Spring017

Dr. Dia Abu Unadi

BY:
Hamza Abedrabbo

Powerunit-ju.com

DSP (Digital Signal Processing) 7/2/2017

Dr. Dia Abuuradi Spring 017

lec #1

*Chapter 2

Discrete-time Signals in time Domain:-

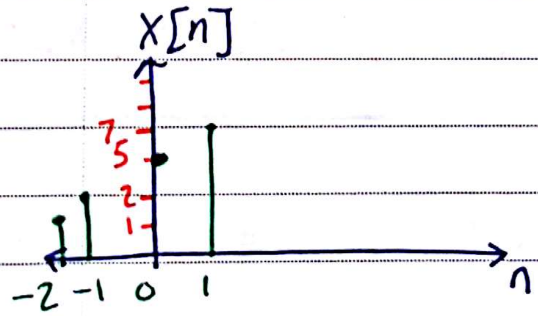
- Continuos time Signal $X(t)$
- Discrete time Signal ≡ Sequence (متتالية)

Representations:

$$\{X[n]\} = \{\ldots, 1, 2, 5, 7, \ldots\}$$

$\ldots -2\ -1\ \underset{n=0}{\uparrow}\ 1\ \ 2 \ldots$

Infinite Sequence, Infinite DT Signal



$$X[n] = X(t)\Big|_{t=nT} = \cos(\Omega t)\Big|_{t=nT} = \cos(\Omega T n)$$

Ω: rad/sec continuos frequency

$$= \cos(\omega n)$$

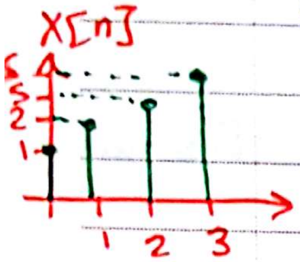
$$\boxed{\omega = \Omega T}$$ rad/sample

DT frequency

FNE APPLE

No.

$x[n]=\{\underset{\uparrow}{1},2,5,6\}$ finite-Sequence



A length-4 Sequence

$x[0]=1$, $x[1]=2$, $x[2]=5$, $x[3]=6$

$x[n]=A\cos(\omega_0 n)$, $-\infty<n<\infty$

$x[n]=\{1,2,\underset{\uparrow}{5},6\}$

$x[-2]=1$, $x[-1]=2$, $x[0]=5$, $x[1]=6$

* Finite Sequence of length N

$N_1 \leq n \leq N_2$ $N=N_2-N_1+1$

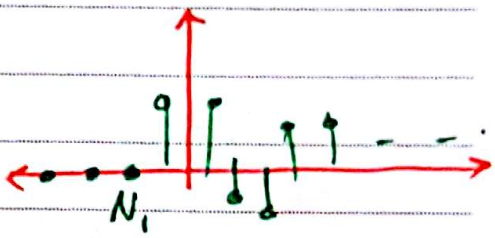
$x[n]=5n+3$, $-2\leq n\leq 6$

$\cancel{x[n]=e^{j\omega_0 n}, -\infty<n<\infty}$

* Infinite Sequences

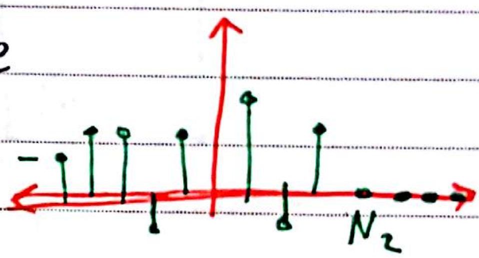
- Right Sided Sequence

$x[n] = 0$ ~~[illegible]~~ for $n < N_1$

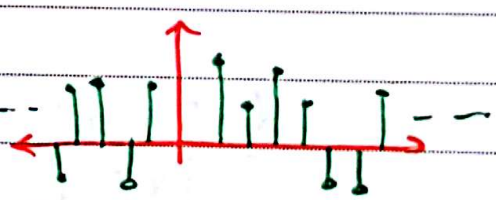


- left Sided Sequence

$x[n] = 0$ for $n > N_2$



- Two sided Sequence



* Complex Sequence

$$x[n] = x_{re}[n] + j\, x_{im}[n]$$

$$x[n] = \{1+j \;,\; \underset{\uparrow}{0.5 - j3} \;,\; 2+j1.5\}$$

$$x_{re}[n] = \{1 \;,\; \underset{\uparrow}{0.5} \;,\; 2\}$$

$$x_{im}[n] = \{1 \;,\; -3 \;,\; 1.5\}$$

No.

Matrix Notation

$$\underline{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} \quad , \quad \underline{x} = [x[0], x[1], x[2] \,..\, x[N-1]]^T$$

Vector

$N \times 1$

Time Domain
(Lower case)

1) Finite Sequence , 2) Starts from $n=0$

$$\underline{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} \quad , \quad \underline{X} = [X[0], X[1], .. X[N-1]]^T$$

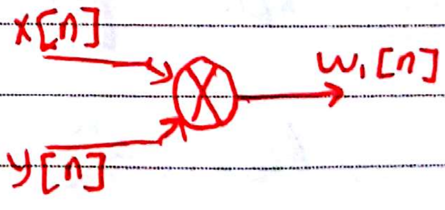
Frequency Domain
(upper case)

* Elementary operations on Sequences

1] Product:

$\cancel{w_1[n] = x[n]\, Y[n]}$

$w_1[n] = x[n]\, y[n]$



No.

$x[n] = \{1, \underset{\uparrow}{2}, 5, 5\} = \{1, \underset{\uparrow}{2}, 5, 5, 0\}$

$y[n] = \{\underset{\uparrow}{2}, 2, 1, 1\} = \{0, 2, 2, 1, 1\}$

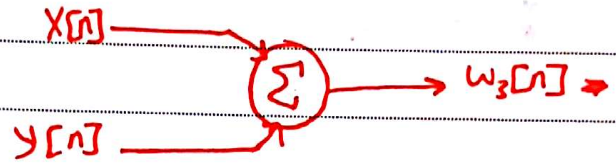
$w[n] = \{0, 4, 10, 5, 0\}$

2] multiplication by constant

$$w_2[n] = A\,x[n]$$

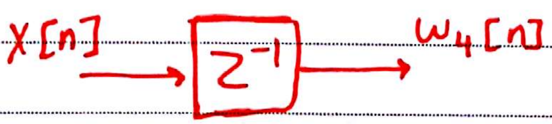
3] Addition

$$w_3[n] = x[n] + y[n]$$

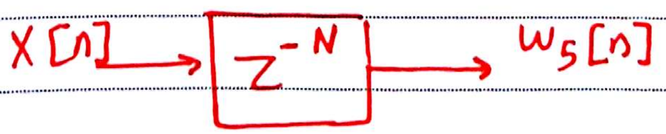


4] Unit delay

$$w_4[n] = x[n-1]$$



$$w_5[n] = x[n-N]$$



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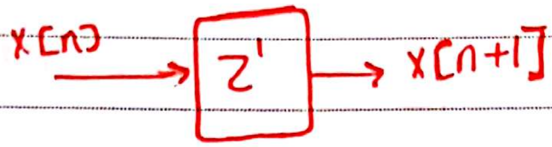
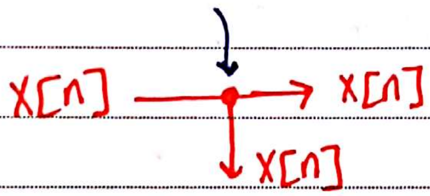
5] Unit advance

$w_6[n] = x[n+1]$



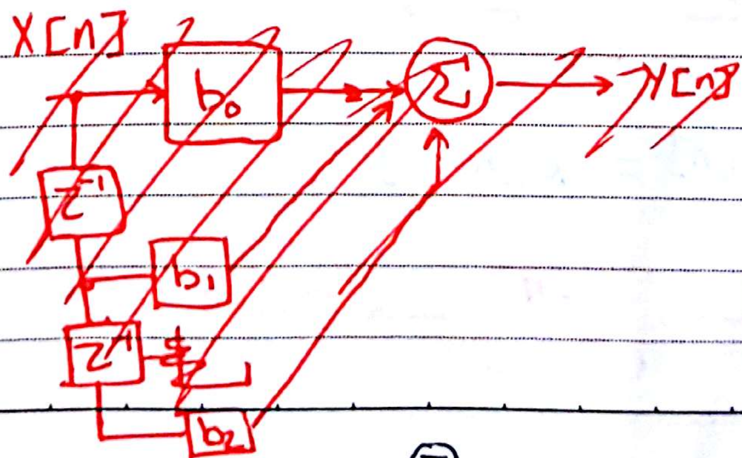
6] Pick of Point



* Combined Operations

$\cancel{y[n] = b_0 x[n] + b_1 x[n+1] + b_2 x[n+2]}$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$



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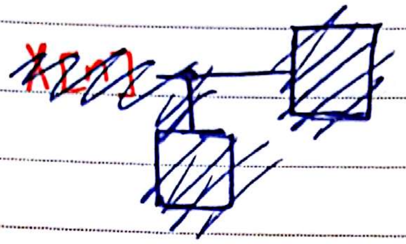

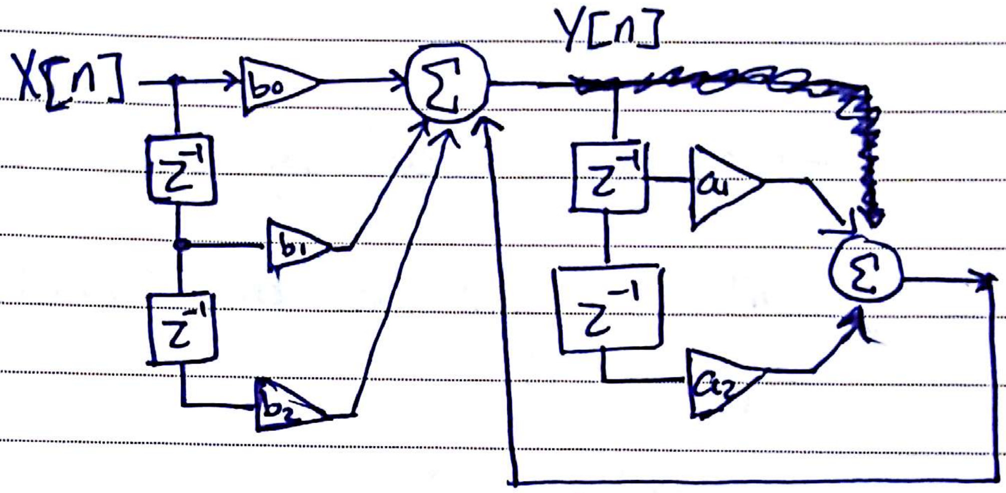

* Classification of Sequences:-

1] Classification based on Symmetry:

- Conjugate Symmetric Sequence:-

X[n] is conjugate Symmetric Sequence if

$$X[n] = X^*[-n]$$

No.

If $X[n]$ is Real.

$X[n] = X[-n]$, even Symmetry

- Conjugate antiSymmetric Sequence :-

$X[n]$ is Conjugate antisymmetric sequence if

$$X[n] = -X^*[n]$$

If $X[n]$ is Real

$X[n] = -X[-n]$, odd Symmetry

Any $X[n]$

$$X[n] = X_{cs}[n] + X_{ca}[n]$$

$$- X_{cs}[n] = \frac{X[n] + X^*[n]}{2}$$

$$- X_{ca}[n] = \frac{X[n] - X^*[-n]}{2}$$

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No.

$$X_{cs}[n] = \frac{X[n] + X^*[-n]}{2}$$

Prove $X[n] = X^*[-n]$ for CS.

$$\cancel{X_{cs}}\ X_{cs}[-n] = \frac{X[-n] + X^*[n]}{2}$$

$$X_{cs}^*[-n] = \frac{X^*[-n] + X[n]}{2}$$

$$X_{cs}[n] = X_{cs}^*[-n]$$

$$X_{ca}[n] = \frac{X[n] - X^*[-n]}{2}$$

Prove $X[n] = -X^*[-n]$ for ca.

$$-X_{ca}[-n] = -\left(\frac{X[n] - X^*[n]}{2}\right)$$

$$-X_{ca}[-n] = \frac{X^*[n] - X[-n]}{2}$$

$$-X_{ca}^*[-n] = \frac{X[n] - X^*[-n]}{2}$$

$$X_{ca}[n] = -X_{ca}^*[-n]$$

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No.

~~Ex: $x[n] = \{0 \quad 1+4j \quad -2+3j$~~ $4-2j$

Ex: $x[n] = \{0 \quad 1+j4 \quad -2+j3 \quad \underset{\uparrow}{4-j2} \quad -5-j6 \quad 2j \quad 3\}$

$x^*[n] = \{0 \quad 1-j4 \quad -2-j3 \quad \underset{\uparrow}{4+j2} \quad -5+j6 \quad -2j \quad 3\}$

~~$x^*[-n] = \{0$~~

$x^*[-n] = \{3 \quad -j2 \quad -5+j6 \quad \underset{\uparrow}{4+j2} \quad -2-j3 \quad 1-j4 \quad 0\}$

$x_{cs}[n] = \{1.5, 0.5+j, -7/2 + j9/2,$

---

* Periodic & Aperiodic Sequences

$\tilde{x}[n] = \tilde{x}[n+kN]$ for all n.

k is any integer

N is the period

N_f : The fundamental period of the least N satisfies the periodicity

No.

$\tilde{X}_a[n]$: Period N_a

$\tilde{X}_b[n]$: Period N_b

$\tilde{X}_c[n] = \tilde{X}_a[n] + \tilde{X}_b[n]$

$N = LCM(N_a, N_b)$

↳ least common multiple

$\cancel{LCM = \frac{N_a * N_b}{GCM}}$

$LCM(N_a, N_b) = \frac{N_a * N_b}{GCD(N_a, N_b)}$

Greatest common Division

* The Same applies for multiplication

No.

* Energy & Power Sequences

The amount of Energy in a sequence $X[n]$

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Ex:

$X[n] = \{1-j \quad 1+j \quad 2j\}$

$$\mathcal{E}_x = \{(1-j)(1+j) + (1+j)(1-j) + (2j)(-2j)\}$$
$$= 2 + 2 + 4$$
$$= 8$$

- For a finite Sequence $\Rightarrow$ finite Energy
- for infinite Sequence $\Rightarrow$:

Ex:

$$X_1[n] = \begin{cases} \frac{1}{n} & , n \geq 1 \\ 0 & , n \leq 0 \end{cases}$$

$$\mathcal{E}_{x_1} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

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No.

Ex:

$$X_2[n] = \begin{cases} 1/\sqrt{n} & , n \geq 1 \\ 0 & , n \leq 0 \end{cases}$$

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^2 = \sum_{n=1}^{\infty} \frac{1}{n} \quad \underline{\text{diverge}}$$

The average power of a sequence

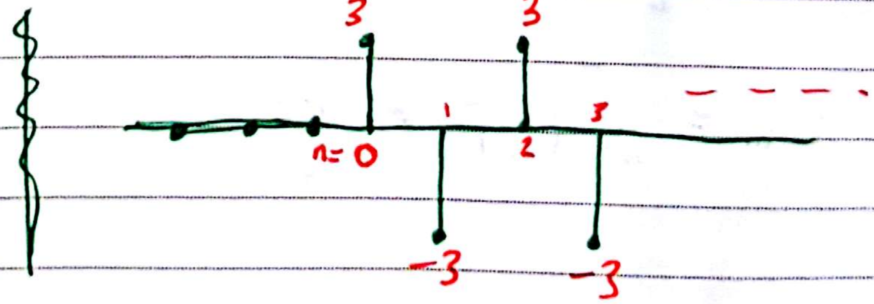
$$P_x = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |X[n]|^2$$

- for a periodic sequence ⇒ :-

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{X}[n]|^2$$

Ex: $X[n] = \begin{cases} 3(-1)^n & , n \geq 0 \\ 0 & , n < 0 \end{cases}$

find P_x ??



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$$P_x = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{n=0}^{k} \left(3(-1)^n\right)^2$$

$$= \lim_{k \to \infty} \frac{9}{2k+1} \boxed{\sum_{n=0}^{k} 1} \longrightarrow (k+1)$$ مجموع وحدات ب

$$= \lim_{k \to \infty} \frac{9(k+1)}{2k+1} = 4.5$$

$$\sum_{n=0}^{k} 9 = 9(k+1)$$

- finite Energy ⇒ Energy Sequence

$P = 0$ [zero average Power]

- finite average Power ⇒ Power Sequence

$E = \infty$ [infinite amount of Energy]

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* Other types of classifications:

1] Bounded Sequence :-

$$|x[n]| \leq B_x < \infty \text{, for all } n.$$

2] Absolutely ~~seq~~ Summable sequence:-

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

3] Square Summable Sequence:-

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

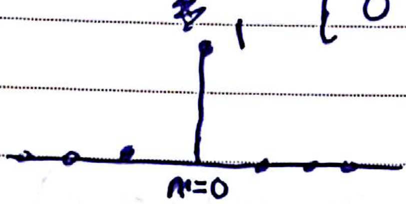
* ~~Typical Sequence~~

No.

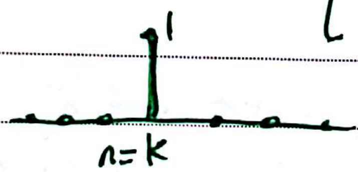
*Typical Sequences:

1] Unit Sample Sequence (unit impulse)

$$\delta[n] = \begin{cases} 1 & , n=0 \\ 0 & , n\neq 0 \end{cases}$$

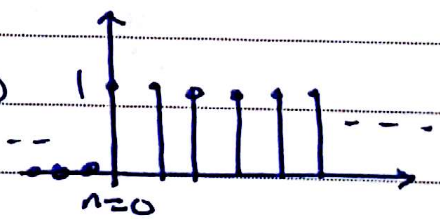


$$\delta[n-k] = \begin{cases} 1 & , n=k \\ 0 & , n\neq k \end{cases}$$

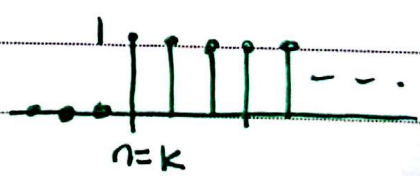


2] Unit Step Sequence

$$\mu[n] = \begin{cases} 1 & , n\geq 0 \\ 0 & , n<0 \end{cases}$$



$$\mu[n-k] = \begin{cases} 1 & , n\geq k \\ 0 & , n<k \end{cases}$$



$$\mu[n] = \sum_{m=0}^{\infty} \delta[n-m] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\mu[n-k] = \sum_{m=k}^{\infty} \delta[n-m]$$

No.

3] Sinusoidal Sequences:

$$x[n] = A\cos(\omega_0 n + \phi) \quad , -\infty < n < \infty$$

The Sinusoidal sequence is periodic if

$$\tilde{x}[n] = \tilde{x}[n+N]$$

$$x[(n+N)] = A\cos(\omega_0 [n+N] + \phi)$$

$$= A\cos((\omega_0 n + \phi) + \omega_0 N)$$

$$= A[\cos(\omega_0 n + \phi) \cancel{\pm} \cos\cancel{\sin}(\omega_0 N)$$

$$- \sin(\omega_0 n + \phi) \sin(\omega_0 N)]$$

$$\boxed{\omega_0 N = 2\pi r} \qquad \boxed{\frac{2\pi}{\omega_0} = \frac{N}{r}}$$

No.

