DSP

NoteBook

Dr. IYAD JA'FAR

By: AHMAD SAEED

POWER UNIT
ELECTRICAL ENGINEERING

CHAPTER 1 (signals & signal processing)

* what is a signal?
it's any physical quantity that varies with time or space or both or any other independent variable(s)

* The signal can be represented mathematically as a function of 1 or more independent variable

→ ex: voltage, current, light, temp.

$\hookrightarrow v(t) = 3 \cos(4t)$ $\hookrightarrow temp(\overset{\text{distance}}{d}) = 3\sqrt{d}$

ex: image → $img(x, y) = x^2 + y^2$ (depend on 2 variables)

Note: the value of the signal at specific point is called the amplitude while the variation of the signal is called the wave form

* classifications of signals:-

① the source
- → single-source (temp, voltage on a resistor) ✓ [in our course]
- → multi-source (speach by more than 1 person, remote sensing, video)

② values
- → real ✓ [in our cource]
- → complex

③ certinity
- → determenistic (can be expressed mathematically or by tables) ✓ [in our cource]
- → random (the values of the signal are generated randomly)

④ Diminsionality (variable(s)ال عدد)
- → 1-D (speach, temp, voltage) $f(t)$
- → 2-D (image) $f(x, y)$
- → 3-D (video) $f(x, y, z)$
- ⋮

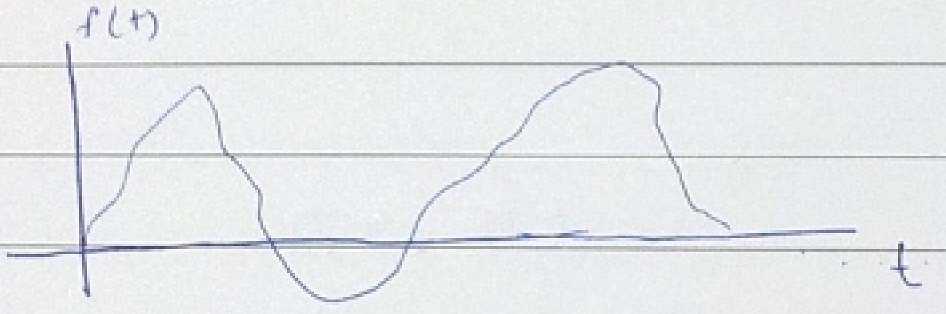
⑤ continuoty of the independant variable(s)
- → continous (having a value at any instant)
- → Discrete (the signal is known at specific values of the independent variable)

Note:- if the ~~var~~ independant variable is time we say cont-time or Discrete-time

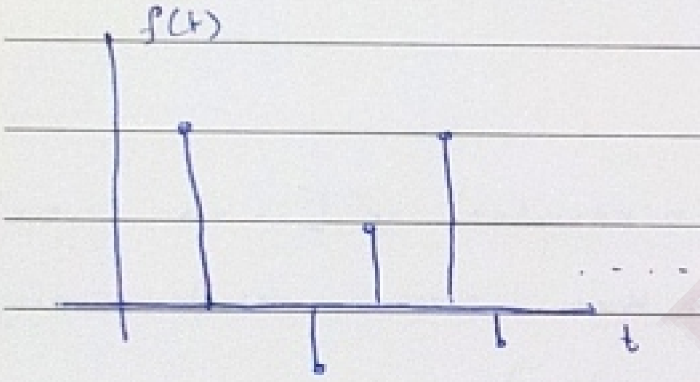
⑥ continouty in the amplitude
- → cont. in amplitude (can take any value)
- → Discrete in amp. (can take specific values)

Note:

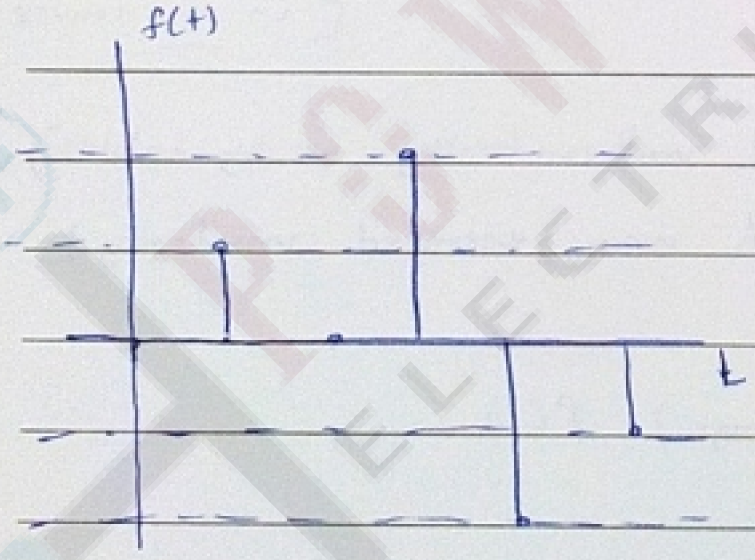
$\boxed{1}$ A signal that is continous in time and amplitude is called Analog



✓ $\boxed{2}$ A signal that is discrete in time & cont. in amplitude is called Sampled-Data-signal

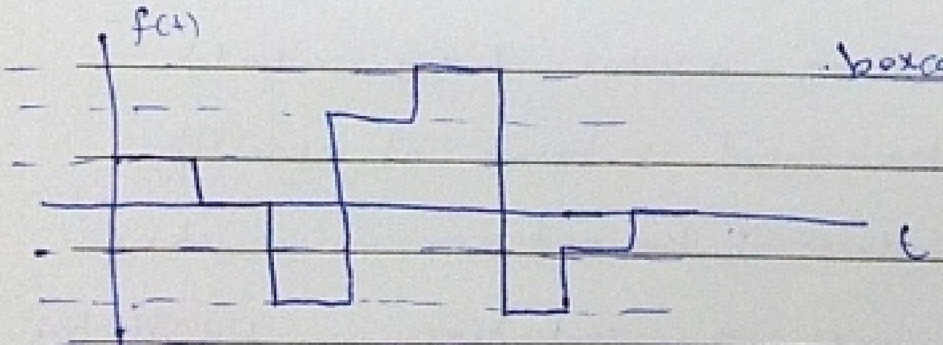


✓ $\boxed{3}$ A signal that is Discrete in time & Amplitude is called a Digital signal



we will focus on this in ~~our course~~

$\boxed{4}$ A signal that is Discrete in amplitude & cont. in time is called boxcar signal



* Signal processing ?

- it's the process of modifying or extracting information from the signal for example: put it in a filter (smoothing), finding it's Avg.)
- the processing can be done in the domain of the independent variable ~~of the original~~ original (time-domain) or in transform domain (laplace, fourier)

* How to obtain digital signal ?

- in real world most signals are Analog !
- Analog signals can't be processed by Digital devices (computers) because we have infinite points & possible values & the computer have a finite memory & storge (locations), finite processing capability, finite memory width.
- so we have to convert it to Digital (A/D conversion) & it has 2 steps:

1] sampling (collect representative samples at discrete time instant)

[cont. time $\longrightarrow$ Discrete - time]

note:- usually the samples are equally spaced with spacing of [T] (uniform sampling)

$$x(t) \xrightarrow{\text{sampling}} x[n] \quad \text{where} \quad x[n] = x_a(t)\Big|_{t=nT} \qquad n \equiv \text{integer} \quad -\infty < n < \infty$$

2] Quantization (the samples may take any value & this will lead to ∞ number of digits to represent the value)

solution:- allow samples' values to take the specific values only. (rounding, truncation)

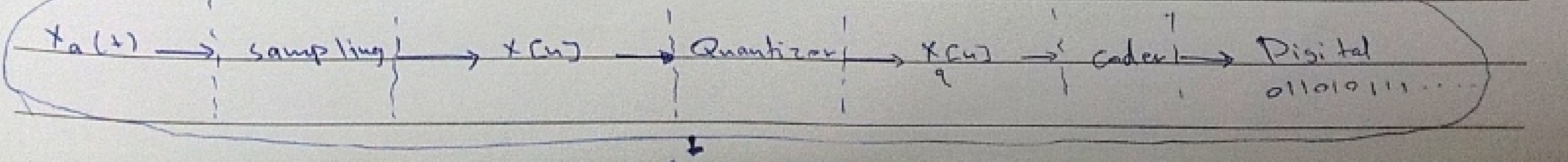
Discrete - time, cont. - Amp $\xrightarrow{\text{Quantization}}$ Discrete - time, " - Amp

remember

$$\boxed{\underset{\#\text{ of bits}}{n} = \log(\underset{\#\text{ of levels}}{L})}$$

$$x[n] \longrightarrow x_q[n]$$

3] Coding (mapping) : Assign binary code to quantized values $x_q[n]$



Summary

* Why DSP ?? (most signals in real world are Analog & A/D → overhead
→ time
→ cost

because there are alot of advantages :-

① Digital signals are less sensitive to noise & the variations in electrical components

② Digital systems can be configured by software (change your code only not the hardware)

③ Digital systems can be easily interfaced with other digital systems

④ Digital systems can be shared by multiple signals

* Disadvantages :-

① overhead (A/D → process → D/A)

② Bandwidth + speed

③ power consumption (Analog systems may consume 0 watt ! ex: LPF)

CHAPTER 2

→ Discrete - time signals in the time domain

1. Time domain representation

$x[n] = X_a(t)\Big|_{t=nT}$ n = integer
T = sampling period

* Based on this, the discrete-time signal can be represented as a sequance of numbers

example :-

$x[n] = \{3, 1.7, -1, 3, 4.1, 7, -9, 10\}$

$n=-2$ $n=-1$ $n=0$ $n=1$ $n=2$ . . .

how to get t → multiply by T so 3 happens at $t=0$
4.1 " " $t=T$
7 " " $t=2T$
⋮

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* General properties of DT-signals :-

1] the sequance could be real or complex
→ so $x[n] = x_{re}[n] + j\,x_{im}[n]$
→ for a real sequance $x_{im}[n] = 0$
→ The conjegate of the sequance is $x^*[n] = x_{re}[n] - j\,x_{im}[n]$

2] length of sequance :

DT-signal could be finite or infinite in length (# of samples)

→ for a finite length sequance, it's defined over a finite period of time : $N_1 < n < N_2$
wher $N_1 > -\infty$ & $N_2 < \infty$

→ for infinite length sequances, we have infinite number of samples and it could be :

cont. next page

a) 2-sided sequance → the sequance is defined for all n

$$x[n] = \{\ldots\ldots, \underset{\uparrow}{1}, 3, 1, 5, \ldots\ldots\}$$

Note

↑ means zero index ≡ n = 0

b) Right-sided sequance → defined for $n \geq 0$

$$x[n] = \{\underset{\uparrow}{3}, 6, -1, \ldots\}$$

c) Left-sided sequance → defined for $n < 0$

$$X[n] = \{\ldots\ldots, 1, 3, 7, \underset{\uparrow}{5}\}$$

2. Operations on sequances:-

A) Elementary Operations:-

Given two sequances $x[n]$ & $y[n]$, we can define the following:-

A.1) scaling:-

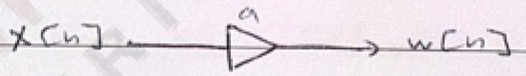
Multiply by a constant → $w[n] = a \cdot x[n]$

if $a > 1$ this called amplification

$a < 1$ " " attenuation

$a = 1$ " " buffering

* Graphical rep.:-

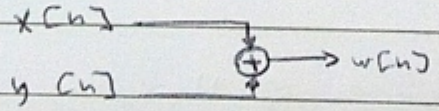


A.2) addition & subtraction:-

$w[n] = x[n] \pm y[n]$ (element by element)

→ given that x & y are of the same length, if not we pad the shorter sequance with zeros & make them of the same length

* Graphical rep.



A.3 Multiplication & division :- (element by element)

$w[n] = x[n] \cdot y[n]$

$w[n] = x[n] / y[n]$

→ example :-

$x[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$

$y[n] = \{-1, \underset{\uparrow}{-2}, -3, -4\}$

find ⓐ $w[n] = 2\,x[n] - y[n]$

$= \{\underset{\uparrow}{2}, 4, 6, 8, 10\} - \{-1, \underset{\uparrow}{-2}, -3, -4\}$

$= \{1, 4, 7, 10, 8, 10\}$

ⓑ $w[n] = \dfrac{x[n] \cdot y[n]}{x[n]+1}$ → we add 1 for each element

$= \dfrac{\{0, \underset{\uparrow}{-2}, -6, -12, 0, 0\}}{\{\underset{\uparrow}{2}, 3, 4, 5, 6\}}$ → to avoid the result Nan, expand x[n] here before adding 1 to it

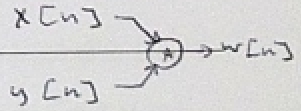
$= \{Nan, -1, -2, -3, 0, 0 ..\}$

$\frac{0}{0}$ (not a number)

* in Matlab :-

```
>> x = [0,1,2,3,4,5];
>> y = [-1,-2,-3,-4,0,0];
>> w = x.*y./(x+1);
```

graphical rep :-



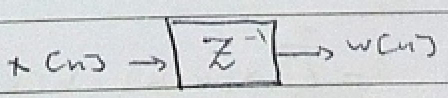
A.4 Time shift :-

$w[n] = x[n - N_0]$

if $N_0 > 0$, it's a right shift (Delay)

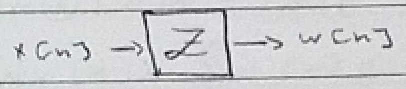
if $N_0 < 0$, " " left " (advanced)

→ graphically :-



Delay

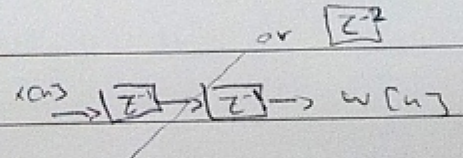
$w[n] = x[n-1]$



advance

$w[n] = x[n+1]$

example :- $w[n] = x[n-2]$ =

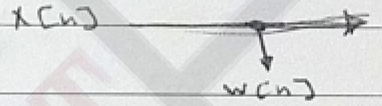


A.5 Time-reversal (folding):-

$w[n] = x[-n]$

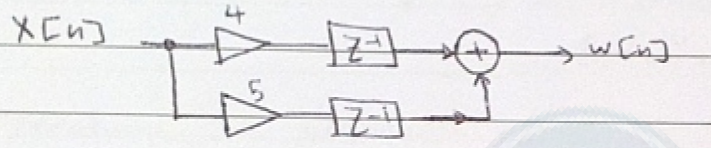
A.6 Branching (pick-off):-

$w[n] = x[n]$ → graphically:-

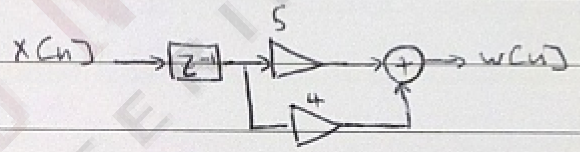


→ example:- $w[n] = 4x[n-1] + 5x[n-2]$

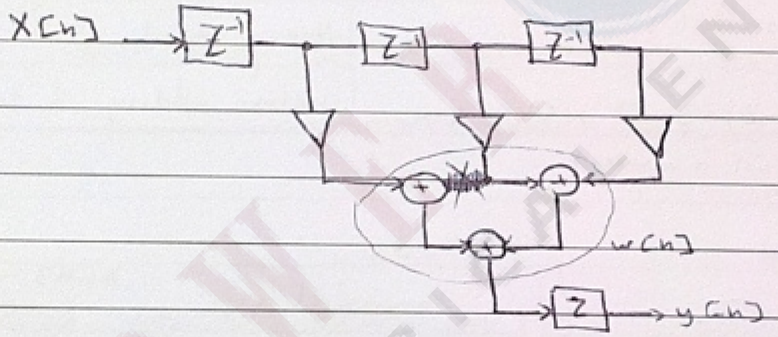
represent this graphically:-



or



→ example:-



what is y[n] as a function of x[n]?

$y[n] = w[n+1]$

$w(n) = -x[n-1] + 4x[n-2] + 3x[n-3]$

$y(n) = -x[n] + 4x[n-1] + 3x[n-2]$

⑬ Convolution:

Convolution between two sequances x[n], h[n] is defined by:-

$y[n] = x[n] \circledast h[n]$

$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot h[k]$

→ $x[n] \circledast h[n] = h[n] \circledast x[n]$

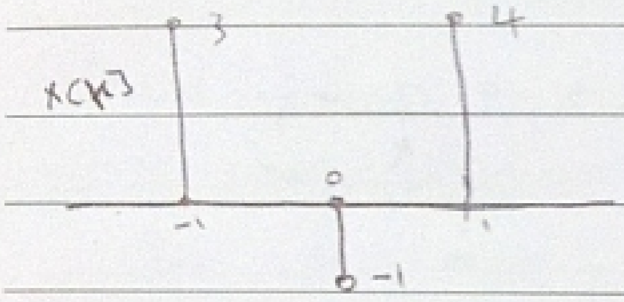
example:-

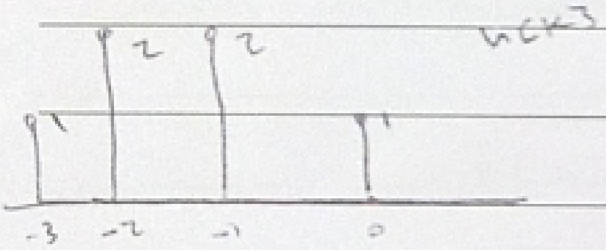
$x[n] = \{3, \underset{\uparrow}{-1}, 4\}$

$h[n] = \{1, 2, 2, \underset{\uparrow}{1}\}$

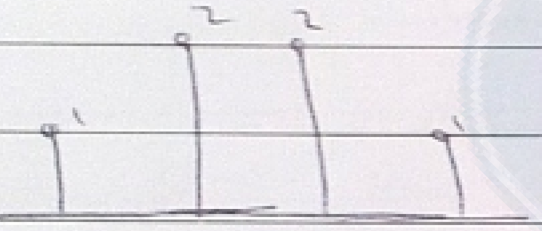
find $y[n] = x[n] \circledast h[n]$

*→ sol:-

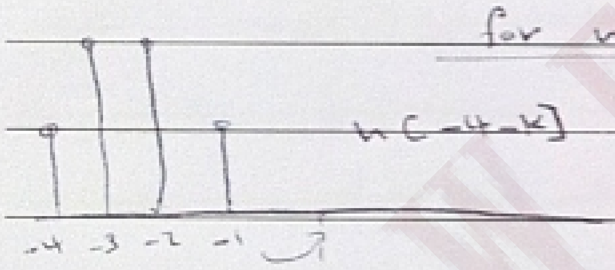




h[-k]



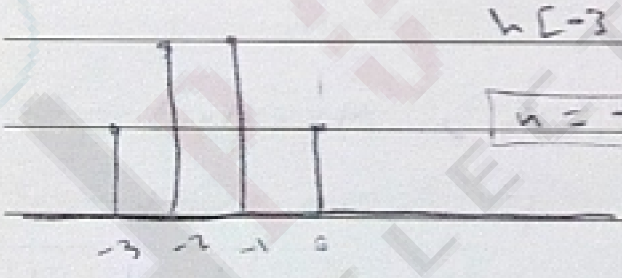
for $n < -4 \rightarrow y[n] = 0$



$h[-4-k]$ $\boxed{n=-4}$

$y[-4] = 1 * 3 = 3$

$h[-3-k]$



$\boxed{n=-3}$

$y[-3] = (1 * -1) + (2 * 3) = 5$

continue it at home

$\boxed{H.W}$

* $y[-2] = 8$

$y[-1] = 9$

$y[0] = 7$

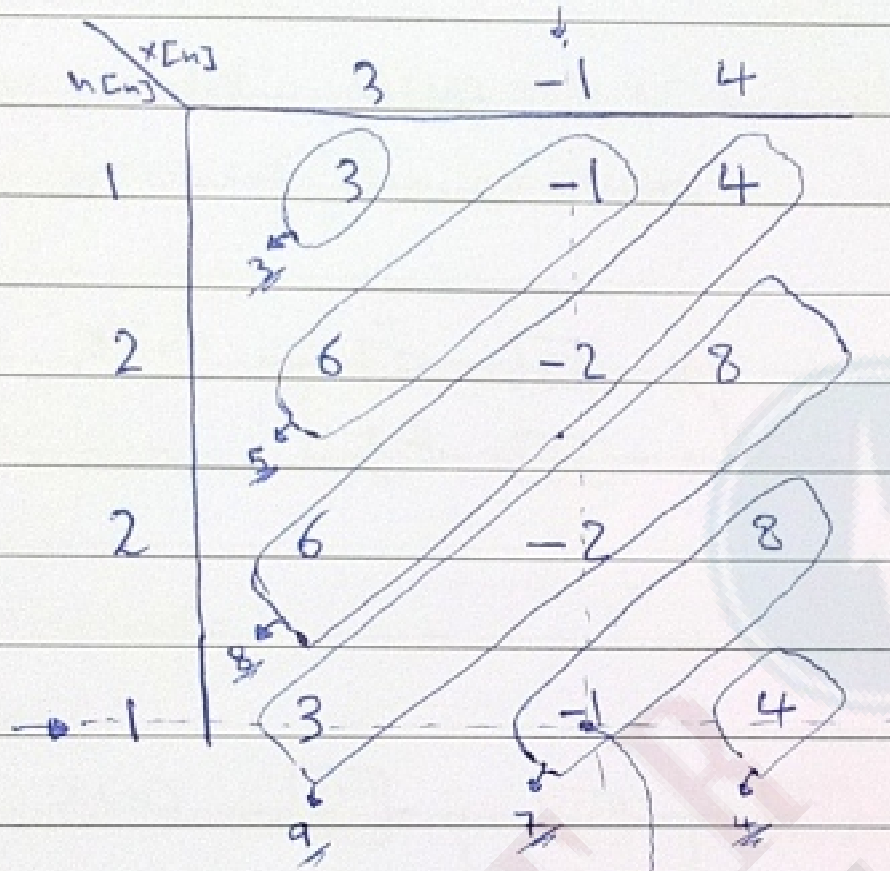
$y[1] = 4$

cont. →

* Graphical convolution is tedious !, instead we can use tabular method

→ $h[n] = \{1 \quad 2 \quad 2 \quad \underset{\uparrow}{1}\}$

$x[n] = \{3 \quad \underset{\uparrow}{-1} \quad 4\}$



h[n] \ x[n]	3	-1	4
1	3	-1	4
2	6	-2	8
2	6	-2	8
1	3	-1	4

$y[n] = \{3, 5, 8, 9, \underset{\uparrow}{7}, 4\}$

this goup represent the zero point $\underline{\underline{n=0}}$

Note :

I] for 2 sequances:

$x[n], \ N_x \leq n \leq M_x$,,, $y[n], \ N_y \leq n \leq M_y$

ⓐ → the range of conv. between x, y is $n \in [N_x + N_y \text{ to } M_x + M_y]$

ⓑ → for two sequances x[n] with length L_x & y[n] with length L_y, the length of the conv. between them is $\boxed{L_x + L_y - 1}$

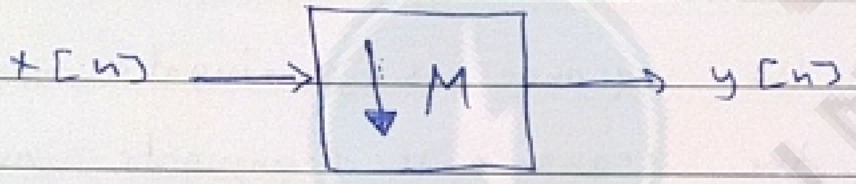
* Sampling Rate alteration:

it's used to generate a sequance y[n] from x[n] with a sampling rate that is higher or lower than that of x[n]

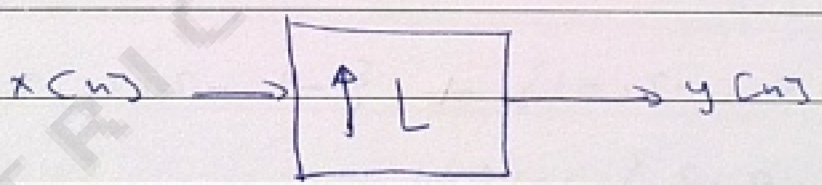
→ if the sampling rate of x[n] is Fx & that of y[n] is Fy so we can define:

$R = F_y / F_x$

① → In case R<1, this is called down-sampling or "decimation"
for decimation with a factor M>1, M is an integer, we remove M-1 samples every M samples



② → In case R>1, this is called up-sampling or "interpolation"
for interpolation with a factor L>1, we insert L-1 zeros between two consequtive samples in x[n]



→ initally we put them zeros then find their values using interpolation method (avg between the previous & Nex sample, or ... or ... [illegible])

H.w

- * Reord 3 seconds of your voice (16 kHz)
- * Downsampling 2, 8, 16 (→ record the modified files)
- * Upsampling by 4 & 8
- (ⓐ→ insert zeros ... ⓑ→ avarege

((iyad_jafar@yahoo.com)) [DSP assignment]

×3 points

next moneday

$\boxed{3}$ * Classification of sequances:-

3.1 based on symmetry

→ for a real sequance $x[n]$

(i) if $x[n] = x[-n]$, the sequance is even

(ii) if $x[n] = -x[-n]$, the sequance is odd

(iii) any sequance $x[n]$ can be expressed as:

$$x[n] = x_{e_{even}}[n] + x_{odd}[n]$$

where ⇒ $x_{even}[n] = (x[n] + x[-n])/2$

$x_{odd}[n] = (x[n] - x[-n])/2$

→ for a complex sequance $x[n]$:-

(i) if $x[n] = x^*[-n]$, the sequance is conjugate symmetry

(ii) if $x[n] = -x^*[n]$, the sequance is conjugate anti-symmetry

(iii) the sequance $x[n]$ can be expressed:-

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

ex:- $x[n] = \{5, 2, \underset{\uparrow}{4}, -2, -1\}$, find $x_{even}[n]$, $x_{odd}[n]$

→ $x_e[n] = (\{5, 2, 4, -2, -1\} + \{-1, -2, 4, 2, 5\})/2$

$= \{2, 0, \underset{\uparrow}{4}, 0, 2\}$ (even)

→ $x_o\{n\} = (\{5, 2, 4, -2, -1\} - \{-1, -2, 4, 2, 5\})/2$

$= \{3, 2, \underset{\uparrow}{0}, -2, -3\}$ (odd)

3.2 * periodicity :-

→ periodic sign

→ a signal x[n] is periodic if $\tilde{x}[n] = x[n+kN]$, for any integer k & with N being the period over which the sequance repeats.

