}} - \frac{0.8889}{1-0.4z^{-1}} + \frac{2.7778}{1-z^{-1}} \quad ; |z|>1$$

$$S[n] = 0.1111\,(0.2)^n \mu[n] - 0.8889\,(0.4)^n \mu[n] + 2.7778\,\mu[n]$$

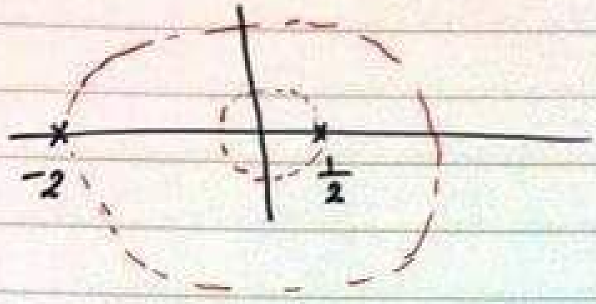
* If: $x[n] = (0.5)^n \mu[n]$

$$X(z) = \frac{1}{1-0.5z^{-1}} \qquad |z|>0.5$$

$$Y(z) = H(z) \cdot X(z)$$

$$= \frac{2 \cdot 1}{(1-0.2z^{-1})(1-0.5z^{-1})}$$

* Stability, Causality, and the ROC :-



$|Z| > 2$ ← causal system, unstable

$1/2 < |Z| < 2$ ← two sided seq., BIBO stable, non causal

$|Z| < \frac{1}{2}$ ← non causal,

WEEK

DSP Notebook

DR . DEYA' ABULNADIE
BY : SAUSAN ALMOHTASEB

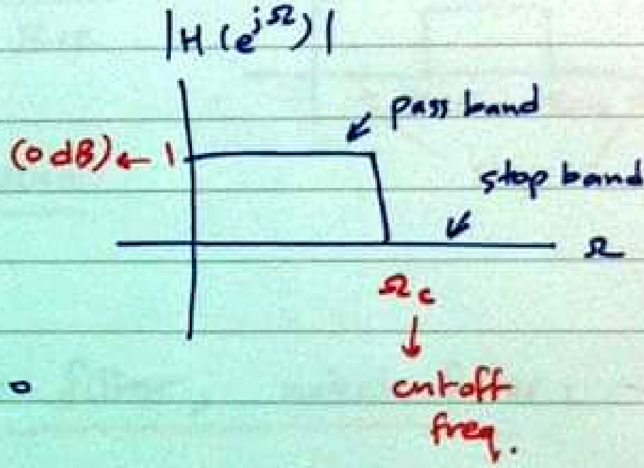
DATE 1/12 Tue

*** Digital filter Design :

** Analog filter design :

* Analog filter based on frequency selectivity :

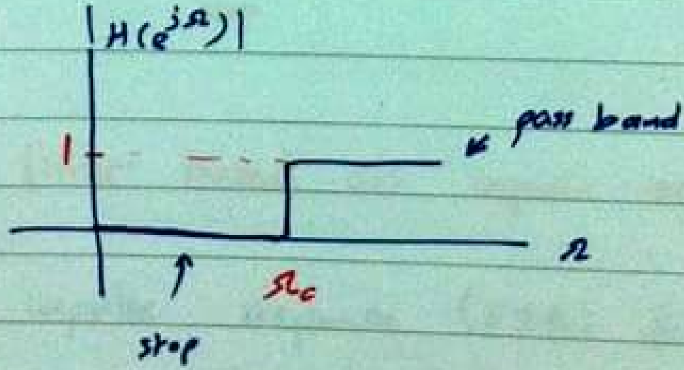
1 Low Pass filter :



$$|H(e^{j\Omega})| = \frac{|X(e^{j\Omega})|}{|Y(e^{j\Omega})|}$$

$$h[n] = \begin{cases} \dfrac{\sin \Omega_c n}{\pi n}, & n \neq 0 \\ \dfrac{\Omega_c}{\pi}, & n = 0 \end{cases}$$

2 High Pass filter ,

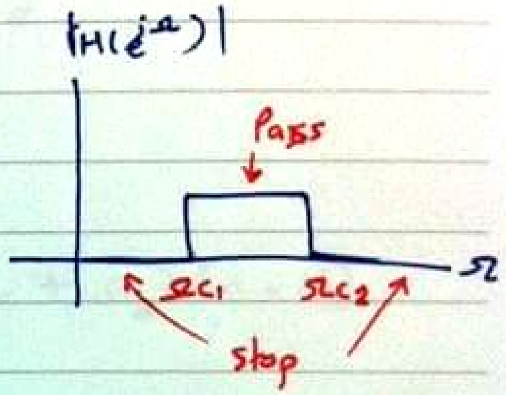


3 Band pass filter:

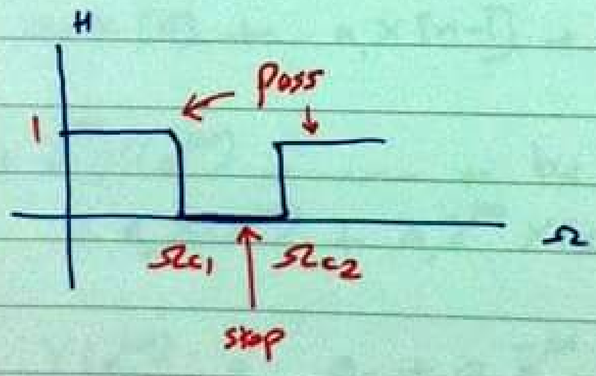
bandwidth (BW) = $\Omega_{c1} - \Omega_{c2}$

center freq. = $\Omega_0 = \sqrt{\Omega_{c1}\Omega_{c2}}$

geometric mean = $\frac{\Omega_{c1} + \Omega_{c2}}{2}$



4 Band stop filter, reject filter, notch filter:



* Analog filter based on input response:

① finite impulse response (FIR) filters:

$\rightarrow$ $Y[n] = a_0 x[n] + a_1 x[n-1] + \dots + a_M x[n-M]$

$\rightarrow$ $H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega})$

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$\rightarrow Y(e^{jw}) = a_0 X(e^{jw}) + a_1 e^{-jw} X(e^{jw}) + \cdots + a_M e^{-jwM} X(e^{jw})$

$\rightarrow H(e^{jw}) = a_0 + a_1 e^{-jw} + \cdots + a_M e^{-jwM}$

$\rightarrow h[n] = \{a_0, a_1, \cdots, a_M\}$

$\Rightarrow h[n] = a_0 \delta[n] + a_1 \delta[n-1] + \cdots + a_M \delta[n-M]$

② infinite - impulse response (IIR) filters:

$\rightarrow y[n] = b_1 y[n-1] + b_2 y[n-2] + \cdots + b_N y[n-N] +$
$a_0 x[n] + a_1 x[n-1] + \cdots + a_M x[n-M]$

$\rightarrow Y(e^{jw}) = b_1 e^{-jw} Y(e^{jw}) + \cdots + b_N e^{-jwN} Y(e^{jw})$
$+ a_0 X(e^{jw}) + a_1 e^{-jw} X(e^{jw}) + \cdots + a_M e^{-jwM} X(e^{jw})$

$$\Rightarrow H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{a_0 + a_1 e^{-jw} + \cdots + a_M e^{-jwM}}{1 - b_1 e^{-jw} + \cdots - b_N e^{-jwN}}$$