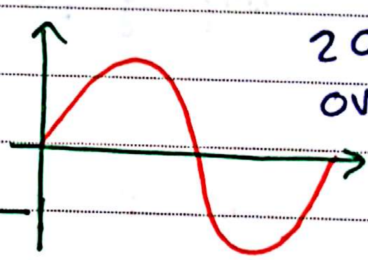
the Sinusoidal sequence is periodic if:

$$\frac{2\pi}{\omega} = \frac{N}{r}$$

N & r are integers.

Ex: $x_1[n] = \cos(.1\pi n)$ periodic

$$\frac{2\pi}{0.1\pi} \stackrel{?}{=} \frac{N}{r}$$



20 Samples over 1 cycle

$$\frac{N}{r} = \frac{20}{1} \frac{\text{samples}}{\text{cycle}}$$

Periodic

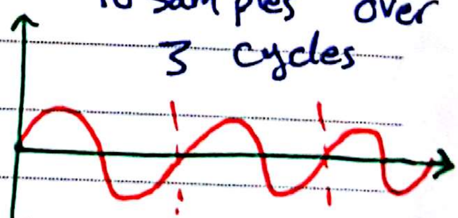
Ex: $x[n] = \cos(0.1n)$

$$\frac{2\pi}{0.1} \stackrel{?}{=} \frac{N}{r}$$

$$\frac{20\pi}{1} \neq \frac{N}{r}$$

Ex: $x[n] = \cos(0.6\pi n)$

10 Samples over 3 cycles



$$\frac{2\pi}{0.6\pi} = \frac{20}{6} = \frac{10}{3}$$

$N_f = 10$ samples over 3 cycles

Periodic

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No.

* The Sampling Process

$X(t) = A\cos(\Omega t + \phi)$

- Sampling frequency F_T

- Sampling period $T = \frac{1}{F_T}$

$\cancel{X(t)\big|_{t=nT} = A\cos(\Omega n t}$

$X(t)\big|_{t=nT} = A\cos(\Omega nT + \phi)$

ω

$X[n] = A\cos(\Omega T n + \phi)$

$X[n] = A\cos(\omega n + \phi)$, $\boxed{\omega = \Omega T}$

$t = [0:0.1:10] \rightarrow 101$ pts

No.

Ex: $g_1(t) = \cos(6\pi t)$, 3Hz

$g_2(t) = \cos(14\pi t)$, 7Hz

$g_3(t) = \cos(26\pi t)$, 13Hz

Sampling rate at $F_T = 10$Hz

Sampling Period $= 0.1$sec T

Sol: • $g_1[n] = \cos(.6\pi n)$

• $g_2[n] = \cos(1.4\pi n)$

$= \cos((2-.6)\pi n)$

$= \cos(2\pi n - .6\pi n)$

$\cancel{= \cos(2\pi n)\cos(.6\pi n)}$

$= \underbrace{\cos(2\pi n)}_{1}\cos(.6\pi n) + \underbrace{\sin(2\pi n)}_{\text{zero}}\sin(.6\pi n)$

$= \cos(.6\pi n) = g_1[n]$

• $g_3[n] = \cos(2.6\pi n)$

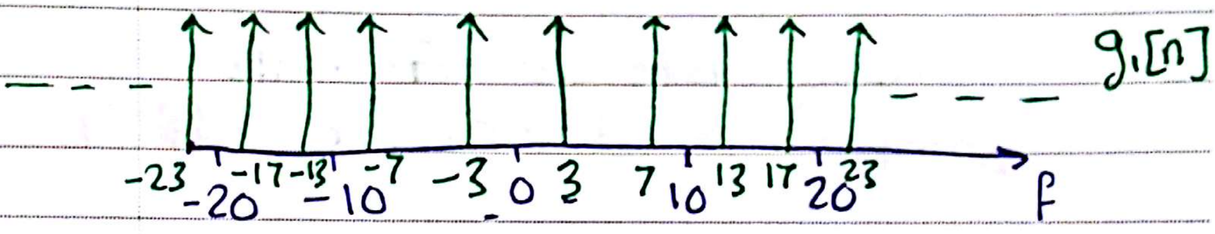
$= \cos((2+.6)\pi n)$

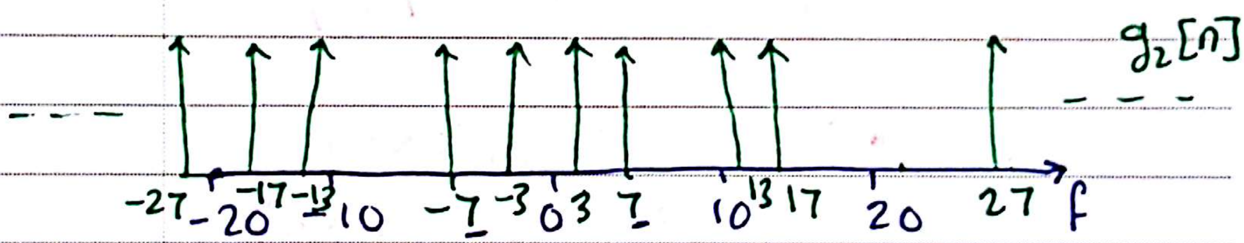
$= \underbrace{\cos(2\pi n)}_{1}\cos(.6\pi n) - \overbrace{\sin(2\pi n)}^{\text{zero}} * \sin(.6\pi n)$

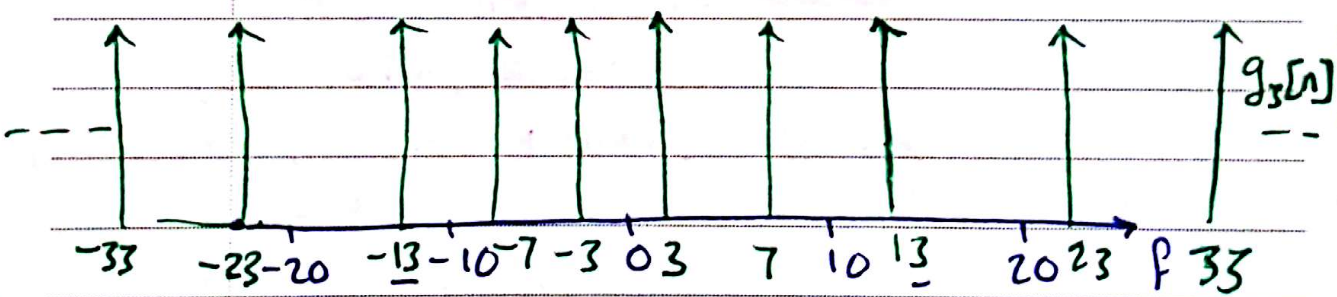
$= \cos(.6\pi n) = g_1[n]$

No.

$$g_1[n] = g_2[n] = g_3[n]$$







Notes:

* All Spectrums are the same

* $g_2[n]$, $g_3[n]$ have over sampling intruders.

* Discrete-time Systems:-

A system is an input-output Relationship

$$X[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n]$$

$$y[n] = T\left[X[n]\right]$$

$$X[n] \xrightarrow{T} y[n]$$

- Classifications of DT systems:

1. Linear & nonlinear systems

$$X[n] = \alpha X_1[n] + \beta X_2[n]$$

The System is linear if

$$T\left[\alpha X_1[n] + \beta X_2[n]\right] = \alpha T\left[X_1[n]\right] + \beta T\left[X_2[n]\right]$$

if we assume that:

$$T\left[X_1[n]\right] = y_1[n]$$

$$T\left[X_2[n]\right] = y_2[n]$$

$$\alpha X_1[n] + \beta X_2[n] \xrightarrow{T} \alpha y_1[n] + \beta y_2[n]$$

No.

Ex: ① $y[n] = n\, x[n]$

② $y[n] = x^2[n]$

③ $y[n] = A\, x[n] + \beta$

Sol:

$\boxed{1}$ assume $x[n] = \alpha x_1[n] + \beta x_2[n]$

For $x_1[n]$

$y_1[n] = n\, x_1[n]$

For $x_2[n]$

$y_2[n] = n\, x_2[n]$

For $x[n] = \alpha x_1[n] + \beta x_2[n]$

$$
\begin{aligned}
y[n] &= n\, x[n] \\
&= n\left[\alpha x_1[n] + \beta x_2[n]\right] \\
&= \alpha n\, x_1[n] + \beta n\, x_2[n] \\
&= \alpha\, y_1[n] + \beta\, y_2[n]
\end{aligned}
$$

The System is linear.

No.

$\boxed{2}$ for $X_1[n]$

$y_1[n] = X_1^2[n]$

for $X_2[n]$

$y_2[n] = X_2^2[n]$

for $X[n] = \alpha X_1[n] + \beta X_2[n]$

$$
\begin{aligned}
y[n] &= X^2[n] \\
&= [\alpha X_1[n] + \beta X_2[n]]^2 \\
&= \alpha^2 X_1^2[n] + \beta^2 X_2^2[n] + 2\alpha\beta X_1[n] X_2[n] \\
&\neq \alpha y_1[n] + \beta y_2[n]
\end{aligned}
$$

The System is nonlinear.

No.

2.* Shift-Invariant Systems & Variant

(Classification #2 of DT Systems)

$x[n] \xrightarrow{T} y[n]$

$x[n-k] \xrightarrow{T} y[n-k]$

- Procedure:-

① Apply $x[n] \xrightarrow{T} y[n]$

② Apply $x_1[n] = x[n-k] \xrightarrow{T} y_1[n]$

③ Shift $y[n]$ by $k \rightarrow y[n-k]$

if $y_1[n] = y[n-k]$ Shift invariant.

Ex: ① $y[n] = x[n] - x[n-1]$

② $y[n] = n\,x[n]$

Sol:

1] $x[n]$:

$y[n] = x[n] - x[n-1]$

~~*Ex~~

$x_1[n] = x[n-k]$

$y_1[n] = x_1[n] - x_1[n-1]$

$y_1[n] = x[n-k] - x[n-k-1]$

$y[n-k] = x[n-k] - x[n-k-1]$

Shift Invariant System.

No.

2] $y[n] = n\,x[n]$

$x_1[n] = x[n-k]$

~~$x_1[n]$~~ $= y_1[n] = n\,x_1[n]$

$y_1[n] = n\,x[n-k]$

$y[n-k] = (n-k)\,x[n-k]$

$= n\,x[n-k] - k\,x[n-k]$

Shift-Variant System.

---

3. Causal & non Causal System:

A Causal System :-

the n_0^{th} output sample $y[n_0]$ depends only on the input samples $x[n]$ for $n \leq n_0$

$$y[n] = T[x[n], x[n-1], x[n-2], \ldots]$$

FIVE APPLE

No.

Ex: ① $y[n] = x[n] - x[n-1]$

② $y[n] = x[-n]$

1] Causal / realisable

2] $y[2] = x[-2]$

$y[-2] = x[2]$

Non Causal

$n=-2$ $\quad$ $n=2$

⇒ Not possible in Real life.

⇒ Non Causal / Unrealisable System

*Note:

If the system depends on a future value ⇒ Non Causal.

4. Stable & UnStable Systems:

BIBO Stability Critirion :-

Bounded Input, Bounded Output

If $\boxed{|x[n]| < B_x}$ for all values of n

then $\boxed{|y[n]| < B_y}$.

No.

Ex: ~~x[n]~~

① $y[n] = e^{x[n]}$

② $y[n] = \cos(x[n])$

~~③ y[n]~~

③ $y[n] = \sum_{k=-\infty}^{n} x[k]$

Sol

1] $y[n] = e^{x[n]}$

$|y[n]| = |e^{x[n]}| \leq e^{|x[n]|} \leq e^{B_x}$ B_y

~~⇒ BIBO B[n]~~

$|y[n]| < B_y$ The System is BIBO stable.

2] $y[n] = \cos[x[n]]$

$|y[n]| = |\cos[x[n]]| \leq \cancel{\cos[|x[n]|] \leq \cos(B_x)}$ 1 B_y

~~$|y[n]| < B_y$~~

⇒ BIBO stable

FIVE APPLE

No.

3] $y[n] = \sum_{k=-\infty}^{n} x[k]$

$|y[n]| = \left|\sum_{k=-\infty}^{n} x[k]\right| \leqslant \sum_{k=-\infty}^{n} |x[k]| \leqslant \underbrace{\sum_{k=-\infty}^{n} B_x}_{B_y}$

$|y[n]| \leqslant B_y$ * Diverge as n increases.

$\Rightarrow$ Unstable system

No.

5. Static and Dynamic System (memoryless)

Static

$y[n] = n\, x[n]$

$y[n] = x^2[n]$

Dynamic

$y[n] = a_1\, x[n] + a_2\, x[n-1]$

6. Passive and lossless System

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2$$

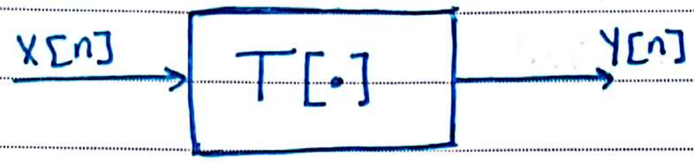
* equality ⇒ lossless

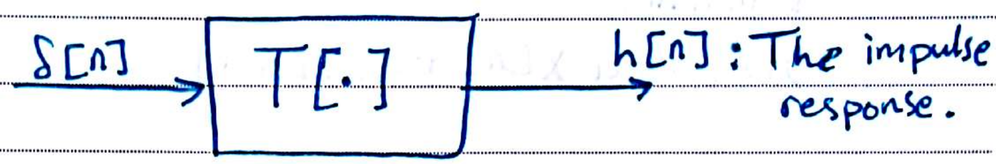
* less than ⇒ passive

No.

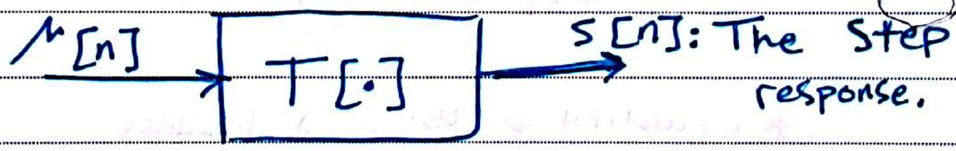
- Linear Time-Invariant DT system:
(LTI)

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$





$$T[\delta[n]] = h[n]$$



$$y[n] = T\left[\sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]\right]$$

for linear System:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] . T[\delta[n-k]]$$

FIVE APPLE

Scanned by CamScanner

Generally

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n, k]$$

~~for time variant~~

for Time Invariant

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}$$

LTI DT system

Convolution Sum

The same

also

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]}$$

$$\Rightarrow y[n] = x[n] * h[n] = h[n] * x[n]$$

~~EX: $x[n] = \{1\ 2\ 8\ 8\}$~~

~~$h[n] = \{1\ 1\ 2\ 2\}$~~

~~Find $y[n] = x[n] * h[n]$~~

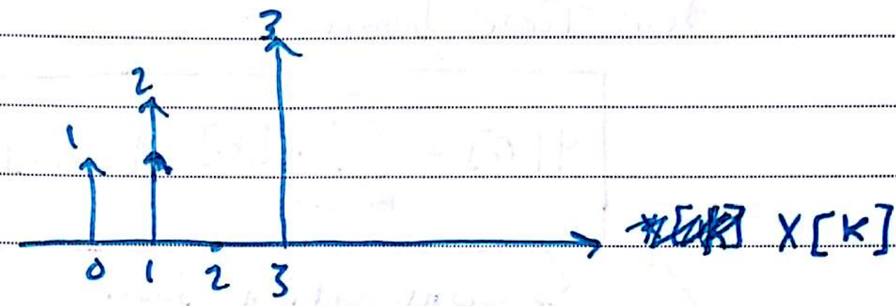
No.

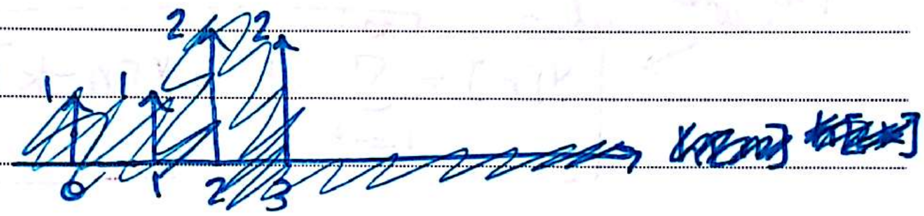
Ex: $x[n] = \{ \underset{\uparrow}{1} \;\; 2 \;\; 0 \;\; 3 \}$

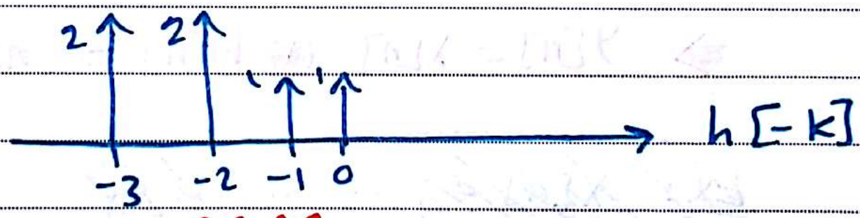
$h[n] = \{ \underset{\uparrow}{1} \;\; 1 \;\; 2 \;\; 2 \}$

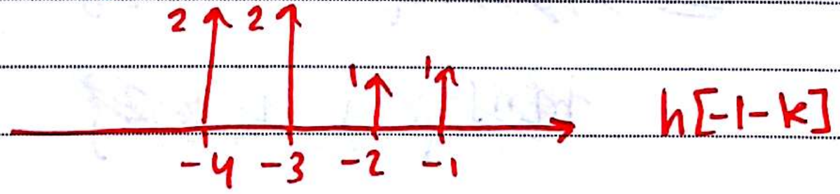
Find $y[n] = x[n] \circledast h[n]$

Sol:

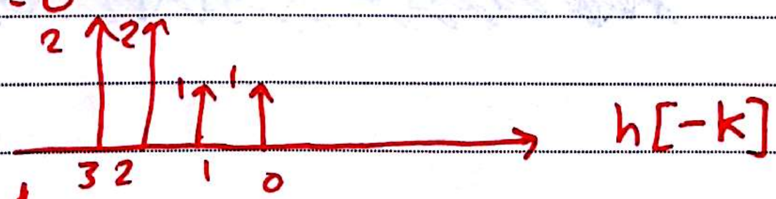








$\Rightarrow y[-1] = 0$

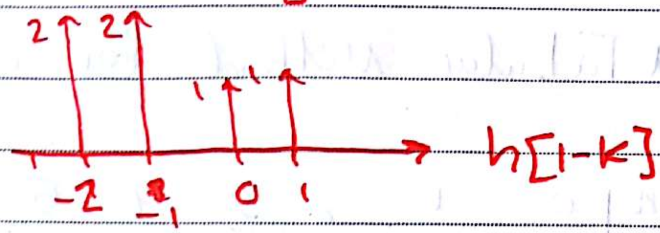


$\Rightarrow y[0] = 1$

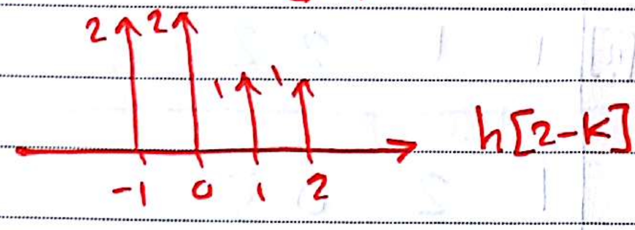
FIVE APPLE

No.