→ the fundamental period N_f, is the smallest N for the sequance the repeats over it.

→ the sum or multiplication of two periodic sequances with periods N_1 and N_2 , is also a periodic sequance with a period of :-

N_3 = Least common multiplier (LCM) between (N_1, N_2)

$$= \frac{N_1 N_2}{GCD(N_1, N_2)}$$

3.3 * Energy & Power :-

→ the energy of a sequance x[n] is given by :-

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

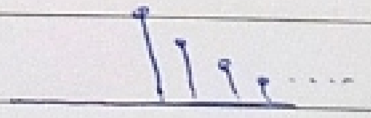
⇒ if $0 < E_x < \infty$, then the sequance is called an energy sequance

↳ any finite sequance is energy sequance, but the infinite sequance it could be energy sequance and maybe not ! .

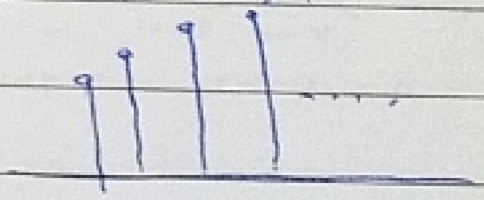
ex :

$$x[n] = \begin{cases} 0 , & n < 0 \\ a^n , & n \geq 0 \end{cases} \Longrightarrow$$

if a<1



if a>1



$$E_x = \sum_{n=-\infty}^{\infty} (a^n)^2 = \sum_{n=-\infty}^{\infty} (a^2)^n = \frac{1}{1-a^n} \quad |a| < 1$$

x(n) is an energy sequance if a<1 , and not if a>1

↳ General Rule ⇒ $$\left[\sum_{n=N_1}^{N_2} r^n = \frac{r^{N_1} - r^{N_2+1}}{1-r} \right]$$ → [illegible] calculas

* The Average power of an aperiodic sequance is given by:

$$\rightarrow P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$$

* while for periodic sequances the average power is given by:

$$\rightarrow P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

* if $0 < P_x < \infty$, then the sequance is a power sequance

* if the signal is finite then $P_x = 0$

ex $x[n] = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$ Find E_x, P_x

Sol:-

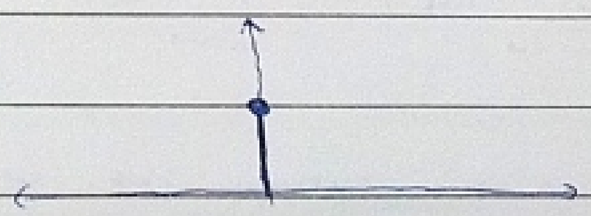
$\rightarrow E_x = \sum_{k=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} (3(-1)^n)^2 = 9 \sum_{n=0}^{\infty} 1 = \infty$; x[n] is not an energy signal

$\rightarrow P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=0}^{\infty} 9(1) = \lim_{K\to\infty} \frac{9(K+1)}{2K+1} = \frac{9}{2} = 4.5$; the sequance is a power sequance

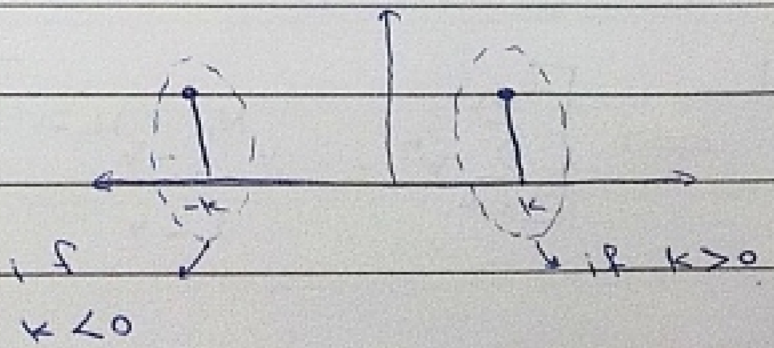
* Typical sequances:-

1) Unit sample sequance:-

$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

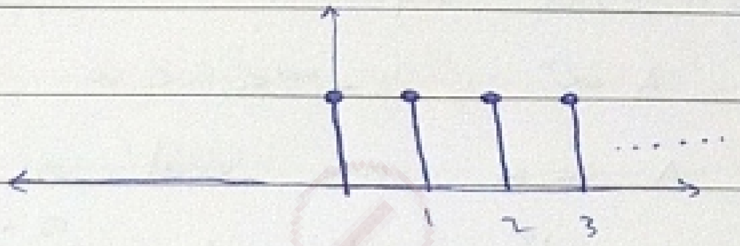


$\rightarrow \delta[n-k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$

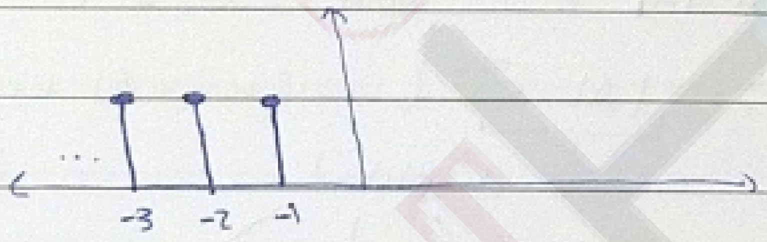


2) Unit step sequance :-

$$u[n] = \begin{cases} 0 & , n<0 \\ 1 & n \geq 0 \end{cases}$$



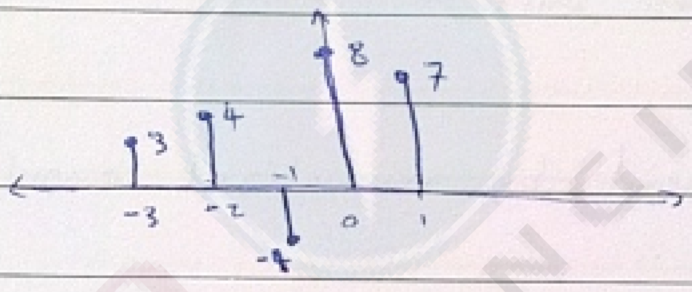
$$u[-n-1] \begin{cases} 0 & , n>-1 \\ 1 & , n \leq -1 \end{cases}$$



ex:-

① $x[n] = \{3, 4, -1, \underset{\uparrow}{8}, 7\}$ (represent this by δ function)

$\rightarrow x[n] = 3\,\delta[n+3] + 4\,\delta[n+2] - \delta[n+1] + 8\,\delta[n] + 7\,\delta[n-1]$



② $x[n] = \{\ldots\ldots, 3,3,3,3, 1,1,1,1, \ldots\ldots\}$

represent this using 2.1 δ function 2.2 u function

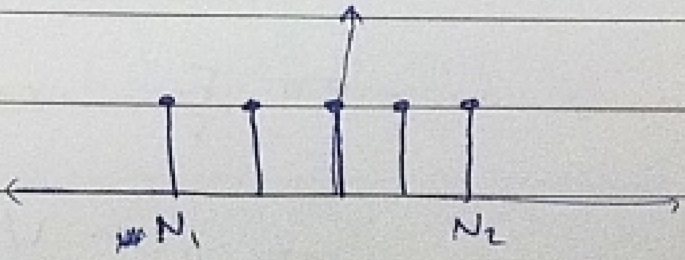
sol:

2.1: $x[n] = 3 \sum_{n=-\infty}^{-3} \delta[n-k] + \sum_{n=-2}^{\infty} \delta[n-k]$

2.2 $x(n) = 3u[-n-3] + u[n+2]$

3) Rectangular window sequance :-

$$w[n] = \begin{cases} 0 & , n<N_1 \\ 1 & , N_1 < n < N_2 \\ 0 & , n>N_2 \end{cases}$$



4) exponential sequance:-

→ $x[n] = A\alpha^n$, $-\infty < n < \infty$

where A & α are real or complex constants

→ if $A = |A| e^{j\phi}$ and $\alpha = e^{\sigma_0 + jw_0}$, then

$$x[n] = \underbrace{|A| e^{\sigma_0 n}}_{\text{Amplitud}} \left(\cos(w_0 n + \phi) + j\sin(w_0 n + \phi) \right)$$

$\sigma_0 = 0$ $\quad$ $\sigma_0 < 0$ decay $\quad$ $\sigma_0 > 0$ grow

5) Sinusoidal:

$$x[n] = \underbrace{A}_{\text{amplitude}} \cos(\underbrace{w_0}_{\text{normalized Angular freq.}} \underbrace{n}_{\text{integer}} + \underbrace{\phi}_{\text{phase shift}})$$

*→ what is w_0 ! , what is meant by normalized angular freq??

* Consider an analog sinusoid $x_a(t)$ with angular frequency

$\Omega_0 = 2\pi F_0$ rad/sec.

$x_a(t) = A\cos(\Omega_0 t + \phi)$

now if the signal is sampled with $F_s = \frac{1}{T}$ sample/s , we obtain

$$x[n] = x_a(t)\Big|_{t=nT} = A\cos(\Omega_0 nT + \phi)$$

$$= A\cos\left(\frac{2\pi F_0}{F_s} n + \phi\right)$$

$$= A\cos(\underbrace{2\pi f_0}_{w_0} n + \phi)$$

so → $w_0 = 2\pi \frac{F_0}{F_s}$ rad / sample (time is hidden)

$= 2\pi \boxed{f_0} \rightarrow$ 1/ sample

⟶

Notes:

① sinusoids in descrete form are periodic with period N if $\frac{2\pi}{w_0} = \frac{N}{r}$, where r is an integer

proof:

periodicity test: $x[n] = x[n+N)$

$\rightarrow \cos(w_0 n + \phi) \overset{?}{=} \cos(w_0(n+N) + \phi)$

$\cos(w_0 n + \phi) \overset{?}{=} \cos(\underbrace{w_0 n + \phi} + \underbrace{w_0 N})$

$\cos(w_0 n + \phi) \overset{?}{=} \cos(w_0 n + \phi) \underbrace{\cos(w_0 N)}_{\text{should be }=1} - \sin(w_0 n + \phi) \underbrace{\sin(w_0 N)}_{\text{should be }=0}$

$\rightarrow \cos(w_0 N) = 1$. so $w_0 N = 2\pi r \rightarrow \boxed{\frac{2\pi}{w_0} = \frac{N}{r}}$ #

ex:

1) $x[n] = \cos(0.1\pi n)$ $0 < n < 100$ is this perodic?

$w_0 = 0.1\pi$

$\rightarrow$ sol:

$\frac{2\pi}{0.1\pi} = 20 = \frac{N}{r} \Longrightarrow r = 1 \rightarrow \underline{N = 20}$ ✓ periodic

2) $x[n] = \cos(\sqrt{2}\, n + \phi)$ $0 < n < 100$ is this perodic?

$w_0 = \sqrt{2}$

$\rightarrow$ sol:

$\frac{2\pi}{\sqrt{2}} \overset{?}{=} \frac{N}{r}$ → integer → integer ✗ a-periodic

$\longrightarrow$ cont.

② two sinusoidal sequences with frequencyies $0 \le w_1 \le \pi$, $\pi \le w_2 \le 2\pi$ are identical if $w_2 = 2\pi - w_1$

Proof:

$x_1[n] = \cos(w_1 n)$,,, $x_2[n] = \cos(w_2 n) = \cos((2\pi - w_1)n)$

→ $x_1[n] \overset{?}{=} x_2[n]$

$\cos(w_1 n) \overset{?}{=} \cos((2\pi - w_1)n)$

$\cos(w_1 n) \overset{?}{=} \underset{1}{\cos(2\pi n)}\cos(-w_1 n) - \overset{0}{\sin(2\pi n)}\sin(-w_1 n)$

$\cos(w_1 n) = \cos(w_1 n)$ ✓ *

→ * All sinusoids in discrete form can be represented in $-\pi \le w \le \pi$

③ Similarly, if $0 \le w_1 \le 2\pi$ & $2\pi(k) \le w_2 \le 2\pi(k+1)$ then the 2 sinusoids are identecal

* Conclusion: the range of oscillation in discrete is:

$w_0 \in [-\pi, \pi]$

$f_0 \in [-1/2, 1/2]$

* Correlation of sequance:

⇒ it's used to test the similarity between sequances.

⇒ the similarity between 2 sequances (energy sequances) $x[n]$, $y[n]$ is given by the cross correlation

$$\Rightarrow r_{xy}\underbrace{[l]}_{\text{lag}} = \sum_{n=-\infty}^{\infty} \underbrace{x[n]}_{\text{referance signal}}\ \underbrace{y[n-l]}_{\text{lag signal}} \qquad -\infty < l < \infty$$

⇒ also,

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} x[n-l]\,y[n]$$

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} x[n-l]\,y[n]$$

let $n = m + l$ $\Rightarrow r_{yx}[l] = \sum_{m=-\infty}^{\infty} x[m+\cancel{l}-\cancel{l}]\,y[m+l]$

$$= \sum x[m]\,y[m+l] = r_{xy}[-l]$$

so $r_{xy}[l] = r_{yx}[-l]$

* The auto correlation of a sequance $x[n]$ is: (جرب على ال Matlab)

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]\,x[n-l] = r_{xx}[-l]$$

<u>Note</u>: $r_{xx}[n]$ is an even function and $r_{xx}[0] = \mathcal{E}_x$

* Relation to convolution:

$$r_{xy} = \sum x[n]\,y[n-l] = \sum_{n=-\infty}^{\infty} x[n]\,y[-(l-n)]$$

$$= x[l] \circledast y[-l]$$

example :

① $x[n] = \alpha^n \, u[n]$, $|\alpha| < 1$ find $r_{xx}[l]$

$$\rightarrow r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]\, x[n-l]$$

$$= \sum_{\substack{\cancel{n=-\infty} \\ n=0}}^{\infty} \alpha^n \cancel{u[n]} \, \alpha^{n-l} \, u[n-l]$$

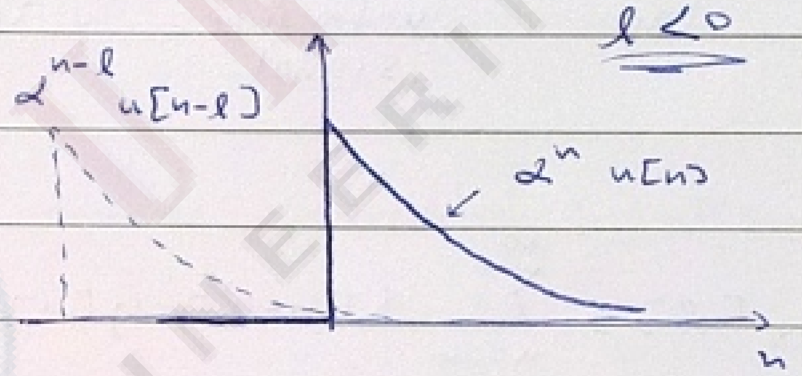
now :

for $l \leq 0$:

$$r_{xx}[l] = \sum_{n=0}^{\infty} \alpha^n \alpha^{n-l}$$

$$= \alpha^{-l} \sum_{n=0}^{\infty} (\alpha^2)^n$$

$$= \frac{\alpha^{-l}}{1-\alpha^2} , |\alpha| < 1$$

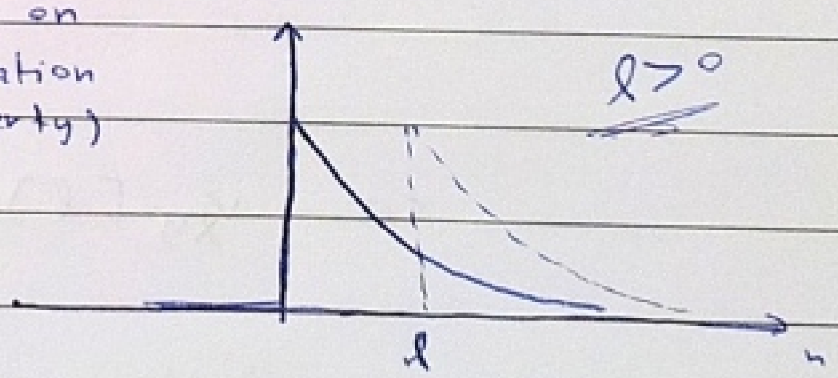


for $l \geq 0$:

$r_{xx}[l] = r_{xx}[-l] = \frac{\alpha^{l}}{1-\alpha^2}$ (based on correleation property)

$$= \sum_{n=l}^{\infty} \alpha^n \alpha^{n-l}$$

$$= \alpha^{-l} \sum_{n=l}^{\infty} (\alpha^2)^n$$



let $n = m + l$

$$\rightarrow r_{xx}[l] = \alpha^{-l} \sum_{m=0}^{\infty} (\alpha^2)^{m+l} = \alpha^{-l} \alpha^{2l} \sum_{m=0}^{\infty} (\alpha^2)^m$$

$$= \frac{\alpha^{l}}{1-\alpha^2} , |\alpha| < 1$$

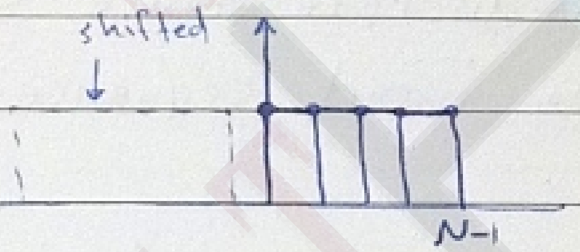
or

$$\rightarrow = \alpha^{-l} \frac{(\alpha^2)^l - 0}{1-\alpha^2}$$

" عدة طرق للحل "

② $x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$, find r_{xx}

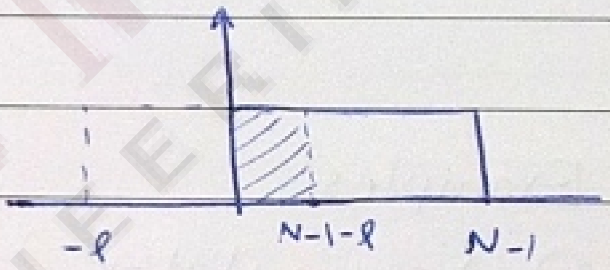
→ $r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]\, x[n-l]$



→ $\boxed{\text{for } l \le N}$ ⇒ $r_{xx}[l] = 0$

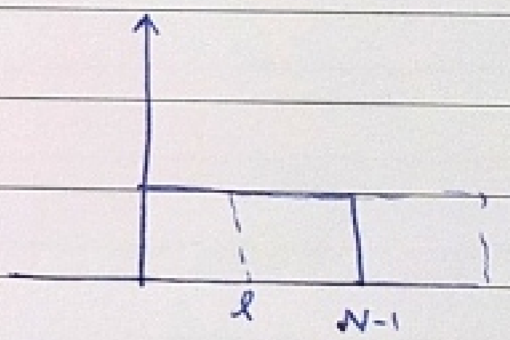
→ $\boxed{\text{for } -N < l \le 0}$ ⇒ $r_{xx}[l] = \sum_{n=0}^{N-1+l} 1$ ⇒

⇒ $\sum_{n=0}^{N-1+l} 1 = N - \cancel{1} + l + \cancel{1} = N + l$

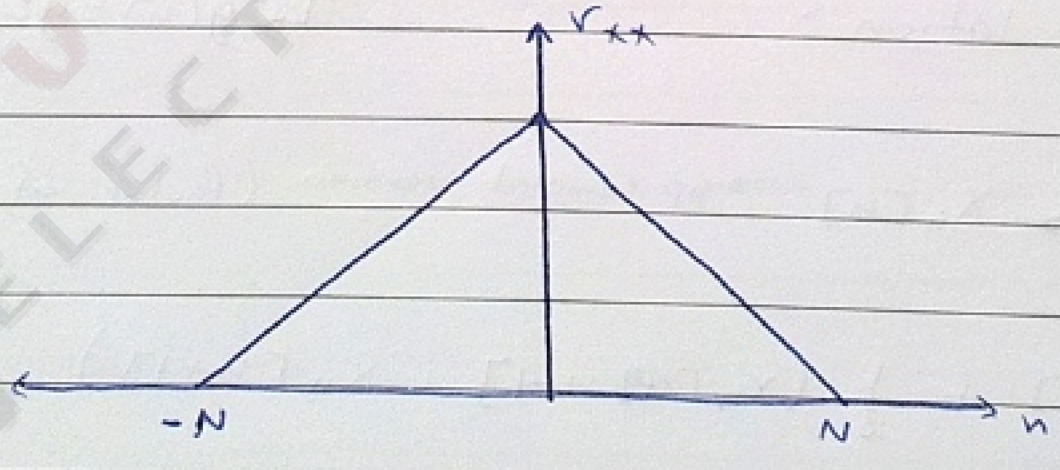


→ $\boxed{\text{for } 0 \le l < N}$ ⇒ $r_{xx}[l] = \sum_{n=l}^{N-1} 1$

$= N - \cancel{1} - l + \cancel{1} = N - l$



→ $\boxed{\text{for } l \ge N}$ ⇒ $r_{xx}[l] = 0$

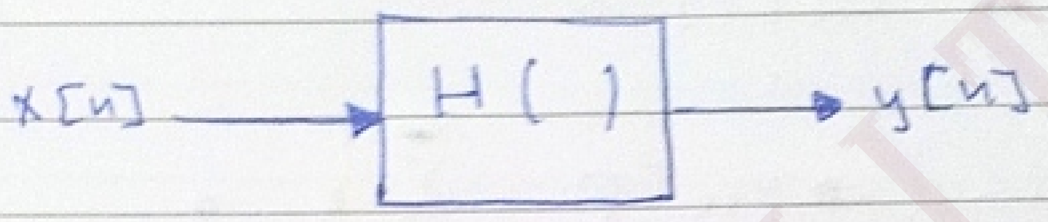


Chapter 4 :

