University of Jordan
College of Engineering & Technology
Department of Electrical Engineering
Fall Term – A.Y. 2012-2013

Name: Solution Student Number:

Q1. For the sequence $x[n] = \{1 \quad 2\}$,

a) Find the D_2 matrix.
b) Find $X[k]$, $k = 0, 1$. (The DFT of $x[n]$).
c) Find the D_2^{-1} matrix.
d) Find the IDFT of $X[k]$.

a) 2) $D_2 = \begin{bmatrix} 1 & 1 \\ 1 & \omega_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

b) 1) $\underline{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c) 2) $D_2^{-1} = \frac{1}{2} D_2^{*} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

d) 1) $\underline{x} = D_2^{-1} \underline{X} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q2. Consider the length-6 sequence, defined $0 \le n \le 5$,

$$x[n] = \{-3 \quad 5 \quad 45 \quad -15 \quad -9 \quad -19\}$$

With a 6-point DFT given by $X[k]$, $0 \le k \le 5$. Evaluate the following functions of $X[k]$ without computing the DFT:

a) $X[0]$ b) $X[3]$ c) $\sum_{k=0}^{5} X[k]$ d) $\sum_{k=0}^{5} |X[k]|^2$

$$X[k] = \sum_{n=0}^{5} x[n] e^{-j\frac{2\pi}{6}nk}$$

a) $\boxed{X[0] = \sum_{n=0}^{5} x[n] e^{j0} = \sum_{n=0}^{5} x[n] = 4}$

.5 b) $X[3] = \sum_{n=0}^{5} x[n] e^{\frac{-j2\pi(3)n}{6}} = \sum_{n=0}^{5} x[n] e^{-j\pi n}$

$X[3] = (-3)(1) + (5)(-1) + (45)(1) + (-15)(-1) + (-9)(1) + (-19)(-1)$

$= -3 - 5 + 45 + 15 - 9 + 19$

$\boxed{X[3] = 62}$

c) $\sum_{k=0}^{5} X[k] = N.(x[0])$ $\quad x[n] = \frac{1}{N} \sum_{k=0}^{5} X[k] e^{j\frac{2\pi}{6}}$

1.5 $= 6(-3)$

$\boxed{\sum_{k=0}^{5} X[k] = -18}$

d) $\sum_{n=0}^{5} |x[n]|^2 = \frac{1}{6} \sum_{k=0}^{5} |X[k]|^2$

1.5 $\sum_{k=0}^{5} |X[k]|^2 = 6[9 + 25 + 2025 + 225 + 81 + 361]$

$= 6(2726) = \boxed{16356.}$

Q3. The time constant K of an LTI stable causal discrete-time system with an impulse response $h[n]$ is given by the value of the total time interval n at which the partial energy of the impulse response is within 95% of the total energy; that is

$$\sum_{n=0}^{K}|h[n]|^2 = \left(\sum_{n=0}^{\infty}|h[n]|^2\right)0.95$$

Determine the time constant K of the first-order causal transfer function

$$H(z)=\frac{1}{1-\beta z^{-1}}\ ,|\beta|<1.$$

$$h[n] = (\beta)^n u[n].$$

$$\sum_{n=0}^{K}(\beta^2)^n = .95\sum_{n=0}^{\infty}((\beta)^2)^n$$

$$\frac{1-(\beta^2)^{(K+1)}}{\cancel{1-\beta^2}} = .95\,\frac{1}{\cancel{1-\beta^2}}.$$

$$\beta^{2(k+1)} = .05$$

$$2(K+1)\ln\beta = \ln(.05).$$

$$\boxed{K = \frac{\ln(.05)}{2\ln\beta} - 1}$$

Q4. A causal LTI discrete-time system is described by the difference equation:

$$y[n] = 0.2y[n-1] + 0.08y[n-2] + 2x[n]$$

Where x[n] and y[n] are the input and the output of the system, respectively.