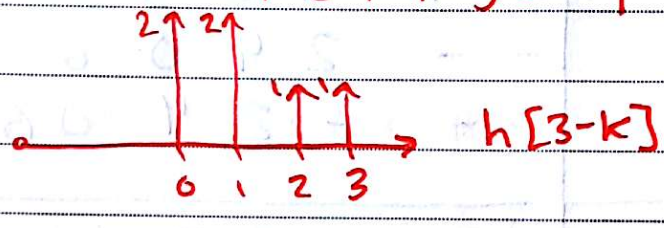
$y[1] = 1*1 + 1*2 = 3$



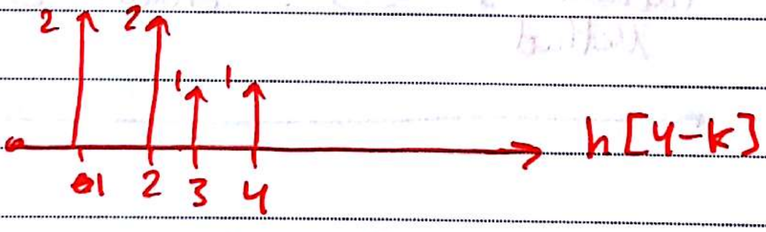
$y[2] = 2*1 + 1*2 = \cancel{3}\,4$



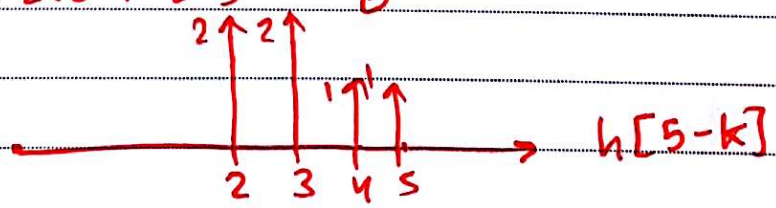
$y[3] = 2*1 + 2*2 + 1*0 + 1*3 = 9$



$y[4] = 2*2 + 2*0 + 1*3 = 7$



$y[5] = 2*0 + 2*3 = 6$



$y[6] = 6$, $y[7]=0$

~~y2[7]=0~~

No.

*Tabular Method for convolution

n	0	1	2	3	4	5	6	7
x[n]	1	2	0	3				
h[n]	1	1	2	2				
	1	2	0	3				
	-	1	2	0	3			
	-	-	2	4	0	6		
	-	-	-	2	4	0	6	
y[n]	1	3	4	9	7	6	6	

ضرب

* حل نفس المثال بطريقة ال Tabular Method

Ex: $x[n] = \{\underset{-1}{1}\ \ \underset{\uparrow 0}{2}\ \ \underset{1}{0}\ \ \underset{2}{3}\}$

$h[n] = \{\underset{-2}{1}\ \ \underset{-1}{1}\ \ \underset{\uparrow 0}{2}\ \ \underset{1}{2}\}$

* $y[n]$ from -3 to 3

find $y[n] = x[n] \circledast h[n]$

Sol: Tabular method

بس عشان السهم بشوف كم رقم ورا السهم ، في المثال 3

$y[n] = \{\underset{-3}{1}\ \ \underset{-2}{3}\ \ \underset{-1}{4}\ \ \underset{\uparrow 0}{9}\ \ \underset{1}{7}\ \ \underset{2}{6}\ \ \underset{3}{6}\}$

$$|y[n]| = |\cos(x[n])| \leq 1$$

B_y

---

*Stability Condition in terms of the Impulse response.

$$y[n] = x[n] \circledast h[n]$$

$$\cancel{= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]}$$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\, |x[n-k]|$$

B_x

S

$$\leq B_x \sum_{k=-\infty}^{\infty} |h[k]|$$

$$\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$$

$$B_x \cdot S = B_y$$

*

No.

* Causality condition in terms of the Impulse Response.

$$y[n_0] = \sum_{k=-\infty}^{\infty} h[k]\, x[n_0-k]$$

$$= \sum_{k=0}^{\infty} h[k]\, x[n_0-k] + \sum_{k=-\infty}^{-1} h[k]\, x[n_0-k]$$

$$= h[0]\, x[n_0] + h[1]\, x[n_0-1] \ldots \;\; ---$$

$$\cancel{h[n_0]\, x[0]} \; +$$

$$h[-1]\, x[n_0+1] + h[-2]\, x[n_0+2] \; ---$$

$$\cancel{+h[n_0]\, x[2n_0]}$$

$\Rightarrow h[k] = 0 \quad ; \; k<0$ (the condition).

* Discrete-time Fourier transform (DTFT)

$\cancel{X(e^{jw})}$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

Ex: $x[n] = \{1 \; \underset{\uparrow}{1} \; 2 \; 2\}$

$$X(e^{jw}) = \sum_{n=-1}^{2} x[n]\, e^{-jwn}$$

$$= 1 * e^{+jw} + 1 \cancel{\;\;} + 2e^{-jw} + 2e^{-2jw}$$

No.

① $X(e^{j\omega})$ is Continuos function in ω.

② $X(e^{j\omega})$ is periodic with a period of 2π.

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \cdot \underbrace{e^{-j2\pi n}}_{1}$$

$$= X(e^{j\omega})$$

Ex: $x[n] = \alpha^n u[n] \qquad |\alpha|<1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

26/2

No.

DTFT: Discrete-time Fourier Transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{X[n]}_{C_n} e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

① Continuos in ω.

② Periodic with a period of 2π

$$X[n] = C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n}\, d\omega$$

$$X[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$X(e^{j\omega}) = \mathcal{F}\{X[n]\}$$

$$X[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

FIVE APPLE

No.

Ex: $x[n] = \delta[n]$

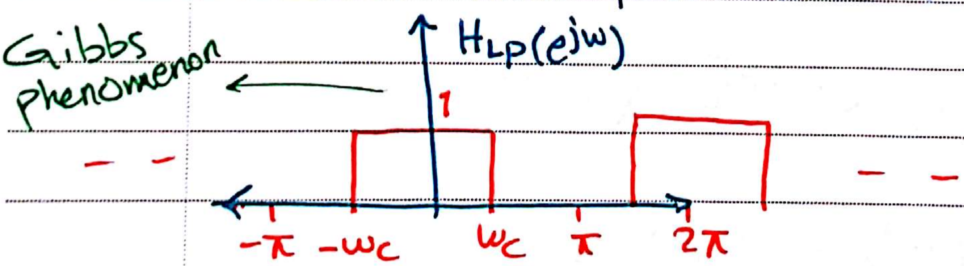
$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = e^{-j\omega(0)} = 1$$

* Compression in t-domain ⇔ expansion in f-domain

* Compression in f-domain ⇔ expansion in t-domain

---

* Ideal Low-pass filter : $H_{Lp}(e^{j\omega})$



Find ~~Hlp~~ $h_{Lp}[n]$

~~$x[n] = A\cos(\omega_0 n)$~~

~~Find Hlp~~

$\cancel{h_{Lp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \cdot e^{j\omega n}\, d\omega}$

FIVE APPLE

$$\cancel{h_{Lp}[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c}}$$

$$h_{Lp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{Lp}(e^{jw})\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} 1.\ e^{jwn}\, dw$$

$$h_{Lp}[n] = \frac{e^{jwn}}{2\pi jn}\Bigg|_{-w_c}^{w_c}$$

$$= \frac{e^{jw_c n} - e^{-jw_c n}}{2j(\pi n)} = \frac{\sin w_c n}{\pi n}$$

$$h_{Lp}[0] = \frac{1}{2\pi}\int_{-w_c}^{w_c} 1.\, dw$$

$$= \frac{\cancel{2}w_c}{\pi}$$

$$h_{Lp}[n] = \begin{cases} \dfrac{\sin w_c n}{\pi n}, & n \neq 0 \\[2ex] \dfrac{w_c}{\pi}, & n = 0 \end{cases}$$

No.

Theorems:

① linearity :

$$\alpha g[n] + \beta x[n] \xleftrightarrow{DTFT} \alpha G(e^{j\omega}) + \beta X(e^{j\omega})$$

② Time - Reversal

$$g[-n] \xleftrightarrow{DTFT} G(e^{-j\omega})$$

③ Ex: $|\alpha| < 1$

$$x[n] = \alpha^n \mu[n]$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

No.

Ex: $X[-n] = \alpha^{(-n)} \mu[-n] \qquad |\alpha| < 1$

Find $X(e^{j\omega})$??

$\Downarrow \quad X(e^{j\omega}) = \dfrac{1}{1-\alpha e^{j\omega}}$

Ex: $X_1[n] = \{1 \ 1 \ 2 \ 2\}$

$X_2[-n] = \{2 \ 2 \ 1 \ 1\}$

$X_1(e^{j\omega}) = \{1 + e^{-j\omega} + 2e^{-j2\omega} + 2e^{-j3\omega}$

$X_2(e^{j\omega}) = 2e^{j3\omega} + 2e^{j2\omega} + e^{j\omega} + 1$

---

③ Time - Shifting

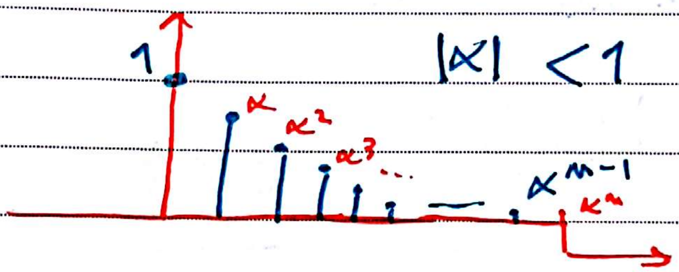
$g[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} G(e^{j\omega})$

~~Ex~~

FIVE APPLE

No.

Ex: $y[n] = \begin{cases} \alpha^n & , 0 \leq n \leq M-1 \\ 0 & , \text{otherwise} \end{cases}$

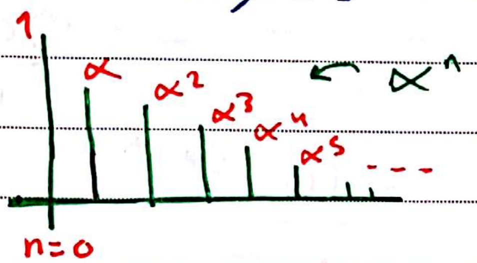


$$y[n] = \alpha^n u[n] - \alpha^n u[n-M]$$

Ex:

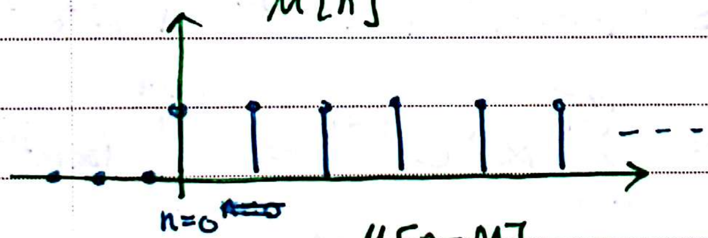
$$y[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases} \qquad |\alpha| < 1$$

$$y[n] = \alpha^n u[n] - \alpha^n u[n-M]$$

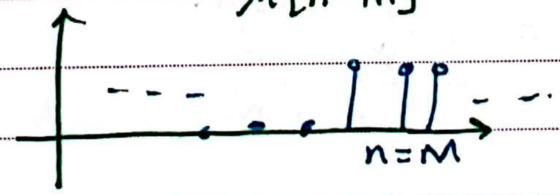


$$y[n] = \alpha^n u[n] - \alpha^n u[n-M]$$
$$= \alpha^n \left[u[n] - u[n-M] \right]$$

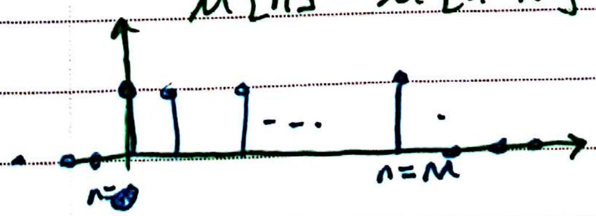
$u[n]$



$u[n-M]$



$u[n] - u[n-M]$



No.

$$= \alpha^n u[n] - \alpha^n u[n-M]$$

$$= \alpha^n u[n] - \alpha^M \cdot \alpha^{n-M} u[n-M]$$

$$\alpha^n u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-\alpha e^{-jw}}$$

$$Y(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} - \frac{\alpha^M \cdot e^{-jwM}}{1-\alpha e^{-jw}}$$

$$= \frac{1-(\alpha e^{-jw})^M}{1-\alpha \cdot e^{-jw}}$$

$$y[n] = \{\underset{\uparrow}{1} \quad \alpha \quad \alpha^2 \cdots \quad \alpha^{M-1}\}$$

$$Y(e^{jw}) = 1 + \alpha e^{-jw} + \alpha^2 e^{-j2w} + \alpha^3 e^{-j3w} + \alpha^{M-1} e^{-j(M-1)w}$$

$$Y(e^{jw}) = \sum_{n=0}^{M-1} \alpha^n e^{-jwn} = \sum_{n=0}^{M-1} (\alpha e^{-jw})^n$$

$$\boxed{\sum_{n=0}^{M-1} r^n = \frac{1-r^M}{1-r}}$$

$$Y(e^{jw}) = \frac{1-(\alpha e^{-jw})^M}{1-\alpha e^{-jw}}$$

FIVE APPLE

No.

~~Ex: $d_0 V[n] + d_1 V[n-1] = P_0 X[n] + P_1 X[$~~

Ex: $d_0 V[n] + d_1 V[n-1] = P_0 X[n] + P_1 X[n-1]$

For $X[n] = \delta[n]$, find $V(e^{jw})$.

$$d_0 V[n] + d_1 V[n-1] = P_0 \delta[n] + P_1 \delta[n-1]$$

$$d_0 V(e^{jw}) + d_1 e^{-jw} V(e^{jw}) = P_0(1) + P_1 e^{-jw} \cdot (1)$$

$$V(e^{jw})[d_0 + d_1 e^{-jw}] = P_0 + P_1 e^{-jw}$$

$$H(e^{jw}) = V(e^{jw}) = \frac{P_0 + P_1 e^{-jw}}{d_0 + d_1 e^{-jw}}$$

$$d_0 V(e^{jw}) + d_1 e^{-jw} V(e^{jw}) = P_0 X(e^{jw}) + P_1 e^{-jw} X(e^{jw})$$

$$V(e^{jw})[d_0 + d_1 e^{-jw}] = X(e^{jw})[P_0 + P_1 e^{-jw}]$$

$$H(e^{jw}) = \frac{V(e^{jw})}{X(e^{jw})} = \frac{P_0 + P_1 e^{-jw}}{d_0 + d_1 e^{-jw}}$$

case1:

$X[n] = \delta[n]$

$X(e^{jw}) = 1$

FIVE APPLE

No.

Case2:

$$X[n] = \alpha^n \mu[n]$$

$$X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}$$

$$V(e^{jw}) = H(e^{jw}) \cdot X(e^{jw})$$

$$= \frac{P_0 + P_1 e^{-jw}}{(d_0 + d_1 e^{-jw})} * \frac{1}{(1-\alpha e^{-jw})}$$

* Frequency Shifting Theorem

$$e^{jw_0 n} g[n] \xleftrightarrow{DTFT} G(e^{j(w-w_0)})$$

Ex: $y[n] = (-1)^n \alpha^n \mu[n] \quad , |\alpha| < 1$

$$y[n] = e^{j\pi n} \alpha^n \mu[n]$$

$$= \frac{1}{1-\alpha e^{-j(w-\pi)}}$$

$$= \frac{1}{1-\alpha e^{j\pi} e^{-jw}} = \frac{1}{1+\alpha e^{-jw}}$$

No.

* Differentiation in frequency

$$n\, g[n] \overset{DTFT}{\longleftrightarrow} j \frac{d\, G(e^{j\omega})}{d\omega}$$

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j\omega n}$$

$$\frac{d\, G(e^{j\omega})}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} g[n]\, e^{-j\omega n}\right]$$

$$j \frac{d\, G(e^{j\omega})}{d\omega} = \cancel{j} \sum_{n=-\infty}^{\infty} g[n]\, (-\cancel{j} n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} n\, g[n]\, e^{-j\omega n}$$

Ex $y[n] = (n+1)\, \alpha^n \mu[n] \quad , \quad |\alpha| < 1$

$$= n\, \alpha^n \mu[n] + \alpha^n \mu[n]$$

let $x[n] = \alpha^n \mu[n]$

$y[n] = n\, x[n] + x[n]$ Find $Y(e^{j\omega})$??

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$j \frac{d\, X(e^{j\omega})}{d\omega} = j \frac{-(\mp \alpha (\pm j)\, e^{-j\omega})}{(1 - \alpha e^{-j\omega})^2}$$

$$\frac{j\,d\,X(e^{j\omega})}{d\omega} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}}$$

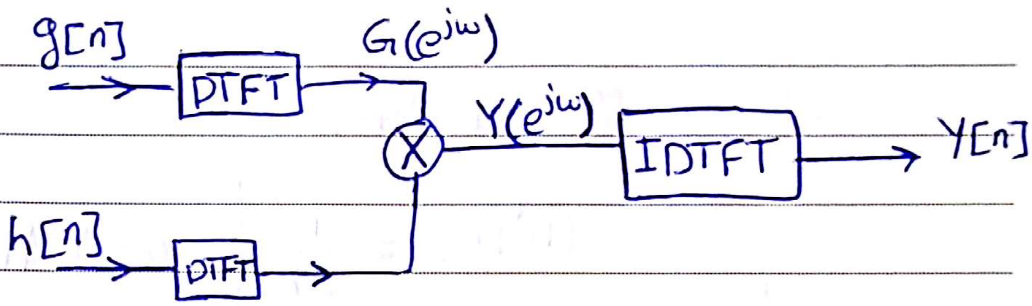
$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega} \pm -(\alpha e^{-j\omega})^2}{(1-\alpha e^{-j\omega})^3}$$

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

No.

* Convolution theorem

$$g[n] \circledast h[n] \xleftrightarrow{DTFT} G(e^{j\omega})\, H(e^{j\omega})$$



Ex: $x[n] = \{1 \quad 2 \quad 0 \quad 3\}$

$h[n] = \{1 \quad 1 \quad 2 \quad 2\}$

$$X(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-j3\omega}$$
$$H(e^{j\omega}) = 1 + e^{-j\omega} + 2e^{-j2\omega} + 2e^{-j3\omega}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$
$$= 1 + e^{-j\omega} + 2e^{-j2\omega} + 2e^{-j3\omega}$$
$$+ 2e^{-j\omega} + 2e^{-j2\omega} + 4e^{-j3\omega} + 4e^{-j4\omega}$$
$$+ 3e^{-j3\omega} + 3e^{-j4\omega} + 6e^{-j5\omega} + 6e^{-j6\omega}$$

$$= 1 + 3e^{-j\omega} + 4e^{-j2\omega} + 9e^{-j3\omega} + 7e^{-j4\omega}$$
$$+ 6e^{-j5\omega} + 6e^{-j6\omega}$$

$$= \{1 \quad 3 \quad 4 \quad 9 \quad 7 \quad 6 \quad 6\}$$

No.