Discrete time systems

1. Definition and examples

* A discrete time system is a process(es) that transforms an input sequance into another output sequance



$$y[n] = H(x[n])$$

* Examples:

① Accumulator → $y[n] = \sum_{l=-\infty}^{n} x[l]$

$$= \sum_{l=-\infty}^{n-1} x[l] + x[n]$$

$$= y[n-1] + x[n]$$

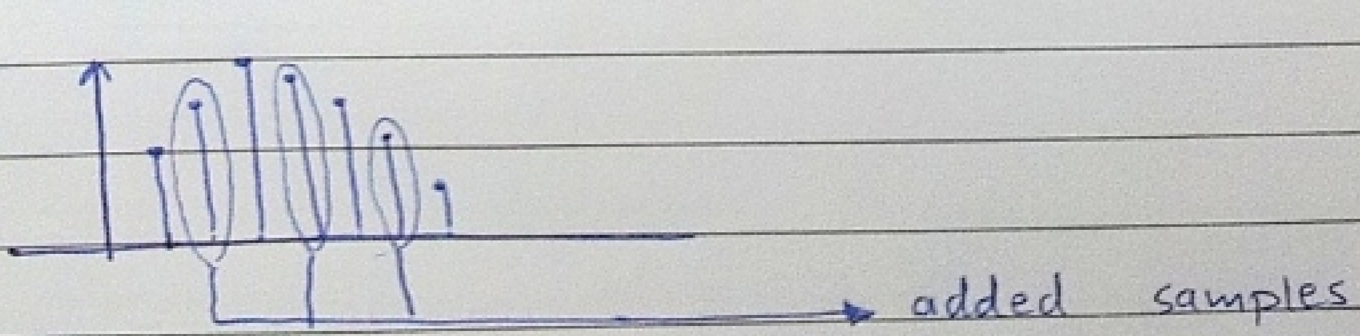
② Moving Average filter → $\boxed{y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]}$

↳ $y[n] = y[n-1] + \frac{1}{M}(x[n] - x[n-M])$

③ Linear interpolation :

$x[n] \xrightarrow{\uparrow L} x_u[n]$ → up sampled version (factor of 2)

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$



$y_p[n]$:

let $\boxed{y_p[n] = K\,3^n\,u[n]}$

$\to K\,3^n u[n] = 3K\,3^{n-1}\,u[n-1] + 4K\,3^{n-2}\,u[n-2] + 3^n u[n]$
$+ (2)(3)^{n-1}\,u[n-1]$

$n \geq 2 \Rightarrow$

$(3^n K = 3^n K + 4K\,3^{n-2} + 3^n + (2)(3)^{n-1})/3^n$

$\boxed{K = -15/4}$

So $y[n] = \underline{A}\,(4)^n + \underline{B}\,(-1)^n - \frac{15}{4}\cdot 3^n \;,\; n \geq 0$

* from diffirence equation:

$y[-1] = 0$
$y[-2] = 0$

$\to y[0] = 3\cancelto{0}{y[-1]} + 4\cancelto{0}{y[-2]} + \cancelto{1}{x[0]} + 2\cancelto{0}{x[-1]} = 1$

$\to y[1] = 3\cancelto{1}{y[0]} + 4\cancelto{0}{y[-1]} + \cancelto{3}{x[1]} + 2\cancelto{1}{x[0]} = 3 + 3 + 2 = 8$

by substitude in the solution

$$\left.\begin{aligned} y[0] &= 1 = A + B - \frac{15}{4} \\ y[1] &= 8 = 4A + (-B) - \frac{45}{4} \end{aligned}\right\} \Rightarrow$$

by solving these equations we find : $A = 24/5$
$B = -1/20$

example:

$y[n] = 4\,y[n-1] - 4\,y[n-2] + x[n]$
$x[n] = 4\,u[n]$
$y[-1] = y[-2] = 1$ (homework)

* we said that $y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$

so $y[n] = \underbrace{\boxed{y_c[n]}}_{\text{system}} + \underbrace{\boxed{y_p[n]}}_{\text{input}}$

$y_c[n] \longrightarrow x[n] = 0$

$y_c[n] = \lambda^n \longrightarrow \boxed{y_c[n] = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \dots \alpha_N \lambda_N^n}$

* what a root λ_1 of L multiplicity ?!

$y[n] = (\alpha_1 \lambda_1^n + \alpha_2 n \lambda_1^n + \alpha_3 n^2 \lambda_1^n + \dots + \alpha_L n^{L-1} \lambda_1^n) + \alpha_{L+1} \lambda_2^n + \dots$

* to compute $y_p[n]$ assume it to take some form of the input:

→ For example:

① $x[n] = A\,u[n] \Longrightarrow y_p[n] = K\,u[n]$

② $x[n] = A^n u[n] \Longrightarrow y_p[n] = K A^n u[n]$

③ $x[n] = A \cos(w_0 n) \Longrightarrow y_p[n] = B \cos(w_0 n]$

* example:

$y[n] = 3y[n-1] + 4y[n-2] + x[n] + 2x[n-1]$

knowing that $y[-1] = y[-2] = 0$, $x[n] = 3^n u[n]$

find $y_c[n]$ & $y_p[n]$

Sol:

$y_c[n]$:

$y[n] - 3y[n] - 4y[n-2] = 0$ take $x[n] = x[n-1] = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0 \Longrightarrow \lambda_1 = 4$

$\lambda_2 = -1$

So $y_c[n] = A(4)^n + B(-1)^n$

example:

$y[n] = -2y[n-1] + 3y[n-2] - x[n-1]$

$y[-1] = y[-2] = 0$

$x[n] = 2(-3)^n u[n]$

sol:

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda+3)(\lambda-1) = 0 \Rightarrow \lambda_1 = -3$

$\lambda_2 = 1$

So $y_c[n] = A(-3)^n + B(1)^n$

$y_p[n] = n K(-1)^n u[n]$

* * *

Note: LTI system is BIBO stable if:

$$\sum_n |h[n]| < \infty$$

$\Rightarrow$ this implies that $|\lambda_i| < 1$ for i

إلى هنا مادة ال First

Lecture 9

* Chapter '3' :

→ Discrete time signals & systems in frequancy domain.

* 1- The continuous time Forier transform (CTFT) :

→ for a certain continous-time signal $X_a(t)$, the CTFT is given by:

$$\boxed{X(j\Omega) = \int_{-\infty}^{\infty} X_a(t)\, e^{-j\Omega t} \, .dt}$$

→ the continous-time signal (in time domain) $X_a(t)$, can obtained from $X_a(j\Omega)$ as follow:

$$\boxed{X_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} \, .d\Omega} \quad , \quad X_a(t) \xleftrightarrow{CTFT} X_a(j\Omega)$$

* The quantity $X_a(j\Omega)$ is complex quantity in general

$$X_a(j\Omega) = \underbrace{|X_a(j\Omega)|}_{\text{magnitude}} \underbrace{e^{j\theta_a(w)}}_{\text{Phase}}$$

* For $X_a(j\Omega)$ to exist, the signal $X_a(t)$ should be absolutely summable:

$$\boxed{\int_{-\infty}^{\infty} |X_a(t)| .dt < \infty}$$

* examples :

① $x_a(t) = \delta(t)$, Find $X_a(j\Omega)$

sol :

$$X_a(j\Omega) = \int_{-\infty}^{\infty} \underbrace{\delta(t)}\ \underbrace{e^{-j\Omega t}} . dt = 1 . \cancel{e^{0j}}^{\,1} = 1$$

② $x_a(t) = \delta(t-t_o)$, Find $X_a(j\Omega)$

sol :

$$X_a(j\Omega) = \int_{-\infty}^{\infty} \delta(t-t_o)\, e^{-j\Omega t} . dt = 1 . e^{-j\Omega t_o}$$

$$= e^{-j\Omega t_o}$$

③ $x_a(t) = e^{-\alpha t}\, u(t)$

sol :

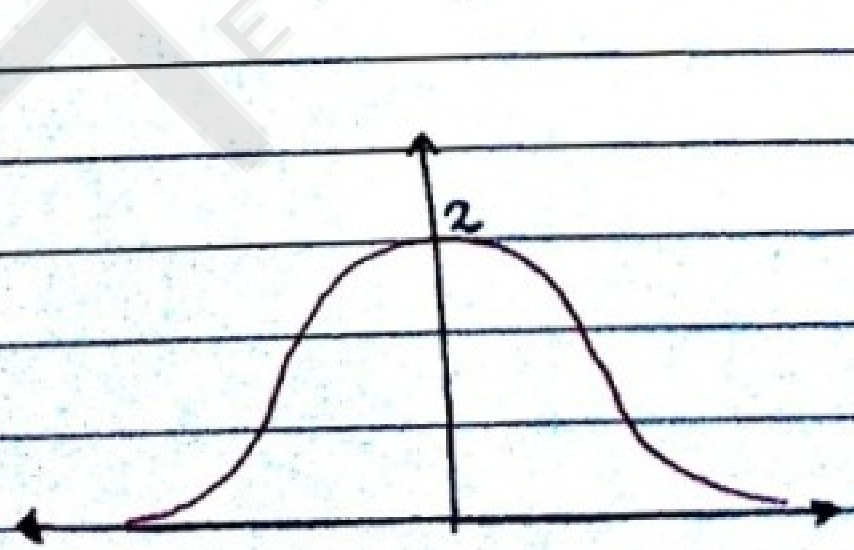
$$X_a(j\Omega) = \int_{-\infty}^{\infty} e^{-\alpha t}\, u(t)\, e^{-j\Omega t} .dt$$

$$= \int_{0}^{\infty} e^{-j(\frac{\alpha}{j}+\Omega)t} .dt = \frac{-1}{\alpha + j\Omega}\ \cancel{e^{-(\alpha+j\Omega)t}}^{\,-1}\Big|_{0}^{\infty}$$

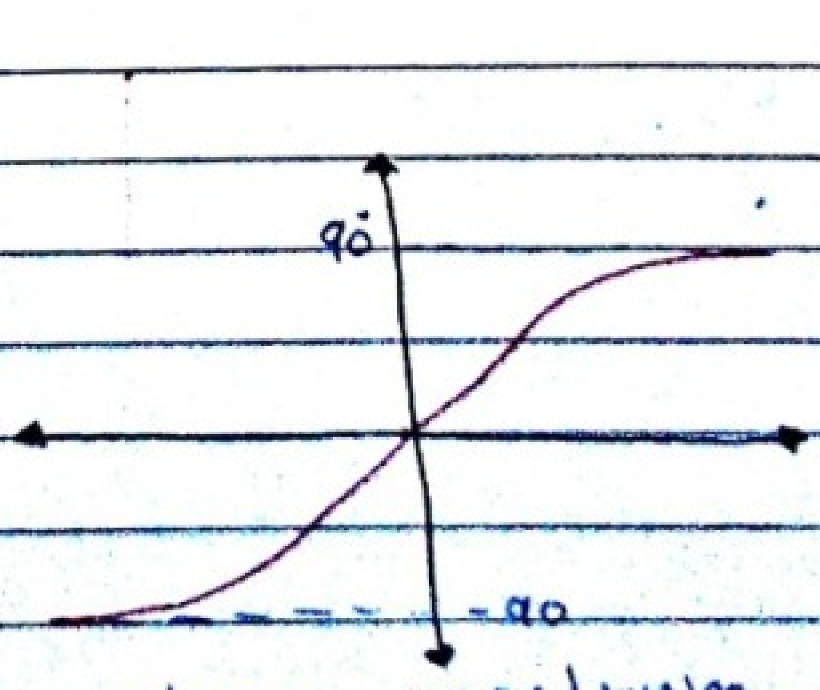
$$= \frac{1}{\alpha + j\Omega}$$

$$= \underbrace{\frac{1}{\sqrt{\alpha^2 + \Omega^2}}}_{\text{magnetude}}\ \underbrace{e^{-j\,Tan^{-1}(\frac{\Omega}{\alpha})}}_{\text{phase}} ,$$

*→ For $\alpha = \frac{1}{2}$ then,



Magnetude spectrum



phase spectrum

* For finite energy signals, the energy is given by:

$$\rightarrow \mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 \cdot dt$$

$$= \int_{-\infty}^{\infty} x_a(t) \cdot x_a^*(t) \cdot dt = \int_{-\infty}^{\infty} x_a(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} \cdot d\Omega\right] \cdot dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) \cdot e^{-j\Omega t} \cdot dt\right] \cdot d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \cdot X_a(j\Omega) \cdot d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \cdot d\Omega$$

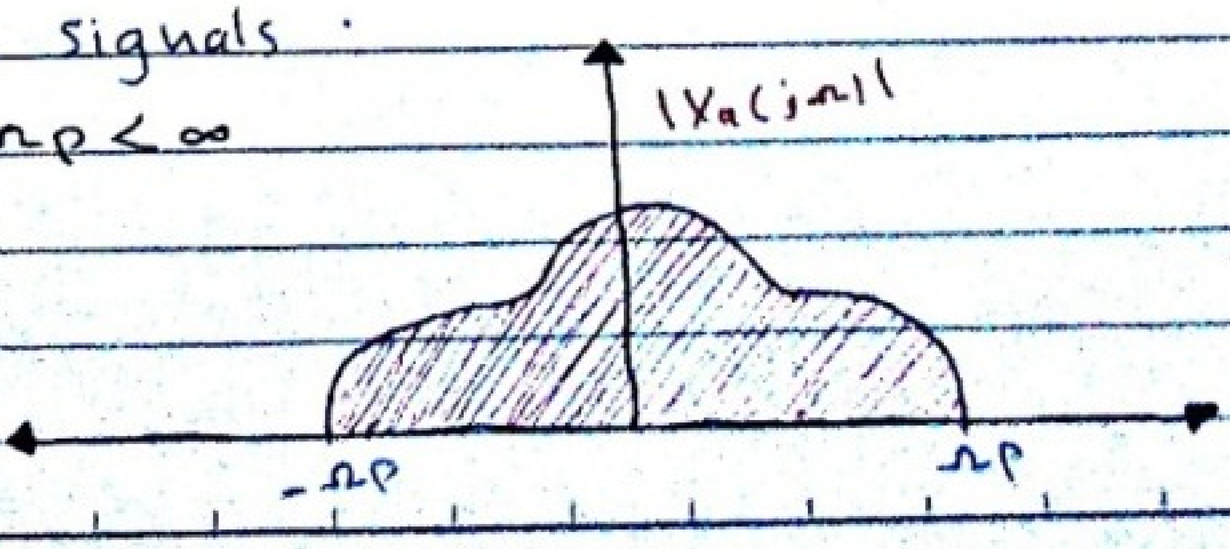
$$= \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{S_{xx}(j\Omega)}_{\text{energy spectrum}} \cdot d\Omega}$$

* Signals can be classified in terms of their span in frequancy domain:

① Full band signal:
the signal's spectrum covers the entire frequancy range $-\infty < \Omega < \infty$.

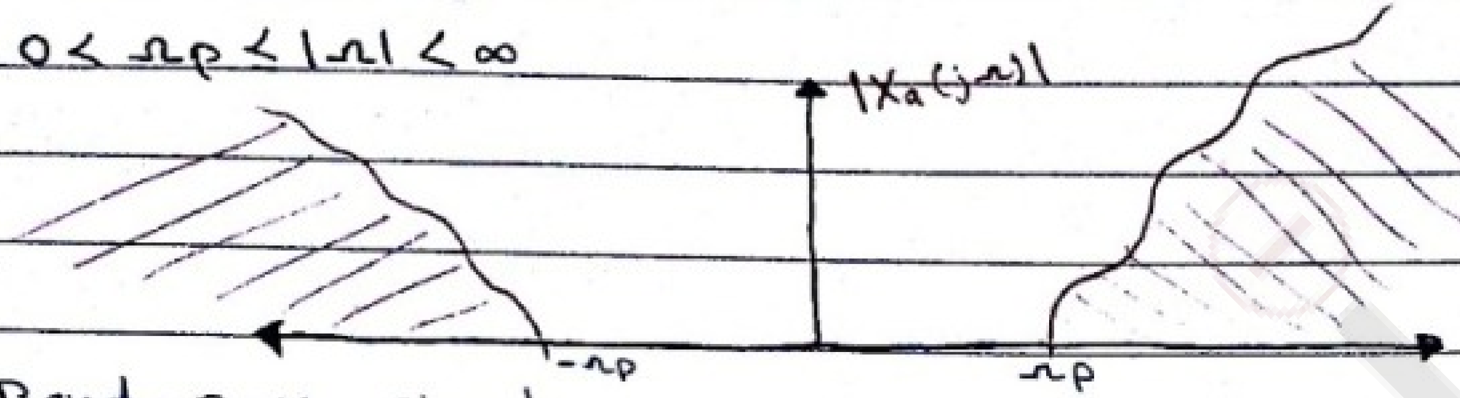
② band limited signal
the spectrum of the signal spans a finite range of frequencies $\Omega_1 \leq |\Omega| \leq \Omega_2$

[a] low-pass signals:
$0 \leq |\Omega| \leq \Omega_P < \infty$



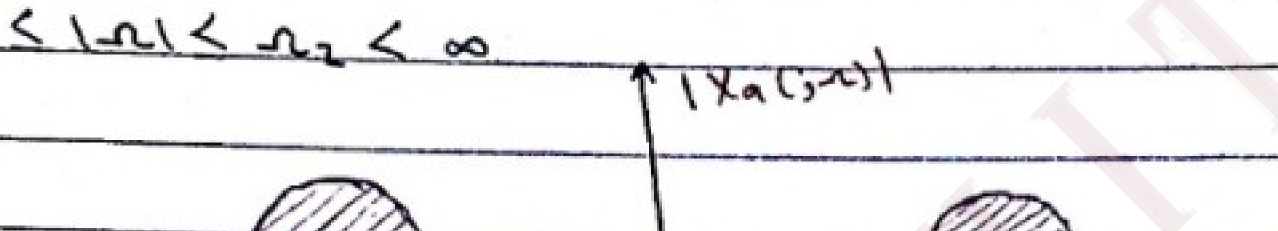
[b] High-pass signals :

$0 < \Omega_p \le |\Omega| < \infty$



[c] Band-pass signals:

$0 < \Omega_1 \le |\Omega| < \Omega_2 < \infty$



2. Discrete-time Forier Transform (DTFT) :

* For discrete-time signal $x[n] = x_a(nT)$, the CTFT becomes:

$$\rightarrow X(j\Omega) = \int_{-\infty}^{\infty} x_a(nT)\, e^{-j\Omega t} \, .dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\boxed{\Omega nT}} \longrightarrow \Omega nT = w = 2\pi \frac{f_0}{f_s} = 2\pi \frac{\Omega_0}{\Omega_s}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$\boxed{X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}} \Longrightarrow \mathcal{F}\{x[n]\}$$

examples:

① $x[n] = \delta[n]$

sol:

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-jwn} = \overset{1}{\cancel{e^{-jw0}}} = 1$$

② $x[n] = \alpha^n u[n]$, $|\alpha| < 1$

Sol:

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] . e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-jwn} \quad = \sum_{n=0}^{\infty} (\alpha e^{-jw})^n$$

$$= \frac{1}{1 - \alpha e^{-jw}}$$

* The inverse DTFT can be computed by:

→ $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) . e^{jwn} . dw$

→ $x[n] = \mathcal{F}^{-1}\{X(e^{jw})\}$,,,, $x[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})$

* why the integration over [-π, π] only ?!

→ The DTFT is periodic with period 2π :

$$X(e^{j(w+2\pi k)}) = \sum_n x[n] e^{-j(w+2\pi k)n}$$

$$= \sum_n x[n] e^{-jwn} \underbrace{e^{-j2\pi \overset{\text{integer}}{k} n \rightarrow \text{integer}}}_{=1}$$

$$= \sum_n x[n] e^{-jwn} = X(e^{jw})$$

3. Basic DTFT properties:

① The DTFT is periodic with a period of 2π

② In general, the DTFT is a complex function:

$$X(e^{jw}) = X_{re}(e^{jw}) + j\,X_{im}(e^{jw})$$
$$= |X(e^{jw})|\, e^{j\phi w}$$

where:

$$|X(e^{jw})| = \sqrt{(X_{re}(e^{jw}))^2 + (X_{im}(e^{jw}))^2}$$

$$\phi(w) = \tan^{-1}\left(X_{im}(e^{jw}) / X_{re}(e^{jw})\right)$$

③ For real sequance $x[n]$, the following symmetry properties hold:

- $X_{ev}[n] \longleftrightarrow X_{re}(e^{jw})$
- $X_{odd}[n] \longleftrightarrow X_{im}(e^{jw})$
- $X(e^{jw}) = X^*(e^{jw})$
- $X_{re}(e^{jw}) = X_{re}(e^{-jw})$
- $X_{im}(e^{jw}) = -X_{im}(e^{-jw})$
  - → $|X(e^{jw})| = |X(e^{-jw})|$
  - → $\arg(X(e^{jw})) = -\arg(X(e^{-jw}))$

* ***Note:*** The DTFT is exist if $\sum_n |x[n]| < \infty$ "absolutly summable"

example:

$x[n] = [\underset{\uparrow}{0}, 1, 2, -1, 0]$, find $X(e^{jw})$

sol:

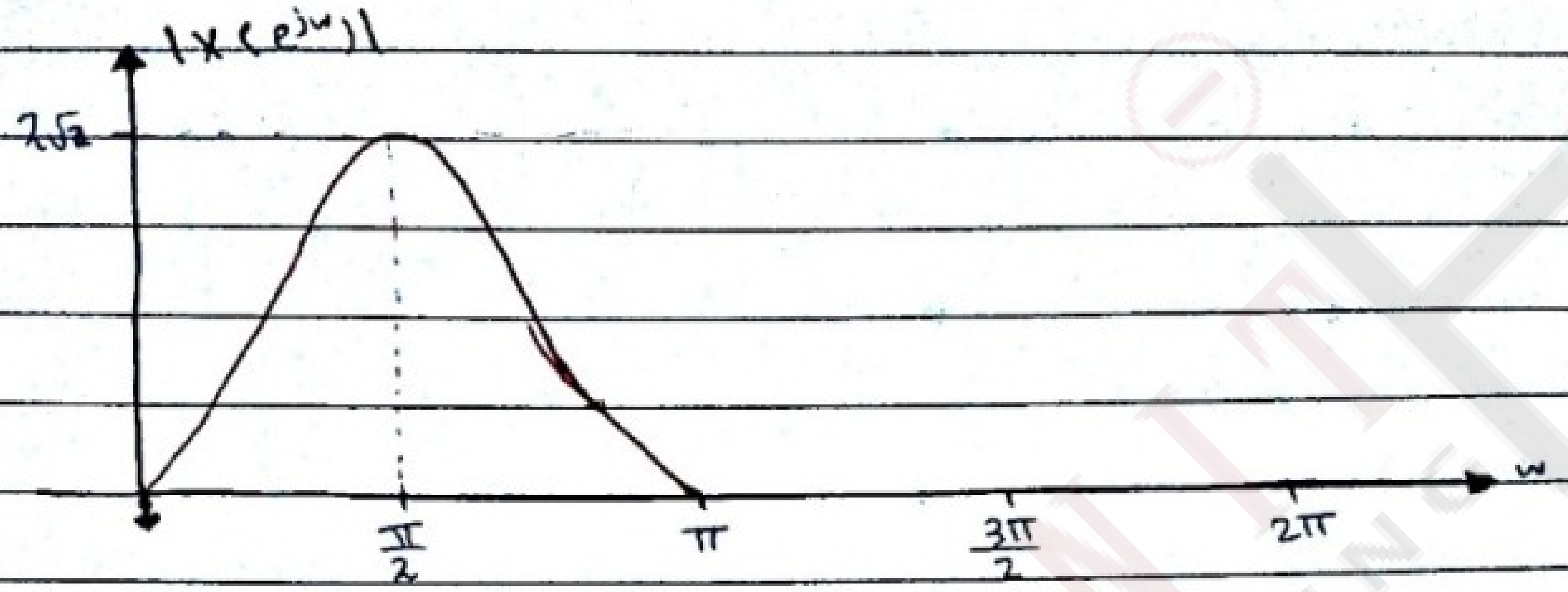
$$X(e^{jw}) = \sum_{n=0}^{4} x[n]\, e^{-jwn}$$