$$= \boxed{\frac{N_M(e^{jw})}{D_N(e^{jw})}}$$

4

* **ex:**

$$H(e^{jw}) = \frac{1}{1+e^{-jw}}$$

$$H(e^{jw}) \rightarrow 1 - e^{-jw} + e^{-j2w} - e^{-jw3} \cdots$$

$$1+e^{-jw} \,\overline{)\,1}$$

$$\underline{-1 \mp e^{-jw}}$$

$$-e^{-jw}$$

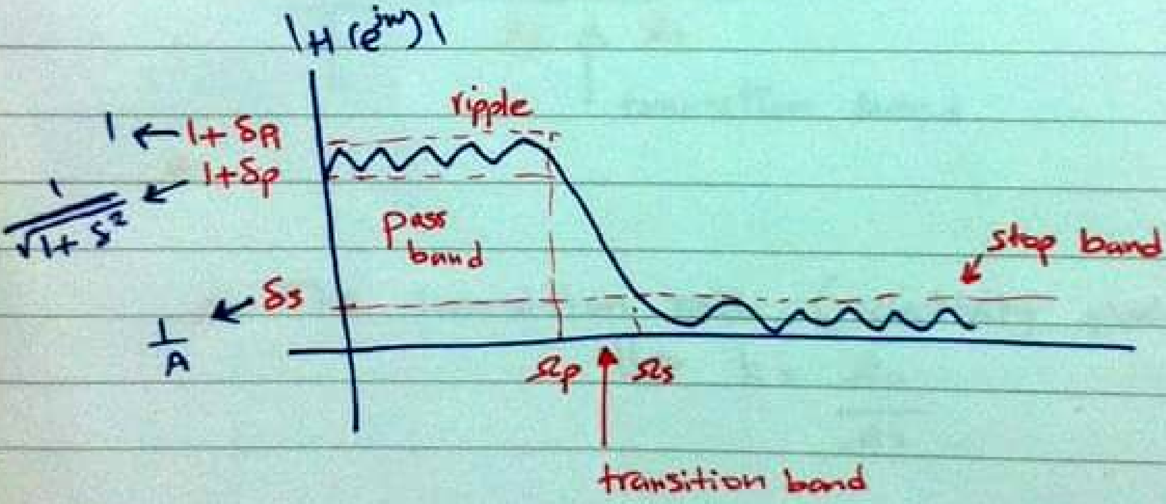
$$\underline{+-e^{-jw} \pm e^{-j2w}}$$

$$e^{-j2w}$$

$$\underline{-e^{-j2w} \mp e^{-j3w}}$$

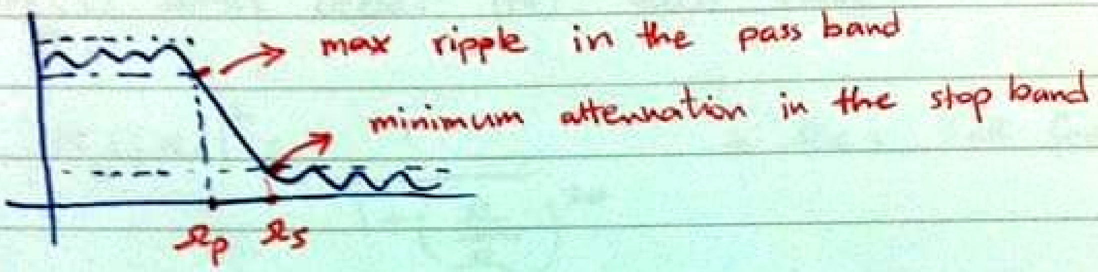
$$-e^{-3jw}$$

$h[n] = \{1, -1, 1, -1\}$

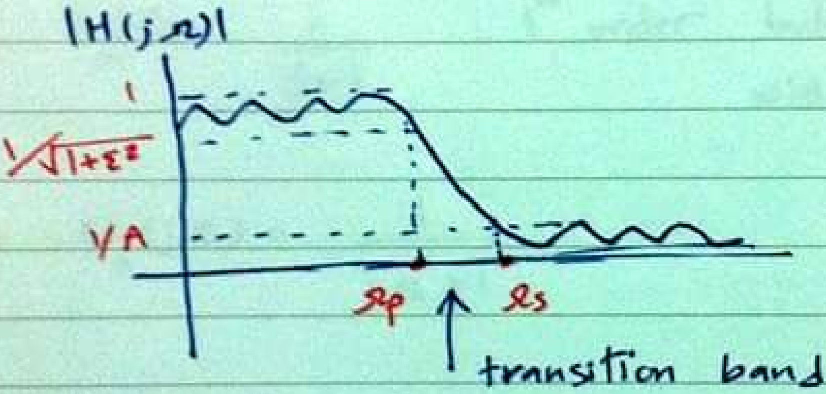


−ve attenuation → −ve dB

* Analog Low Pass filter Design:



⇓



*

* The transition ratio (selectivity parameters)

$\Rightarrow k = \dfrac{\Omega_p}{\Omega_s}$

* The descrimination parameter: $k_1 = \dfrac{\varepsilon}{\sqrt{A^2 - 1}}$, $k_1 \ll 1$

* Butterworth Approximation:

$H_a(s)$ with order (N) such that:

→ $|H_a(j\Omega)|^2 = \dfrac{1}{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$ * Ω_c: 3dB freq

→ $H_a(s) = \dfrac{1}{s+1}$: 1^{st} order butterworth filter with $\Omega_c = 1$

→ $H_a(j\Omega) = \dfrac{1}{j\Omega+1}$

→ $H_a\left(j\dfrac{\Omega}{\Omega_c}\right) = \dfrac{1}{j\left(\dfrac{\Omega}{\Omega_c}\right)+1} = \dfrac{\Omega_c}{j\Omega+\Omega_c}$

→ $H_a(s) = \dfrac{1}{s^2+\sqrt{2}\,s+1}$: 2^{nd} order butterworth filter with $\Omega_c = 1$

→ $H_a\left(j\left(\dfrac{\Omega}{\Omega_c}\right)\right) = \dfrac{1}{\left(j\left(\dfrac{\Omega}{\Omega_c}\right)\right)^2+\sqrt{2}\,j\,\dfrac{\Omega}{\Omega_c}+1}$

$H_a(j\Omega') \cdot H_a(-j\Omega) = \dfrac{1}{\left(j\left(\frac{\Omega}{\Omega_c}\right)\right)^2+\sqrt{2}\,j\left(\frac{\Omega}{\Omega_c}\right)+1} \cdot \dfrac{1}{\left(-j\frac{\Omega}{\Omega_c}\right)^2-\sqrt{2}\,j\,\frac{\Omega}{\Omega_c}+1}$

↳

$$\Rightarrow \frac{1}{\left(\frac{\Omega}{\Omega_c}\right)^4 \cancel{- \sqrt{2}\, j \left(\frac{\Omega}{\Omega_c}\right)^3} \cancel{- \left(\frac{\Omega}{\Omega_c}\right)^2} \cancel{+ \sqrt{2}\, j \left(\frac{\Omega}{\Omega_c}\right)^3} + 2 \cancel{- \left(\frac{\Omega}{\Omega_c}\right)^2} \cancel{+ 2 \left(\frac{\Omega}{\Omega_c}\right)^2} \cancel{- \sqrt{2}\, j \frac{\Omega}{\Omega_c}} \cancel{+ \sqrt{2}\, j \left(\frac{\Omega}{\Omega_c}\right)} + 1}$$

$$= \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^4}$$

* Maximally flat $(2N-1)$ derivatives at $(\Omega = 0)$ is $= 0$

$$* \quad f(x+c) = f(c) + \frac{1}{1!} f'(x)\Big|_{x=c} \cdot \overbrace{(x-c)}^{\Delta x} + \frac{1}{2!} f''(x)\Big|_{x=c} (\Delta x)^2 \text{ - - - - -}$$

$$* \quad |H_a(j\Omega_p)|^2 = \frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = \boxed{\frac{1}{1+\varepsilon^2}} \text{ - - - - } ①$$

$$* \quad |H_a(j\Omega_s)|^2 = \frac{1}{1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \boxed{\frac{1}{A^2}} \text{ - - - - } ②$$

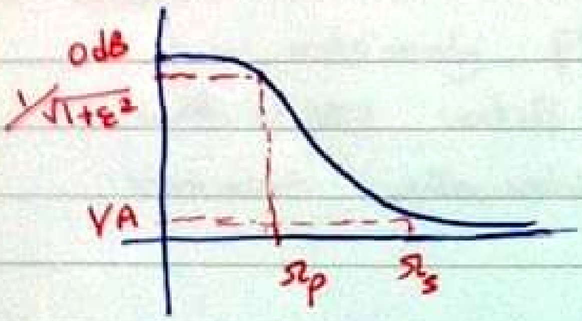
$\rightarrow \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = A^2 - 1$