* Modulation theorem

$$g[n]\,h[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\, H(e^{j(\omega-\theta)})\, d\theta$$

- Parseval's relation:

$$\sum_{n=-\infty}^{\infty} g[n]\,h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega}) \cdot H^*(e^{j\omega})\, d\omega$$

&

If $g[n] = h[n]$

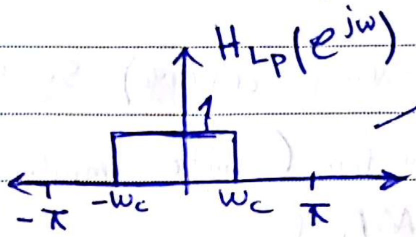
$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2\, d\omega$$

- The quantity $S_{gg}(e^{j\omega}) = |G(e^{j\omega})|^2$ is ~~the~~ called the energy density spectrum of the sequence $g[n]$

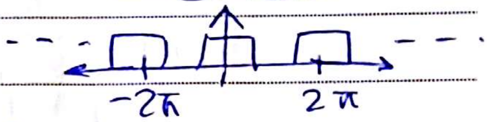
- The area under this curve in the range between $-\pi < \omega < \pi$ divided by 2π is the energy of the sequence.

FIVE APPLE

Ideal low pass filter :-



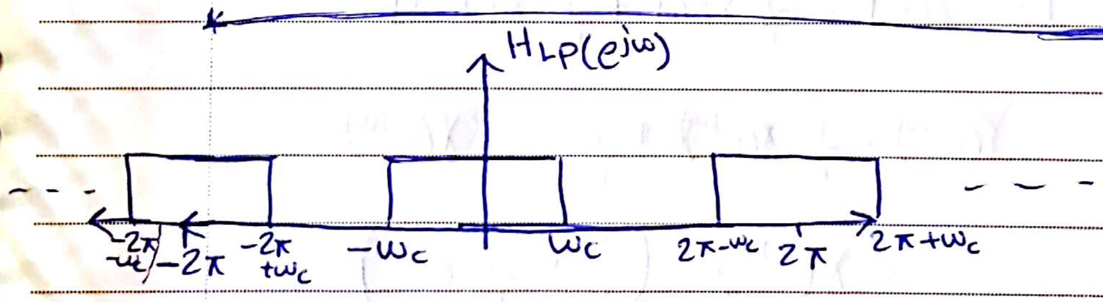
يكرر نفسه
كل 2π

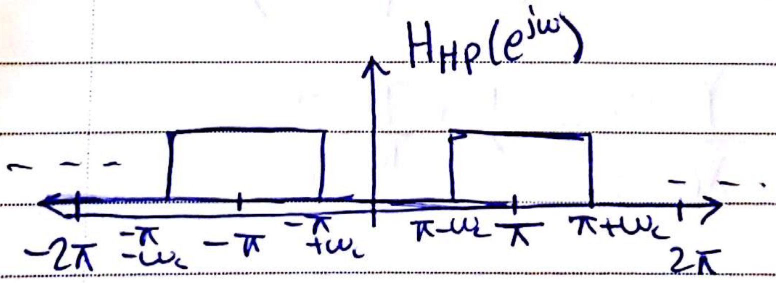


$$h_{LP}[n] = \begin{cases} \omega_c/\pi & , n=0 \\ \dfrac{\sin \omega_c n}{\pi n} & , n \neq 0 \end{cases}$$

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{LP}[n]|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot d\omega = \frac{\omega_c}{\pi} < \infty \text{, finite}$$





No.

✱ Frequency Response of LTI Discrete-time Systems:

MA (moving average) System

① LTI FIR System (Finite Impulse response)

~~MA (~~

$$y[n] = \sum_{k=N_1}^{N_2} h[k]\, \underline{x[n-k]} \quad , N_1 < N_2$$

$$Y(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k] \boxed{e^{-j\omega k}\, X(e^{j\omega})}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \sum_{k=N_1}^{N_2} h[k]\, e^{-j\omega k}$$

✱ **<u>Ex</u>:** $y[n] = \frac{1}{2} x[n] + \frac{7}{10} x[n-1]$ ✱

$$Y(e^{j\omega}) = \frac{1}{2} X(e^{j\omega}) + \frac{7}{10} e^{-j\omega} X(e^{j\omega})$$

$$= X(e^{j\omega}) \cdot \left(\frac{1}{2} + \frac{7}{10} e^{-j\omega}\right)$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{2} + \frac{7}{10} e^{-j\omega}$$

$$h[n] = \left\{ \frac{1}{2}, \frac{7}{10} \right\}$$

② LTI IIR System (Infinite Impulse response)

$$\sum_{k=0}^{N} d_k \, y[n-k] = \sum_{k=0}^{M} P_k \, x[n-k]$$

$$\sum_{k=0}^{N} d_k \, e^{-j\omega k} \, Y(e^{j\omega}) = \sum_{k=0}^{M} P_k \, e^{-j\omega k} \, X(e^{j\omega})$$

$$Y(e^{j\omega}) \cdot \sum_{k=0}^{N} d_k \, e^{-j\omega k} = X(e^{j\omega}) \cdot \sum_{k=0}^{M} P_k \, e^{-j\omega k}$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} P_k \, e^{-j\omega k}}{\sum_{k=0}^{N} d_k \, e^{-j\omega k}}$$

No.

IIR Example :-

$$y[n] = x[n] + \alpha\, y[n-1]$$

let $x[n] = \delta[n]$ All $y[n] = 0$, $n<0$

$$y[0] = x[0] + \alpha(0)$$

$$y[0] = 1$$

$$y[1] = x[1] + \alpha\, y[0]$$

$$y[1] = 0 + \alpha(1) = \alpha$$

$$y[2] = x[2] + \alpha\, y[1]$$

$$y[2] = 0 + \alpha\alpha = \alpha^2$$

$$\boxed{h[n] = \alpha^n \mu[n]}$$

$$Y(e^{jw}) = X(e^{jw}) + \alpha e^{-jw}\, Y(e^{jw})$$

$$Y(e^{jw}) * [1 - \alpha e^{-jw}] = X(e^{jw})$$

$$H(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}$$

FIVE APPLE

$$1-\alpha e^{-j\omega} \;\overline{)\;1\quad}\quad 1+\alpha e^{-j\omega}+\alpha^2 e^{-j2\omega}+\alpha^3 e^{-j3\omega}+\dots$$

$$-1 \pm \alpha e^{-j\omega}$$

$$\alpha e^{-j\omega}$$

$$-\alpha e^{-j\omega} \pm \alpha^2 e^{-j2\omega}$$

$$\alpha^2 e^{-j2\omega}$$

$$-\alpha^2 e^{-j2\omega} \pm \alpha^3 e^{-j3\omega}$$

$$h[n] = \{1 \quad \alpha \quad \alpha^2 \quad \alpha^3 \dots\}$$

$$h[n] = \alpha^n u[n]$$

---

* $$x[n] = \sum_{i=1}^{L} A_i \cos(\omega_i n + \phi_i) \quad , -\infty < n < \infty$$

$$x[n] \longrightarrow \overset{h[n]}{\boxed{H(e^{j\omega})}} \longrightarrow y[n]$$

$$y[n] = \sum_{i=1}^{L} A_i \cdot |H(e^{j\omega})| \cdot \cos(\omega_i n + \phi_i + \theta(e^{j\omega}))$$

No.

Ex: $H(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\omega}}$

$y[n] = x[n] + \frac{1}{2}\, y[n-1]$

$\dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1}{1 - \frac{1}{2}e^{-j\omega}}$

$x[n] = 10 - 5\sin\frac{\pi}{2}n + 20\cos(\pi n)$, $-\infty < n < \infty$

Find $y[n]$??

- $x_1[n] = 10 \Rightarrow \omega = 0$

  $H(e^{j0}) = \dfrac{1}{1 - \frac{1}{2}e^{-j0}} = 2 \quad : \quad 2\angle 0^\circ$

  $y_1[n] = 2 * 10 = 20$

- $x_2[n] = -5\sin\frac{\pi}{2}n$

  $H(e^{j\omega=\frac{\pi}{2}}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\pi/2}} = \cancel{\dfrac{1}{1 - \frac{1}{2}[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}}}$

  $= \dfrac{1}{1 - \frac{1}{2}(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2})}$

  $= \dfrac{1}{1 + \frac{1}{2}j} = \dfrac{2}{\sqrt{5}}\angle -26.6^\circ$

  $y_2[n] = -\dfrac{10}{\sqrt{5}}\sin\left(\dfrac{\pi}{2}n - 26.6\right)$

No.

• $X_3[n] = \cancel{20} \; 20\cos(\pi n) \Rightarrow \omega = \pi$

$$H(e^{j\pi}) = \frac{1}{1-\frac{1}{2}e^{-j\pi}} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}\angle 0^\circ$$

$\cancel{y_3[n]}$

$$y_3[n] = \frac{40}{3}\cos\pi n$$

$$\cancel{y[n] = 20 - \frac{10}{\sqrt{5}} \sin\frac{\pi}{2} + \frac{40}{3}\cos\pi n}$$

$$y[n] = 20 - \frac{10}{\sqrt{5}}\sin\left(\frac{\pi}{2}n - 26.6^\circ\right) + \frac{40}{3}\cos\pi n$$

---

* Discrete Fourier Transform (DFT)

No.

~~DFT~~

*Finite - length Discrete Transforms:-

* Finite-length Sequence $x[n]$ of length N

~~Index~~

* The index of the sequence starts at $n=0$ until $n=N-1$.

• Orthogonal Transforms:

$x[n]$ denote a length-N Sequence with $X[k]$ denoting the coefficients of its N-point orthogonal transform such that.

Analysis equation
$$X[k] = \sum_{n=0}^{N-1} x[n]\, \psi^*[k,n] \quad , \quad 0 \leq k \leq N-1$$

Synthesis equation
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, \psi[k,n] \quad , \quad 0 \leq n \leq N-1$$

$\psi[k,n]$ are called the basis sequences.

The basis sequences satisfy the condition

$$\boxed{\frac{1}{N} \sum_{n=0}^{N-1} \psi[k,n] \cdot \psi^*[\ell,n] = \begin{cases} 1 & , \ell = k \\ 0 & , \ell \neq k \end{cases}}$$

orthogonality condition

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No.

- ~~Orthogonality condition.~~

- ~~The Discrete Fourier Transform~~

- The Discrete Fourier Transform (DFT)

$$- X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi n k}{N}}$$

$$- \psi[k,n] = e^{j\frac{2\pi n k}{N}}$$

orthogonality condition

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi n k}{N}} \, e^{-j\frac{2\pi n l}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(k-l)}$$

$$= \frac{1}{N} \cdot \left[\frac{1 - e^{-j\frac{2\pi (k-l) \cancel{N}}{\cancel{N}}}}{1 - e^{\frac{j2\pi (k-l)}{N}}} \right]$$

- for $k \neq l$ $= 0$

for $k = l$ $= 1$

No.

$$-X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-j2\pi nk}{N}} \quad , 0 \leq k \leq N-1$$

$$-x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{\frac{j2\pi nk}{N}} \quad , 0 \leq n \leq N-1$$

let $W_N = e^{\frac{-j2\pi}{N}}$

$W_8 = e^{-j\frac{2\pi}{8}}$, $W_4 = e^{-j\frac{2\pi}{4}}$

$$-X[k] = \sum_{n=0}^{N-1} x[n] \, W_N^{nk} \quad , 0 \leq k \leq N-1$$

$$-x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{-nk} \quad , 0 \leq n \leq N-1$$

- $x[n] \xleftrightarrow{DFT} X[k]$

$$X[0] = \sum_{n=0}^{N-1} x[n] \, W_N^{0} = x[0] + x[1] + \dots x[N-1]$$

$$X[1] = \sum_{n=0}^{N-1} x[n] \, W_N^{n} = x[0] + x[1] W_N^{1} + x[2] W_N^{2} + \dots + x[N-1] \, W_N^{N-1}$$

$\vdots$

$$X[2] = \sum_{n=0}^{N-1} x[n] W_N^{2n} = x[0] + x[1] W_N^2 + x[2] W_N^4 + \dots + x[N-1] W_N^{2(N-1)}$$

$$\vdots$$

$$X[N-1] = \sum_{n=0}^{N-1} x[n] W_N^{(N-1)n}$$

$$= x[0] + x[1] W_N^{N-1} + x[2] W_N^{2(N-1)} + \dots + x[N-1] W_N^{(N-1)^2}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^1 & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ 1 & W_N^3 & W_N^6 & \dots & W_N^{3(N-1)} \\ \vdots & & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)^2} \end{bmatrix}}_{N \times N} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

D_N matrix

~~Find N & you'll know~~ ~~D_N~~

function in matlab

dftmtx(N)

No.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} \quad , \; 0 \le k \le N-1$$

$$x[n] = \sum_{n=0}^{N-1} X[k]\, W_N^{-nk} \quad , \; 0 \le n \le N-1$$

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{N \times 1} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{1} & W_N^{2} & \cdots & W_N^{N-1} \\ \vdots & & & & \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}}_{N \times N} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{N \times 1}$$

$$\underline{X} = D_N \cdot \underline{x}$$

Inverse

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ \vdots & & & & \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & \cancel{W_N^{-(N-1)}} \end{bmatrix} \cdot \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$W_N^{-(N-1)^2}$

$$\underline{x} = \frac{1}{N} D_N^{*} \, \underline{X}$$

No.

$$A^{-1}\left[\underline{X} = A \cdot \underline{b}\right]$$

$$A^{-1}X = A^{-1}A \cdot b$$

$$b = A^{-1}X$$

$$\underline{x} = D_N^{-1}\,\underline{X}$$

$$\boxed{D_N^{-1} = \frac{1}{N} D_N^{*}}$$

$$D_I = \frac{\text{Conj}(D)}{N}$$

Ex: $X[n] = \{1 \quad 2 \quad 0 \quad 3\}$

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$W_4 = e^{-j\frac{2\pi}{4}(n)}$$
$$= e^{-j\pi/2}$$
$$= -j$$

FIVE APPLE

No.

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 1+2+0+3 \\ 1-2j+0+3j \\ 1-2+0-3 \\ 1+2j+0-j3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1+j \\ -4 \\ 1-j \end{bmatrix}$$

$$D_4^{-1} = \frac{1}{4} D_4^{*}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\cancel{x[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \end{bmatrix}}$$

$$X[n] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ 1+j \\ -4 \\ 1-j \end{bmatrix}$$

$$X[n] = \frac{1}{4}\begin{bmatrix} 6 + 1 + \cancel{j} - 4 + 1 - \cancel{j} \\ 6 + \cancel{j} - 1 + 4 - j - 1 \\ 6 - 1 - \cancel{j} - 4 - 1 + \cancel{j} \\ 6 - \cancel{j} + 1 + 4 + \cancel{j} + 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 0 \\ 12 \end{bmatrix}$$

$$X[n] = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

The Relationship between the DTFT and the DFT.

$$X(e^{jw}) = \sum_{n=0}^{N-1} x[n]\, e^{-jwn}$$; $x[n]$ is a length N sequence.

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi kn}{N}}$$

let $0 \leq w \leq 2\pi$

$$W_k = \frac{2\pi k}{N} \quad , \quad k=0 \rightarrow k=N-1$$

0 ———— 2π

0 ———— $N-1$

- $$X(e^{jw})\Big|_{W_k=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi kn}{N}}$$

$W_k = 2\pi k / M$ where $M \gg N$

$0 \leq k \leq M-1$

$$X(e^{jw})\Big|_{w_k=\frac{2\pi k}{M}} = \sum_{n=0}^{M-1} x[n]\, e^{-jw_k n}$$

$$= \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{M} n}$$

No.

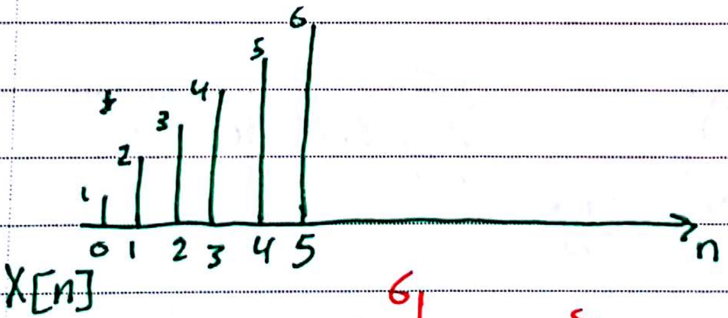
let $x_e[n] = \begin{cases} x[n] & , 0 \leq n \leq N-1 \\ 0 & , N \leq n \leq M-1 \end{cases}$

$\Rightarrow$ $x_e[n]$ is a length-M Sequence.

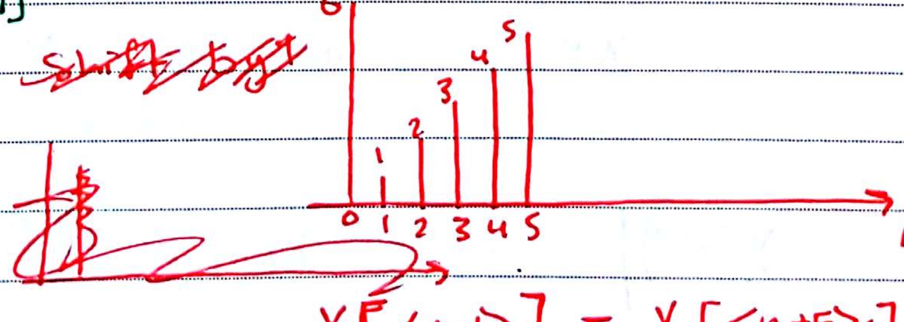
$$X_e[k] = \sum_{n=0}^{M-1} x_e[n] \cdot e^{-j\frac{2\pi kn}{M}}$$

$$= \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi kn}{M}} + 0$$

Circular Shift of a Sequence:



~~Shift by 1~~



$x[\langle n-1 \rangle_6] = x[\langle n+5 \rangle_6]$

* Shift by 1 to the right
or Shift by 5 to the left.

No.

$$X[\langle n-2\rangle_6] = X[\langle n+4\rangle_6]$$

No.