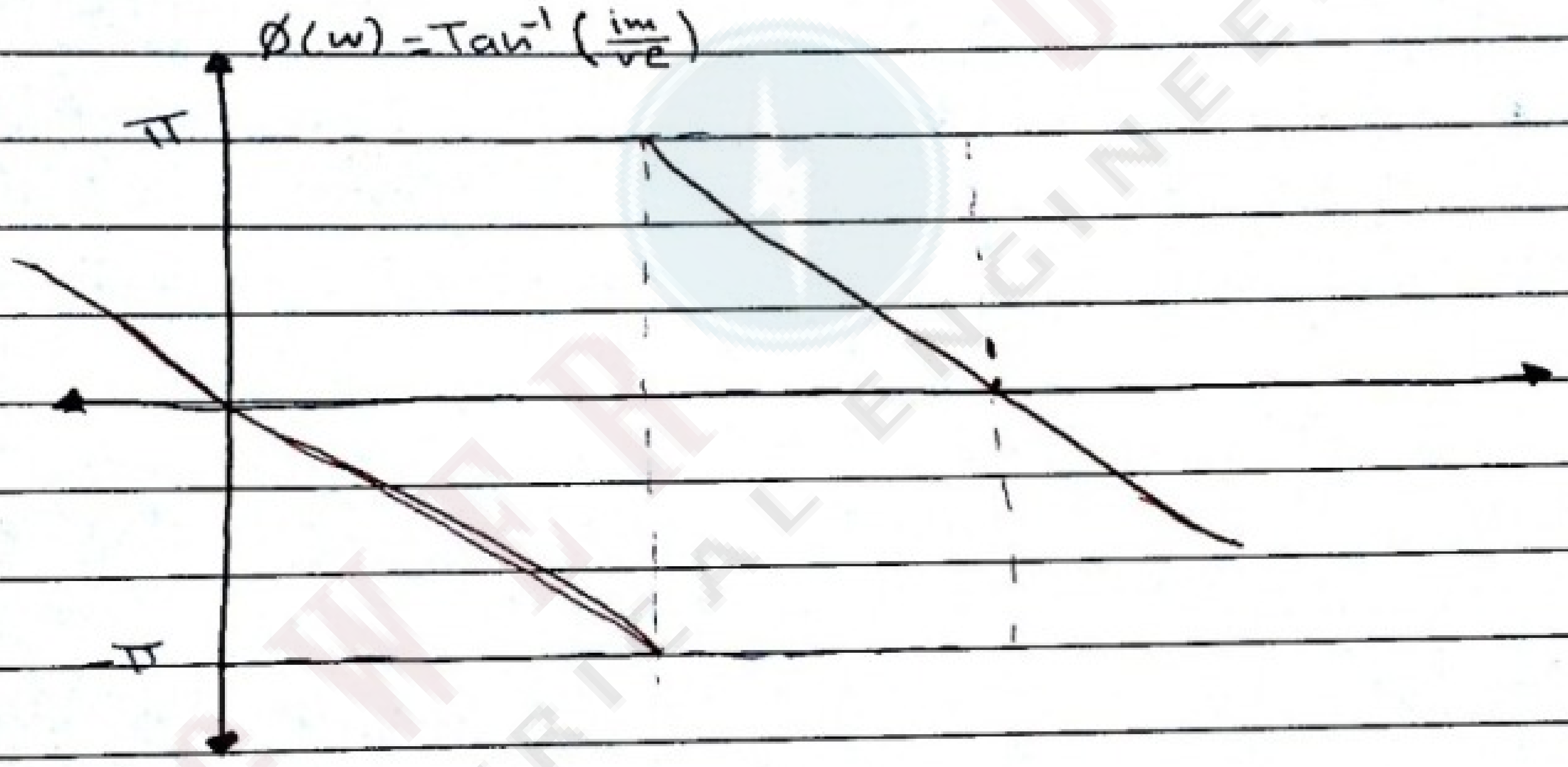
$$= \cancelto{0}{0 \cdot e^{-jw0}} + e^{-jw} + 2e^{-j2w} - e^{-j3w} + \cancelto{0}{0 \cdot e^{-j4w}}$$

$X(e^{jw}) = e^{-jw} + 2e^{-j2w} - e^{-j3w}$

$= \cos(w) - j\sin(w) + 2\cos(2w) - j2\sin(2w) - \cos(3w) + j\sin(3w)$

$= (\cos(w) + 2\cos(2w) - \cos(3w)) - j(\sin(w) + 2\sin(2w) - \sin(3w))$





lecture 10

* Common DTFT pairs:

① $\delta[n] \longleftrightarrow 1$

② $u[n] \longleftrightarrow \frac{1}{1-e^{-jw}} + \sum_{K=-\infty}^{\infty} \pi\, \delta(w-2\pi K)$

③ $1 \longleftrightarrow \sum_{K} 2\pi\, \delta(w-2\pi K)$

④ $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-jw}}, |a|<1$

⑤ $\cos(w_o n) \longleftrightarrow \sum_{k} \pi(\delta(w+w_o-2\pi K)+\delta(w-w_o-2\pi K))$

⑥ $\sin(w_o n) \longleftrightarrow \sum_{k} j\pi(\delta(w+w_o-2\pi K)-\delta(w-w_o-2\pi K))$

* DTFT theorems:

① Linearity:

$\alpha g[n] + \beta h[n] \longleftrightarrow \alpha G(e^{jw}) + \beta H(e^{jw})$

② Time reversal:

$g[-n] \longleftrightarrow G(e^{-jw})$

③ Time shift:

$g[n-n_o] \longleftrightarrow e^{-jwn_o} G(e^{jw})$

④ Frequancy shift:

$e^{jw_o n} g[n] \longleftrightarrow G(e^{j(w-w_o)})$

⑤ Differantiation in frequancy:

$n g[n] \longleftrightarrow j \partial \frac{G(e^{jw})}{\partial w}$

⑥ Convolution:

$$g[n] \circledast h[n] \longleftrightarrow G(e^{jw}) \cdot H(e^{jw})$$

⑦ Modulation:

$$g[n]\,h[n] \longleftrightarrow \frac{1}{2\pi} G(e^{jw}) \circledast H(e^{jw})$$

⑧ parseval theorem:

$$\sum_n g[n]\,h^*[n] \longleftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{jw})\,H^*(e^{jw})\,.dw$$

→ if $g[n] = h[n]$, then:

$$\mathcal{E}_x = \sum_n |g[n]|^2 \longleftrightarrow \mathcal{E}_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|G(e^{jw})|^2}_{S_{xx} = \text{energy spectrum}}\,.dw$$

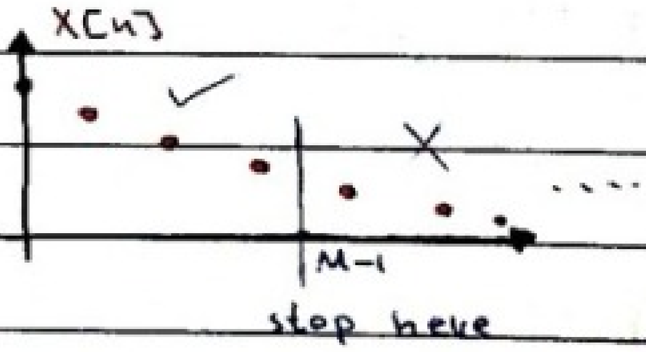
examples:

① $x[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$, find $X(e^{jw})$

sol:

$$X(e^{jw}) = \sum_{n=0}^{M-1} \alpha^n e^{-jwn}$$

$$= \sum_{n=0}^{M-1} (\alpha e^{-jw})^n$$



directly

or:

$$x[n] = \alpha^n u[n] - \alpha^n u[n-m]$$

$$= \underbrace{\alpha^n u[n]} - \alpha^m \underbrace{\alpha^{n-m} u[n-m]}$$

$$X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} - \alpha^m \cdot \frac{1}{1-\alpha e^{-jw}} e^{-jwm}$$

$$\to = \frac{1-\alpha^m e^{-jwm}}{1-\alpha e^{-jw}}$$

example :

Determine $Y(e^{jw})$ if $y[n] + 6\,y[n-1] = \delta[n] + 3\,\delta[n-1]$

solution :-

$$y[n] + 6\,y[n-1] = \delta[n] + 3\,\delta[n-1]$$

$$\xrightarrow{DTFT} \quad Y(e^{jw}) + 6\,Y(e^{jw})\,e^{-jw} = 1 + 3\,e^{-jw}$$

$$\rightarrow \quad Y(e^{jw}) = \frac{1 + 3\,e^{-jw}}{1 + 6\,e^{-jw}}$$

example :

Find $X(e^{jw})$ for $x[n] = (-1)^n \alpha^n u[n]$, $|\alpha| < 1$

solution :

$$x[n] = (-1)^n \alpha^n u[n]$$

$$= (e^{j\pi})^n \alpha^n u[n]$$

$$= (\alpha e^{j\pi})^n u[n]$$

$$\xrightarrow{DTFT} \quad X(e^{jw}) = \frac{1}{1 - \alpha e^{j\pi} e^{-jw}} = \frac{1}{1 - \alpha e^{-j(w-\pi)}}$$

example :

$y[n] = (n+1)\,\alpha^n u[n]$, $|\alpha| < 1$, Find $Y(e^{jw})$

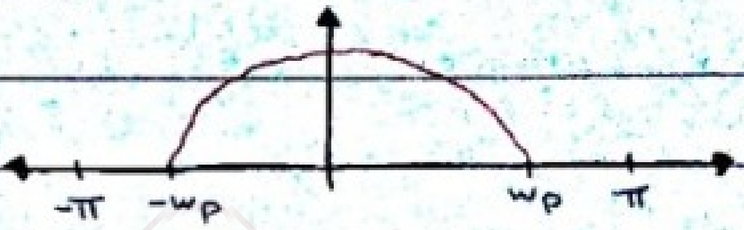
solution:

$$Y(e^{jw}) = j\,\frac{\partial}{\partial w}\left(\frac{1}{1 - \alpha e^{-jw}}\right) + \frac{1}{1 - \alpha e^{-jw}}$$

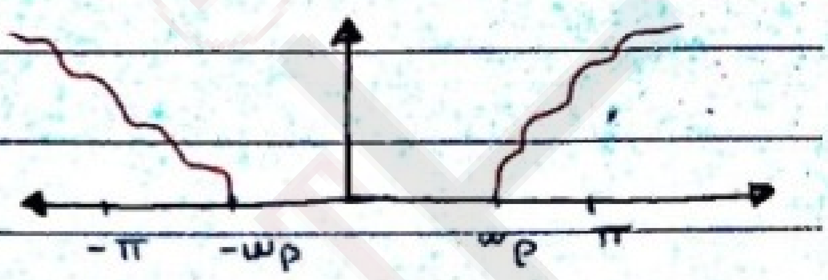
$$= \frac{1}{(1 - \alpha e^{-jw})^2}$$

* Band-limited Discrete-time signals :

① Low-pass , $-\pi < -w_p \leq w \leq w_p < \pi$



② High-pass , $w_p \leq |w < \pi$



③ Band-pass , $0 < w_L \leq |w| \leq w_H < \pi$

*3. Digital proccessing of continous-time signals :

- $$\underset{g_a(t)}{\text{Analog}} \longrightarrow \boxed{\text{Sampling}} \longrightarrow \boxed{Q} \longrightarrow \underset{g[n]}{\text{Digital}}$$

- $$\underset{G(j\Omega)}{\text{CTFT}} \longleftarrow \boxed{???} \longrightarrow \underset{G(e^{jw})}{\text{DTFT}}$$

$\to$ given $g_a(t) \xrightarrow{\mathcal{F}} G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t)\, e^{-j\Omega t} . dt$

$\to$ given $g[n] \xrightarrow{\mathcal{F}} G(e^{jw}) = \sum_n g[n]\, e^{-jwn}$

* if $g_a(t)$ is sampled at $f_s = \frac{1}{T}$, the sampling process can be expressed by :

- $$g_p(t) = g_a(t) \overset{\text{train of impulses}}{p(t)}$$

$$= \sum_n g_a(t)\, \delta(t - nT)$$

$$= \sum_n g_a(nT)\, \delta(t - nT)$$

→ now, computing the CTFT of $g_p(t)$

→ $G_p(j\Omega) = \int_{-\infty}^{\infty}\sum_n g_a(nT)\,\delta(t-nT)\,e^{-j\Omega t}\,.dt$

$$= \sum_n g_a(nT)\int_{-\infty}^{\infty}\delta(t-nT)\,e^{-j\Omega t}\,.dt$$

$$\boxed{G_p(j\Omega) = \sum_n g_a(nT)\,e^{-j\Omega nT}} \quad .... ①$$

→ The Pason's summation formula is given by:

$$\sum_n \phi(t+nT) = \frac{1}{T}\sum_n \underbrace{\Phi(jk\Omega_s)\,e^{jk\Omega_s t}}_{\text{CTFT of } \phi(t)} \quad \text{, where } \Omega_s = \frac{2\pi}{T}$$

& $\Phi(jk\Omega_s)$ is the CTFT of $\phi(t)$. [redacted]

→ now for $t=0$.:

$$\sum_n \phi(nT) = \frac{1}{T}\sum \Phi(jk\Omega_s)$$

→ let $\phi(t) = g_a(nT)$.

$$\sum_n g_a(nT) = \frac{1}{T}\sum Ga(jk\Omega_s)$$

→ multiply $g_a(nT)$ by $e^{-j\Omega nT}$

$$\boxed{\underbrace{\sum_n g_a(nT)\,e^{-j\Omega nT}} = \frac{1}{T}\sum_k Ga(j[k\Omega_s+\Omega])} \quad .... ②$$

→ $G_p(j\Omega) = \frac{1}{T}\sum_k Ga(j[k\Omega_s+\Omega])$

→ i want $G(j\Omega) \xleftrightarrow{?} G(e^{jw})$

→ but i have $G(j\Omega) \longleftrightarrow G_p(j\Omega)$

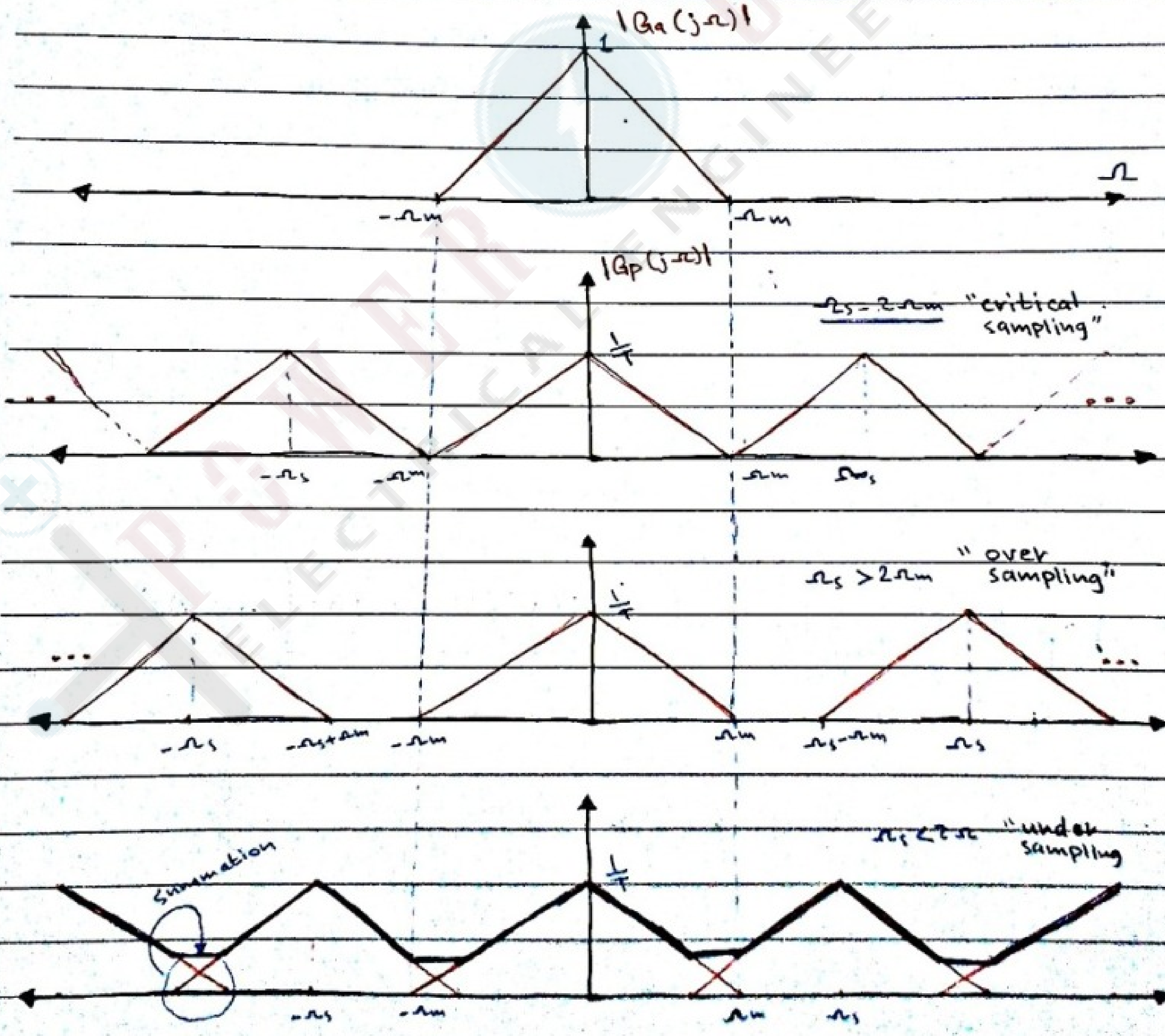
& we know $\Omega t = w$

* Notes :

① $G_p(j\Omega)$ is a periodic function with a period Ω_s, and it's basically a sum of shifted copies of $G_a(j\Omega)$ scaled by $\frac{1}{T}$

② for $K=0$, $G_p(jw) = \frac{1}{T} G_a(jw)$.
and this is called the base-band portion of $G_p(j\Omega)$.

③ for $K \neq 0$, the copies of $G_a(j\Omega)$ are called the frequancy-translated portion of $G_p(j\Omega)$



lecture
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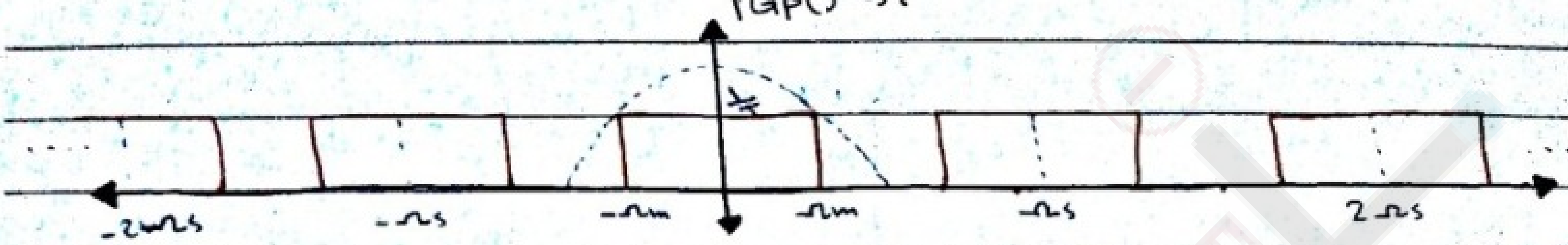
* $G(j\Omega) \propto G_a(j\Omega)$

→ $G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega + k\Omega_s))$



- $H(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_m \\ 0, & |\Omega| \leq \Omega_m \end{cases}$

- $\mathcal{F}^{-1}\{R(j\Omega)\} = G_p(j\Omega) \cdot H(j\Omega)$

* Sampling theorem:

→ Let $g_a(t)$ be a band-limited signal with $G_a(j\Omega) = 0$ when $|\Omega| > \Omega_m$, then $g_a(t)$ uniquely determined by it's samples $g_a(nT)$, ∀n if:

$\boxed{\Omega_s \geq 2\Omega_m}$ → [$w_s = 2\Omega_m$] ≡ nyquest rate

- $G(e^{jw}) \propto G_a(e^{jw})$

Proof:

→ $G_p(j\Omega) = \frac{1}{T}\sum_{k} G_a(j(\Omega + k\Omega_s))$

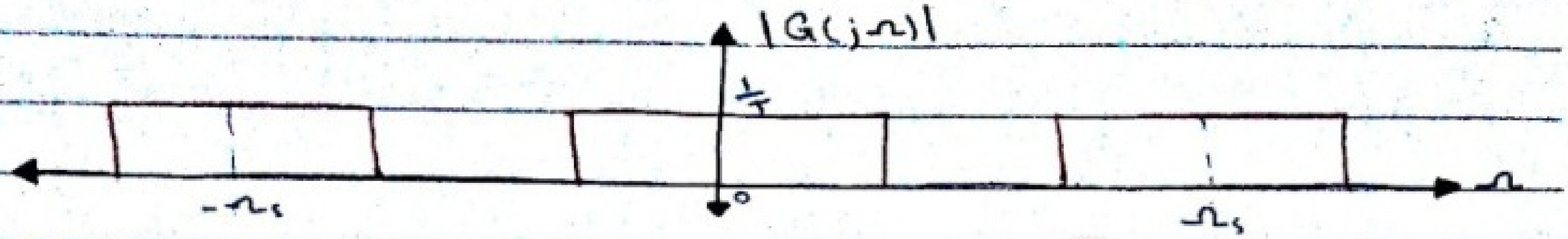
but,

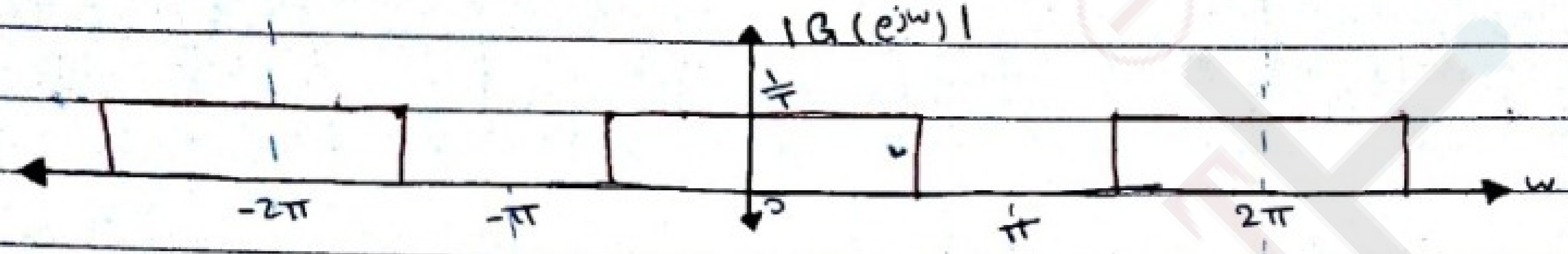
$\Omega T = w$, then:

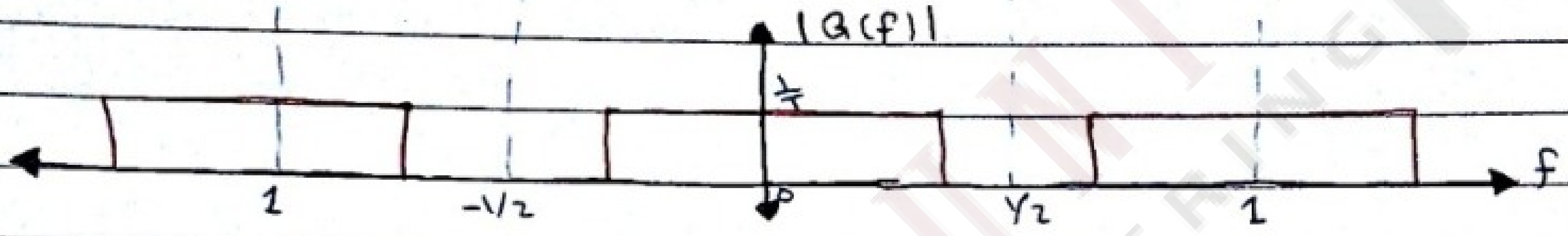
→ $G_p(e^{jw}) = \frac{1}{T}\sum_{k} G_a(j(\frac{w}{T} + \underbrace{k2\pi}_{\text{how?!}}))$

if $\Omega = \Omega_s$, then:

$\boxed{\Omega_s T = w = \frac{2\pi}{\cancel{T}}\cancel{T} = 2\pi}$







* Frequancy domain vepresentation of discrete LTI systems :

- for an LTI h[n], the input-output relationship is:

$$y[n] = \sum_{k} h[K]\, x[n-K]$$

let $x[n] = e^{jw_o n}$, $-\infty < n < \infty$

$$y[n] = \sum_{k} h[K]\, e^{jw_o[n-K]}$$

$$= \left(\sum_{k} h[K]\, e^{-jw_o K}\right) e^{jw_o n}$$

→ $$\boxed{y[n] = H(e^{jw_o})\, e^{jw_o n}}$$

→ $$y[n] = |H(e^{jw_o})|\, e^{j\theta} \,.\, \underbrace{e^{jw_o n}}_{\text{phase shift}}$$

* Hence, the output of an LTI system is also a complex exponential of the same frequancy scaled by a complex constant that is function of the input frequancy.