$\rightarrow \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \varepsilon^2$

$* \quad \log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \log_{10} \frac{A^2-1}{\varepsilon^2}$

$$\cancel{2}N \log\left(\frac{\Omega_s}{\Omega_p}\right) = \cancel{2} \log_{10}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)$$

$$* \quad N = \frac{\log_{10}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\frac{1}{k_1}\right)}{\log_{10}(1/k)} \quad *$$



* $|H_a(j\Omega)|^2 = \dfrac{1}{1+\left(\frac{\Omega}{\Omega_s}\right)^2}$

* $|H_a(j\Omega_p)|^2 = \dfrac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = \dfrac{1}{1+\varepsilon^2}$

* $|H_a(j\Omega_s)|^2 = \dfrac{1}{1+\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}} = \dfrac{1}{A^2}$

$\rightarrow \left(\dfrac{\Omega_p}{\Omega_c}\right)^{2N} = \varepsilon^2 \quad , \rightarrow \left(\dfrac{\Omega_s}{\Omega_{p_c}}\right)^{2N} = A^2-1$

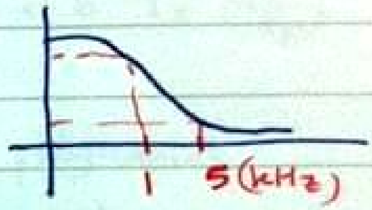
$\rightarrow \log_{10}\left(\dfrac{\Omega_s}{\Omega_p}\right)^{2N} = \dfrac{A^2-1}{\varepsilon^2} = \log_{10}\left(\dfrac{\sqrt{A^2-1}}{\varepsilon}\right)^2$

$\rightarrow \cancel{2}N \log_{10}\left(\dfrac{\Omega_s}{\Omega_p}\right) = \cancel{2} \log_{10}\left(\dfrac{\sqrt{A^2-1}}{\varepsilon}\right)$

$\Rightarrow N = \dfrac{\log_{10}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$

$\Rightarrow N = \dfrac{\log_{10}(1/k_1)}{\log_{10}(1/k)}$

* Ex: Determine the lowest order of $H_a(s)$ having a maximaly flat low pass filter characteristics with (1 dB) cutoff freq. at (1 kHz) and minimum attenuation of (40 dB) at (5 kHz) ?



Sol: $10 \log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -1$ dB

solve : $\varepsilon^2 = 0.25895$

$$10^{-1} = \frac{1}{\varepsilon^2+1} \Rightarrow 10^{0.1} = 1+\varepsilon^2$$

→ $10 \log_{10}\left(\frac{1}{A^2}\right) = -40$ dB

→ $\log_{10}(1/A^2) = -4$

$$1/A^2 = 10^{-4} \Rightarrow A^2 = 10^4 = 10000$$

$$A = 100$$

* $\frac{1}{k_1} = 196.5$

* $k = \frac{5000}{1000} = 5$

* $N = \frac{\log_{10}(196.5)}{\log_{10}(5)} = 3.28 \Rightarrow \boxed{N = 4}$

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* Chebyshev Approximation:

Type (1): chebychev filter:

$H_a(s)$ such that $\rightarrow |H_a(j\Omega)|^2 = \dfrac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$

* where $T_N(\Omega)$ is a chebyshev polynomial of order (N)

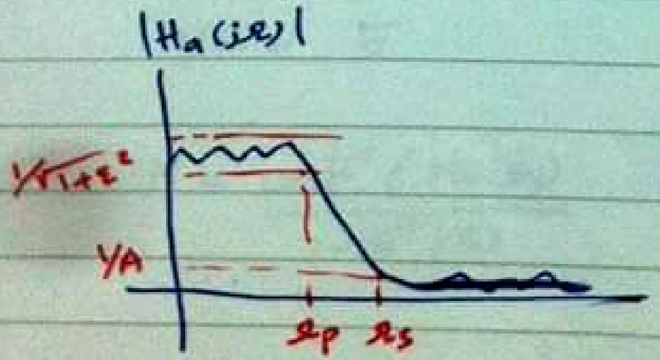
* $$T_N(\Omega) = \begin{cases} \cos\left(N \cos^{-1}(\Omega)\right) & |\Omega| \leq 1 \\ \cosh\left(N \cosh^{-1}(\Omega)\right) & |\Omega| > 1 \end{cases}$$

* Recursive formula:

$T_r(\Omega) = 2\Omega\, T_{r-1}(\Omega) - T_{r-2}(\Omega)$. $\quad r \geq 2$

such that:

$T_0(\Omega) = 1$ and $T_1(\Omega) = \Omega$



* Sub in this:

→ $$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{1}{A^2}$$

*

$\Omega_s > \Omega_p$ → $\boxed{\frac{\Omega_s}{\Omega_p} > 1}$ then we sub in:

→ $\cosh(N \cosh^{-1}(\Omega))$ $|\Omega| > 1$

⇒ $$\sqrt{\left(\cosh\left(N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)\right)^2} = \sqrt{\frac{A^2-1}{\varepsilon^2}}$$

→ $$\cosh\left(N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right) = \frac{\sqrt{A^2-1}}{\varepsilon}$$

* take $\cosh^{-1}$ for the equation.

$$N = \cosh^{-1}\frac{\sqrt{A^2-1}}{\varepsilon} \Big/ \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{\cosh^{-1}\left(\frac{1}{k_1}\right)}{\cosh^{-1}\left(\frac{1}{k}\right)}$$

* Ex:

$1/k_1 = 196.5$

$1/k = 5$

$$N = \frac{\cosh^{-1}(196.5)}{\cosh^{-1}(5)} = 2.6$$

$$\boxed{N = 3}$$

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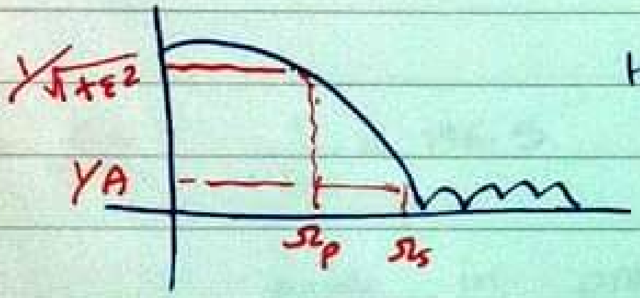
* Type (2): chebyshev filter:
(inverse chebyshev filter)

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{T_N\left(\frac{\Omega_s}{\Omega_p}\right)}{T_N\left(\frac{\Omega_s}{\Omega}\right)}\right)^2} = \frac{1}{A^2}$$

put Ω_s



$$\therefore \ \varepsilon^2 \frac{T_N\left(\frac{\Omega_s}{\Omega_p}\right)}{\cancel{T_N\left(\frac{\Omega_s}{\Omega_s}\right)}_{1}} = A^2 - 1$$

* Elliptic Approximation: (Cauer filter)

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(R_N^2\left(\frac{\Omega}{\Omega_p}\right)\right)}$$

$R_N(\Omega) \Rightarrow$ is a rational number such that:

$$R_N\left(\frac{1}{\Omega}\right) = \frac{1}{R(\Omega)}$$

* $N = 2 \log_{10}\left(\frac{4}{k_1}\right) / \log_{10}(1/\rho)$

where: → $k' = \frac{1}{\sqrt{1-k^2}}$

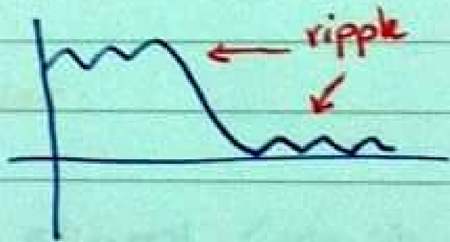
→ $\rho_o = \frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}$

→ $\rho = \rho_o + 2\rho_o^5 + 15\rho_o^9 + 150\rho_o^{13}$

* Ex: $\frac{1}{k_1} = 196.5$, $\frac{1}{k} = 5$

apply in previous equation: $N = 2.2$

$\boxed{N = 3}$



* Matlab codes:

Butterworth filter:

```
>> [N, Wn] = buttord (Wp, Ws, Rp, Rs, 's') ;
        Ws = 2π * 1000 (rad /sec)
        Rp (in dB)
        S: analog filter
>> [num, den] = butter (N, Wn , 'S') ;
```

DATE

15 5

* Type (1): chebyshev :

```
>> [N, Wn] = cheb1ord (wp, ws, Rp, Rs, 's');
```

→ w_p not w_c

```
>> [num, den] = cheby1 (N, Rp, Wn, 's');
```

* type (2) chebyshev :

```
>> [N, Wn] = cheb2ord (Wp, Ws, Rp, Rs, 's');
>> [num, den] = cheby2 ( N , Rs, Wn, 's' );
```

* Elliptic filter :

```
>> [N, Wn] = ellipord (Wp, Ws, Rp, Rs, 's'];
```

→ with two values (w_p, w_s)

```
>> [num, den] = ellip (N, Rp, Rs, Wn, 's');
```

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* Design of analog HPF, BPF, BSF :

→ $s = F(\hat{s})$

$$H_D(\hat{s}) = H_a(s)\Big|_{s = F(\hat{s})}$$

$$H_a(\hat{s}) = H_D(\hat{s})\Big|_{\hat{s} = F^{-1}(s)}$$

$$s = F(\hat{s}) = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$$

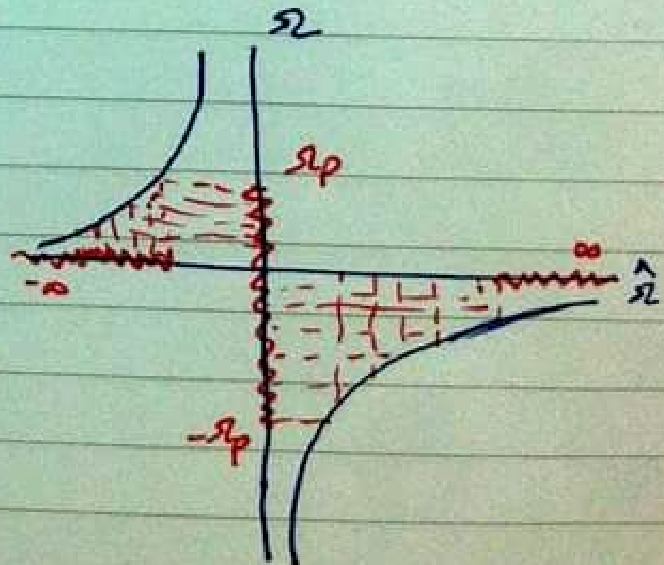
* 1^{st} order low pass Butterworth : $\frac{1}{s+1} = H_a(s)$

$$H_D(\hat{s}) = \left(\frac{\hat{s}}{\hat{s}}\right)\frac{1}{\frac{1}{\hat{s}}+1} = \frac{\hat{s}}{1+\hat{s}}$$

$$j\Omega = \frac{\Omega_p \hat{\Omega}_p}{j\hat{\Omega}}$$

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

$$\boxed{\Omega = \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}}$$



* **Ex:** Design an analog filter: (butterworth High pass) with the following specifications:

$\hat{f}_p = 4\,kHz$ $\quad\quad$ $R_p = 0.1\,dB$

$\hat{f}_s = 1\,kHz$ $\quad\quad$ $R_s = 40\,dB$

sol:

$\Rightarrow \Omega_s = \dfrac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s}$

* **let:** $\Omega_p = 1$ rad/sec

$$\Omega_s = \frac{(1)(2\pi)(4000)}{(2\pi)(1000)} = 4 \text{ rad/sec}$$

* Low Pass Prototype specifications:

$\Omega_p = 1$ rad / sec $\longrightarrow R_p = 0.1$ dB

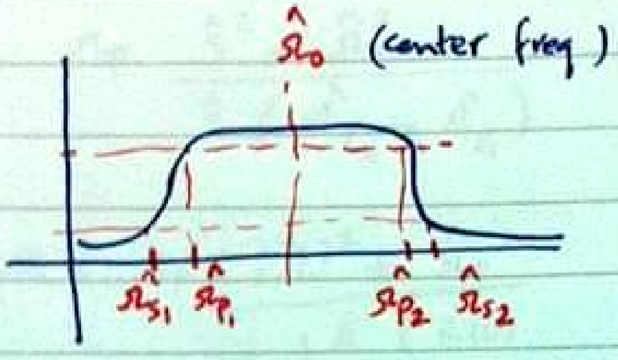
$\Omega_s = 4$ rad / sec $\longrightarrow R_s = 40$ dB

```
>> [N, Wn] = buttord (1, 4, 0.1, 40, 's');
>> [B, A] = butter (N, Wn, 's');
>> [num, den] = lp2hp (B, A, (2*pi*4000));
```

$H_a(s)$

$H(\hat{s})$

* The analog BPF design : (band pass)

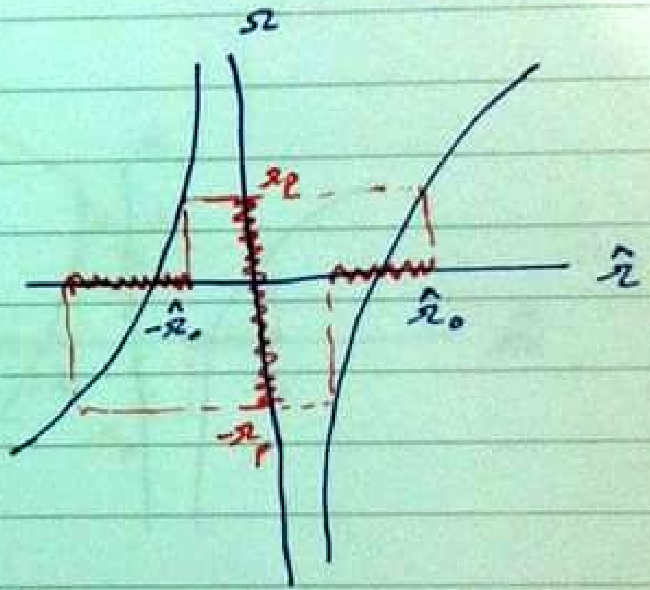


$$BW = \hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}$$

→ $$S = \Omega_p \frac{\hat{S}^2 + \hat{\Omega}_0^2}{\hat{S}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$$

$$j\Omega = \Omega_p \frac{(j\hat{\Omega})^2 + \hat{\Omega}_0^2}{j\hat{\Omega}\,(BW)}$$

$$\Omega = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\,(BW)}$$



* $$\underbrace{\hat{\Omega}_{p_1} \cdot \hat{\Omega}_{p_2}}_{\text{fixed}} = \hat{\Omega}_{s_1} \hat{\Omega}_{s_2} = \hat{\Omega}_0^2$$

* $$\hat{\Omega}_0 = \sqrt{\hat{\Omega}_{p_1} \hat{\Omega}_{p_2}}$$

* if $\hat{\Omega}_{s_1} \hat{\Omega}_{s_2} > \hat{\Omega}_{p_1} \hat{\Omega}_{p_2}$ ⇒
* $\downarrow \Omega_{s_1} \rightarrow$ transition $\uparrow$
$\downarrow \Omega_{s_2} \rightarrow$ transition $\downarrow$ ✓ أحسن

$$\therefore \hat{\Omega}_{s_2} = \frac{\hat{\Omega}_{p_1} \hat{\Omega}_{p_2}}{\hat{\Omega}_{s_1}}$$

* if $\hat{\Omega}_{s_1} \hat{\Omega}_{s_2} < \hat{\Omega}_{p_1} \hat{\Omega}_{p_2}$

$$\therefore \hat{\Omega}_{s_1} = \frac{\hat{\Omega}_{p_1} \hat{\Omega}_{p_2}}{\hat{\Omega}_{s_2}}$$