* Circular Shift & Circular Convolution :-

* $x[n] \xleftrightarrow{DFT} X[k]$

length-N length-N

$h[n] \xleftrightarrow{DFT} H[k]$

length-N length-N

$y[n] = x[n] \circledast h[n]$

2N-1 points

The modulo operation

$r = \langle m \rangle_N$ m modulo N

$r = m + lN$ where l is an integer chosen such that $0 \leq r \leq N-1$

$r = \langle 25 \rangle_7 = 25 + (-3)(7)$

$r = 4$

$r = \langle -16 \rangle_7 = -16 + 3(7)$

$r = 5$

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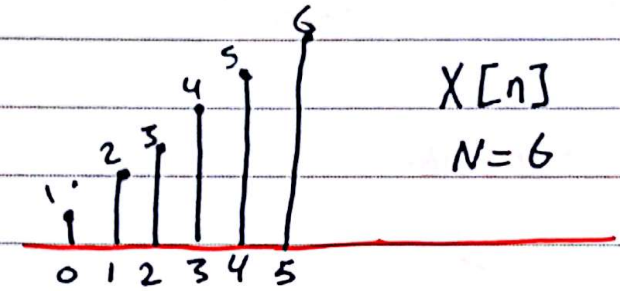
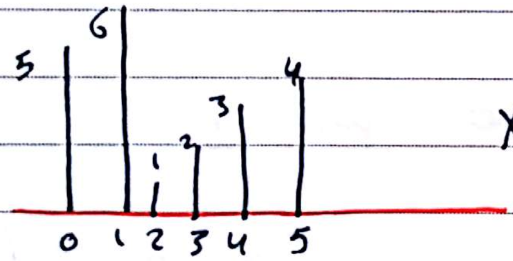
$$\langle 28\rangle_7 = 28-4(7)$$
$$= 0$$

$$\langle -50\rangle_7 = -50+8(7)$$
$$= 6$$



$X[n]$

$N=6$



$$X_c[n] = X[\langle n-2\rangle_6]$$

$$X_c[2] = X[\langle 2-2\rangle_6]$$
$$= X[0]$$
$$X_c[3] = X[\langle 3-2\rangle_6]$$
$$= X[1]$$
$$X_c[4] = X[2]$$
$$X_c[5] = X[3]$$

$$X_c[0] = X[\langle 0-2\rangle_6]$$
$$= X[4]$$
$$X_c[1] = X[\langle 1-2\rangle_6]$$
$$= X[5]$$

No.

- $X_c[n] = X[\langle n-2\rangle_6] = X[\langle n+4\rangle_6]$

$X_c[0] = X[\langle 0+4\rangle_6] = X[4]$

$X_c[1] = X[\langle 1+4\rangle_6] = X[5]$

$X_c[2] = X[\langle 2+4\rangle_6] = X[0]$

⋮

* Circular Convolution

Linear Convolution

$$Y_L[n] = \sum_{m=0}^{N-1} X[m]\,h[n-m] \quad , \quad 0 \leq n \leq 2N-2$$

$$= X[n] \circledast h[n]$$

$$= \sum_{m=0}^{N-1} h[m]\,X[n-m]$$

$$= h[n] \circledast X[n]$$

No.

The Circular Convolution

$$Y_c[n] = \sum_{m=0}^{N-1} X[m]\, h[\langle n-m \rangle_N], \quad 0 \le n \le N-1$$

$$Y_c[n] = X[n] \; (N) \; h[n]$$

Ex $X[n] = \{1 \;\; 2 \;\; 0 \;\; 3\}$

$h[n] = \{1 \;\; 1 \;\; 2 \;\; 2\}$

$Y_c[n]$??

$Y_c[n] = X[n] \; (N) \; h[n]$

$= X[n] \; (4) \; h[n]$

n	0	1	2	3	$\langle 4 \rangle_4$	$\langle 5 \rangle_4$	$\langle 6 \rangle_4$ …
$X[n]$	1	2	0	3			
$h[n]$	1	1	2	2	2		
	1	2	0	3			
	3	1	2	0			
	0	6	2	0			
	4	0	6	2			
	8	9	10	9			

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No.

$$Y_c[0] = \sum_{m=0}^{3} x[m]\, h[\langle -m \rangle_4]$$

$$Y_c[0] = x[0]\cdot h[\langle 0 \rangle_4] + x[1]\cdot h[\langle -1 \rangle_4] + x[2]\, h[\langle -2 \rangle_4] + x[3]\cdot h[\langle -3 \rangle_4]$$

$$= 1\times 1 + 2(2) + 0(2) + 3(1) = 8$$

$$\cancel{Y_c[n] =} \begin{bmatrix} \cancel{h[0]} & \cancel{h[N-1]} \\ h[1] & h[0] \\ \cancel{h[2]} & \\ \vdots & \\ \cancel{h[N-1]} & \end{bmatrix} \begin{bmatrix} \cancel{x[0]} \\ \cancel{x[1]} \\ \cancel{x[2]} \\ \vdots \\ \cancel{x[N-1]} \end{bmatrix}$$

$$Y_c[n] = \underbrace{\begin{bmatrix} h[0], & h[N-1], & h[N-2] \cdots h[1] \\ h[1], & h[0], & h[N-1] \cdots h[2] \\ h[2], & h[1], & h[0] \cdots h[3] \\ \vdots & \vdots & \vdots \\ h[N-1], & h[N-2], & h[N-3] \cdots h[0] \end{bmatrix}}_{\text{Circulant matrix}} \cdot \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

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No.

Ex:

$$Y_c[n] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \\ 9 \end{bmatrix}$$

19/3

No.

$X[n] = \{1 \quad 2 \quad 0 \quad 3\}$

$h[n] = \{1 \quad 1 \quad 2 \quad 2\}$

$X_e[n] = \{1 \quad 2 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0\}$

$h_e[n] = \{1 \quad 1 \quad 2 \quad 2 \quad 0 \quad 0 \quad 0\}$

n	0	1	2	3	4	5	6	$\langle 7 \rangle_7$
	1	2	0	3	0	0	0	
	1	1	2	2	0	0	0	
	1	2	0	3	0	0	0	$\leftarrow X_e[n] \; ⑦ \; h_e[n]$
	0	1	2	0	3	0	0	
	0	0	2	4	0	6	0	
	0	0	0	2	4	0	6	
	1	3	4	7	9	6	6	

No.

*Discrete Fourier Transform Theorems

$$x[n] \xleftrightarrow{DFT} X[k]$$
$$h[n] \xleftrightarrow{DFT} H[k]$$

① linearity

$$\alpha x[n] + \beta h[n] \xleftrightarrow{DFT} \alpha X[k] + \beta H[k]$$

② Circular Time Shifting

$$g[\langle n-n_o\rangle_N] \xleftrightarrow{DFT} W_N^{kn_o} G[k]$$

③ Circular frequency Shifting

$$W_N^{-kn_o} g[n] \xleftrightarrow{DFT} G[\langle k-k_o\rangle_N]$$

④ ~~Circul~~ Circular Convolution.

$$g[n] \; Ⓝ \; h[n] \xleftrightarrow{DFT} G[k] \cdot H[k]$$

⑤ Modulation

$$g[n] \cdot h[n] \xleftrightarrow{DFT} \frac{1}{N} \sum_{l=0}^{N-1} G[l] \cdot H[\langle k-l\rangle_N]$$

FIVE APPLE

No.

⑥ Parseval's Relation

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X[k]|^2$$

$$\sum_{n=0}^{N-1} g[n]\, h^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} G[k]\, H^*[k]$$

If $g[n] = h[n] = x[n]$

<u>Ex</u>: $x[n] = \{1 \quad 2 \quad 0 \quad 3\}$

$X[k] = \{6 \quad 1+j \quad -4 \quad 1-j\}$

Find $x[0]$, $x[2]$, $\sum_{k=0}^{3} X[k]$ and $\sum_{k=0}^{3} |X[k]|^2$, Do not evaluate the DFT.

<u>Solution</u>

$$X[k] = \sum_{n=0}^{3} x[n]\, W_4^{nk}$$

$$X[0] = \sum_{n=0}^{3} x[n]\, W_4^{n(0)} = 1+2+0+3 = 6$$

$$X[2] = \sum_{n=0}^{3} (-1)^n\, x[n] = 1-2+0-3 = -4$$

$$W_4^{n(2)} = e^{-\frac{j2\pi(2)n}{4}} = e^{-j\pi n} = (-1)^n$$

No.

$$X[n] = \frac{1}{N} \sum_{k=0}^{3} X[k] \cdot W_4^{-nk}$$

$$= \frac{1}{4} \sum_{k=0}^{3} X[k] \, W_4^{-nk}$$

$$X[0] = \frac{1}{4} \sum_{k=0}^{3} X[k]$$

$$4X[0] = \sum_{k=0}^{3} X[k] \Rightarrow \sum_{k=0}^{3} X[k] = 4$$

$$\sum_{n=0}^{N-1} |X[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

$$4[1^2+2^2+0^2+3^2] =$$

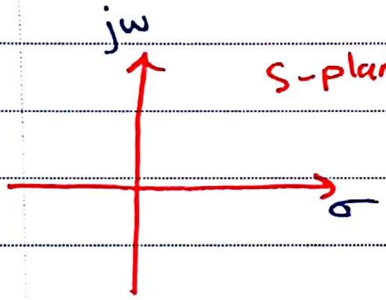
$$\sum_{k=0}^{3} |X[k]|^2 = 56$$

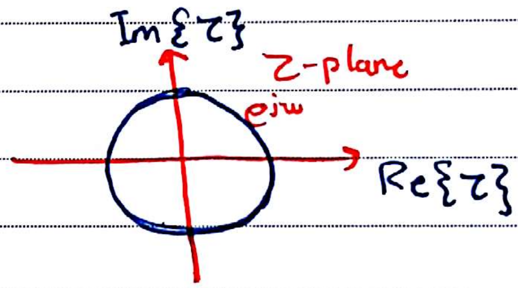
$$\sum_{k=0}^{3} |X[k]|^2 = 6^2 + (1+j)(1-j) + (-4)^2 + (1-j)(1+j) = 56$$

$h[n] = 2^n \mu[n]$ Fourier Transform ما الها

$x[n] = \mu[n]$

$\mu[n] \xleftarrow{DTFT} \frac{1}{1-e^{-jw}} + \pi \sum \delta(n$



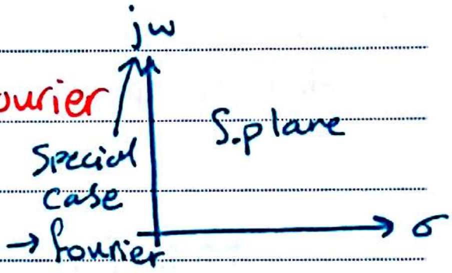


No.

Laplace Transform

$h[n] = 2^n \mu[n] \rightarrow$ no fourier

S-plane $\longleftrightarrow$ Z-plane
cont. discrete

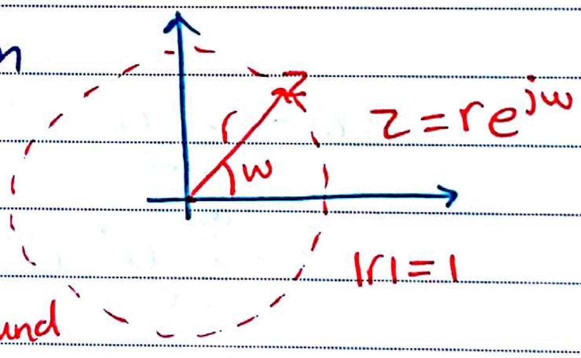


jw axis $\leftrightarrow$ unit Circle

mag → 1 phase → e^{jw}

the Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



the Inverse is found by inspection.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] (r e^{jw})^{-n} = \sum_{n=-\infty}^{\infty} x[n] . r^{-n} . e^{-jwn}$$

<u>Ex</u>: $x[n] = a^n \mu[n]$ find $X(z)$

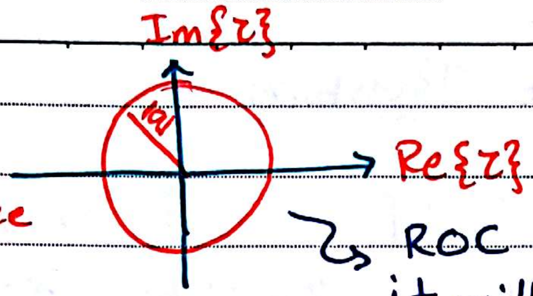
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}} , |a z^{-1}| < 1$$

thus $|z| > |a|$ → ROC Region of Convergence

No.

this is a right sided sequence



ROC

it will be rings/disks

Re{z}, Im{z} discrete & σ, $j\omega$ continuos

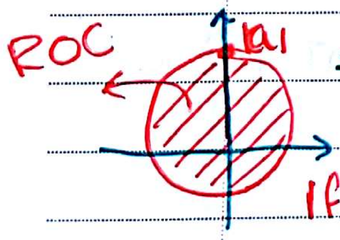
Ex: $X[n] = -a^n u[-n-1]$

Starts from -1 goes to $-\infty$ (left sided sequence).

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} \quad (\text{let } m=-n)$$

$$= -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} (a^{-1}z)^m$$

we find term of zero & subtract it from $\sum_{m=0}^{\infty}$



$$= 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m = 1 - \frac{1}{1-a^{-1}z}$$

if $|a^{-1}z| < 1$ thus $|z| < |a|$ inside the circle

$$= \frac{1-a^{-1}z-1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} \div a^{-1}z$$

$$= \frac{1}{1-az^{-1}} \quad \text{same as prev. ex}$$

FIVE APPLE

No.

* the transform of RSS & LSS is the Same but the ROC is different that's why it is Important to put it.

- what happens if i have a combination of both ??

Ex: $x[n] = \left(-\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$

its $-$ $-$

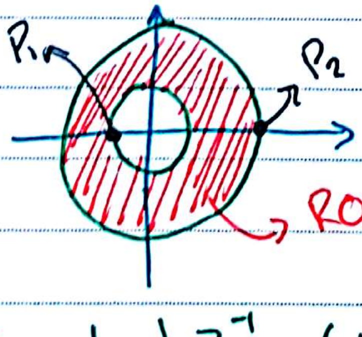
$= x_1[n] - x_2[n]$

$x_1[n] = \left(-\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1+\frac{1}{3}z^{-1}}, |z| > \frac{1}{3}$

$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2}$

$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$, $\cancel{\frac{1}{2} < |z| < \frac{1}{3}}$

$\frac{1}{3} < |z| < \frac{1}{2}$



ROC: Intersection of both (ring)

$$X(z) = \frac{1-\frac{1}{2}z^{-1} - \left(1+\frac{1}{3}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{-\frac{5}{6}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

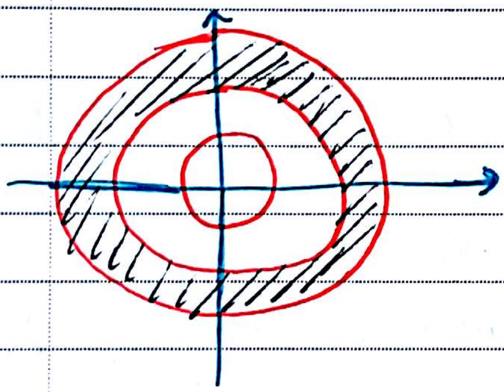
poles: $z = -\frac{1}{3} = P_1$ $\quad z = \frac{1}{2} = P_2$

FIVE APPLE

No.

⇒ the ROC cannot contain any poles because if it did $\frac{1}{0} = \infty$ it will diverge.

Ex I have 3 poles



thus ROC 1st circle
⇒ RSS ↗
ROC 2nd circle
⇒ RSS ↗
ROC 3rd circle
⇒ LSS ↙
we will get this ring. it doesn't contain poles.

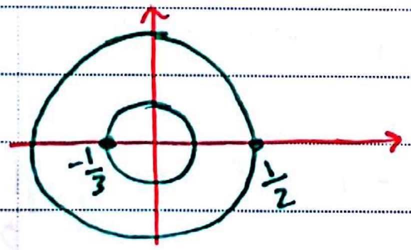
* Projection of the Z-Transform

1) linearity

$$x[n] \overset{z}{\longleftrightarrow} X(z)$$
$$h[n] \overset{z}{\longleftrightarrow} H(z)$$

$$\alpha x[n] + \beta h[n] \overset{z}{\longleftrightarrow} \alpha X(z) + \beta H(z)$$

ROC: $R_1 \cap R_2$



Ex: Find $X(z)$ in both cases

a) $x[n] = \cos(\omega_0 n)\mu[n]$

b) $x[n] = \sin(\omega_0 n)\mu[n]$

Solution:

a) $x[n] = \dfrac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\mu[n]$

$= \frac{1}{2}e^{j\omega_0 n}\mu[n] + \frac{1}{2}e^{-j\omega_0 n}\mu[n]$

No.

$$e^{j\omega_0 n}\mu[n] \xleftrightarrow{z} \frac{1}{1-e^{j\omega_0}\cdot z^{-1}} \qquad ROC: |z|>1$$

$$e^{-j\omega_0 n}\mu[n] \xleftrightarrow{z} \frac{1}{1-e^{-j\omega_0}\cdot z^{-1}}$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j\omega_0}z^{-1} + 1-e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})\cdot(1-e^{-j\omega_0}z^{-1})}\right]$$

$$= \left[\frac{2-e^{-j\omega_0}z^{-1}-e^{j\omega_0}z^{-1}}{1-e^{j\omega_0}z^{-1}-e^{-j\omega_0}z^{-1}+z^{-2}}\right]\cdot\frac{1}{2}$$

$$= \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

* $\cos(\omega_0 n)\mu[n] \xleftrightarrow{z} \dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$

  $ROC: |z|>1$

* $\sin(\omega_0 n)\mu[n] \xleftrightarrow{z} \dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$

  $ROC: |z|>1$

FIVE APPLE

No.

2) Time Shifting:

$x[n] \xleftrightarrow{Z} X(z)$ R.O.C : R_1

$x[n-k] \xleftrightarrow{Z} z^{-k} X(z)$ except possibly the addition or deletion of $z=0$ or $z=\infty$

Ex: $x[n] = \{1\ 2\ 0\ 3\}$ ROC: The z-plane except $z=0$.

$X(z) = 1 + 2z^{-1} + 3z^{-3}$

$x[n] = \{1\ \underset{\uparrow}{2}\ 0\ 3\}$ ROC: The z-plane except $z=0$ and $z=\infty$

$X(z) = z^{1} + 2 + 3z^{-3}$

$x[n] = \{1\ 2\ 0\ \underset{\uparrow}{3}\}$ ROC: z-plane except $z=\infty$

$X(z) = z^{3} + 2z^{2} + 3$

FIVE APPLE

No.

3) Scaling in the Z-domain

$x[n] \xleftrightarrow{Z} X(Z)$ ROC: $r_1 < |Z| < r_2$

$a^n x[n] \xleftrightarrow{Z} X(a^{-1}Z) = X\left(\frac{Z}{a}\right)$

ROC : $|a| r_1 < |Z| < |a| r_2$

Ex : a) $a^n \cos(\omega_0 n)\, u[n]$

b) $a^n \sin(\omega_0 n)\, u[n]$

a) $a^n \cos(\omega_0 n)\, u[n] \xleftrightarrow{Z} \dfrac{1 - \cos(\omega_0)\left(\frac{Z}{a}\right)^{-1}}{1 - 2\cos(\omega_0)\left(\frac{Z}{a}\right)^{-1} + \left(\frac{Z}{a}\right)^{-2}}$

$a^n \cos(\omega_0 n)\, u[n] \xleftrightarrow{Z} \dfrac{1 - \frac{a}{Z}\cos(\omega_0)}{1 - \frac{2a}{Z}\cos(\omega_0) + \frac{a^2}{Z^2}}$

ROC: $|Z| > |a|$

b) $a^n \sin(\omega_0 n)\, u[n] \xleftrightarrow{Z} \dfrac{\frac{a}{Z}\sin(\omega_0)}{1 - \frac{2a}{Z}\cos(\omega_0) + \frac{a^2}{Z^2}}$

ROC: $|Z| > |a|$

No.

- $\alpha^n u[n] \xleftrightarrow{z} \dfrac{1}{1-\alpha z^{-1}}$, $|z| > |\alpha|$

- $\alpha^n u[-n-1] \xleftrightarrow{z} \dfrac{1}{1-\alpha z^{-1}}$, $|z| < |\alpha|$

FIVE APPLE

No.