→ The term $H(e^{jw})$ is called the frequancy response of the system, it describes the system in the frequancy domain.

- Consider :

$$y[n] = \sum_{k} h[k]\, x[n-k]$$

based on the convolution theorem:

$$Y(e^{jw}) = H(e^{jw})\, X(e^{jw})$$

then,

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

- For a <u>FIR</u> LTI system with $h[n] = 0$ for $\left\{\begin{matrix} n < N_1 \\ n > N_2 \end{matrix}\right\}$

$$y[n] = \sum_{k=N_1}^{N_2} h[k]\, x[n-k]$$

$$Y(e^{jw}) = \underbrace{\left(\sum_{k=N_1}^{N_2} h[n]\, e^{-jwn}\right)}_{H(e^{jw})} X(e^{jw})$$

$$\boxed{H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \sum_{k=N_1}^{N_2} h[n]\, e^{-jwn}}$$

Note that $\underbrace{H(e^{jw})}_{\text{frequancy response}}$ is simply a polynomial of (e^{jw})

* For IIR LTI:

DTFT

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$ } → Difinition of vecursive system

$$\sum_{k=0}^{N} a_k\, Y(e^{j\omega})\, e^{-j\omega k} = \sum_{k=0}^{M} b_k\, X(e^{j\omega})\, e^{-j\omega k}$$

$$Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{-j\omega k} = X(e^{j\omega}) \sum_{k=0}^{M} b_k\, e^{-j\omega k}$$

$$\rightarrow \boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k\, e^{-j\omega k}}}$$

frequancy response is a rational function in $e^{j\omega}$

example:

$h[n] = [-3, \underset{\uparrow}{10}, 3]$, what is the output $y[n]$ when the input is $x[n] = 5\cos(0.2\pi n)$?

solution:

$$y[n] = H(e^{j0.2\pi})\,.\,5\cos(0.2\pi n) \rightarrow 5\,\frac{e^{j0\,2\pi} + e^{-j0.2\pi}}{2}$$

$$H(e^{j\omega}) = \sum_{n=-1}^{1} h[n]\, e^{-j\omega n}$$

$$= -3\, e^{j\omega} + 10\, e^{j\omega(0)} + 3\, e^{-j\omega}$$

$$H(e^{j0.2\pi}) = 10 - 6j\sin(0.2\pi)$$

$$= 10 - 6\, e^{-j19.4^\circ}$$

$$\boxed{y[n] = 53\cos(0.2\pi n - 19.4^\circ)}$$

* The concept of filtering :

- Digital filters are used to pass certain frequancies & block others.

example :

$$H(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & |w| > w_c \ \& \ |w| < \pi \end{cases}$$

- if the input is $x[n] = A\cos(w_1 n) + B\cos(w_2 n)$ with $0 < w_1 < w_c < w_2 < \pi$, then the output is :

$$y[n] = A|H(e^{jw})|\cos(w_1 n)e^{j\theta(w_1)} + \underbrace{B|H(e^{jw})|\cos(w_1 n)e^{j\theta(w_2)}}_{=0}$$

$$y[n] = A|H(e^{jw})|\cos(w_1 n)e^{j\theta(w_1)}$$

example:

Design a simple filter [high pass filter], that passes frequancies at 0.4 rad/sample, and block these at 0.1 rad/sample - such that :

$$h[n] = \{\underset{\uparrow}{\alpha_0}, \alpha_1, \alpha_2\}$$

solution :

$$H(e^{jw}) = \sum_{n=0}^{2} h[n]e^{-jwn} = \alpha_0 e^{j0} + \alpha_1 e^{-jw} + \alpha_2 e^{-2jw}$$

→ to simplify the equation let $\alpha_0 = \alpha_2$:

$H(e^{jw}) = \alpha_0(1+e^{-j2w}) + \alpha_1 e^{-jw}$

→ $H(e^{j0.1}) = \alpha_0(1+e^{-j0.2}) + \alpha_1 e^{-j0.1} = 0$ "reject"

→ $H(e^{j0.4}) = \alpha_0(1+e^{-j0.8}) + \alpha_1 e^{-j0.4} = 1$ "pass"

→ also to simplify the equation we will consider the magnitude.

$$H(e^{jw}) = (2\alpha_0 \cos(w) + \alpha_1) e^{-jw}$$

$$|H(e^{jw})| = 2\alpha_0 \cos(w) + \alpha_1$$

$$\left.\begin{array}{l} \rightarrow |H(e^{j0.1})| = 2\alpha_0 \cos(0.1) + \alpha_1 = 0 \\ \quad |H(e^{j0.4})| = 2\alpha_0 \cos(0.4) + \alpha_1 = 1 \end{array}\right\} \begin{array}{l} \alpha_0 = -6.761 \\ \alpha_1 = 13.456 \\ \alpha_2 = -6.761 \end{array}$$

→ $h[n] = \{-6.761, 13.456, -6.761\}$

$$y[n] = h[n] \circledast x[n]$$

$$= \sum_k h[K]\, x[n-K]$$

$$= \sum_K x[n-k]\, h[K]\, \delta[k-\underset{\overset{\downarrow}{0-2}}{m}]$$

$$= \sum_m x[n-m]\, h[m]$$

→ $$\boxed{y[n] = -6.761\, x[n] + 13.456\, x[n-1] - 6.761\, x[n-2]}$$

lecture 12

* Chapter 5: Finite length discrete Transform :-

① Discrete Fourier Transform "DFT":

→ in practise, most sequancies are finite in length

$$x[n] = 0 \ , \ n < N_1 \ \& \ n > N_2$$

→ the frequancy domain representation of such sequence is

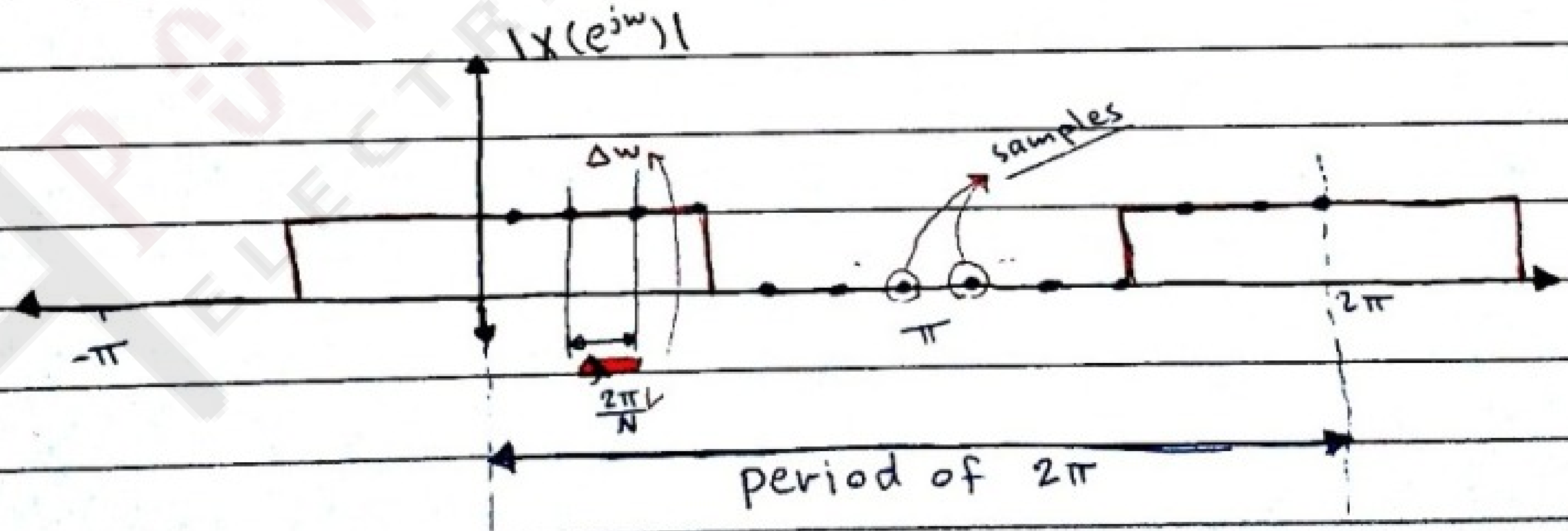
$$X(e^{jw}) = \sum_{n=0}^{N-1} x[n] \, e^{-jwn}$$

→ However, $X(e^{jw})$ is a continous function in w !!
Also, it's periodic and extends from $-\infty$ to ∞.

→ This makes it impossible to process the signal in the frequancy domain using computer!

→ solution:

→ Insted of working with finite number of points, sample the spectrum of $X(e^{jw})$ using N-point over the interval $[0, 2\pi]$



* The spacing between successive samples is:

$$\Delta w = \frac{2\pi}{N}$$

* the frequancy of each sample is:

$$w = \frac{2\pi}{N} k \quad , \quad 0 \leq k \leq N-1$$

* Accordingly, the DTFT of $x[n]$ becomes:

$$\underline{\underline{X(e^{jw})}} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad , \quad \underline{\underline{X[K]}} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

* $X[K]$ is called N-point DFT and it's inverse is computed by:

$$\boxed{x[n] = \frac{1}{M} \sum_{k=0}^{N-1} X[K]\, e^{j\frac{2\pi}{N}kn}}$$

example:

$x[n] = \{2, 0, -2, 1\}$, find it's 4-point DFT

solution

→ the 4-point DFT is $X[K] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn}$:

→ $X[0] = \sum_{n=0}^{3} x[n] = 1$

→ $X[1] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}n} = 2e^{-j\frac{\pi}{2}(0)} + 0e^{-j\frac{\pi}{2}} - 2e^{-j\pi} + e^{-j\frac{3\pi}{2}}$

$= 4 + j$

→ $X[2] = -1$

→ $X[3] = 4 - j$

$|X[K]| = \{1, 4.12, 1, 4.12\}$

$\measuredangle X[K] = \{0, 0.245, 3.14, 6.03\}$

$w = \{0, \pi/2, \pi, 3\pi/2\}$

*** Notes:**

① when computing the DFT, all samples in $x[n]$ are used to compute each sample in $X[k]$

② The DFT always exist !!

↳ since it's a finite sum $X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$

*** The Matrix form of DFT :**

- $X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$ let $W_N = e^{-j\frac{2\pi}{N}}$

$X[K] = \sum_{n=0}^{\infty} x[n]\, W_N^{kn}$, $0 \leq X \leq N-1$

$$X[K] = \begin{bmatrix} 1 & W_N^{k} & W_N^{2k} & \dots\dots\dots & W_N^{(N-2)k} & W_N^{(N-1)k} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

→ In general:

$$X[K] = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \dots\dots & 1 \\ 1 & W_N & W_N^{2} & \dots\dots & W_N^{N-1} \\ 1 & W_N^{2} & W_N^{4} & \dots\dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \dots\dots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \dots\dots & W_N^{(N-1)^2} \end{bmatrix}}_{D_N} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{X}$$

$$\boxed{X[K] = \underline{D_N} X} \longrightarrow (N \times N)\ \text{Matrix}$$

* For the inverse N-point DFT :

- $X[k] = D_N \, x[n]$

  $\mathbb{X} = D_N \, x$

→ $x[n] = \boxed{x = \mathbb{X} \, D_N^{-1}}$, where $D_N^{-1} = \frac{1}{N} D_N^{*}$

$$D_N^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & W_N^{-(N-1)} \\ \vdots & \vdots & \vdots \cdots & \vdots \\ 1 & \cdots & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

example:

$x[n] = \{-2, 1, 7, 3\}$, $N=4$, find the 4-point DFT

solution:

$W_N = W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$

$$\rightarrow D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \rightarrow x = \begin{bmatrix} -2 \\ 1 \\ 7 \\ 3 \end{bmatrix}$$

$$\rightarrow X[k] = D_4 \, x = \begin{bmatrix} 9 \\ -9+j2 \\ 1 \\ -9-j2 \end{bmatrix}$$

* Note that :

D = det. Matrix "could be big"

$\mathbb{X} = \underbrace{\text{FFt}}$ (x, m) → number of points

↳ fast fourier transform

$x = \underbrace{\text{I}}\text{FFt}$ $(\mathbb{X}, M)$

↳ Inverse

*** Notes :**

① $X[K] = X(e^{jw})\Big|_{w=\frac{2\pi}{N}K}$, $0 < K \leq N-1$

② For a finite length sequance $x[n]$, $0 \leq n < N-1$, the N-point IDFT is periodic with period N, on other words, if $y[n] = \mathcal{F}^{-1}\{X[K]\}$, then:

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN]$$

Proof :

let $X(e^{jw})$ be the discrete-time Fourier transform of $x[n]$, $0 \leq n \leq N-1$, the M-point DFT $Y[K]$ is :

$$Y[K] = X(e^{jw})\Big|_{w=\frac{2\pi}{N}K} = \left(\sum_{l=-\infty}^{\infty} x(l)\,W_M^{Kl}\right) \quad , \; 0 \leq K \leq M-1$$

now, the M-point IDFT is

$$y[n] = \frac{1}{M}\sum_{K=0}^{M-1} Y[K]\,W_M^{-Kn}$$

$$= \frac{1}{M}\sum_{K=0}^{M-1}\left(\sum_{l=-\infty}^{\infty} x(l)\,W_M^{Kl}\right)W_M^{-Kn}$$

$$= \sum_{l=-\infty}^{\infty} x(l)\left(\frac{1}{M}\sum_{K=0}^{M-1} W_M^{-K(n-l)}\right)$$

However, $\frac{1}{M}\sum_{K=0}^{M-1} W_M^{-K(n-l)} = \begin{cases} 1 & , \; l = n+mM \\ 0 & , \; \text{othewise} \end{cases}$

$$\rightarrow \quad y[n] = \sum_{m=-\infty}^{\infty} x[n+mM] \quad \#\#$$

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*** Finite length sequance:**

→ $X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$ (Discrete Forier transform)

$0 \le K \le N-1$

now the inverse Forier transform:

$$\boxed{x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X[K]\, e^{j\frac{2\pi Kn}{N}}} \qquad 0 \le n \le N-1$$

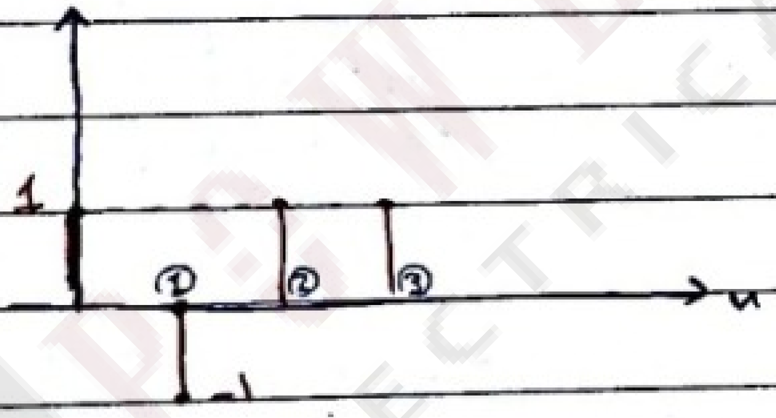
→ $X[K] = X[K+\ell N]$ where ℓ is an integer

So DFT is periodic with period "N"

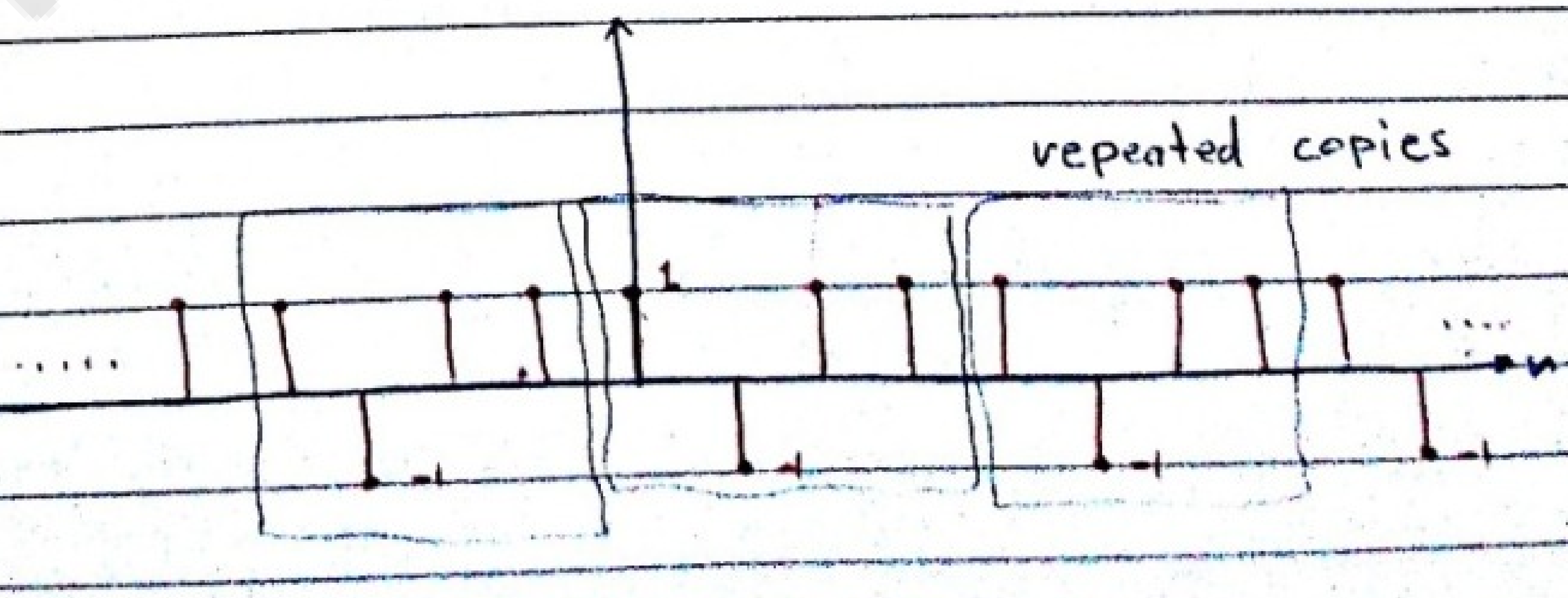
IDFT is periodic

→ $\boxed{y[n] = IDFT(X[K]) = \sum_{\ell} x[n+\ell \textcircled{N}]}$ → number of points in frequancy

example:

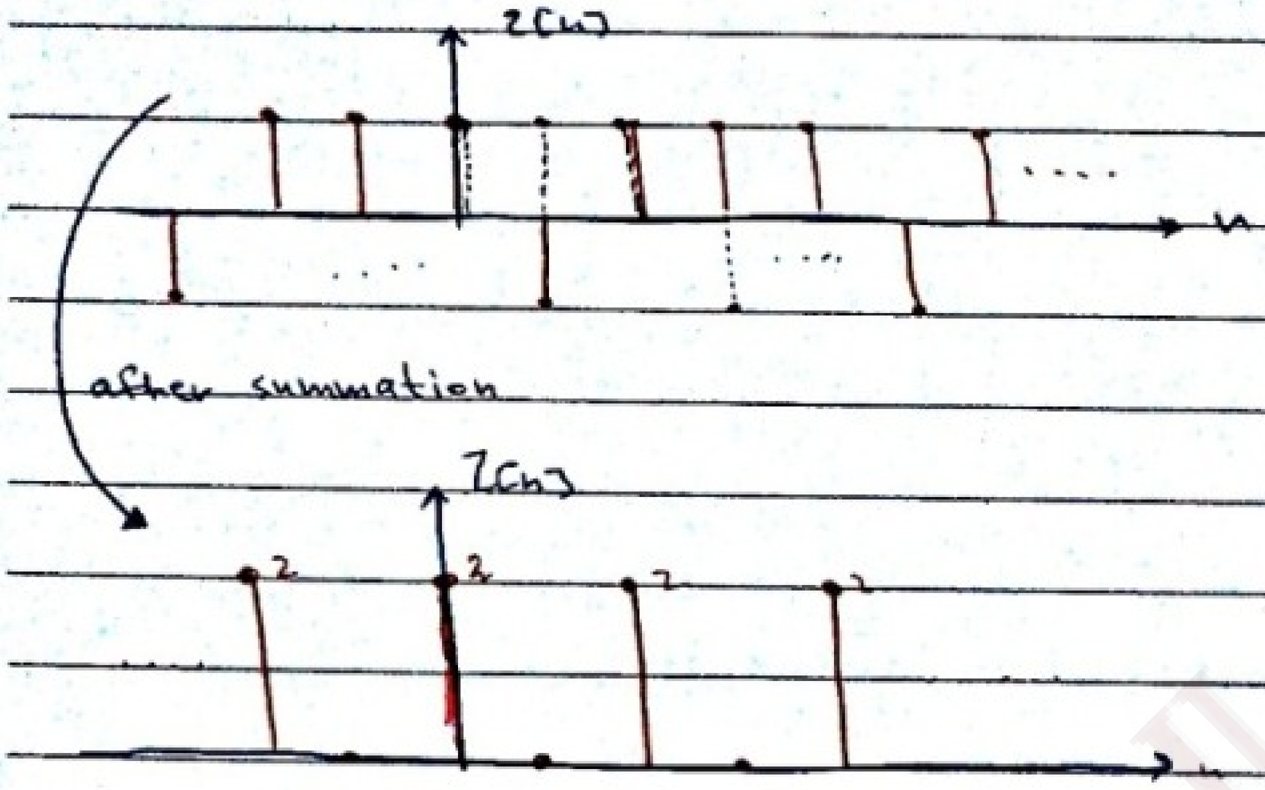


$y[n] = IDFT(X[K])$ where $X[K]$ is the 4-point DFT of $x[n]$



$\leftrightarrow z[n] = IDFT\,(X[K])$

where $X[K]$ is the 2-point DFT of $x[n]$



Note:

In order to avoid time-domain aliasing when computing the DFT in IDFT, the number of points used to compute the DFT should be grater than or equal to the length of the sequance.