a) Determine the transfer function $H(z) = \frac{Y(z)}{X(z)}$ of the system.

b) Determine the impulse response $h[n]$ of the system.

c) Determine the step response $s[n]$ of the system.

$$Y(z) = 0.2z^{-1}Y(z) + .08z^{-2}Y(z) + 2X(z)$$

a) $$\frac{Y(z)}{X(z)} = \frac{2}{1 - 0.2z^{-1} - .08z^{-2}} = \frac{2}{(1-.4z^{-1})(1+.2z^{-1})}$$

b) $$\frac{Y(z)}{X(z)} = H(z) = \frac{A_1}{1-.4z^{-1}} + \frac{A_2}{(1+.2z^{-1})}$$

$$A_1 = \frac{2}{1+\frac{.2}{.4}} = \frac{2}{1.5} = \frac{4}{3}$$

$$A_2 = \frac{2}{1-\frac{.4}{-.2}} = \frac{2}{3}$$

$$h[n] = \left(\frac{4}{3}\right)(.4)^n u[n] + \frac{2}{3}(-.2)^n u[n]$$

c) $$S(z) = \frac{2}{(1-.4z^{-1})(1+.2z^{-1})(1-z^{-1})}$$

$$= \frac{2.7778}{1-z^{-1}} - \frac{.8889}{1-.4z^{-1}} + \frac{.1111}{1+.2z^{-1}}$$

$$s[n] = 2.777u[n] - .8889(.4)^n u[n] + 0.1111(-0.2)^n u[n]$$

$$\frac{25}{9} \qquad -\frac{8}{9} \qquad \frac{1}{9}$$

Q5. Let $x[n]$ be a length-N complex sequence with an N-point DFT $X[k]$. Determine the N-point DFTs of the following N-length sequences in terms of $X[k]$:

a) $x^*[\langle -n \rangle_N]$ b) $x_{re}[n]$

a) $\mathrm{DFT}\{x^*[\langle -n\rangle_N]\} = X^*[k]$

b) $\mathrm{Re}\{x[n]\} = \frac{1}{2}\{x[n] + x^*[n]\}$.

$\mathrm{DFT}\{\mathrm{Re}\{x[n]\}\} = \frac{1}{2}\{X[k] + X^*[\langle -k\rangle_N]\}$

Solution

University of Jordan
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Department of Electrical Engineering
1^{st} Term – A.Y. 2012-2013
Digital Signal Processing – EE 424 First Exam

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Q1.

2 a) The odd part of a real-valued sequence $x[n]$ is given by $x_{od}[n] = (\frac{1}{3})^3 u[n]$. If the average power of x[n] is $P_x = 10$, determine the average power of its even part $x_{ev}[n]$.

2 b) Compute the energy of length-N sequence $x[n] = \sin(2\pi kn/N)$, $0 \le n \le N-1$.

1 c) Determine if the system described by the following input-output equation $y[n] = y^2[n-1] + x[n]$ is stable or not.

a) $P_{od} = \lim_{k\to\infty} \frac{1}{2K+1} \sum_{0}^{K} (\frac{1}{3})^6$

$= (\frac{1}{3})^6 \lim_{K\to\infty} \frac{K+1}{2K+1} = (\frac{1}{3})^6 \cdot \frac{1}{2} = \frac{1}{2(3)^6} = \frac{1}{1458}$

$= 6.86 \times 10^{-4}$

$P_{even} = P_x - P_{od}$

$= 10 - 6.86 \times 10^{-4}$

$= 9.9993$ W.

$P_{even} = 10 - \frac{1}{2}(\frac{1}{3})^6$

b) $x[n] = \sin(\frac{2\pi kn}{N})$ $0 \le n \le N-1$

$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} \sin^2(2\pi k\frac{n}{N}) = \sum_{n=0}^{N-1} (\frac{1}{2} - \frac{1}{2}\cos\frac{4\pi kn}{N})$

$= \frac{N}{2} - \frac{1}{2} \underbrace{\cancel{\sum_{n=0}^{N-1} \cos(4\pi kn/N)}}_{C}$

$= \frac{N}{2}$.