~~Exam~~ First Exam:

Q1 $y[n] = [5(\frac{1}{2})^n - 2(\frac{1}{4})^n]\, u[n]$

$$E_y = \sum_{n=0}^{\infty} \left(5(\tfrac{1}{2})^n - 2(\tfrac{1}{4})^n\right)^2$$

$$= \sum_{n=0}^{\infty} 25(\tfrac{1}{4})^n + 4(\tfrac{1}{16})^n - 20(\tfrac{1}{2})^n(\tfrac{1}{4})^n$$

$$= \sum_{n=0}^{\infty} 25(\tfrac{1}{4})^n + 4(\tfrac{1}{16})^n - 20(\tfrac{1}{8})^n$$

$$= 25 \cdot \frac{1}{1-\frac{1}{4}} - 20 \cdot \frac{1}{1-\frac{1}{2}} + 4 \cdot \frac{1}{1-\frac{1}{16}}$$

$$= \frac{516}{35} = 14.74$$

$P_y = 0 \Rightarrow$ finite Energy (Energy Sequence)

Q2 $y[n] = 4x[n] + 2x[n-1] + \frac{8}{10} y[n-1]$

$H(e^{j\omega})$??

$$Y(e^{j\omega}) = 4X(e^{j\omega}) + 2e^{-j\omega} X(e^{j\omega}) + \frac{8}{10} Y(e^{j\omega}) e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{4 + 2e^{-j\omega}}{1 - 0.8e^{-j\omega}}$$

No.

$y[n]$??

$$x[n] = (0.4)^n u[n]$$

$$X(e^{jw}) = \frac{1}{1-0.4e^{-jw}}$$

$$Y(e^{jw}) = H(e^{jw}) \cdot X(e^{jw})$$

$$= \frac{4+2e^{-jw}}{(1-0.8e^{-jw})} \cdot \frac{1}{(1-0.4e^{-jw})}$$

$$= \frac{A}{1-0.8e^{-jw}} + \frac{B}{1-0.4e^{-jw}}$$

$$= \frac{A-0.4Ae^{-jw}+B-0.8Be^{-jw}}{(1-0.8e^{-jw})(1-0.4e^{-jw})}$$

$$A+B=4 \qquad A=13$$

$$-0.4A-0.8B=2 \qquad B=-9$$

$$Y(e^{jw}) = \frac{13}{1-0.8e^{-jw}} - \frac{9}{1-0.4e^{-jw}}$$

$$y[n] = 13 * (.8)^n u[n] - 9(0.4)^n u[n]$$

FIVE APPLE

No.

Q_3. $x[n] = \cos(0.55\pi n + 30^\circ)$

$\omega = 0.55\pi$ rad/sample

$\frac{2\pi}{0.55\pi} = \frac{200}{55} = \frac{40}{11} = \frac{N}{T}$ rational number

$N_f = 40$ Sample

$$X(e^{j\omega}) = 3 + 2\sum_{l=0}^{N} \cos(\omega l)$$

Find $x[n]$??

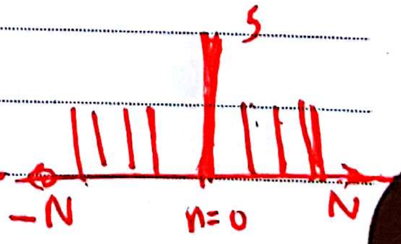
$$X(e^{j\omega}) = 3 + 2\sum_{l=0}^{N} \frac{e^{j\omega l} + e^{-j\omega l}}{2}$$

$$= 3 + \sum_{l=0}^{N} e^{j\omega l} + \sum_{l=0}^{N} e^{-j\omega l}$$

$\cancel{x[n] = \{ 5 \;,\; l=0}$

$$= 3 + [1 + e^{j\omega} + e^{j2\omega} + \dots e^{jN\omega}] + [1 + e^{-j\omega} + e^{-j2\omega} + \dots e^{-jN\omega}]$$

$$x[n] = \begin{cases} 5 & , n=0 \\ 1 & , 0 < |n| < N \\ 0 & , \text{otherwise} \end{cases}$$



No.

Q4 $h[n] = 2^{n-5}\, u[n-5]$

Impulse Response $\rightarrow$ Linear, Time Invariant

$\Downarrow$ always

<u>LTI System</u>

Causal Since $h[n]=0$ for $n<0$

Unstable

$$\sum_{k=0}^{N} |h[k]| < \infty \rightarrow \text{to be Stable}$$

$$\sum_{n=5}^{\infty} (2)^n$$

BIBO Unstable

• The Inverse of Z-Transform

Inspection method :-

- Table of Z-transform pairs.

$$X(Z) = \frac{\sum_{K=0}^{M} a_K Z^{-K}}{\sum_{K=0}^{N} b_K Z^{-K}}$$

If $N > M$ (Proper Polynomial)

Partial fraction expansion.

~~Distinct~~

a) Distinct real roots:-

$$X(Z) = \frac{\sum_{K=0}^{M} a_K Z^{-1}}{b_0 (1-d_1 Z^{-1})(1-d_2 Z^{-1}) \cdots (1-d_N Z^{-1})}$$

$$X(Z) = \sum_{K=1}^{N} \frac{A_K}{(1-d_K Z^{-1})}$$ (By partial fraction expansion)

$$X(Z) = \left(\frac{A_1}{1-d_1 Z^{-1}} + \frac{A_2}{1-d_2 Z^{-1}} + \cdots \frac{A_K}{1-d_K Z^{-1}} \cdots + \frac{A_N}{1-d_N Z^{-1}} \right)$$

$$* (1 - d_K Z^{-1}) \Big|_{Z = d_K} = A_K$$

Ex : $X(z) = \dfrac{1}{(1-\frac{1}{2}z^{-1}).(1-\frac{1}{4}z^{-1})}$, $|z| > \frac{1}{2}$

$z^2 - \frac{3}{4}z + \frac{1}{8} = 0$

$\frac{1}{2}, \frac{1}{4}$

$$= \frac{A_1}{(1-\frac{1}{2}z^{-1})} + \frac{A_2}{(1-\frac{1}{4}z^{-1})}$$

$$A_k = X(z)\,(1-d_k z^{-1})\Big|_{z=d_k}$$

$$A_1 = \frac{1}{1-\frac{1}{4}(2)} = 2$$

$$A_2 = \frac{1}{1-\frac{1}{2}(4)} = -1$$

No.

$$X(z) = \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Ex $\frac{1}{4} < |z| < \frac{1}{2}$

$$x[n] = -2\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{4}\right)^n u[n]$$

$|z| < \frac{1}{4}$

$$x[n] = -2\left(\frac{1}{2}\right)^n u[-n-1] + \left(\frac{1}{4}\right)^n u[-n-1]$$

If $M \geq N$ (Improper function)

Use long division.

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} A_k / (1 - d_k z^{-1})$$

No.

Ex $X(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$, $|z|>1$

$$X(z) = B_0 + \frac{A_1}{1-d_1z^{-1}} + \frac{A_2}{1-d_2z^{-1}}$$

$$= \frac{1+2z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$= B_0 + \frac{A_1}{(1-\frac{1}{2}z^{-1})} + \frac{A_2}{(1-z^{-1})}$$

By long division

By partial fraction expansion.

$$\begin{array}{r|l} & 2 \\ \hline \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & z^{-2}-3z^{-1}+2 \\ \hline & 5z^{-1}-1 \end{array}$$

$$X(z) = 2 + \frac{5z^{-1}-1}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1} \Rightarrow B_0 = 2$$

$$A_1 = X(z)\cdot(1-\tfrac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}}$$

$$= \frac{5(2)-1}{1-2}$$

$$A_1 = -9$$

No.

$$A_2 = X(z) \cdot (1 - z^{-1}) \Big|_{z=1}$$

$$= \frac{5(1) - 1}{1 - \frac{1}{2}(1)}$$

$$A_2 = +8$$

$$X(z) = 2 + \frac{8}{1 - z^{-1}} - \frac{9}{1 - \frac{1}{2} z^{-1}} \quad , |z| > 1$$

Causal System $\Rightarrow$ Right Sided Sequence

$$x[n] = 2\delta[n] + 8\mu[n] - 9\left(\frac{1}{2}\right)^n \mu[n]$$

Ex: A causal LTI discrete-time System is described by the difference equation

$$Y[n] = 2X[n] + Y[n-1] - 0.5Y[n-2]$$

a) Determine the transfer function $H(z)$

b) Determine the impulse response $h[n]$

c) Determine the step response $S[n]$

Solution:

a) $Y(z) = 2X(z) + z^{-1}Y(z) - 0.5z^{-2}Y(z)$

$$Y(z) - z^{-1}Y(z) + 0.5z^{-2}Y(z) = 2X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - z^{-1} + 0.5z^{-2}}$$

b) $z^2 - z + 0.5 = 0$

$$z_{1,2} = \frac{1}{2} \pm j\frac{1}{2}$$

$$a^n \cos(\omega n)\mu[n] \overset{z}{\longleftrightarrow} \frac{1 - a\cos(\omega)z^{-1}}{1 - 2a\cos(\omega)z^{-1} + a^2z^{-2}}$$

No.

$$a^n \sin(\omega n)\,\mu[n] \overset{z}{\longleftrightarrow} \frac{a \sin(\omega) z^{-1}}{1 - 2a\cos(\omega) z^{-1} + a^2 z^{-2}}$$

$a^2 = 0.5$

$a = \frac{1}{\sqrt{2}}$

$2a\cos(\omega) = 1$

$\sqrt{2}\cos(\omega) = 1$

$\boxed{\cos(\omega) = \frac{1}{\sqrt{2}}}$ $\boxed{\sin(\omega) = \frac{1}{\sqrt{2}}}$

$\omega = \pi/4$

$$\frac{2}{1 - z^{-1} + 0.5z^{-2}} = \frac{2 - z^{-1} + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

~~a cos(ω)~~ $a\sin(\omega)$

$a\cos(\omega)$ ~~a cos(ω)~~

$$= 2\frac{\left(1 - \frac{1}{2}z^{-1}\right)}{1 - z^{-1} + 0.5z^{-2}} + 2\frac{\left(\frac{1}{2}z^{-1}\right)}{1 - z^{-1} + 0.5z^{-2}}$$

$$= 2 * \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n\right)\mu[n]$$

$$+ 2 * \left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi}{4}n\right)\mu[n]$$

FIVE APPLE

No.

$a^n \cos(\omega n)\, u[n] + a^n \sin(\omega n)\, u[n] \overset{Z}{\longleftrightarrow} \dfrac{1}{1-2a\cos(\omega) z^{-1} + a^2 z^{-2}}$

c) $S[n] =$

$$S(z) = H(z) \cdot U(z)$$

$$= \frac{2}{1-z^{-1}+0.5z^{-2}}$$

$$= \frac{A_1 + A_2 z^{-1}}{1-z^{-1}+0.5z^{-2}} + \frac{A_3}{1-z^{-1}}$$

$$A_3 = \frac{2}{1-1+0.5} = 4$$

$$S(z) = \frac{A_1 + A_2 z^{-1}}{1-z^{-1}+0.5z^{-2}} + \frac{4}{1-z^{-1}}$$

$$= \frac{A_1 - A_1 z^{-1} + A_2 z^{-1} - A_2 z^{-2} + 4 - 4z^{-1} + 2z^{-2}}{} = 2$$

$A_1 + 4 = 2 \rightarrow \boxed{A_1 = -2}$

$-A_1 + A_2 - 4 = 0 \rightarrow \boxed{A_2 = 2}$

FIVE APPLE

No.

$$S(z) = \frac{-2+2z^{-1}}{1-z^{-1}+0.5z^{-2}} + \frac{4}{1-z^{-1}}$$

$$= \frac{-2\left[1-\frac{1}{2}z^{-1}\right]}{1-z^{-1}+0.5z^{-2}} + \frac{2\left(\frac{1}{2}z^{-1}\right)}{1-z^{-1}+0.5z^{-2}}$$

$$+ \frac{4}{1-z^{-1}}$$

$$= -2\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n\right) u[n]$$

$$+ 2\left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$$

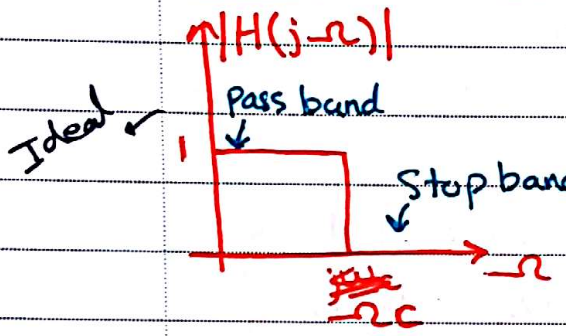
$$+ 4\, u[n]$$

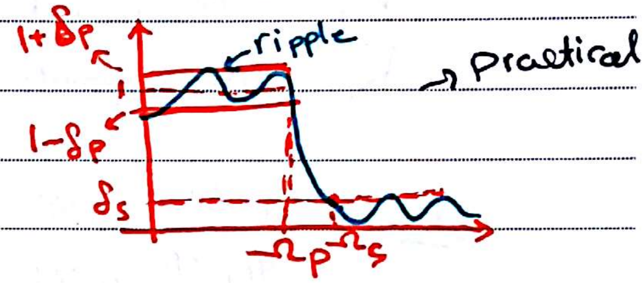
لهون السكشن

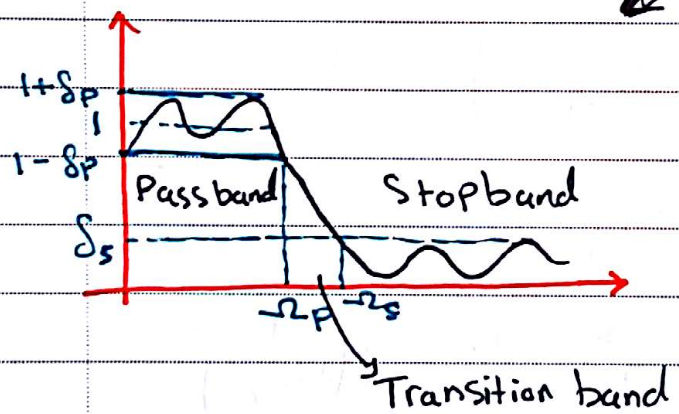
FIVE APPLE

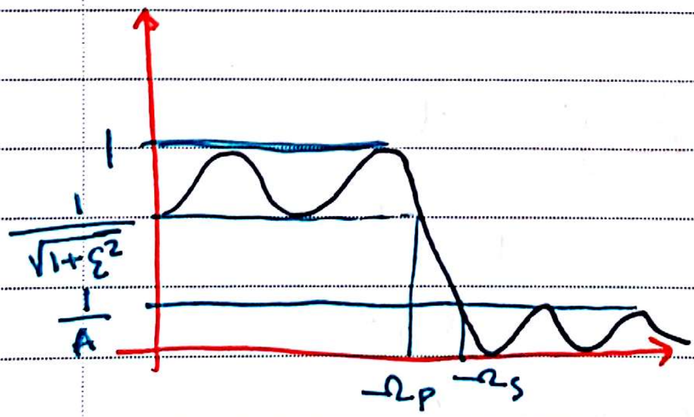
Discrete-time IIR filter design:

- Analog low pass filter design:









No.

Specifications:

~~Rp~~ Co

R_p corresponds to Ω_p
R_s corresponds to Ω_s

Ex: -1 dB with $\Omega_p = 1$ kHz
-40 dB with $\Omega_s = 5$ kHz

$$10 \log\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)^2 = -1$$

$$10 \log\left(\frac{1}{A}\right)^2 = -40$$

Note: $20 \log\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)^{\cancel{2}} = 10 \log\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)^2$

$$10 \log\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)^2 = -1$$

$$\frac{1}{1+\varepsilon^2} = 10^{-0.1}$$

$$\varepsilon^2 = 0.25895$$

$$10 \log\left(\frac{1}{A}\right)^2 = -40$$

$$\log \frac{1}{A} = -2$$

$$A = 10^2 = 100$$

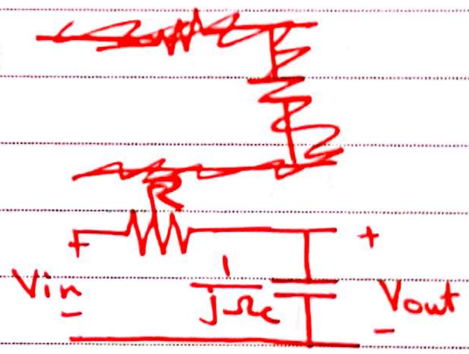
No.

The Butterworth Approximation:

$$|H_a(j\omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

~~[illegible] Pass~~

- 1st Order Butterworth low Pass filter:



$$H(s) = \frac{1}{s+1}$$

$$H(j\Omega) = \frac{1}{1+j\Omega}$$

$$H\left(j\left(\frac{\Omega}{\Omega_c}\right)\right) = \frac{1}{1+j\frac{\Omega}{\Omega_c}}$$

$$H(j\Omega) = \frac{\Omega_c}{\Omega_c + j\Omega}$$

$$= \frac{\left(\frac{1}{j\Omega_c}\right) j\Omega C}{\left(\frac{1}{j\Omega_c} + R\right) j\Omega C}$$

$$= \frac{(j\Omega_c/(RC))}{\frac{1}{RC} + j\Omega}$$

$$\{H(j\Omega)\} = \frac{\Omega_c}{\Omega_c + j\Omega}$$

$$H(jw) \cdot H^*(jw) = \frac{\Omega_c}{\Omega_c + j\Omega} * * \frac{\Omega_c}{\Omega_c - j\Omega}$$

$$= \frac{\Omega_c^2}{\Omega_c^2 + j\Omega\Omega_c - j\Omega\Omega_c + \Omega^2}$$

$$= \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^2}$$

~~2nd order Butterworth~~

$$\cancel{H(S) = \frac{1}{S^2 + \sqrt{2}S + 1}}$$

$$|Ha(jw)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^4}$$

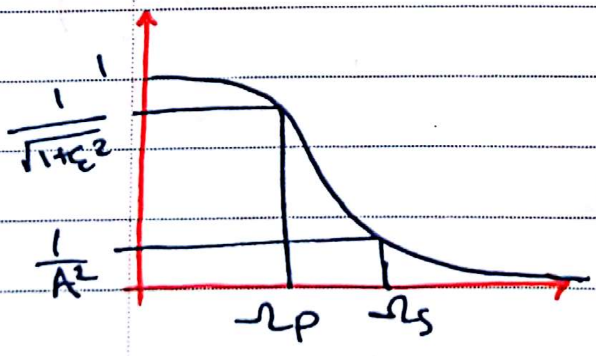
2nd order Butterworth:

$$H(S) = \frac{1}{S^2 + \sqrt{2}S + 1}$$

~~Re~~ * Required:

find N

find Ω_c



$$|Ha(j\omega\Omega_p)|^2 = \frac{1}{1+\varepsilon^2}$$

$$\frac{1}{1+\varepsilon^2} = \frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}}$$

No.

$$|H_a(j\Omega_c)|^2 = \frac{1}{A^2} = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}}$$

$$\varepsilon^2 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \quad — (1)$$

$$A^2 = 1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \Rightarrow \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = A^2 - 1 \quad — (2)$$

let

$K = \frac{\Omega_p}{\Omega_s}$: Transition Parameter (Selectivity)

$K_1 = \frac{\varepsilon}{\sqrt{A^2-1}}$: The ~~selectivity~~ discrimination Parameter

Divide (2) over (1)

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{A^2-1}{\varepsilon^2}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{N} = \frac{\sqrt{A^2-1}}{\varepsilon}$$

$$\cancel{N \log_{10}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}$$

$$N \log_{10}\left(\frac{\Omega_s}{\Omega_p}\right) = \log_{10}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)$$

No.

$$\boxed{N = \frac{\log_{10}\left(\frac{1}{k_1}\right)}{\log\left(\frac{1}{k}\right)}}$$

Ex $-1dB$ with $f_p = 1kHz$
$-40dB$ with $f_s = 5kHz$

$\varepsilon = 0.25895$

$A = 100$

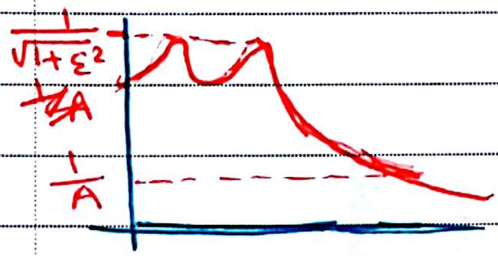
$$N = \frac{\log_{10}\left(\frac{1}{k_1}\right)}{\log_{10}\left(\frac{1}{k}\right)} = \frac{\log(196)}{\log(5)} = 3.28$$

$$\boxed{N = 4}$$

No.