Length-N sequance → M-points DFT → $\boxed{M \geq N}$

* 2. Circular convolution:

$x[n]$ finite length $\xrightarrow{DTFT} X(e^{jw}) \xrightarrow[\text{Points}]{N} X[K]$

$h[n]$ finite length $\xrightarrow{DTFT} H(e^{jw}) \xrightarrow[\text{Points}]{N} H[K]$

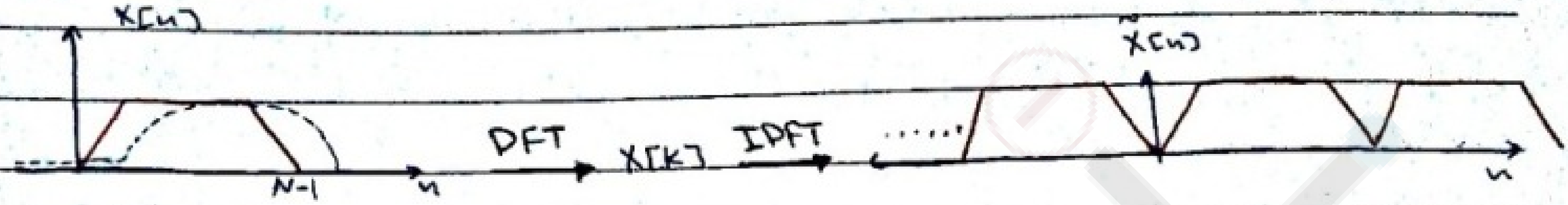
* if $h[n]$ represents the impulse response of an LTI system, then the output $y[n]$ is:

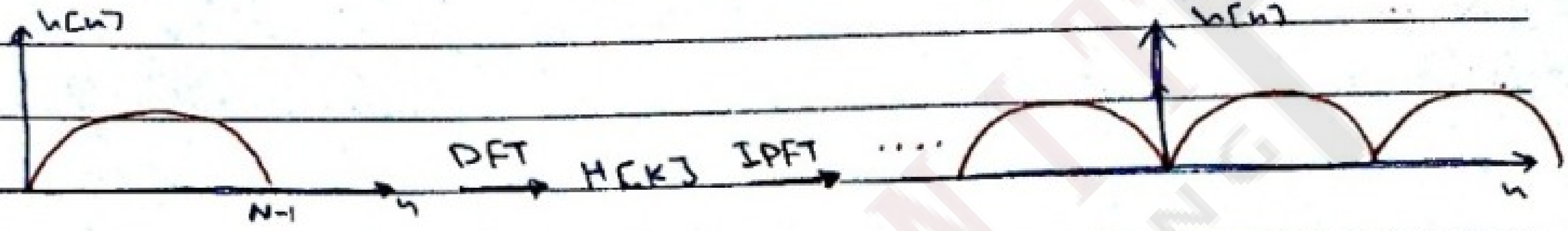
$\boxed{y[n] = x[n] \circledast h[n]}$ and $\boxed{Y(e^{jw}) = X(e^{jw}) \cdot H(e^{jw})}$

$$x[n] \circledast h[n] \Longleftrightarrow X(e^{jw}) \cdot H(e^{jw})$$

However,

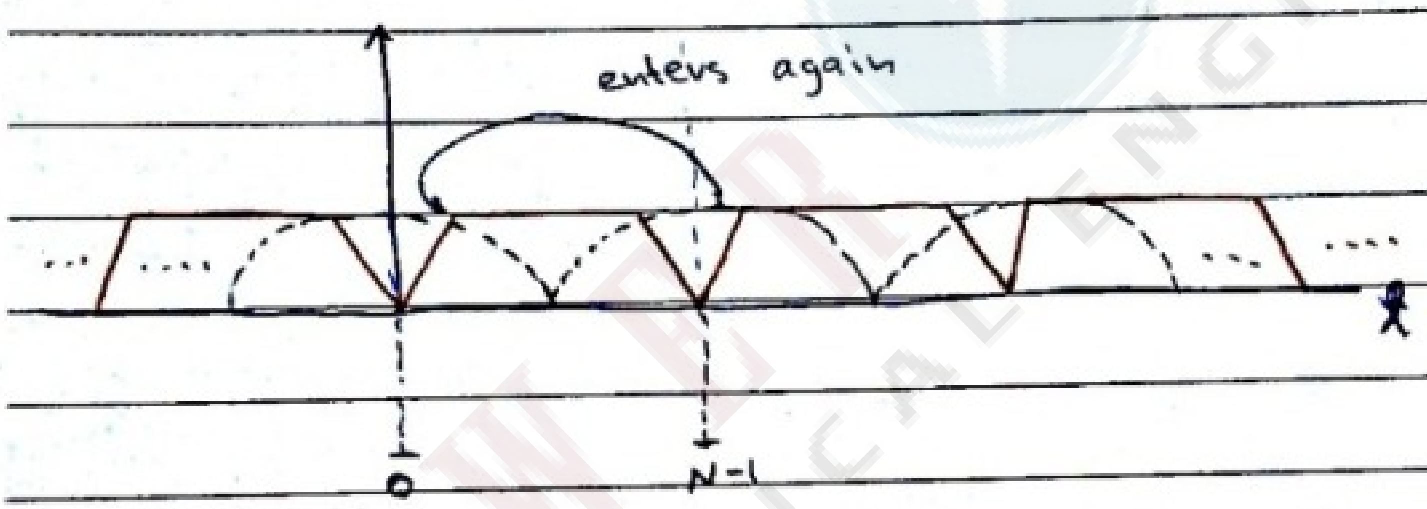
$x[n] \circledast h[n] \Longleftrightarrow X[K]\cdot H[K]$??





→ $\tilde{x}[n] \circledast \tilde{h}[n] \longleftrightarrow X[K]\,H[K]$

→ $y[n] = \tilde{x}[n] \circledast \tilde{h}[n] = \sum_{l=0}^{N-1} \tilde{x}[l]\,\tilde{h}[n-l]$

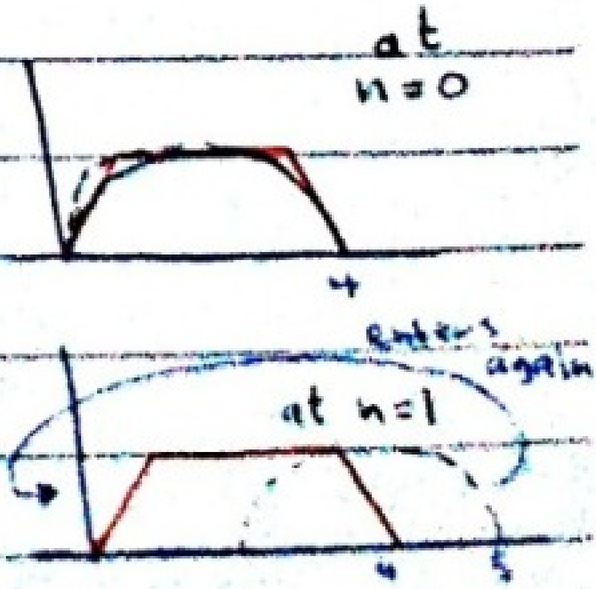


* This operation is not actually linear convolution!
This is called N-point circualr convolution.

→ $y_c[n] = x[n] \textcircled{N} h[n] = \boxed{\sum_{m=0}^{N-1} x[m]\, h[\langle n-m \rangle_N]}$

→ This requires:

① Circular time reversal (Modulas)
② Circular time shift (Modulas)
③ Multiply
④ Add



* The Modular operation :

for a set of N integers in [0, N-1], then the modulas of an integer m by N is called the residue and it's an Integer in [0, N-1].

* for positive m :

→ the residue is straight forward → example! $\langle 25 \rangle_7 = 4$

* for negative m :

→ let $r = m + \ell N$, where ℓ is an integer, then the residue r is found by finding the smallest ℓ that makes r in the range [0, N-1].

example :

$r = \langle -15 \rangle_7 \Rightarrow r = -15 + 7\ell$

$\boxed{\ell = 3}$ $\boxed{r = 6}$

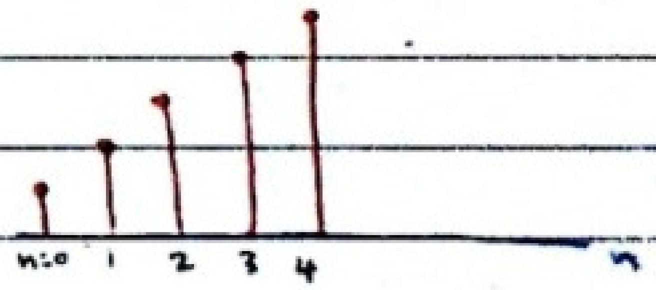
→ $\ell = 3$ is the smallest value of ℓ that makes $7\ell > 15$, then r is +ve

* Circular time reversal

→ $y[n] = x[\langle -n \rangle_N]$

example :

$x[n] = [\underset{\uparrow}{1}\ 2\ 3\ 4\ 5]$



$y[0] = x[\langle -0 \rangle_5] = x[0] = 1$

$y[1] = x[\langle -1 \rangle_5] = x[4] = 5$

$y[2] = x[\langle -2 \rangle_5] = x[3] = 4$

$y[3] = x[\langle -3 \rangle_5] = x[2] = 3$

$y[4] = x[\langle -4 \rangle_5] = x[1] = 2$

remember

$\boxed{y[n] = x[\langle -n \rangle_N]}$

$$\rightarrow x[\langle -n \rangle_N] = \begin{cases} x[n-N], & 1 \le n \le N-1 \\ x[n], & n = 0 \end{cases}$$

* Circular time shift:

→ $y[n] = x[\langle n-\underset{\uparrow}{n_o}\rangle_N]$

shift magnitude

* example:

$x[n] = [-1, 0, 1, 2]$ $\quad 0 \le n \le 3$

find $x[\langle n-2\rangle_4] = y[n]$

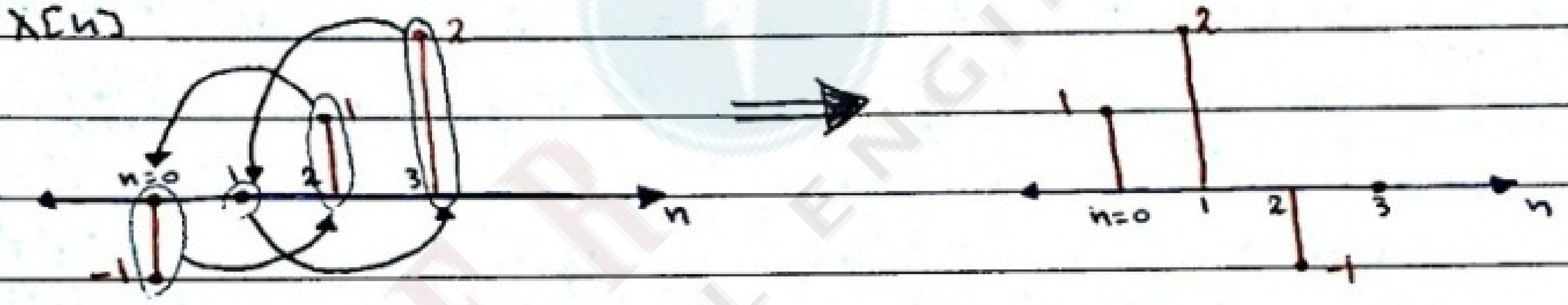
solution:

$y[0] = x[\langle 0-2\rangle_4] = x[\langle -2\rangle_4] = x[2] = 1$

$y[1] = x[\langle 1-2\rangle_4] = x[\langle -1\rangle_4] = x[3] = 2$

$y[2] = x[\langle 2-2\rangle_4] = x[\langle 0\rangle_4] = x[0] = -1$

$y[3] = x[\langle 3-2\rangle_4] = x[\langle 1\rangle_4] = x[1] = 0$



***example:**

$x[n] = [1,2,0,1]$, $h[n] = [2,2,1,1]$ $0 < n \le 3$

find $y_c[n] = x[n] \circledast h[n]$

solution

→ $y_c[n] = \sum_{m=0}^{3} x[m]\, h[\langle n-m\rangle_4]$

$\boxed{n=0}$ → $y_c[0] = \sum_{m=0}^{3} x[m]\, h[\langle -m\rangle_4]$

$y_c[0] = x[0]h[0] + x[1]h[\langle -1\rangle_4] + x[2]h[\langle -2\rangle_4] + x[3]h[\langle -3\rangle_4]$

$= x[0]h[0] + x[1]h[3] + x[2]h[2] + x[3]h[1]$

$= 2 + 2 + 0 + 2$

$= \underline{6}$

$\boxed{n=1}$ → $y_c[1] = \sum_{m=0}^{3} x[m]\, h[\langle 1-m\rangle_4]$

$y_c[1] = x[0]h[\langle 1\rangle_4] + x[1]h[\langle 0\rangle_4] + x[2]h[\langle -1\rangle_4] + x[3]h[\langle -2\rangle_4]$

$= x[0]h[1] + x[1]h[0] + x[2]h[3] + x[3]h[2]$

$= 2 + 4 + 0 + 1$

$= 7$

* Similarly

for $\boxed{n=2}$ → $y[2] = 6$

$\boxed{n=3}$ → $y[3] = 5$

* note that length of $y_c[n]$, $0 \le n \le 3$, the same as $x[n]$ & $h[n]$

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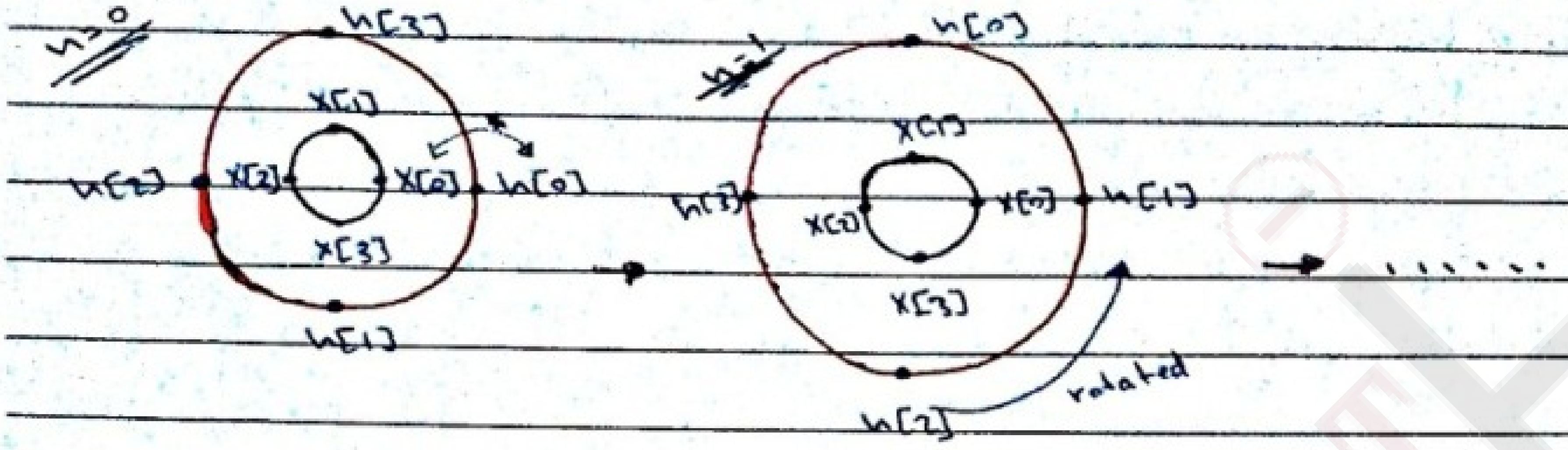
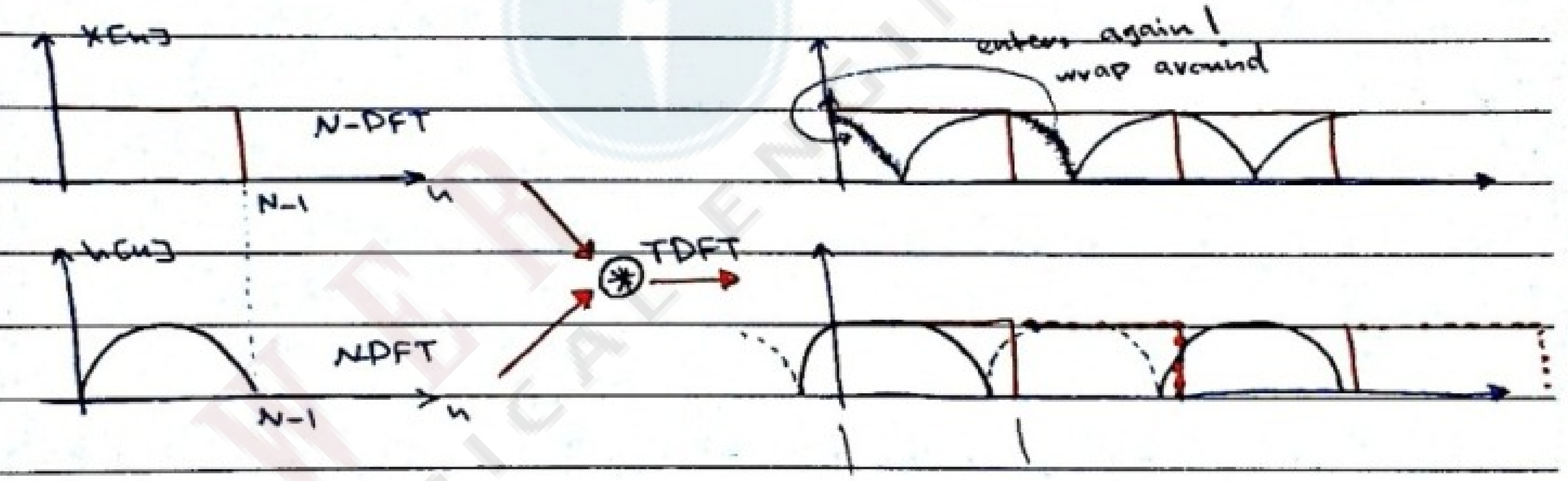
* representing Circular convolution graphicly :



- the black circle is fixed & the red one is rotating
- to find $y_c[0]$ for example → multiply & add:

$$y_c[0] = h[0]x[0] + x[1]h[3] + x[2]h[2] + x[3]h[1]$$

* Can we use circular convolution to implement linear convolution ?!



→ Simply yes just we need to cancel out the wrap around error, this can be done by padding the 2 sequances with K zeros such that K ≥ N-1, where N is the length of sequance.

3. Symmetry of finite length sequances:

→ For a complex length N sequance:

ⓐ the sequance is called circular conjugate symmetry if $x[n] = x^*[\langle -n\rangle_N]$

ⓑ the sequance is called conjigate anti-symmetry if $x[n] = -x^*[\langle -n\rangle_N]$.

⇒ * the sequance can be written as $x[n] = \underbrace{x_{cs}[n]}_{\text{conjigate symmetric part}} + \underbrace{x_{ca}[n]}_{\text{anti-conj symetric part}}$

~> $x_{cs}[n] = \frac{1}{2}\left(x[n] + x^*[\langle -n\rangle_N]\right)$

~> $x_{ca}[n] = \frac{1}{2}\left(x[n] - x^*[\langle -n\rangle_N]\right)$

→ For a real length N sequance:

ⓐ the sequance is circular even if $x[n] = x[\langle -n\rangle_N]$

ⓑ the sequance is circular odd if $x[n] = -x[\langle -n\rangle_N]$

⇒ * the sequance can be written as $x[n] = \underbrace{x_e[n]}_{\text{even part}} + \underbrace{x_o[n]}_{\text{odd part}}$

~> $x_e[n] = \frac{1}{2}\left(x[n] + x[\langle -n\rangle_N]\right)$

~> $x_o[n] = \frac{1}{2}\left(x[n] - x[\langle -n\rangle_N]\right)$.

example:

$x[n] = \{1+j4, -2+j3, 4-j2, -5-j6\}, \quad 0 \le n < 3$

find $x_{cs}[n]$ & $x_{ca}[n]$

solution:

$x^*[n] = \{1-j4, -2-j3, 4+j2, -5+j6\}$

$x^*[\langle -n\rangle_N] = \{1-j4, -5+j6, 4+j2, -2-j3\}$

→ $x_{cs}[n] = \frac{1}{2}\left[\{1+j4, -2+j3, 4-j2, -5-j6\} + \{1-j4, -5+j6, 4+j2, -2-j3\}\right]$

$= \{1, -3.5+j4.5, 4, -3.5-j4.5\} \quad 0 \le n \le 3$

→ $x_{ca}[n] = \frac{1}{2}\left[x[n] - x^*[\langle -n\rangle_N]\right] = \{j4, 1.5-j1.5, -j2, -1.5-j1.5\}$

$0 \le n < 3$

4. DFT symmetry relations:

* For a length-N real sequance $x[n] = x_{even}[n] + x_{odd}[n]$, if the N-point DFT is $X[K] = X_{real}[K] + j\,X_{imag.}[K]$, the following hold:

- $x^*[n] \xleftrightarrow{DFT} X^*[\langle -K\rangle_N]$
- $x^*[\langle -n\rangle_N] \xleftrightarrow{DFT} X^*[K]$
- $x_{re}[n] \xleftrightarrow{DFT} X_{cs}[K]$
- $x_{im}[n] \xleftrightarrow{DFT} X_{ca}[K]$
- $x_{cs}[n] \xleftrightarrow{DFT} X_{re}[K]$
- $x_{ca}[n] \xleftrightarrow{DFT} jX_{im}[K]$

* For a length-N real sequance $x[n]$ such that:

$$x[n] = x_{even}[n] + x_{odd}[n]$$

if N-point DFT is:

$$X[K] = X_{re}[K] + j\,X_{im}[K]$$

then the following is hold:

- $x_{even}[n] \xleftrightarrow{DFT} X_{re}[K]$
- $x_{odd}[n] \xleftrightarrow{DFT} jX_{im}[K]$
- $X[K] = X^*[\langle -K\rangle_N]$
- $X_{re}[K] = X_{re}[\langle -K\rangle_N]$
- $X_{im}[K] = X_{im}[\langle -K\rangle N]$
- $|X[K]| = |X[\langle -K\rangle_N]|$
- $\arg(X[K]) = -\arg(X[\langle -K\rangle_N])$

5. DFT theorems:

① Linearity

② Circular time shif:

$$g[\langle n-n_0\rangle_N] \xleftrightarrow{DFT} W_N^{Kn_0}\, G[K]$$

③ Circular frequancy shif:

$$g[n]\, W_N^{-K_0 n} \xleftrightarrow{DFT} G[\langle K-K_0\rangle_N]$$

④ Convolution:

$$x[n] \circledN h[n] \xleftrightarrow{DFT} X[K]H[K]$$

⑤ Modulation:

$$G[K] \circledN H[K] \xleftrightarrow{DFT} g[n]h[n]$$

⑥ Parseval's theorem:

$$\sum_{n=0}^{N-1} g[n]\,h^*[n] = \frac{1}{N}\sum_{K=0}^{N-1} G[K]\,H^*[K]$$

example:

let $g[n] = \{1,2,0,1\}$ & $h[n] = \{2,2,1,1\}$

compute $G[K]$ & $H[K]$ using a single 4-point DFT.

solution:

$x[n] = g[n] + j\,h[n]$

$= [\,1+j2,\ 2+j2,\ j1,\ 1+j1\,]$

$X[K] = [\,4+j6,\ 2,\ -2,\ j2\,]$

→ $g[n] = x_{re}[n] \xleftrightarrow{DFT} G[K] = X_{cs}[K]$

$G[K] = [\,4,\ 1-j,\ -2,\ 1+j\,]$

→ $h[n] = x_{im}[n] \xleftrightarrow{DFT} H[K] = \dfrac{X_{ca}[K]}{j}$

$H[K] = [\,6,\ 1-j,\ 0,\ 1+j\,]$

example:

$X[k] = [4 , -1+j7.33 , -1-j3.36 , -1+j3.36 , -1-j7.33]$

what is $G[k]$ if $g[n] = x[\langle n-12\rangle_5]$.