$C = \sum_{n=0}^{N-1} \cos(\frac{4\pi nk}{N})$

$S = \sum_{n=0}^{N-1} \sin(\frac{4\pi nk}{N})$

$C + jS = \sum_{n=0}^{N-1} e^{-j\frac{4\pi nk}{N}} \cdot \frac{1 - e^{-j4\pi nk}}{1 - e^{-j4\pi k}}$

c) - Unstable

let $x[n] = u[n]$

$x[n]$ i

$y[-1] = 0$

$y[0] = y^2[-1] + u[0]$

$y[0] = 1$

$y[1] = 1^2 + 1 = 2$

$y[2] = 2^2 + 1 = 5$

$y[3] = 5^2 + 1 = 26$

$y[n]$ is Unbounded.

Q2. Determine the DTFT of the following signals:

a) $x[n]=\cos(\omega_o n)u[n]$.

b) $x[n]=u[n]-u[n-6]$.

a) $x[n] = \cos(w_o n)u[n]$

$$= \frac{e^{jw_o n} + e^{-jw_o n}}{2} u[n]$$

$$= \frac{1}{2} e^{jw_o n} u[n] + \frac{1}{2} e^{-jw_o n} u[n]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{-j(w-w_o)}} + \frac{1}{1-e^{-j(w+w_o)}}\right]$$

$$= \frac{1}{2} \frac{1-e^{-j(w+w_o)} + 1 - e^{-j(w-w_o)}}{\left[1-e^{-j(w-w_o)} - e^{-j(w+w_o)} + e^{-j2w}\right]}$$

$$= \frac{1-e^{-jw}\left[\frac{e^{jw_o}+e^{-jw_o}}{2}\right]}{1-2e^{-jw}\left[\frac{e^{jw_o}+e^{-jw_o}}{2}\right]+e^{-j2w}}$$

$$= \frac{1-e^{-jw}\cos w_o}{1-2e^{-jw}\cos w_o + e^{-j2w}}$$

$w \neq \pm w_o + 2\pi k$

$k = 0, 1, \ldots$

b) $x[n] = u[n] - u[n-6]$

$$= \frac{1}{1-e^{-jw}} - \frac{e^{-j6w}}{1-e^{-jw}} = \frac{1-e^{-j6w}}{1-e^{-jw}}.$$

Q4. Two LTI systems are connected in cascade. Their impulse responses are:

$$h_1[n] = (\frac{1}{2})^n u[n] \qquad\qquad h_2[n] = (\frac{1}{4})^n u[n]$$

a) Prove that the impulse response of the cascaded system $h[n] = \sum_{k=-\infty}^{\infty} h_1[k] h_2[n-k]$.

b) Find the impulse response of the cascaded system h[n]. page 82

$$x[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{y_1[n]} \boxed{h_2[n]} \longrightarrow y[n]$$

$$y[n] = y_1[n] * h_2[n]$$

$$y_1[n] = x[n] * h_1[n]$$

$$\therefore y[n] = x[n] * h_1[n] * h_2[n]$$

$$= x[n] * h[n]$$

where $h[n] = h_1[n] * h_2[n]$.

$$h[n] = \sum_{k=-\infty}^{\infty} h[k] h_2[n-k]$$

b)

$$\boxed{h[n] = 0 \ , \ n < 0}$$

$$h[n] = \sum_{k=0}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^{n-k}$$

$$= (\tfrac{1}{4})^n \sum_{k=0}^{n} 2^k$$

$$= (\tfrac{1}{4})^n \left[\frac{1 - 2^{n+1}}{1 - 2}\right] = (\tfrac{1}{4})^n (2^{n+1} - 1).$$

$$\boxed{h[n] = (\tfrac{1}{2})^n \left[2 - (\tfrac{1}{2})^n\right] , \ n \geq 0}$$