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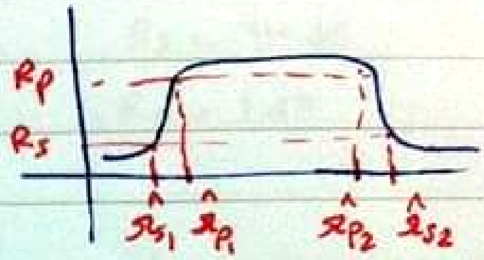
* Band pass filter design:

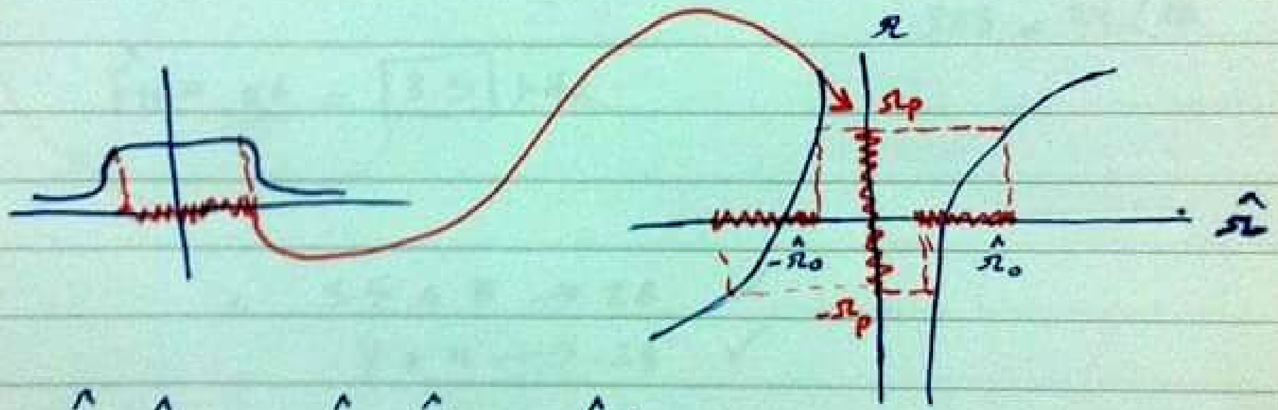
$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_0^2}{\hat{s}\,(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$$

$$j\Omega = \Omega_p \frac{(j\hat{\Omega})^2 + \hat{\Omega}_0^2}{j\hat{\Omega}\,(BW)}$$



$BW = \hat{\Omega}_{p2} - \hat{\Omega}_{p1}$

$$\boxed{\Omega = \cancel{j}\,\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\,(BW)}}$$



* $\underbrace{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}}_{\text{fixed}} = \hat{\Omega}_{s_1}\hat{\Omega}_{s_2} = \hat{\Omega}_0^2$

$$\hat{\Omega}_{p_1}\hat{\Omega}_{p_2} = \hat{\Omega}_0^2 \quad \text{(fixed)}$$

* if $\hat{\Omega}_{s_1}\hat{\Omega}_{s_2} > \hat{\Omega}_{p_1}\hat{\Omega}_{p_2} \rightarrow \hat{\Omega}_{s_2} = \dfrac{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}}{\hat{\Omega}_{s_1}}$

* if $\hat{\Omega}_{s_1}\hat{\Omega}_{s_2} < \hat{\Omega}_{p_1}\hat{\Omega}_{p_2} \rightarrow \hat{\Omega}_{s_1} = \dfrac{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}}{\hat{\Omega}_{s_2}}$

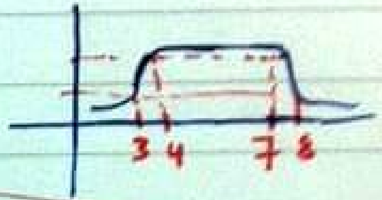
20

* Design a Butterworth BPF with the following specifications:

$\hat{f}_{s1} = \cancel{3}\ kHz$ ∴ 3.5 ✓ $\qquad \hat{f}_{s2} = 8\ kHz \qquad R_s = 40\ dB$

$\hat{f}_{p1} = 4\ kHz \qquad \hat{f}_{p2} = 7\ kHz \qquad R_p = 1\ dB$

⇒ $\hat{\Omega}_{p1}\,\hat{\Omega}_{p2} = (2\pi)^2\,(28 * 10^6) = \hat{\Omega}_0^2$



$BW = 2\pi * 3000$ rad/sec

$3 \times 8 = 24 < 28$

$\hat{f}_{s1} = \frac{28}{8} = \boxed{3.5}\ kHz$

∴ $3.5 * 8 \rightarrow 28$ ✓

$7 * 4 \rightarrow 28$ ✓

~~* Bandpass filter design~~

* Low Pass prototype specifications:

→ $\Omega_s = \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\,(BW)} \rightarrow \Omega_s = \frac{\hat{\Omega}_0^2 - \hat{\Omega}_s^2}{\hat{\Omega}_s\,(BW)}$

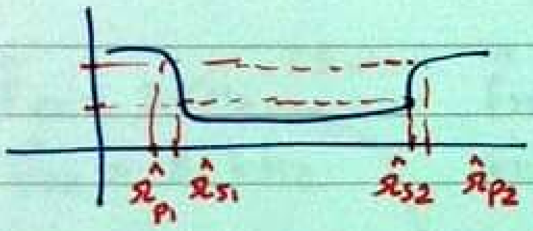
* let $\Omega_p = 1$ rad/sec $\rightarrow \Omega_s = \frac{(1)(28) - (3.5)^2}{3.5\,(3)} = \frac{3.5\,(8 - 3.5)}{3.5\,(3)} = 1.5$

$\left\{ \begin{array}{l} \Omega_p = 1 \text{ rad/sec} \rightarrow R_p = 1\,dB \\ \Omega_s = 1.5 \rightarrow R_s = 40\,dB \end{array} \right.$

```
>> BW = 2*pi*3000;
>> wo = 2π sqrt(28)*1000;
>> [N, Wn] = buttord(1, 1.5, 1, 40, 's');
>> [B, A] = butter(N, Wn, 's');
>> [num, den] = lp2bp(B, A, wo, BW);
```

$H_n(s)$ $H_0(s)$

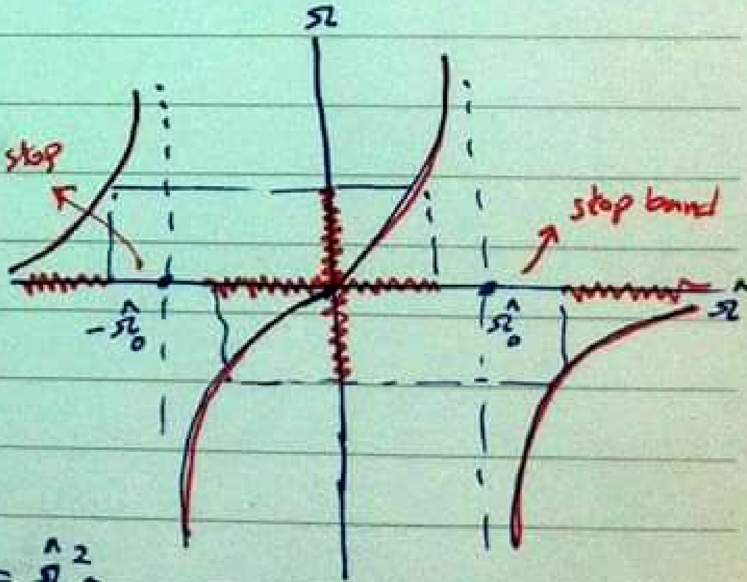
* Analog Bandstop filter:



$\rightarrow$ $BW = \hat{\Omega}_{s2} - \hat{\Omega}_{s1}$ (fixed)

$\rightarrow$ $s = F(\hat{s})$

$\rightarrow$ $s = \dfrac{\hat{\Omega}\,(BW)}{\hat{s}^2 + \hat{\Omega}_0^2}$



* $\cancel{j}\Omega = \Omega_s \cdot \cancel{j}\,\dfrac{\hat{\Omega}\,(BW)}{\hat{\Omega}_0^2 - \hat{\Omega}^2}$

* $\hat{\Omega}_{s1}\,\hat{\Omega}_{s2} = \hat{\Omega}_{p1}\,\hat{\Omega}_{p2} = \hat{\Omega}_0^2$

fixed

$\rightarrow$

* if $\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} > \hat{\Omega}_{s_1} \hat{\Omega}_{s_2}$

$$\therefore \quad \hat{\Omega}_{p_2} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_1}}$$

* if $\hat{\Omega}_{p_1} \hat{\Omega}_{p_2} < \hat{\Omega}_{s_1} \hat{\Omega}_{s_2}$

$$\therefore \quad \hat{\Omega}_{p_1} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_2}}$$

* Ex: Design ~~analon~~ analog butterworth band stop filter.