Solution:

$x[\langle n-12\rangle_5] = x[\langle n-2\rangle_5]$

$G[k] = X[k]\, W_5^{2k}$; $0 \le k \le 4$

[redacted]

→ $G[k] = [4 , 5.118-j5.34 , 2.88-j1.98 , 2.88+j1.98 , 5.11+j5.34]$

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example:

The 134-point DFT for x[n] is zero except for the following K if x[n] is real, then determine the constants:

① K_1, K_2, K_3, K_4, α, β, ε, δ, γ

② The DC value of $x[n]$

③ The energy of $x[n]$

K	X[K]
0	$342 - j\alpha$
17	$4.52 + j7.91$
K_1	$-3.6 + j5.46$
K_2	$0 + j3.78$
47	$0 - j3.87$
67	$45 + j\beta$
K_3	$\gamma - j7.91$
79	$-3.6 + j5$
108	$\varepsilon + j7.56$
K_4	$-3.7 - j7.56$

solution:

→ since x[n] is real then we can apply the property:

$$X[K] = X^*[\langle -K \rangle_N]$$

→ $\boxed{K=17}$ → $X[17] = X^*[\langle -17 \rangle_{134} \rangle]$

$$X^*[17] = X[117] = 4.52 - j7.91$$

$$\boxed{K_3 = 117} \quad \& \quad \boxed{\gamma = 4.52}$$

→ $X[47] = X^*[\langle -47 \rangle_{134}]$

$$X^*[47] = X[87]$$

$$\boxed{K_2 = 87}$$

→ $\boxed{k=0}$

$X[0] = X^{*}[<134>_{134}] = X^{*}[0]$

$\alpha = 0$

→ continue to find:

① $k_1 = 55$

$k_2 = 26$

$k_3 = 117$

$\delta = 5.46$

$\gamma = 4.52$

$\varepsilon = -3.7$

$\alpha = 0$

$\beta = 0$

② DC value of $x[n]$

→ $X[0] = 3.42$

③ $\mathcal{E}_x$!!

→ using parsaval theorem:

$$\mathcal{E}_x = \frac{1}{134} \sum_{k=0}^{134} |X[k]|^2$$

$$= 3.376$$

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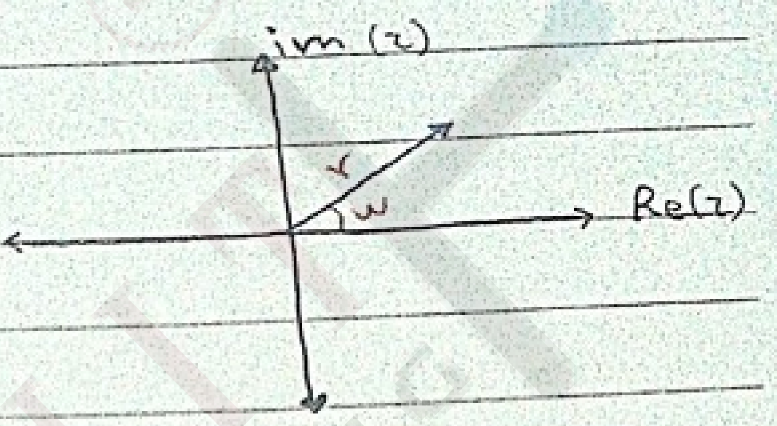
Chapter 6 : Z - Transform :

* Definition :

→ for a sequance g[n] , the Z-transform is given by:

$$\boxed{G(Z) = \sum_{n=-\infty}^{\infty} g[n]\, Z^{-n}}$$



where Z is complex variable

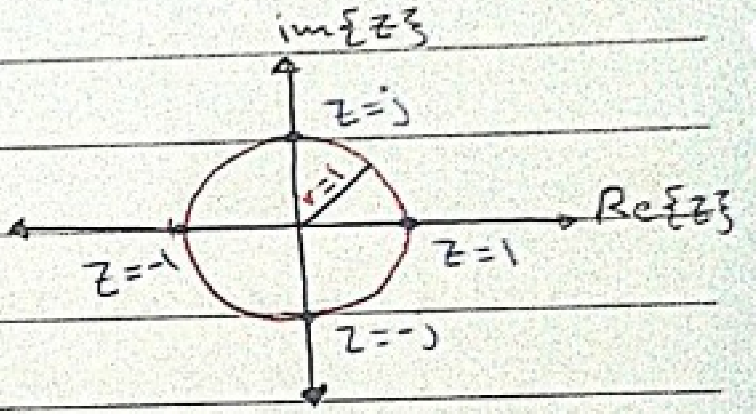
→ $Z = Re\{Z\} + j Im\{Z\}$

*→ The Z-transfor is a generelization of the DTFT?!

→ $Z = Re\{Z\} + j\, Im\{Z\}$
$= re^{jw}$

→ $G(z) = \sum_{n=-\infty}^{\infty} g[n]\, Z^{-n} = \sum_{n=-\infty}^{\infty} g[n]\, (re^{jw})^{-n}$

$G(z) = \sum_{n=-\infty}^{\infty} r^{-n}\, g[n]\, e^{-jwn}$



for $\boxed{r=1}$:

$G(z)\Big|_{Z=re^{jw}} = \sum_{n=-\infty}^{\infty} g[n]\, e^{-jwn}$ (DTFT)

⇒ So, the DTFT is essentially the Z-transform evaluated on the unit circle "r=1"

- $G(z)\Big|_{Z=1} = G(e^{j0})$
- $G(z)\Big|_{Z=j} = G(e^{j\pi/2})$

*→ For the Z-transform to exist , the following condition must hold :

$$\boxed{\sum_{n=-\infty}^{\infty} |g[n]\, Z^{-n}| < \infty}$$

So $\sum_{n=-\infty}^{\infty} |g[n](re^{jw})^{-n}| < \infty$

$$\boxed{\sum_{n=-\infty}^{\infty} |g[n]\, r^{-n}| < \infty}$$

* This implies that even if g[n] is not absolutly summable, we can make the z-transform converge by specifying "r"

* The values of "r" for which the z-transform converge called the region of convergance (ROC)

* In general, the ROC takes the form:

$$R_{g_1} \leq |Z| \leq R_{g_2}$$

where, $0 \leq R_{g_1} < R_{g_2} < \infty$

example:

$X[n] = \alpha^n u[n]$, find the ROC

solution:

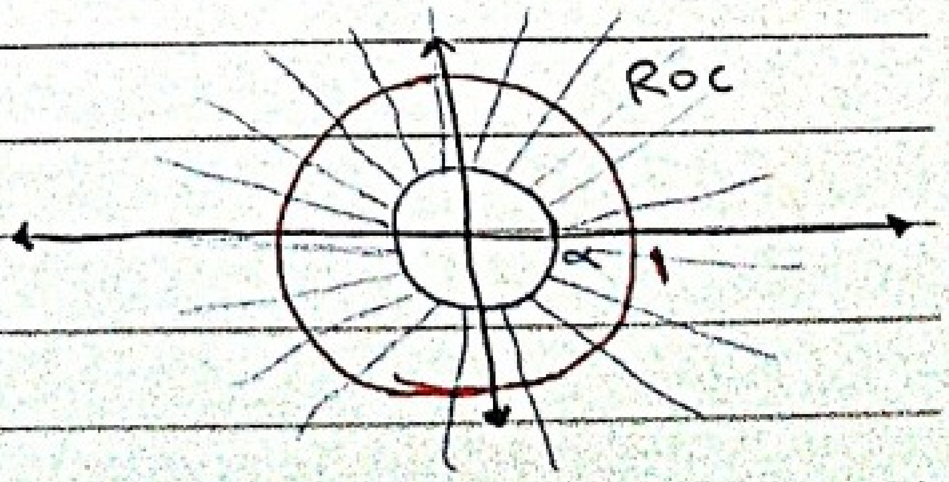
$$G(Z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] Z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n Z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha Z^{-1})^n = \frac{1}{1-\alpha Z^{-1}} , \boxed{|\alpha Z^{-1}| < 1}$$

$\rightarrow |\alpha Z^{-1}| < 1 \rightarrow |Z^{-1}| < |\alpha^{-1}| \rightarrow |Z| > |\alpha|$

$\therefore \boxed{ROC \rightarrow |Z| > |\alpha|}$



* ROC must include th unit circle, because DTFT exist

example:

$x[n] = -\alpha^n u[-n-1]$ (anti-causal), Find ROC

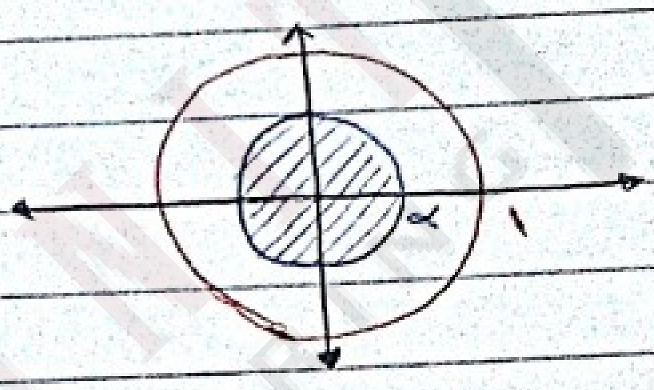
solution:

$X[z] = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$, now let $m=-n$

$= \sum_{m=1}^{\infty} -\alpha^{-m} z^{m} = \sum_{k=0}^{\infty} -\alpha^{-(k+1)} z^{(k+1)}$

$= -\alpha z^{-1} \sum_{k=0}^{\infty} (\alpha^{-1} z)^k$

$= \frac{-\alpha z^{-1}}{1-\alpha^{-1} z} = \frac{1}{1-\alpha^{-1} z}$



ROC: $|z| < \alpha$

* note that if $\alpha > 1$, then DTFT exist.

* Note that if the region of convergance of the Z-transform includes $r=1$, then DTFT exist

* Some of Z-transform pairs:

① $\delta[n] \longleftrightarrow 1$, ROC: for all z

② $u[n] \longleftrightarrow \frac{1}{1-z^{-1}}$, ROC: $|z|>1$

[redacted]

③ $\alpha^n u[n] \longleftrightarrow \frac{1}{1-\alpha z^{-1}}$, ROC: $|z|>|\alpha|$

④ $n\alpha^n u[n] \longleftrightarrow \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, ROC: $|z|>|\alpha|$

⑤ $(n+1)\alpha^n u[n] \longleftrightarrow \frac{1}{(1-\alpha z^{-1})^2}$, ROC: $|z|>|\alpha|$

⑥ $r^n \cos(w_0 n) u[n] \longleftrightarrow \frac{1-(r\cos w_0)z^{-1}}{1-(2r\cos w_0)\underset{z^{-1}}{+}r^2 z^{-2}}$, ROC: $|z|>|r|$

⑦ $r^n \sin(w_0 n) u[n] \longleftrightarrow \frac{r(\sin w_0)z^{-1}}{1-(2r\cos w_0)z^{-1}+r^2 z^{-2}}$, ROC: $|z|>|r|$

2. Rational Z-transformer

- Most LTI systems have a rational z-transform of the form.

$$H(z) = \frac{P(z)}{D(z)} = \frac{P_0 + P_1 Z^{-1} + P_2 Z^{-2} + \dots P_M Z^{-M}}{d_0 + d_1 Z^{-1} + d_2 Z^{-2} + \dots d_N Z^{-N}}$$

Z-transform

where $P(z)$ & $D(z)$ are 2 polynomials in Z^{-1} of degrees M & N respectively.

- $H(z)$ can be written as :

$$H(z) = Z^{N-M} = \frac{P_0 Z^{M} + P_1 Z^{M-1} + \dots P_M}{d_0 Z^{N} + d_1 Z^{N-1} + \dots d_N}$$

or

$$H(z) = \frac{P_0}{d_0} \frac{\prod_{\ell=1}^{M} (1 - \mathfrak{z}_\ell Z^{-1})}{\prod_{\ell=1}^{N} (1 - \lambda_\ell Z^{-1})} = \frac{P_0}{d_0} \frac{\prod_{\ell=1}^{M} (Z - \mathfrak{z}_\ell)}{\prod_{\ell=1}^{N} (Z - \lambda_\ell)} Z^{N-M}$$

- when $Z = \mathfrak{z}_\ell$, $H(z) = 0 \Rightarrow \mathfrak{z}_\ell$ are called the zeros of $H(z)$

- when $Z = \lambda_\ell$, $H(z) = \infty \Rightarrow \lambda_\ell$ are called the poles of $H(z)$

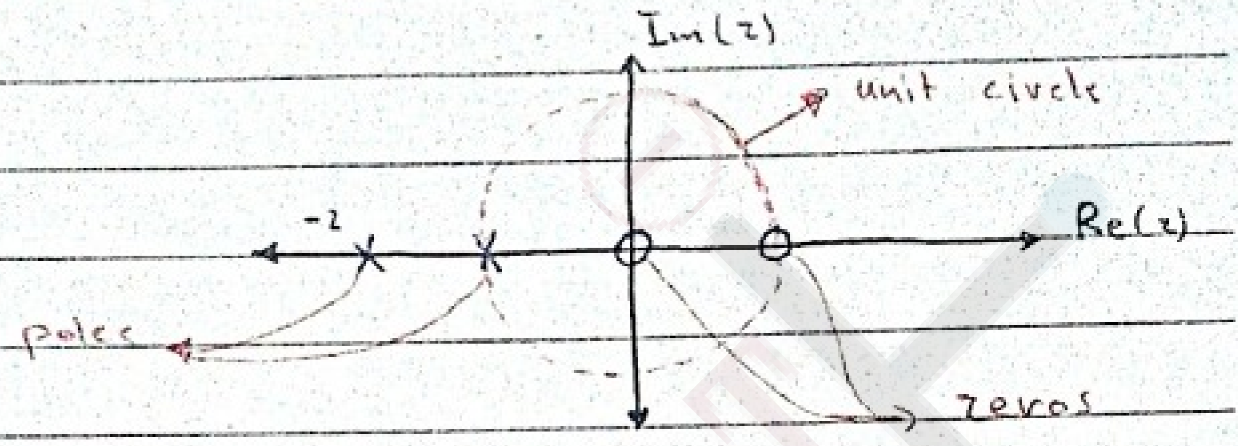
* if $N > M$, this implies that there are additional $N-M$ zeros at $Z = \infty$

* if $N < M$, this implies that there are additional $M-N$ poles at $Z = 0$

example:

→ $H(z) = Z \cdot \frac{Z-1}{Z^2+3Z+2}$, find it's poles & zeros.

solution:



→ the zeros are at:

$\boxed{Z=0}$, $\boxed{Z=1}$

→ the poles are at:

$\boxed{Z=-2}$, $\boxed{Z=-1}$ "roots of Denomenator"

example

→ $H(z) = \frac{1}{1-Z^{-1}}$, $Roc = |Z|>1$ $\xrightarrow{}$ $= \frac{Z}{1-Z}$ بعد الترتيب

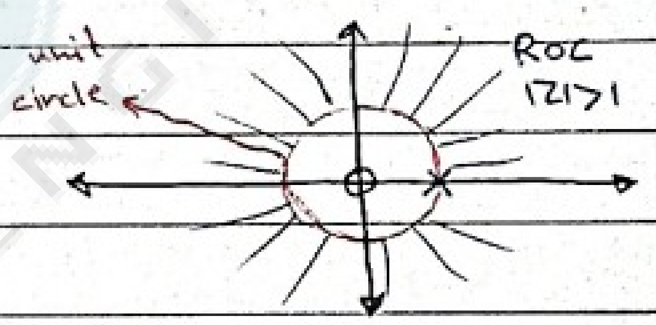
solution:

→ the zeros are at:

$\boxed{Z=0}$

→ the poles are at:

$\boxed{Z=1}$



دائماً في الـ Roc نستثني

الـ poles لأنه عند الـ poles

الـ Z-transform ⇐ "doesn't exist"

example:

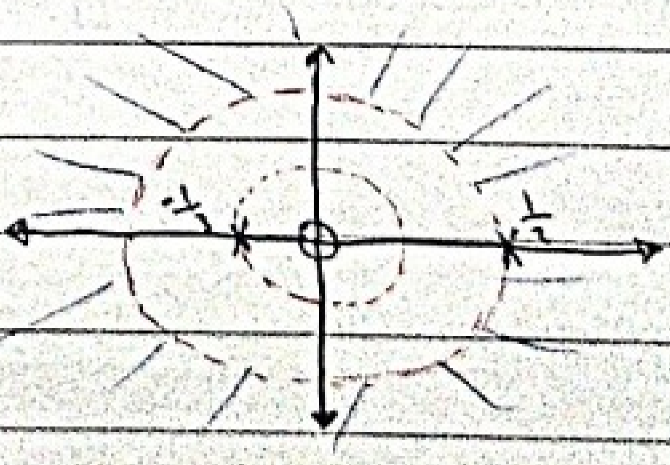
$X[n] = (1/2)^n u[n] + (-1/3)^n u[n]$, find the Roc of it's Z-trans.

solution:

→ $X(z) = \frac{1}{1-\frac{1}{2}Z^{-1}} + \frac{1}{1+\frac{1}{3}Z^{-1}}$

$\lambda = \frac{1}{2} \leftarrow 1-\frac{1}{2}Z^{-1}$; $1+\frac{1}{3}Z^{-1} \rightarrow \lambda = -\frac{1}{3}$

↓ $Roc1 : |Z| > \frac{1}{2}$ ↓ $Roc2 : |Z| > \frac{1}{3}$



→ Roc of the whole X(z) should not include any pole so Roc1 & Roc2 should be satisfied → $\boxed{Roc = Roc1 \cap Roc2 = |Z| > \frac{1}{2}}$

Notes:

① for a rational Z-transform it can be completely defined using the poles, zeros, & the gain $K = \frac{p_0}{d_0}$

② the region of convergence of a rational Z-transform should not includ any poles

example

$$H(z) = \frac{Z^2 + 2Z - 3}{Z^3 - 3Z^2 + Z + 5}$$

determine all possible regions of convergence

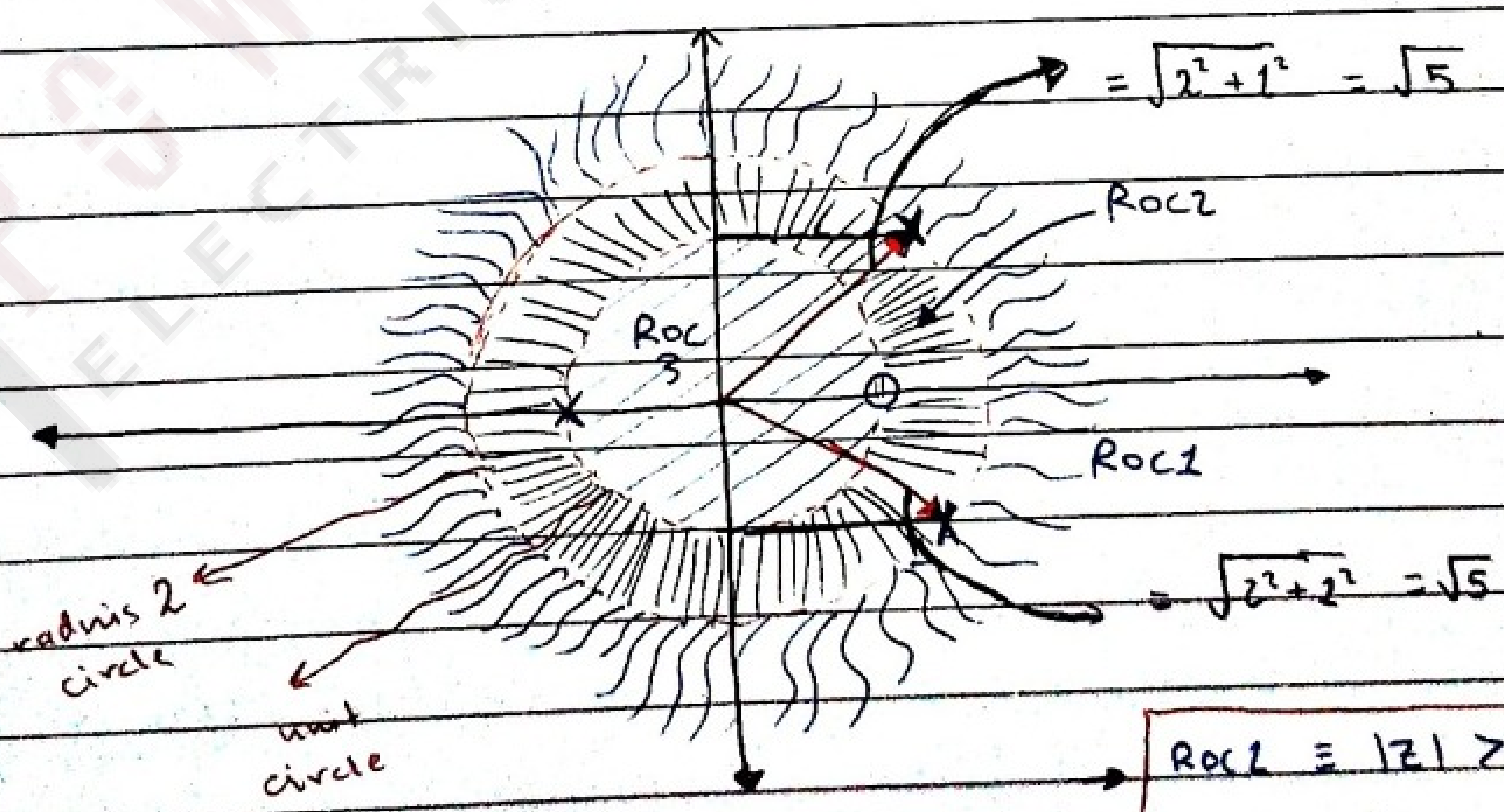
solution

→ zeros are at:

$\boxed{Z = -3}$, $\boxed{Z = 1}$, $\boxed{Z = \infty}$

→ poles are at:

$\boxed{Z = -1}$, $\boxed{Z = 2 + j}$, $\boxed{Z = 2 - j}$



Roc1 ≡ $|Z| > \sqrt{5}$

Roc2 ≡ $1 < |Z| < \sqrt{5}$

Roc3 ≡ $|Z| < 1$

3. The inverse z-transform

→ $g[n] = \frac{1}{2\pi j} \oint_{C} G(z)\, z^{n-1}\, dz$

→ colsed region in complex plane

→ this is complex integration, alternatively, we can use look up table or/and partial fractions

① Using look up table:

example

$X(z) = \frac{0.5Z}{Z^2 - Z + 0.25} \quad |Z| > 0.5$

find $X[n]$

note:
لما ROC: $|Z| > C$ حيث C = constant لما
يكون causal
ولما $Z < C$ يكون anti-causal

solution

$X(z) = \frac{0.5Z}{Z^2(1 - Z^{-1} + 0.25Z^{-2})} = \frac{0.5Z^{-1}}{1 - Z^{-1} + 0.25Z^{-2}} = \frac{0.5Z^{-1}}{(1 - 0.5Z^{-1})^2} \quad |Z| > 0.5$

من ال table

→ $X[n] = n(0.5)^n u[n]$

② Using partial fractions:

→ A rational z-transform can be expressed as:

$G(z) = \frac{P(z)}{d(z)}$, where $P(z)$ & $d(z)$ are polynomials with degrees M & N respectively

→ Now, if $M < N$, we can write $G(z)$ as:

$G(z) = \sum_{i=1}^{N} \frac{\rho_i}{1 - \lambda_i Z^{-1}}$, where ρ_i are called the residence of $G(z)$ & λ_i are the poles

→ the residence ρ_i are computing using:

$$\boxed{\rho_i = (1 - \lambda_i Z^{-1})\, G(z)\Big|_{z = \lambda_i}}$$

example

$H(z) = \frac{Z(Z+2)}{(Z-0.2)(Z+0.6)}$ for a causal sequance

$\rightarrow$ Roc = $|Z| > 0.6$

find $h[n]$.