*Chebysev Approximation

1. type (1) Chebysev Approximation



$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

where $T_N(\Omega)$ is a chebysev polynomial of order N.

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}(\Omega)) & , |\Omega| \leq 1 \\ \cosh(N\cosh^{-1}(\Omega)) & , |\Omega| > 1 \end{cases}$$

- recurence relation:

$$T_r(\Omega) = 2\Omega T_{r-1}(\Omega) - T_{r-2}(\Omega) \qquad r \geq 2$$

$T_0(\Omega) = 1$, $T_1(\Omega) = \Omega$, ~~$T_2(\Omega) =$~~

$$T_2(\Omega) = 2\Omega T_1(\Omega) - T_0(\Omega)$$
$$= 2\Omega^2 - 1$$

$$|H_a(j\Omega_p)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_p}{\Omega_p}\right)} = \frac{1}{1+\varepsilon^2}$$

→ It stayed the same, No information gained.

FIVE APPLE

No.

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\varepsilon^2 T_N(\frac{\Omega_s}{\Omega_p})} = \frac{1}{A^2}$$

$$\therefore A^2 = 1+\varepsilon^2 T_N(\Omega_s/\Omega_p)$$

$$\cancel{T_N^2 = A^2 - 1/\varepsilon^2}$$

$$T_N^2\left(\frac{\Omega_s}{\Omega_p}\right) = (A^2-1)/\varepsilon^2 \Rightarrow T_N\left(\frac{\Omega_s}{\Omega_p}\right) = \sqrt{A^2-1}/\varepsilon$$

$$\cosh(N\cosh^{-1}(\Omega_s/\Omega_p)) = \frac{\sqrt{A^2-1}}{\varepsilon}$$

$\Omega_s > \Omega_p$
$\cosh(N\cosh^{-1}(\Omega))$
ratio > 1

$$N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right) = \cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)$$

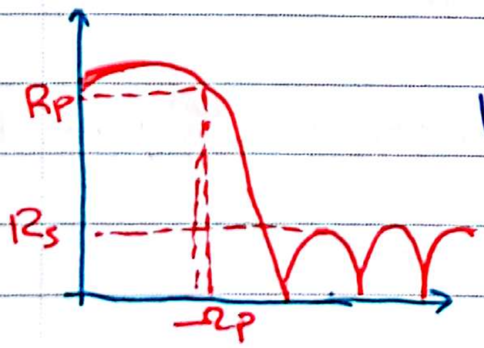
$$N = \cosh^{-1}(1/k_1)/\cosh^{-1}(1/k)$$

<u>Ex</u>: $\frac{1}{k_1} = 196$, $\frac{1}{k} = 5$

$$N = \cosh^{-1}(196)/\cosh^{-1}(5) = 2.6 \Rightarrow N = 3$$

---

2. Type (2) chebysev Approximation



$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\frac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}\right]^2}$$

FIVE APPLE

No.

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\varepsilon^2\left[\frac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega_s)}\right]^2} = \frac{1}{A^2}$$

* $H_a(j\Omega_s) = \frac{1}{A^2}$

$N = \cosh^{-1}(1/k_1) \,/\, \cosh^{-1}(1/k)$

• Analog Filter Design Using Matlab

1) Butterworth Filter

$[N, \omega_n] = \text{buttord}(\Omega_p, \Omega_s, R_p, R_s, \text{'S'})$

$\Omega_p = 2*pi*1000$

$\Omega_s = 2*pi*5000$

$R_p = 1$

$R_s = 40$

$\omega_n = \omega_0 = 3dB$ frequency.

Vectors

$[num, den] = \text{butter}(N, \omega_n, \text{'S'}) \leftarrow H(S)$

No.

2) Type(2) chebysev

[N, wn] = Cheb1ord(ω_p, ω_s, Rp, Rs, 'S')
[num, den] = cheby1 (N, Rp, wn, 'S')

3) Type (3) Chebysev

[N, wn] = cheb2ord (ω_p, ω_s, Rp, Rs, 'S')
[num, den] = cheby2 (N, Rs, wn, 'S')

18/4

No.

*Design of Analog Highpass, Bandpass and Bandstop Filters

Mapping of $S \rightarrow \hat{S}$

$$S = F(\hat{S})$$

lowpass　　Desired filter

$$H_D(\hat{S}) = H_{LP}(S)\Big|_{S=F(\hat{S})}$$

$$H_{LP}(S) = H_D(\hat{S})\Big|_{\hat{S}=F^{-1}(S)}$$

① Analog Highpass Filter Design :-

$$H_{LP}(S) \rightarrow H_{HP}(\hat{S})$$

$$S = F(\hat{S}) = \frac{\Omega_p \cdot \hat{\Omega}_p}{\hat{S}} \quad , \quad S = \frac{1}{\hat{S}}$$

$$H_{HP}(\hat{S}) = H(S)\Big|_{S=F(\hat{S})} = \frac{1}{\left(\frac{1}{\hat{S}}\right)+1} \quad , \quad H(S) = \frac{1}{S+1}$$

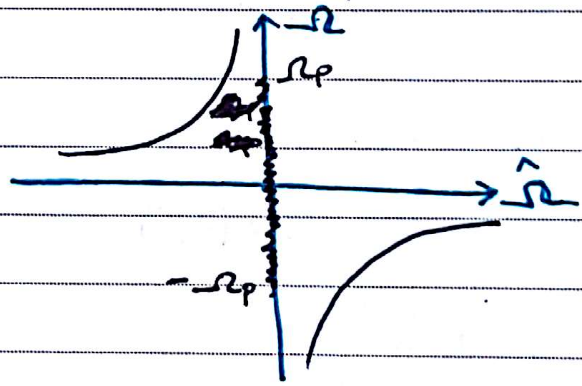
$$= \frac{\hat{S}}{1+\hat{S}}$$

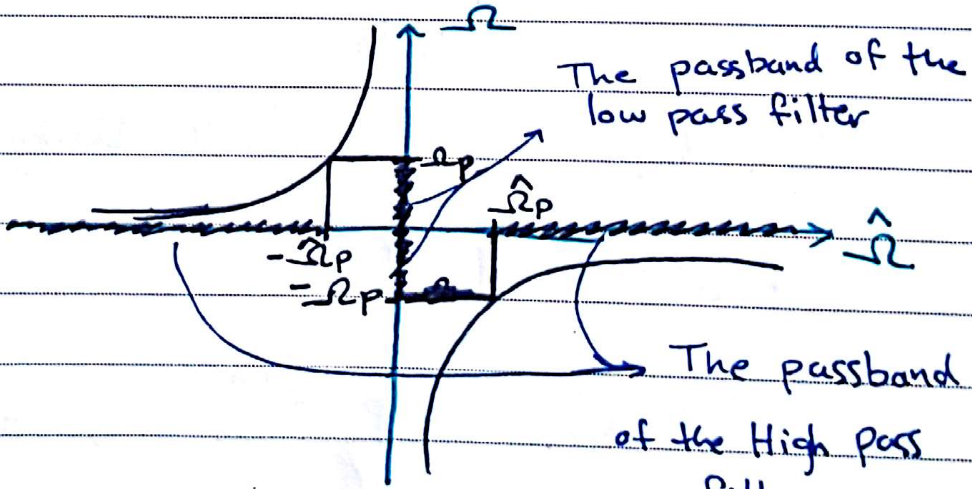
No.

*On the Imaginary axis

$$j\Omega = \frac{\Omega_p \hat{\Omega}_p}{j\hat{\Omega}}$$

$$\boxed{\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}}$$





No.

Ex: Design of an analog Butterworth Highpass filter using matlab.

Maximally

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

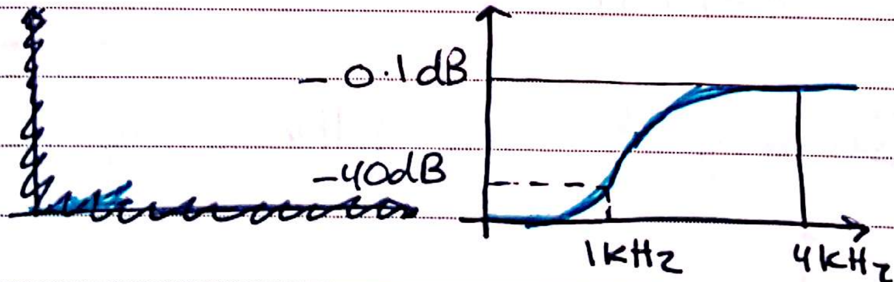
$(2N-1)$ Derivatives of $\Omega = 0 \Rightarrow 0$

Maximally Flat

Specifications

$\hat{f}_p = 4\,kHz$ with $0.1\,dB$ ripple

$\hat{f}_s = 1\,kHz$ with $40\,dB$ attenuation.



$$\Omega_s = \frac{\Omega_p \cdot \hat{\Omega}_p}{\hat{\Omega}_s} \quad , \quad S = \frac{\Omega_p \cdot \hat{\Omega}_p}{\hat{S}}$$

let $\Omega_p = 1$,

$$\Omega_s = \frac{(1)(4000)}{1000} = 4$$

No.

low pass Specifications

$\Omega_p = 1$ with $R_p = -.1dB$
$\Omega_s = 4$ with $R_s = -40dB$

ε, A, N
Ω_c, K', K
all of those can be found

>> [N, Wn] = buttord(1,4, 0.1,40,'S')

(Ωp, Ωs, Rp, Rs, 'S')

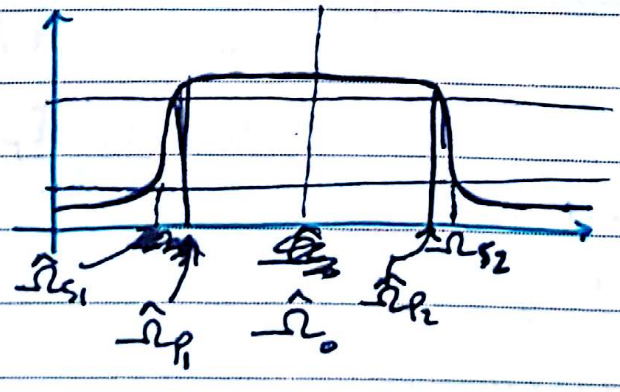
N: order of filter
Wn: cutoff frequency

>> [B,A] = butter(N, Wn, 'S')

>> [num,den] = Lp2hp(B,A, 2*pi*4000)

B = [1]
A = [1,1]

② Analog Band pass filter design :-



No.

$\hat{\Omega}_{s_1}$ & $\hat{\Omega}_{s_2} \rightarrow R_s$

$\hat{\Omega}_{p_1}$ & $\hat{\Omega}_{p_2} \rightarrow R_p$

$BW = \hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}$

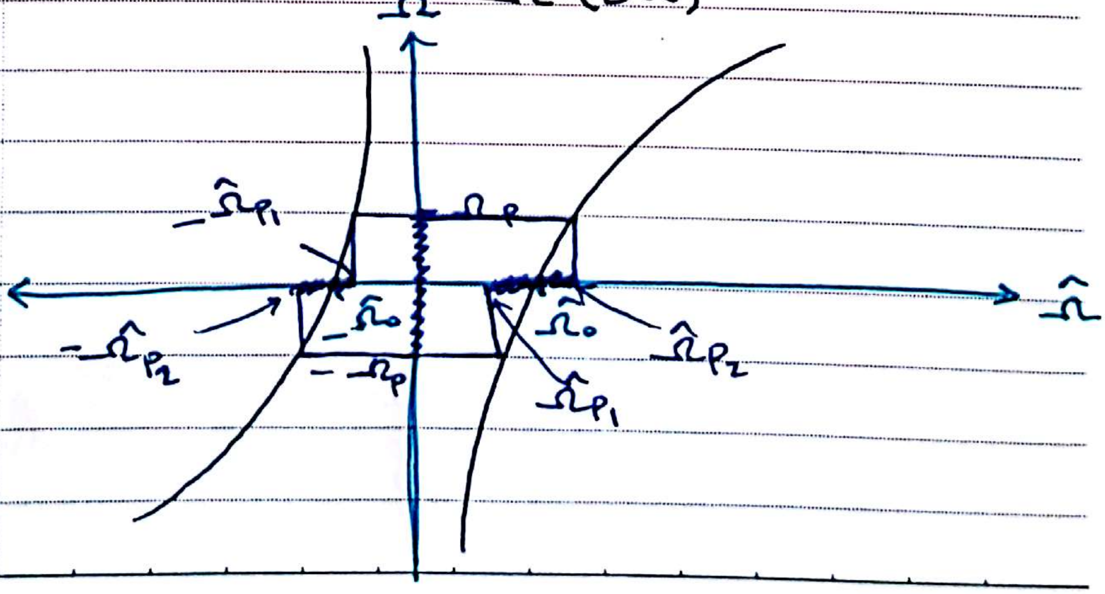
$$s = F(\hat{s})$$

$$s = \Omega_p \cdot \frac{\hat{s}^2 + \hat{\Omega}_o^2}{\hat{s}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$$

on the Imaginary axis

$$j\Omega = \Omega_p \frac{(j\hat{\Omega})^2 + \hat{\Omega}_o^2}{j\hat{\Omega}(BW)}$$

$$\Omega = -\Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega}(BW)}$$



No.

$$\boxed{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}=\hat{\Omega}_{s_1}\hat{\Omega}_{s_2}=\hat{\Omega}_0^2}$$

~~[illegible]~~ ~~[illegible]~~ شرط يجب يتحقق

Fix $\hat{\Omega}_{p_1}$ & $\hat{\Omega}_{p_2}$, $\hat{\Omega}_{p_1}\cdot\hat{\Omega}_{p_2}=\hat{\Omega}_0^2$

If $\hat{\Omega}_{s_1}\hat{\Omega}_{s_2}<\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}\rightarrow$ I will increase $\hat{\Omega}_{s_1}$.

~~So~~ $$\boxed{\hat{\Omega}_{s_1}=\frac{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}}{\hat{\Omega}_{s_2}}}$$

If $\hat{\Omega}_{s_1}\hat{\Omega}_{s_2}>\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}\rightarrow$ I will decrease $\hat{\Omega}_{s_2}$

$$\boxed{\hat{\Omega}_{s_2}=\frac{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}}{\hat{\Omega}_{s_1}}}$$

2014

No.

Ex: Design an analog Bandpass type 2 ~~Chebysev~~ Chebyshev filter using matlab.

$\hat{f}_p = 4\,kHz$ $\quad\hat{f}_{p_2} = 7\,kHz$ $\quad R_p = 1dB$ ripple

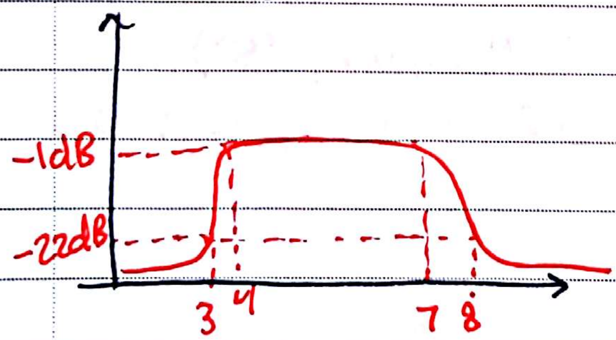
$\hat{f}_s = 3\,kHz$ $\quad\hat{f}_{s_2} = 8\,kHz$ $\quad R_s = 22dB$ attenuation

$$\hat{f}_0^{\,2} = 4\times10^3 \times 7\times10^3 = 28\times10^6$$

$$\hat{f}_0 = \sqrt{28}\,kHz \qquad BW = \hat{f}_{p_2} - \hat{f}_{p_1} = 3\,kHz$$

$$\hat{f}_{s_1} \cdot \hat{f}_{s_2} = 3\times8 = 24\times10^6$$

$$\hat{f}_{s_1} = \frac{28\times10^6}{8\times10^3} = 3.5\,kHz$$



$$\Omega = \Omega_p \cdot \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\,BW}$$

let $\Omega_p = 1$

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No.

$$\Omega_s = (1) \cdot \frac{28\times10^6 - (3.5\times10^3)^2}{(3.5)\times10^3\times(3\times10^3)}$$

$$\Omega_s = 1.5$$

<u>low pass Spec.</u>

$\Omega_p = 1$ with $R_p = 1dB$

$\Omega_s = 1.5$ with $R_s = 22dB$

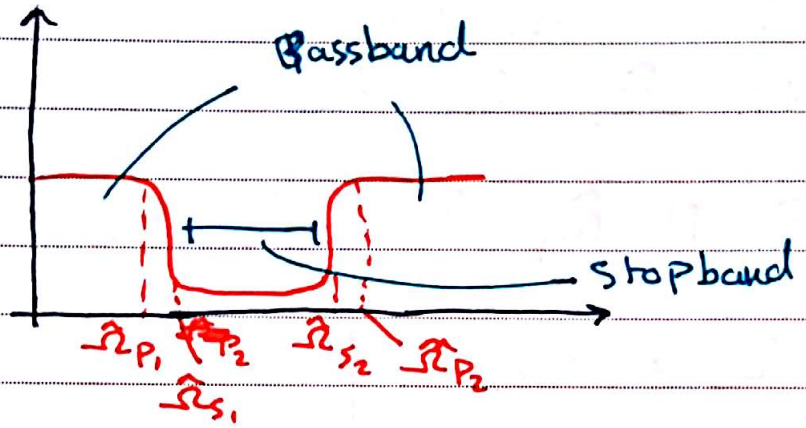
$$N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)}$$

```
wo= 2*pi* Sqrt(28)*1000;
BW= 2*pi*3000
[N,Wn] = cheb2ord(1,1.5,1,22,'S');
[B,A] = cheby2 (N,22,Wn,'S');
[num,den] = lp2bp(B,A,wo,BW);
```

No.