$\hat{f}_{p_1} = \cancel{3}\ kHz$ 3.5 ✓ $\quad \hat{f}_{s_1} = 4\ kHz \quad R_p = 1\,dB$

$\hat{f}_{p_2} = 8\ kHz \quad \hat{f}_{s_2} = 7\ kHz \quad R_s = 40\,dB$

⇒ $4 \times 7 = 28$

$$\hat{f}_{p_1} = \frac{28}{8} = 3.5\ kHz$$

→ $3.5 * 8 = 28$ ✓

$7 * 4 = 28$ ✓

→ $BW = 3\,KHZ$

$$\Omega_0^2 = (2\pi)^2 \, (28 * 10^6)$$

→

$\Rightarrow$ $\Omega = \Omega_s \cdot \dfrac{\hat{\Omega}_{p_1} \times BW}{\hat{\Omega}_0^2 - \hat{\Omega}_{p_1}^2}$ * let: $\Omega_s = 1$

$$= \frac{1\,(3.5 * 3)}{28 - (3.5)^2} = \frac{3}{4.5} = \frac{2}{3}$$

$$\left\{ \begin{array}{l} \Omega_p = 1 \\ \Omega_s = \frac{3}{2} = 1.5 \end{array} \right\}$$

>> w_0 - - -

>> BW - - -

>> [N, Wn]

>> [B, A]

the same

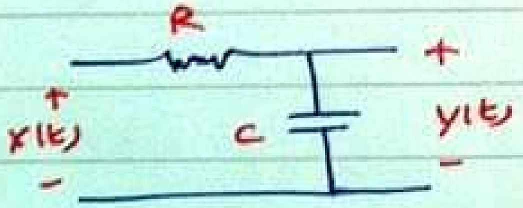
>> [num , den] = Lp2bs (B, A, w_0, BW)

band stop

* Obtaining discrete time equivalent of a continuous time filters:

$$\frac{Y(s)}{X(s)} = H(s) = \frac{a}{s+a}$$

$$H(s) = \frac{\omega_n}{s+\omega_n}$$

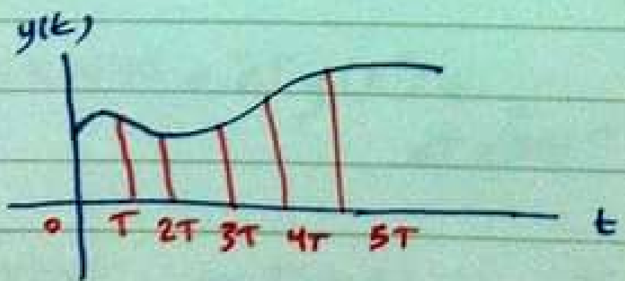


$\rightarrow$ $$H(s) = \frac{1/RC}{1/RC + s}$$

$\Rightarrow$ $$sY(s) + aY(s) = aX(s)$$

$$\frac{dY(t)}{dt} = -a\,y(t) + a\,x(t)$$

$$\int_0^t \frac{dy}{dt}\,.\,dt = -a\int_0^t y(t)\,dt + a\int_0^t x(t)\,dt$$



$\rightarrow$

$$\int_0^{kt} \frac{dy(t)}{dt}\,.dt = -a\int_0^{kT} y(t)\,dt + a\int_0^{kT} x(t)\,dt$$

$$= y(kT) - y(0)$$

$$\int_0^{(k-1)T} \frac{dy(t)}{dt}\,.dt = -a\int_0^{(k-1)T} y(t).\,dt + a\int_0^{(k-1)T} x(t)\,dt$$

$\rightarrow$ $y((k-1)T) - y(0) =$

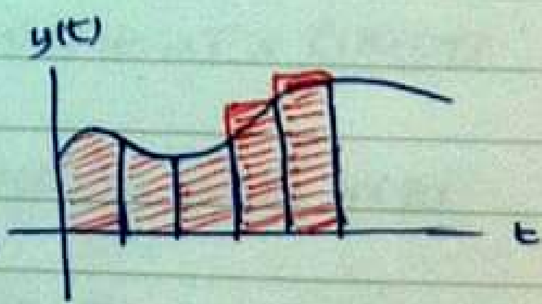
$\rightarrow$ $$y(kT) - y((k-1)T) = -a\int_{(k-1)T}^{kT} y(t)\,dt + \int_{(k-1)T}^{kT} x(t)\,dt$$

① backward difference method:

* $\int_{(k-1)T}^{kT} y(t)\,dt = T\,y(kT)$

* $\int_{(k-1)T}^{kT} x(t)\,dt = T\,x(t)$

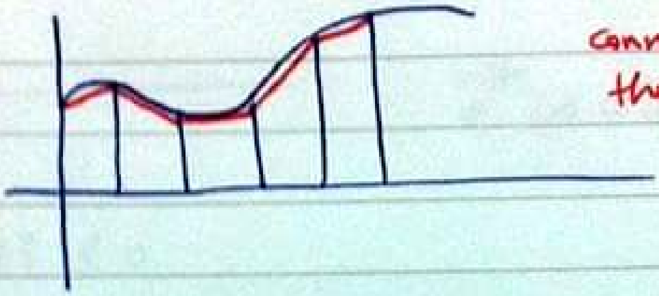
$$y(kT) - y(k-1)T) = -aT\,y(kT) + aT\,x(kT)$$



$$Y(z) - z^{-1}Y(z) = -aT\,Y(z) + aT\,X(z)$$

$$Y(z)\,[1 - z^{-1} + aT] = aT\,X(z)$$

(3) Bilinear method: ✓



connect ~~between~~ them with lines

$$* \quad \int_{(k-1)T}^{kT} y(t)\,dt = \frac{T}{2}\left[y(kT) + y((k-1)T)\right]$$

$$* \quad \int_{(k-1)T}^{kT} x(t)\,dt = \frac{T}{2}\left[x(kT) + x((k-1)T)\right]$$

$$\rightarrow \quad y(kT) - y((k-1)T) = -\frac{aT}{2}\left[y(kT) + y(k-1)T\right] + \frac{aT}{2}\left[x(kT) + x((k-1)T)\right]$$

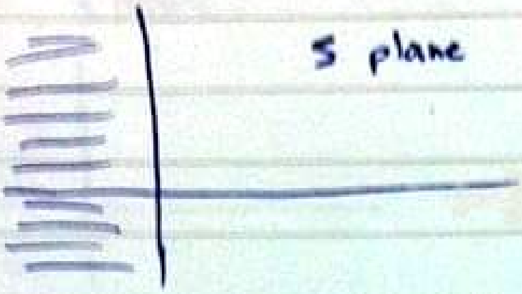
$$\Rightarrow \quad Y(z) - z^{-1}Y(z) = -\frac{aT}{2}\left[Y(z) + z^{-1}Y(z)\right] + \frac{aT}{2}\left[X(z) + z^{-1}X(z)\right]$$

$$Y(z)\left[1 - z^{-1} + \frac{aT}{2} + \frac{aT}{2}z^{-1}\right] = \frac{aT}{2}\left(1 + z^{-1}\right)X(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{a}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

$$\rightarrow \quad \boxed{S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$ bilinear transf.