Analog BandStop Filter design:-

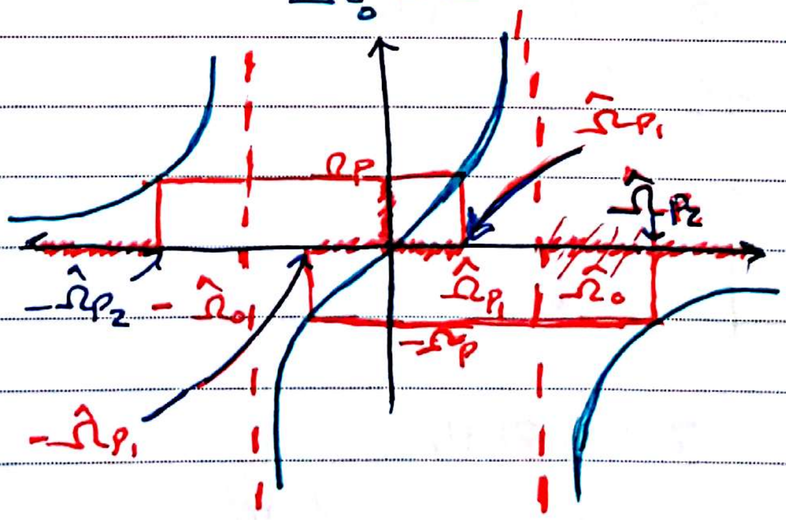
$S \equiv F(\hat{S})$



$BW = \hat\Omega_{s_2} - \hat\Omega_{s_1}$

$$S = \Omega_s \cdot \frac{\hat{S}(\hat\Omega_{s_2} - \hat\Omega_{s_1})}{\hat{S}^2 + \hat\Omega_0^2}$$

$$\Omega = \Omega_s \cdot \frac{\hat\Omega \, BW}{\hat\Omega_0^2 - \hat\Omega^2}$$



No.

$$\hat{\Omega}_{s_1} \cdot \hat{\Omega}_{s_2} = \hat{\Omega}_{p_1} \cdot \hat{\Omega}_{p_2} = \hat{\Omega}_o^2$$

$BW = \hat{\Omega}_{s_2} - \hat{\Omega}_{s_1}$

Fix $\hat{\Omega}_{s_1} \cdot \hat{\Omega}_{s_2} = \hat{\Omega}_o^2$

If $\hat{\Omega}_{p_1} \cdot \hat{\Omega}_{p_2} > \hat{\Omega}_{s_1} \cdot \hat{\Omega}_{s_2}$

$$\boxed{\hat{\Omega}_{p_2} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_1}}}$$

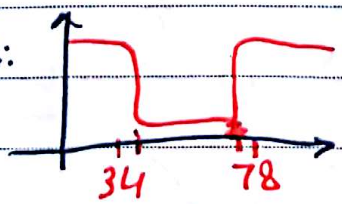
If $\hat{\Omega}_{p_1} \cdot \hat{\Omega}_{p_2} < \hat{\Omega}_{s_1} \cdot \hat{\Omega}_{s_2}$

$$\boxed{\hat{\Omega}_{p_1} = \frac{\hat{\Omega}_{s_1} \hat{\Omega}_{s_2}}{\hat{\Omega}_{p_2}}}$$

~~Ex~~ ~~BW = 7-4 = 3kHz~~

~~$\hat{\Omega}_{s_2} - \hat{\Omega}_{s_1}$~~

Ex:



$BW = 7 - 4 = 3 \text{ kHz}$

$\hat{\Omega}_o^2 = \hat{\Omega}_{s_1} \cdot \hat{\Omega}_{s_2} = 28 \times 10^6 \Rightarrow \hat{\Omega}_o = \sqrt{28} \text{ kHz}$

No.

$\hat{\Omega}_{p_1} \cdot \hat{\Omega}_{p_2} = 24\times10^6$

$\hat{\Omega}_{p_1} = \dfrac{28\times10^6}{8\times10^3} = 3.5\,kHz$

---

let $\Omega_s = 1$

$\Omega_p = (1)\,\dfrac{(3.5)(3)}{28-(3.5)^2} = \dfrac{2}{3}$

~~let $\Omega_s = 1$~~

~~Ω_s~~

matlab

```
lp2bs(B,A,2*pi*wo, 2*pi*BW);
```

No.

*Discretization of Analog filter

$$H(S) = \frac{a}{S+a} \quad , \quad \frac{Y(S)}{X(S)} = \frac{a}{S+a}$$

$$S\,Y(s) + a\,Y(s) = a\,X(S)$$

$$\frac{dy(t)}{dt} = -a\,y(t) + a\,x(t)$$

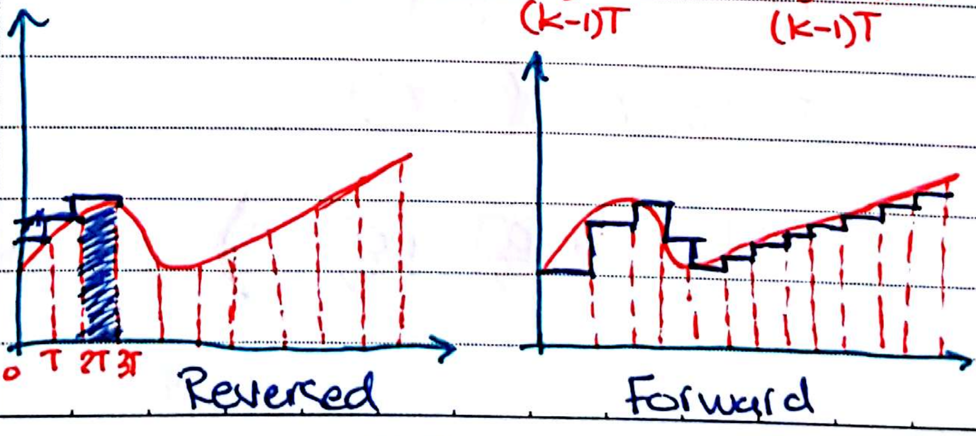
$$\int_0^{KT} \frac{dy(t)}{dt} = \int_0^{KT} -a\,y(t) + \int_0^{KT} a\,x(t) \; dt$$

$$y(KT) - y(0) = -a \int_0^{KT} y(t)\,dt + a \int_0^{KT} x(t)\,dt$$

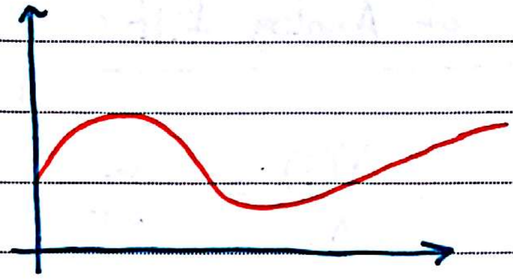
$$\int_0^{(k-1)T} \frac{dy(t)}{dt}\,dt = -a \int_0^{(k-1)T} y(t)\,dt + a \int_0^{(k-1)T} x(t)\,dt$$

$$y((k-1)T) - y(0) =$$

$$y(KT) - y((k-1)T) = -a \int_{(k-1)T}^{KT} y(t)\,dt + a \int_{(k-1)T}^{KT} x(t)\,dt$$



No.



$$\boxed{\frac{1}{2}\left((y(kT)+y((k-1)T)\right)T}$$ ⇝ Trapezoid Approximation Bilinear Transformation

$$y(kT)-y((k-1)T) = -a\left[\frac{1}{2}\left(y(kT)+y((k-1)T)\right)\right].T$$

$$+a\left[\frac{1}{2}\left(x(kT)+x((k-1)T)\right)\right].T$$

$$Y(z)-z^{-1}Y(z) = -a\left[\frac{1}{2}\left(Y(z)+z^{-1}Y(z)\right)\right].T$$

$$+a\left[\frac{1}{2}\left(X(z)+z^{-1}X(z)\right)\right].T$$

$$Y(z)\left(1-z^{-1}\right)+\frac{aT}{2}\left(Y(z)+z^{-1}Y(z)\right)$$

$$= \frac{aT}{2}X(z)\left(1+z^{-1}\right)$$

$$Y(z)\left(1-z^{-1}+\frac{aT}{2}+\frac{aT}{2}z^{-1}\right) = \frac{aT}{2}X(z)\left(1+z^{-1}\right)$$

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$$\frac{Y(z)}{X(z)} = \frac{(aT/2)(1+z^{-1})}{\left(1-z^{-1}+\frac{aT}{2}(1+z^{-1})\right)}$$

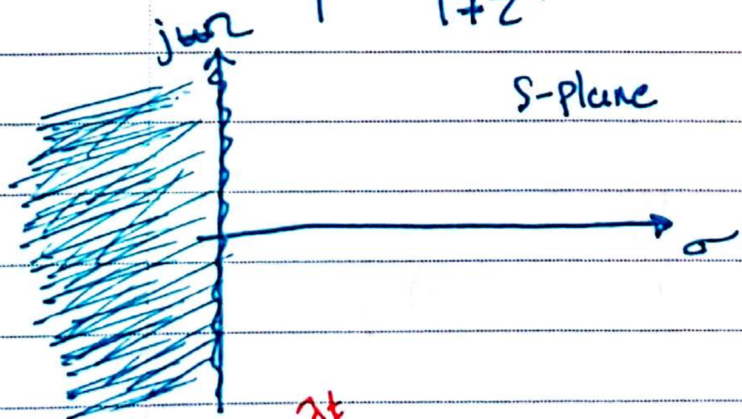
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{aT}{2}+\frac{aT}{2}z^{-1}}{1+\frac{aT}{2}-z^{-1}+\frac{aT}{2}z^{-1}}$$

$$= \frac{a\left[\frac{T}{2}+\frac{T}{2}z^{-1}\right]}{1-z^{-1}+a\left[\frac{T}{2}+\frac{T}{2}z^{-1}\right]}$$

$$\boxed{H(z) = \frac{a}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}}$$

* The Bilinear Transformation :-

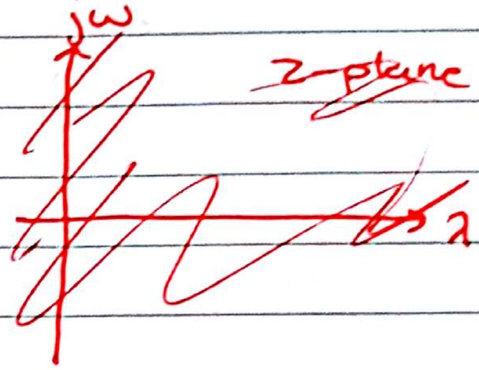
$$S = \frac{2}{T} * \frac{1-z^{-1}}{1+z^{-1}}$$



$e^{-\lambda t}$

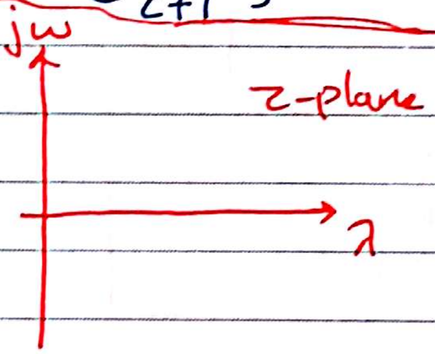
$(\lambda)^n$

$$Re\{s\} < 0$$



$$Re\left\{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\} < 0$$

$$Re\left\{\frac{z-1}{z+1}\right\} < 0$$



$$Re\left\{\frac{\lambda + jw - 1}{\lambda + jw + 1}\right\} < 0$$

$$Re\left\{\frac{(\lambda-1) + jw}{(\lambda+1) + jw}\right\} < 0$$

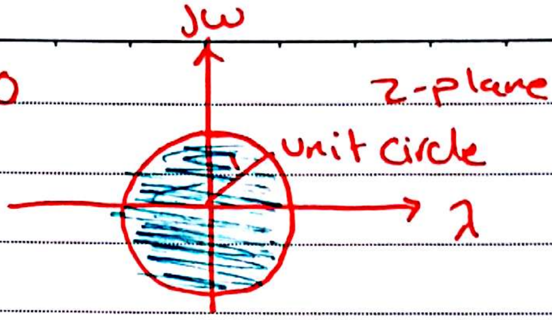
$$Re\left\{\frac{((\lambda-1)+jw) * ((\lambda+1) - jw)}{((\lambda+1)+jw) * ((\lambda+1) - jw)}\right\}$$

$$Re\left\{\frac{\lambda^2 - 1 \; + jw(\lambda+1) - (\lambda-1)jw + w^2}{(\lambda+1)^2 + w^2}\right\}$$

No.

$\lambda^2+\omega^2-1<0$

$\lambda^2+\omega^2<1$



let T=2

$$s = \frac{1-z^{-1}}{1+z^{-1}}, \quad z=e^{j\omega}$$

$$j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}}$$

$$= \frac{j e^{-j\omega/2}\left[e^{j\omega/2}-e^{-j\omega/2}\right]/2j}{e^{-j\omega/2}\left[\frac{e^{+j\omega/2}+e^{-j\omega/2}}{2}\right]}$$

$$\boxed{\Omega = \tan\left(\frac{\omega}{2}\right)}$$

No.

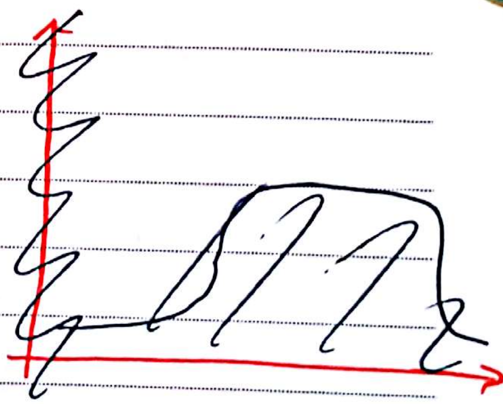
Q. Using the bilinear transformation method, design a discrete-time type 1 chebyshev bandpass filter operating at a sampling rate of 10 KHz with the following Specifications:
pass band edges at 1.2 kHz and 2.2 kHz with peak passband ripple of 0.8 dB and Stop band edges of 0.65 kHz and 3 kHz with minimum Stopband attenuation of 31 dB.

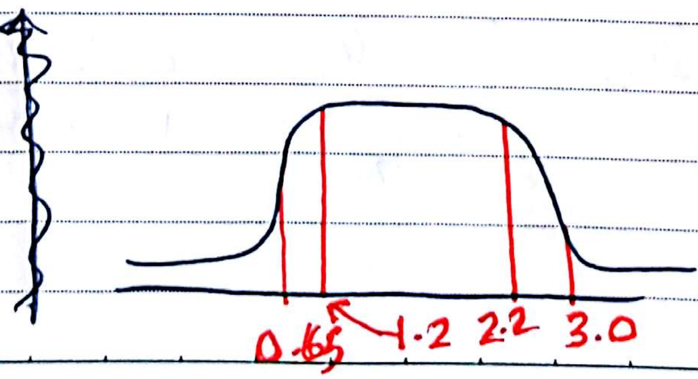
$$\hat{\omega}_{s_1} = \frac{2\pi(0.65)}{10} = 0.13\pi$$

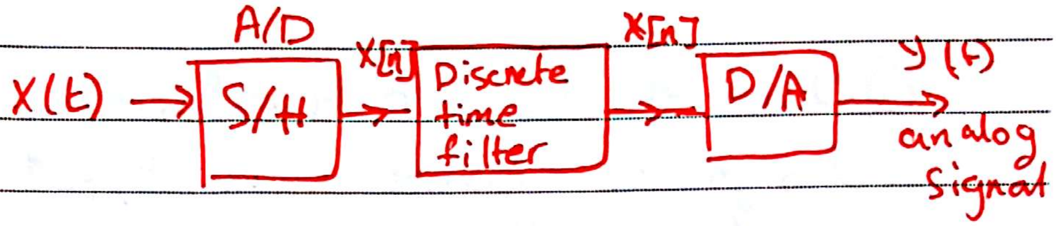
$$\hat{\omega}_{p_1} = \frac{2\pi(1.2)}{10} = 0.24\pi$$

$$\hat{\omega}_{s_2} = \frac{2\pi(3)}{10} = 0.6\pi$$

$$\hat{\omega}_{p_2} = \frac{2\pi(2.2)}{10} = 0.44\pi$$







$$\hat{\Omega}_{s_1} = \tan\left(\frac{\omega_{s_1}}{2}\right) = \tan\left(\frac{.13\pi}{2}\right) = 0.2071 \text{ rad}$$

$$\hat{\Omega}_{p_1} = \tan\left(\frac{\omega_{p_1}}{2}\right) = \tan\left(0.12\pi\right) = 0.3959 \text{ rad}$$

$$\hat{\Omega}_{p_2} = \tan\left(\frac{\omega_{p_2}}{2}\right) = 0.8273 \text{ rad}$$

$$\hat{\Omega}_{s_2} = \tan\left(\frac{\omega_{s_2}}{2}\right) = 1.3764 \text{ rad}$$

$$BW = \hat{\Omega}_{p_2} - \hat{\Omega}_{p_1} = 0.4314$$

$$\hat{\Omega}_0^2 = \hat{\Omega}_{p_1} \hat{\Omega}_{p_2} = 0.32753$$

$$\hat{\Omega}_0 = 0.5723$$

$$\hat{\Omega}_{s_1} \hat{\Omega}_{s_2} = 0.28505$$

$$\hat{\Omega}_{s_1} = \frac{\hat{\Omega}_{p_1} \hat{\Omega}_{p_2}}{\hat{\Omega}_{s_2}} = \frac{0.32753}{1.3764} = 0.23796$$

FIVE APPLE

No.

$$\left.\begin{array}{l}\hat{\Omega}_{p_1} = 0.3959 \\ \hat{\Omega}_{p_2} = 0.8273\end{array}\right\} R_p = 0.8\ dB$$

$$\left.\begin{array}{l}\hat{\Omega}_{s_1} = 0.23796 \\ \hat{\Omega}_{s_2} = 1.3764\end{array}\right\} R_s = 31dB$$

Lowpass Spec.:

$\Omega_p = 1$ with $R_p = 0.8dB$

$$\Omega_s = \Omega_p \cdot \frac{\hat{\Omega}_0^2 - \hat{\Omega}_{s_1}^2}{\hat{\Omega}_{s_1}\ BW}$$

$$\Omega_s = (1)\ \frac{0.32753 - (0.23796)^2}{(0.23796)(0.4314)}$$

$\Omega_s = 2.639$ with $R_s = 31dB$

$$10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.8 \Rightarrow \varepsilon^2 = \cancel{2.02264}\ 0.202264$$

$$\varepsilon = 0.44974$$

$$10\log_{10}\left(\frac{1}{A^2}\right) = -31dB \Rightarrow A^2 = 1258.9254$$

$$A = 35.48$$

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 2.639$$

$$\frac{1}{k_1} = \frac{\sqrt{A^2-1}}{\varepsilon} = 78.86$$

No.

$\cancel{N = \cos^{-1}}$

$$N = \frac{\cosh^{-1}\left(\frac{1}{k_1}\right)}{\cosh^{-1}\left(\frac{1}{k}\right)} = \frac{5.0608}{1.6256} = 3.11$$

$$\boxed{N = 4}$$

The order of the discrete-Time bandpass filter $\Rightarrow 2N = 8$

<u>matlab code</u>

```
[N,wn] = cheb1ord(1,2.639,0.8,31,'S')
[B,A] = cheby1(N,0.8,wn,'S')

[BT,AT] = lp2bp(B,A,0.5723,0.4314)

[num,den] = bilinear(BT,AT,0.5)
```