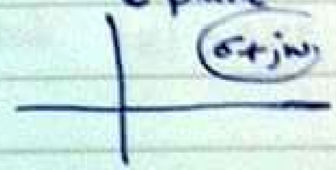
*



$Re\{s\} < 0$

$e^{-\lambda t}(A \cos\omega_0 t + B \sin\omega_0 t)$

$Re\{s\} < 0$



$$= Re\left\{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\} < 0$$

$$= Re\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\} < 0$$

$$= Re\left\{\frac{z-1}{z+1}\right\} < 0$$

$$= Re\left\{\frac{\sigma+j\omega-1}{\sigma+j\omega+1}\right\} < 0$$

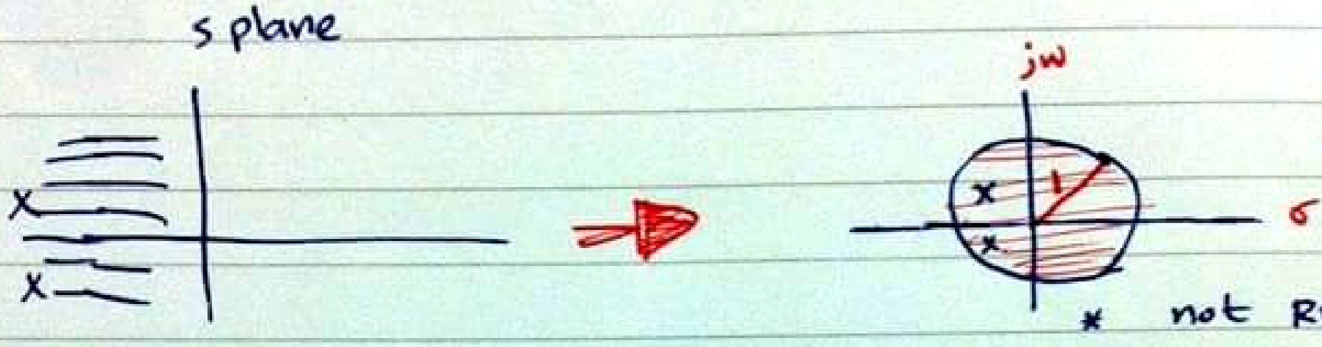
complex conjugate

$$= Re\left\{\frac{\sigma+j\omega-1}{\sigma+j\omega+1}\cdot\frac{(\sigma+1)-j\omega}{(\sigma+1)-j\omega}\right\} < 0$$

$$= Re\left\{\frac{\sigma^2-1+\omega^2+j\omega 2}{(\sigma+1)^2+\omega^2}\right\} < 0$$

$$\sigma^2-1+\omega^2 < 0$$

$$\boxed{\sigma^2+\omega^2 < 1}$$



* not ROC

it is mapping

$|z| > |\alpha|$

$$H(z) = \frac{1}{1 - 2z^{-1}} \qquad |z| > 2$$

$$h[n] = 2^n u[n]$$

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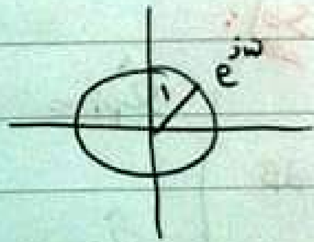
* Bilinear transformation:

$$S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

* Prewarping process:

$T = 2$

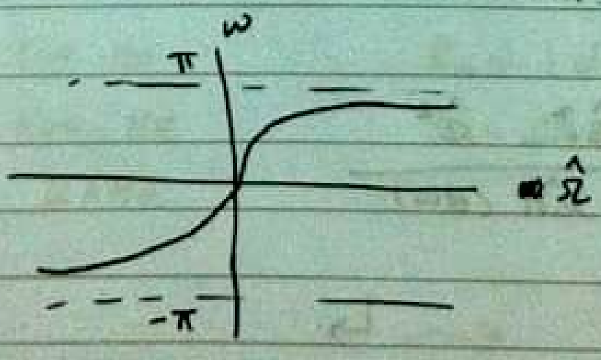
$$S = \frac{1-z^{-1}}{1+z^{-1}}$$



$$\boxed{j\hat{\Omega} = \frac{1-e^{-jw}}{1+e^{-jw}}} \rightarrow \text{used in prewarping process}$$

$$j\hat{\Omega} = \frac{\left[e^{\frac{jw}{2}} - e^{-\frac{jw}{2}}\right]\cancel{e^{-\frac{jw}{2}}}}{\frac{\left[e^{\frac{jw}{2}} + e^{-\frac{jw}{2}}\right]}{2}\cancel{e^{-\frac{jw}{2}}}}$$

$$\hat{\Omega} = \frac{\sin\left(\frac{w}{2}\right)}{\cos\left(\frac{w}{2}\right)} \rightarrow \boxed{\hat{\Omega} = \tan\left(\frac{w}{2}\right)}$$



DATE

* **Ex:** Design of band pass FIR digital filter (Butterworth)

$$\left.\begin{aligned} \omega_{p1} &= 0.45\pi \\ \omega_{p2} &= 0.65\pi \end{aligned}\right\} R_b = 1\,dB$$

$$\left.\begin{aligned} \omega_{s1} &= 0.3\pi \\ \omega_{s2} &= 0.75\pi \end{aligned}\right\} R_s = 40\,dB$$

Sol:

$R_p = 1dB$

$$\hat{\Omega}_{p1} = \tan\left(\frac{\omega_{p1}}{2}\right) = \tan\left(\frac{0.45\pi}{2}\right) = 0.854$$

$$\hat{\Omega}_{p2} = \tan\left(\frac{\omega_{p2}}{2}\right) = \tan\left(\frac{0.65\pi}{2}\right) = 1.63185$$

$R_s = 40dB$

$$\hat{\Omega}_{s1} = \tan\left(\frac{\omega_{s1}}{2}\right) = \tan\left(\frac{0.3\pi}{2}\right) = \cancel{0.5905} \quad 0.577$$

$$\hat{\Omega}_{s2} = \tan\left(\frac{\omega_{s2}}{2}\right) = \tan\left(\frac{0.75\pi}{2}\right) = 2.9142$$

→ $BW = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 0.7777$

→ $\hat{\Omega}_0^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = 1.3937 \longrightarrow \hat{\Omega}_0 = 1.1805$

→ $\hat{\Omega}_{s1}\hat{\Omega}_{s2} = 1.230$

→ $$\hat{\Omega}_{s1} = \frac{1.3937}{2.4142} = \boxed{0.577}$$

→ $$\Omega = \Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}_{s1}(BW)}$$

↳

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$$\Rightarrow \Omega = \Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}_{s_1} (BW)}$$

→ let ($\Omega_p = 1$): $\Omega s = \frac{(1)(1.3937 - 0.3333)}{(0.577)(0.7777)} = 2.3617$

* low pass prototype: $\Omega_p = 1$ $R_p = 1dB$

$\Omega_s = 2.3617$ $R_s = 40\,dB$

* $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k_2)}$, $k = \frac{\Omega_p}{\Omega_s}$, $\Omega = \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$

* matlab:

```
>> [N, Wn] = buttord (1, 23617, 1, 40, 'S') ;
Hlp(s) >> [B, A] = butter (N, Wn, 'S') ;
Hbp(s) >> [BT, AT] = lp2bp ( B, A, 1.1805 , 0.7777)
Hbp(s) >> [num, den] = bilinear (BT, AT, 0.5) ;
```

* Ex: Design of highpass ~~IIR~~ FIR digital filter:

chebyshev type 1:

$F_p = 700$ Hz $R_p = 1\,dB$

$F_s = 500$ Hz $R_s = 32\,dB$

$F_T = 2\,kHz$

↝

4

$x(t)$ — sampling — $x[n]$ — DTs

sol: $\omega_p = \frac{2\pi f_p}{F_T} = \frac{2\pi(700)}{2000} = 0.7\pi$, $R_p = 1dB$

$\omega_s = \frac{2\pi f_s}{F_T} = \frac{2\pi(500)}{2000} = 0.5\pi$, $R_s = 32dB$

$\hat{\Omega}_p = \tan\left(\frac{0.7\pi}{2}\right) = 1.9626$

$\hat{\Omega}_s = \tan\left(\frac{0.5\pi}{2}\right) = 1$

$\Omega_s = \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s}$; (let $\Omega_p = 1$)

$\Omega_s = \frac{(1)(1.9626)}{(1)}$

$\boxed{\Omega_s = 1.9626}$

* matlab.

```
>> [N,wn] = cheb1ord (1, 1.9626, 1.32, 's');
>> [B,A] = cheby1 (N2, 1, wn, 's');
>> [BT,AT] = lp2hp (B, A, 1.9676);
>> [num,den] = bilinear (BT, AT, 0.5);
```

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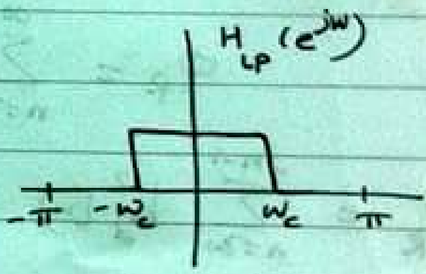
* FIR filter design based on windowed fourier series :

$$h[n] = [\alpha_0 \ \alpha_1 \ \cdots \ \alpha_N]$$

$$\frac{Y(z)}{X(z)} = H(z) = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} \cdots + \alpha_N z^{-N}$$

* least integral - squared error design on FIR filters:

→ let $H_d(e^{jw})$ denote the desired frequency response function



→ $H_d(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] \, e^{-jwn}$

periodic in w
with period of 2π

→ $h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{jw}) \cdot e^{jwn} \, dw \qquad -\infty < n < \infty$

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* infinite length and noncausal :

one objective is to find a finite - duration impulse response $\{h_t[n]\}$ of length $(2M+1)$ whose DTFT $H_t(e^{jw}) = \sum_{-M}^{M} h_t[n]\, e^{-jwn}$ which approximation the desired DTFT $H_d(e^{jw})$ in some sense

* $$\Phi_R = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_t(e^{jw}) - H_d(e^{jw}) \right|^2 dw$$

→ using parseval's theorem :

$$\Phi_R = \sum_{n=-\infty}^{\infty} \left| h_t[n] - h_d[n] \right|^2$$

$$\Phi_R = \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=-M}^{M} \left| h_t[n] - h_d[n] \right|^2 + \sum_{M+1}^{\infty} h_d^2[n]$$

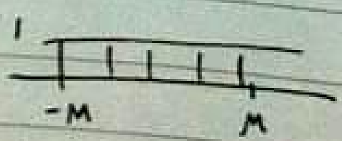
* $$\boxed{h_t[n] = h_d[n] \qquad -M \le n \le M}$$

window

rectangular

* $$h_t[n] = h_d[n]\, W_R[n]$$

where : $$w_R[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$



* $h[n] = h_t[n-M]$ causal system

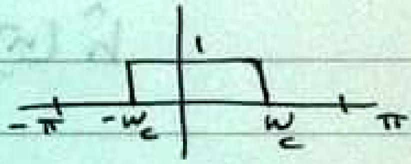
→ the magnitude will not change
→ the phase will change by wM radians

$$\Rightarrow \hat{H}(e^{jw}) = H_t(e^{jw}) \cdot e^{jwM}$$

Impulse response of ideal filters :

① ideal LPF :

$$H_{LP}(e^{jw}) = \begin{cases} 1 & |w| \leq w_c \\ 0 & w_c < |w| \leq \pi \end{cases}$$



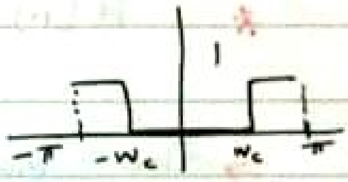
$$h_{LP}[n] = \frac{\sin w_c n}{\pi n} \qquad -\infty < n < \infty$$

$$h_t[n] = \frac{\sin(w_c n)}{\pi n} \qquad -M \leq n \leq M$$

$$\hat{h}[n] = h_t[n-M] = \frac{\sin w_c[n-M]}{\pi[n-M]} \qquad 0 \leq n \leq 2M$$

Scanned by CamScanner

② ideal HPF :

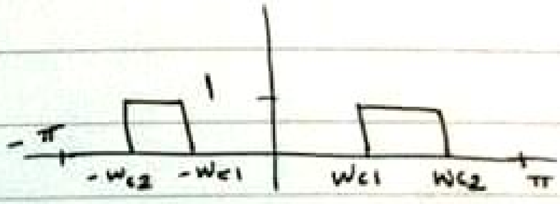


$$H_{HP}(e^{jw}) = \begin{cases} 0 & |w| < w_c \\ 1 & w_c \leq |w| \leq \pi \end{cases}$$

$$h_{hp}[n] = \begin{cases} 1 - \dfrac{w_c}{\pi} & n = 0 \\ \dfrac{-\sin(w_c n)}{\pi n} & |n| > 0 \end{cases}$$

$$\hat{h}[n] = \begin{cases} \dfrac{-\sin(w_c[n-M])}{\pi[n-M]} & 0 \leq n \leq M-1 \\ 1 - \dfrac{w_c}{\pi} & M = 1 \\ \dfrac{-\sin(w_c[n-M])}{\pi[n-M]} & M+1 \leq n \leq 2M \end{cases}$$

③ ideal BPF : (band pass)



$$H_{BP}(e^{jw}) = \begin{cases} 0 & |w| < w_{c1} \\ 1 & w_{c1} \leq |w| \leq w_{c2} \\ 0 & w_{c2} < |w| \leq \pi \end{cases}$$

$$h_{BP}[n] = \frac{\sin w_{c2} n}{\pi n} - \frac{\sin w_{c1} n}{\pi n} \qquad |n| \geq 0$$

$\hat{h} = \ldots$

④ ideal BSF : (band stop)

$$H_{BS}(e^{jw}) = \begin{cases} 1 & |w| \leq w_c \\ 0 & w_{c1} \leq |w| \leq w_{c2} \\ 1 & w_{c2} \leq |w| \leq \pi \end{cases}$$

$$h_{BS}[n] = \begin{cases} 1 - \dfrac{w_{c2} - w_{c1}}{\pi} & n = 0 \\ \dfrac{\sin(w_{c1} n)}{\pi n} - \dfrac{\sin(w_{c2} n)}{\pi n} & |n| > 0 \end{cases}$$

$\hat{h} = \ldots$

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* Hann :

$$w[n] = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \le n \le M$$

* Hamming :-

$$w[n] = 0.54 + 0.46\cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \le n \le M$$

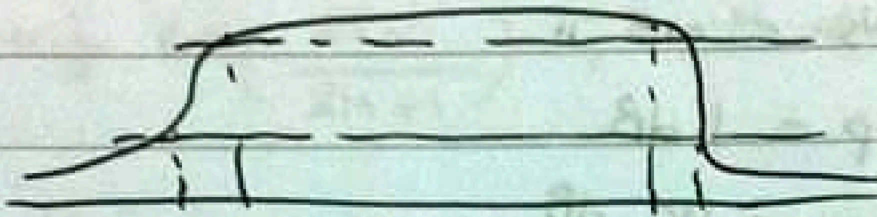
* Blackman :

$$w[n] = 0.42 + 0.5\cos\left(\overset{w}{\frac{2\pi n}{2M+1}}\right) + 0.08\cos\left(\frac{4\pi n}{2M+1}\right)$$

$$-M \le n \le M$$

* Type of window	minimum stop band attenuation	Transition Bandwidth (Δw)
1) rectangular window	20.9 dB	$\frac{0.92\pi}{M}$
2) Hann	43.9 dB	$\frac{3.11\pi}{M}$
3) Hamming	54.5 dB	$\frac{3.32}{M}\pi$
4) Blackman	75.3 dB	$\frac{5.56}{M}\pi$

$\simeq \frac{C}{M}$



$$\omega_{c1} = \frac{\omega_{s1} + \omega_{p1}}{2} \qquad \frac{\omega_{p2} + \omega_{s2}}{2} = \omega_{c2}$$

$$\omega_{p1} + \omega_{s1} = \boxed{\Delta W_1} \qquad \boxed{\Delta W_2} = \omega_{s2} - \omega_{p2}$$