لو ما حددلي بدي اخذ كل الاحتمالات

solution:

$$H(z) = \frac{Z^2(1+2Z^{-1})}{Z^2(1-0.2Z^{-1})(1+0.6Z^{-1})}$$

$$= \frac{P_1}{(1-0.2Z^{-1})} + \frac{P_2}{(1+0.6Z^{-1})}$$

$$\rightarrow P_1 = (1-0.2Z^{-1})\,H(z)\Big|_{Z=0.2} = \frac{1+2Z^{-1}}{1+0.6Z^{-1}}\Big|_{Z=0.2} = 2.75$$

$$\rightarrow P_2 = (1+0.6Z^{-1})\,H(z)\Big|_{Z=-0.6} = \frac{1+2Z^{-1}}{1-0.2Z^{-1}}\Big|_{Z=-0.6} = -1.75$$

now, $H(z) = \underset{\text{Roc: } |Z|>0.2}{\frac{2.75}{1-0.2Z^{-1}}} + \underset{\text{Roc: } |Z|>0.6}{\frac{-1.75}{1+0.6Z^{-1}}}$

$$h[n] = 2.75(0.2)^n u[n] + -1.75(0.6)^n u[n]$$

- for a proper fraction with multiple poles, the fraction can be written as:

$$G(z) = \sum_{l=1}^{N-L} \frac{P_l}{1-\lambda_l Z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1-rZ^{-1})^i}$$

where r is a pole with L multiplicity, & γ_i are the associated residues & computed by

$$\gamma_i = \frac{1}{(L-i)!\,(-r)^{L-i}} \cdot \frac{d^{L-i}}{d(Z^{-1})^{L-i}}\left(1-rZ^{-1}G(z)\right)\Big|_{Z=r} , \; 1 \le i \le L$$

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Example

$G(z) = \frac{z}{(z-0.5)(z-1)^2}$

find $g[n]$

* لاحظ انه السؤال مش محددلي causal ولا anti-causal لذلك ال ROC يجب أن اخذ جميع الاحتمالات الممكنه

solution:

by partial fractions:

$$G(z) = \frac{4}{1-0.5z^{-1}} + \frac{-6}{1-z^{-1}} + \frac{2}{(1-z^{-1})^2}$$

$\downarrow$ $\downarrow$ $\downarrow$

ROC: $|z|>0.5$ ROC: $|z|>1$ ROC: $|z|>1$ → causal region

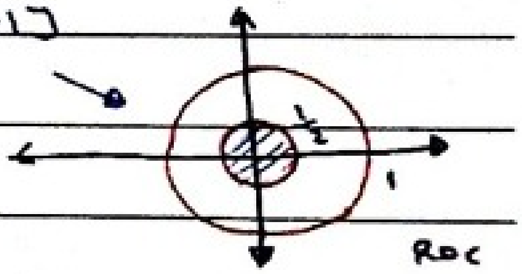
ROC: $|z<0.5$ ROC: $|z|<1$ ROC: $|z|<1$ → anti-causal region

① if we take $|z|>0.5$ & $|z|>1$ "causal"

→ $g[n] : 4(0.5)^n u[n] - 6u[n] + 2u[n]$ " this represent causal system "

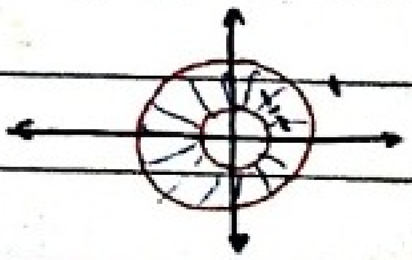
② if we take $|z|<0.5$ & $z<1$ " anti-causal"

→ $g[n] = -4(0.5)^n u[n-1] + 6u[-n-1] - 2(n)u[-n-1]$



③ if $|z|>0.5$ & $z<1$ "two sided"

→ $g[n] = \underbrace{4(0.5)^n u[n]}_{\text{causal part}} \underbrace{+6u[-n-1] - 2(n)u[-n-1]}_{\text{anti-causal part}}$ "



④ if $|z|<0.5$ & $z>1$

→ $ROC_1 \cap ROC_2 = \phi$ " we cant define such system"

example

$$X(z) = \frac{1/3\, Z^{-3} + 1/6\, Z^{-2} + 3 Z^{-1} + 1}{1/6\, Z^{-2} + 5/6\, Z^{-1} + 1}$$

find $x[n]$.

solution

→ بقسمة البسط على المقام نجد → $X(z) = 1 + 2Z^{-1} + \underbrace{\frac{1/6\, Z^{-1}}{1/6 Z^{-2} + 5/6 Z^{-1} + 1}}_{\tilde{X}(z)}$

على

→ $\tilde{X}(z) = \frac{1/6\, z^{-1}}{1/6 z^{-2} + 5/6 z^{-1} + 1} = \frac{1/6\, z^{-1}}{1 + 5/6 z^{-1} + 1/6 z^{-2}}$

$= \frac{1/6\, z^{-1}}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{A}{(1+\frac{1}{2}z^{-1})} + \frac{B}{(1+\frac{1}{3}z^{-1})}$

- find A, B " كما في المثال السابق " → $\boxed{A=-1}$, $\boxed{B=1}$

now, $X(z) = \underbrace{1} + \underbrace{2\, Z^{-1}} + \frac{-1}{(1+\frac{1}{2}z^{-1})} + \frac{1}{(1+\frac{1}{3}z^{-1})}$

ROC: all Z

ROC: all Z exept zero

ROC: $|z| > \frac{1}{2}$ if causal, $|z| < \frac{1}{2}$ if anti-causal

ROC: $|z| > \frac{1}{3}$ if causal, $|z| < \frac{1}{3}$ if anti-causal

now,

① take $|z| > 0$ & $|z| > \frac{1}{2}$ & $|z| < \frac{1}{3}$

→ $ROC_1 \cap ROC_2 \cap ROC_3 \cap ROC_4 = \phi$

② $|z| > 0$ & $|z| < \frac{1}{2}$ & $|z| > \frac{1}{3}$ $\equiv$ $\frac{1}{3} < |z| < \frac{1}{2}$ "2-sided"

$g[n] = \delta[n] + 2\delta[n-1] + \underbrace{(1/2)^n u[-n-1]}_{\text{anti-causal part}} + \underbrace{(-1/3)^n u[n]}_{\text{causal part}}$

جرب باقي الحالات ;

* when you try to take $|z| > 0$, & $|z| < \frac{1}{2}$ & $|z| < \frac{1}{3}$ → تعني anti-causal

point circle ← $0 < |z| < \frac{1}{3}$ → it's 2-sided

4. Z-transform Theorems

if $g[n] \xleftrightarrow{Z} G(z)$, ROC: R_g

$h[n] \xleftrightarrow{Z} H(z)$, ROC R_h

Theorems:

① Time reversal:

$g[-n] \xleftrightarrow{Z} G(1/Z)$, ROC: $1/R_g$ → "مثلا" $|z| < \alpha$ تصبح $|z| < \frac{1}{\alpha} \Rightarrow |z| > \alpha$

② Time shift:

$g[n-n_o] \xleftrightarrow{Z} G(z)\, z^{-n_o}$, ROC: R_g excluding possibly $(z=0)$, or $(z=\infty)$,

if n_o +ve   if n_o -ve

③ linearity:

$\alpha g[n] + \beta h[n] \longleftrightarrow \alpha G(z) + \beta H(z)$, ROC: $R_g \cap R_h$

example

$\alpha X[n] + \beta X[n-1] = C\delta[n] + D\delta[n-1]$

find $X(z)$

solution

take Z-transform for both sides:

$\alpha X(z) + \beta X(z) z^{-1} = C + D z^{-1}$

$$X(z) = \frac{C + D z^{-1}}{\alpha + \beta z^{-1}}$$ " zeros: $z = -d/c$ poles: $z = -\beta/\alpha$ "

④ Multiplication by exponential sequance:

$\alpha^n g[n] \xleftrightarrow{Z} G(z/\alpha)$, ROC: $|\alpha| R_g$

⑤ Diffirentiation:

$n g[n] \xleftrightarrow{Z} -z \frac{d G(z)}{dz}$, ROC: R_g excluding possibly $z=0$ or $z=\infty$

example

$x[n] = (n+1)\,\alpha^n\,u[n]$

find $X(z)$

solution

take Z-transform for both sides

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) + \underbrace{\frac{1}{1-\alpha z^{-1}}}_{ROC: |z|>\alpha}$$

$$\rightarrow \frac{z}{z-\alpha}$$

$$= \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} + \frac{1}{1-\alpha z^{-1}}$$

$$= \frac{1}{(1-\alpha z^{-1})^2}, \quad \underline{|z|>\alpha}$$

⑥ convolution:

$$g[n] \circledast h[n] \xleftrightarrow{z} G(z)H(z), \quad ROC: R_g \cap R_h$$

example:

$$\left.\begin{aligned} G(z) &= \frac{2+1.2z^{-1}}{1-0.2z^{-1}}, \quad ROC1: |z|>0.2 \\ H(z) &= \frac{3}{1+0.6z^{-1}}, \quad ROC2: |z|>0.6 \end{aligned}\right\} \begin{aligned} &ROC = ROC1 \cap ROC2: \\ &|z|>0.6 \end{aligned}$$

what is $y[n] = g[n] \circledast h[n]$

solution:

$$\rightarrow Y(z) = G(z) \cdot H(z)$$

$$\left.= \frac{2\,\cancel{+1.2}\,z^{-1}}{1-0.2z^{-1}} \cdot \frac{3}{\cancel{1+0.6z^{-1}}}\right\} \text{pole-zero cancellation}$$

(2 written above the cancelled numerator)

$$= \frac{6}{1-0.2z^{-1}}, \quad |z|>0.2$$

example

$x[n] = \{0, 3, 4, 5\}$, $0 \leq n \leq 3$

$h[n] = \{2, 7, 1\}$, $0 \leq n \leq 2$

find $y[n] = x[n] \circledast h[n]$

solution

$$\rightarrow Y(z) = X(z) \cdot H(z)$$

$$= (3z^{-1} + 4z^{-2} + 5z^{-3})(2 + 7z^{-1} + z^{-2})$$

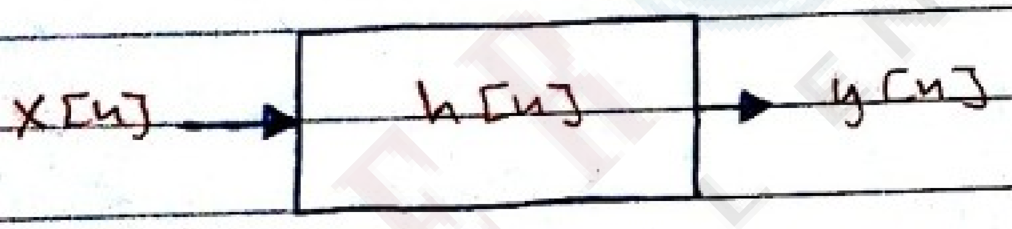
$$= 6z^{-1} + 29z^{-2} + 41z^{-3} + 39z^{-4} + 5z^{-5}$$

$$\rightarrow y[n] = \{\underset{\uparrow}{0}, 6, 29, 4, 39, 5\} \quad , \; 0 \leq n \leq 5$$

or

$$y[n] = \{6, 29, 4, 39, 5\} \quad , \; 1 \leq n \leq 5$$

5. LTI systems and the z-transform :



$$y[n] = x[n] \circledast h[n] = \sum_k x[k]\, h[n-k]$$

z-transform

$$Y(z) = \sum_n \Big(\sum_k x[k]\, h[n-k] \Big) z^{-n}$$

$$= \sum_k x[k] \Big(\sum_n h[n-k]\, z^{-n} \Big)$$

shift

$$= \sum_k x[k]\, H(z)\, \underbrace{z^{-k}}$$

$$= H(z) \sum_k x[k]\, z^{-k}$$

$$\boxed{Y(z) = H(z)\, X(z)} \rightarrow \text{Transfer fun.} = \frac{Y(z)}{X(z)}$$

* For an FIR LTI system, $h[n]$, $N_1 \leq n \leq N_2$, the transfer function is:

$$H(z) = \sum_{n=N_1}^{N_2} h[n] z^{-n}$$

if $h[n]$ is causal "$N_1 \geq 0$", then:

$$H(z) = \sum_{n=0}^{N_2} h[n] z^{-n}$$

$$= h[0] z^{0} + h[1] z^{-1} + h[2] z^{-2} + \dots$$

$\rightarrow$ all poles are at zero

then, the ROC is the entire Z-plane except zero

example

Draw the pole-zero diagram for the system represented by the following difference equation:

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$

solution:

Consider the Moving Avg filter:

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$ "not recursive & FIR!"

Z transform both sides

$$Y(z) = \frac{1}{M} \sum_{l=0}^{M-1} X(z) z^{-l}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{M} \sum_{l=0}^{M-1} z^{-l} = \left(\frac{1}{M} \frac{1-z^{-M}}{1-z^{-1}}\right) = \frac{1}{M} \frac{(z^{M}-1)}{(z-1)z^{M-1}}$$

"rational"

Cont. $\longrightarrow$ next page

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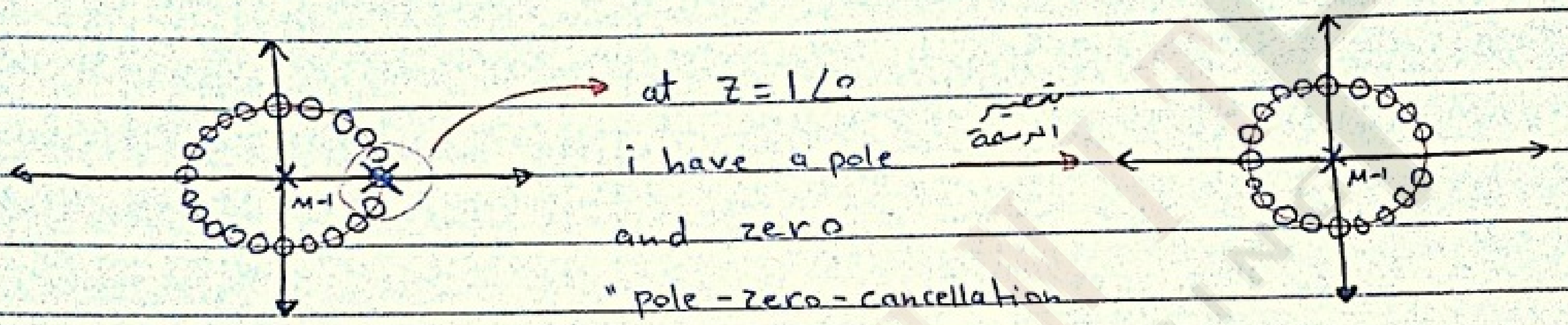
→ poles: $\boxed{z=1}$, $\boxed{z=0}$ → we have M-1 poles at zero

→ zeros:

$z^M - 1 = 0$

$|z| e^{jw} = 1 e^{j2\pi}$

$|z| = 1$ at $w = \frac{2\pi k}{M}$ "M-zeros" at the unit circle on diffirent angles.



at $z = 1\angle 0$
i have a pole
and zero
"pole-zero-cancellation

* Notes

① for a causal FIR system, the ROC is entire z-plane except at $z=0$

② for an IIR LTI system, it can be described by:

$$\sum_{k=0}^{N} d_k \, y[n-k] = \sum_{k=0}^{N} P_k \, x[n-k]$$

z-transform

$$\sum d_k \, Y(z) \, z^{-k} = \sum P_k \, X(z) \, z^{-k}$$

so, $$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} P_k \, z^{-k}}{\sum_{k=0}^{N} d_k \, z^{-k}}}$$

* if the system is causal, then ROC is exterior of the circle passing through the furthast pole from the origin:

ROC: $|z| > \max\{\overset{\text{pole}}{\lambda_k}\}$

* Frequancy response and transfer function :

→ $H(e^{jw}) = H(z)\Big|_{z=e^{jw}}$, Given that ROC includes the unit circle

↳ "unit vector → magnitud = 1 "

→ For a rational Z-transform : $H(z) = \frac{P_o}{d_o} Z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - \lambda_k)}$

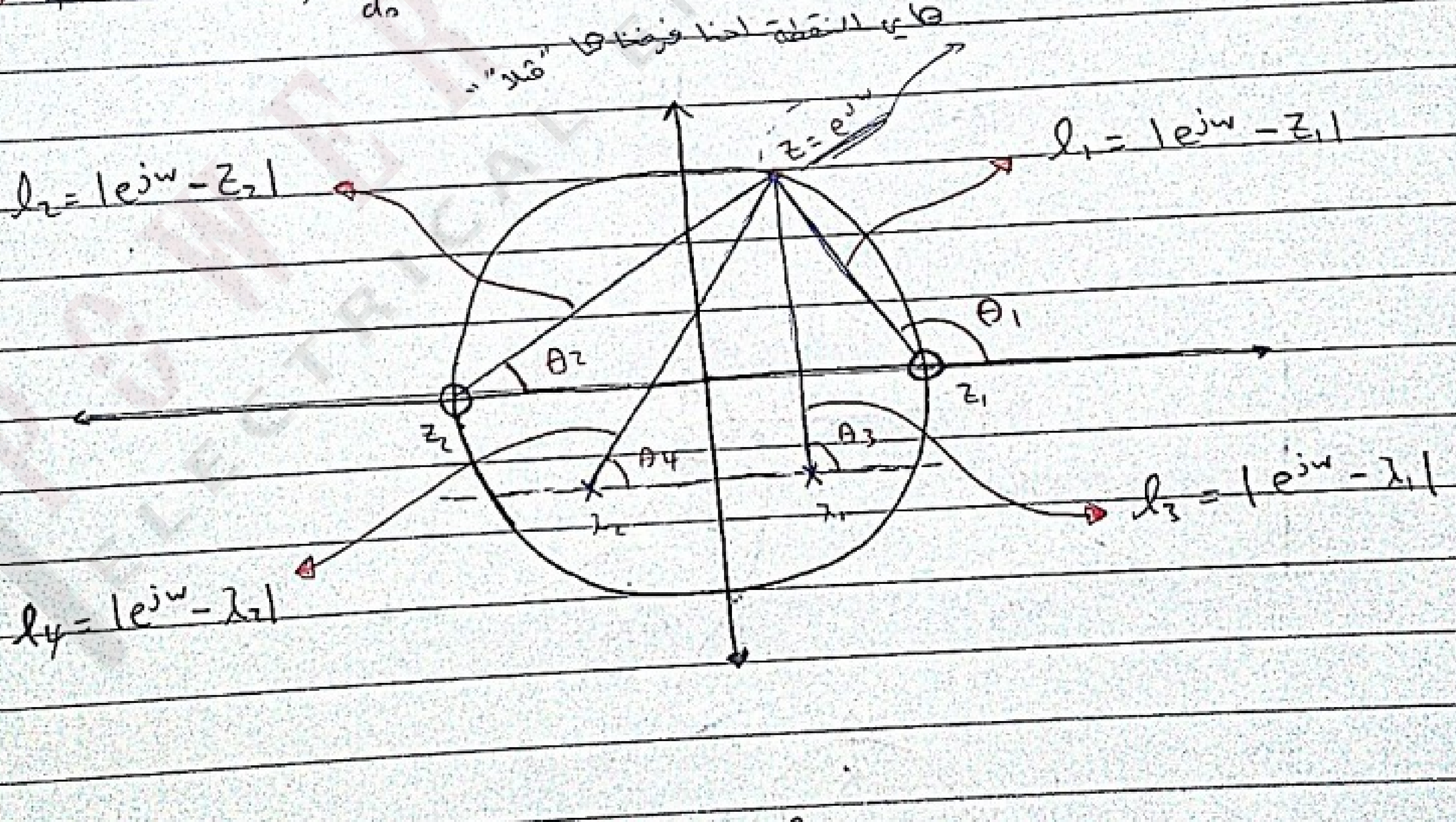
& $H(e^{jw}) = H(z)\Big|_{z=e^{jw}}$ so:

$$H(e^{jw}) = \frac{P_o}{d_o} e^{jw(N-M)} \frac{\prod (e^{jw} - z_k)}{\prod (e^{jw} - \lambda_k)}$$

→ $|H(e^{jw})| = \left|\frac{P_o}{d_o}\right| \cdot \left|\cancel{e^{jw(N-M)}}\right| \cdot \left|\frac{\prod (e^{jw} - z_k)}{\prod (e^{jw} - \lambda_k)}\right|$

$$= \left|\frac{P_o}{d_o}\right| \frac{\prod |e^{jw} - z_k|}{\prod |e^{jw} - \lambda_k|}$$

→ $\measuredangle H(e^{jw}) = \measuredangle \frac{P_o}{d_o} + w(N-M) + \sum \measuredangle (e^{jw} - z_k) - \sum \measuredangle (e^{jw} - \lambda_k)$

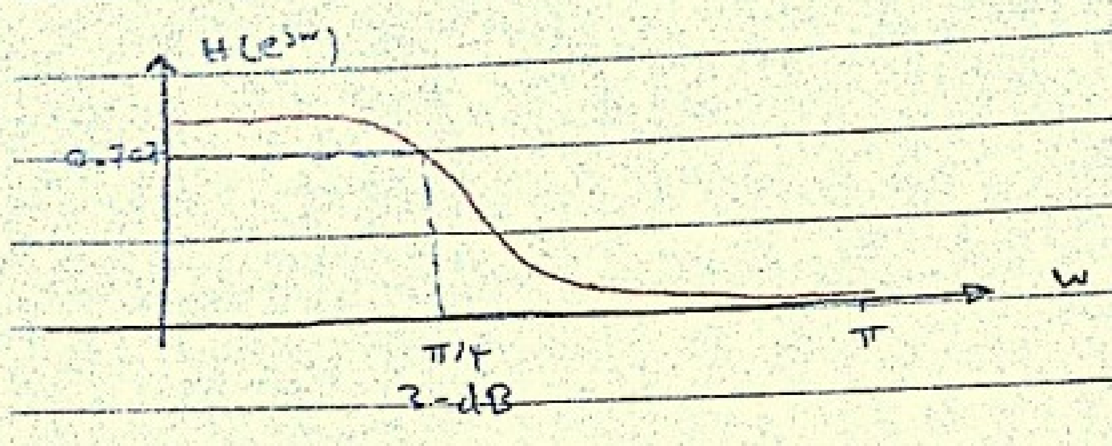


→ $|H(e^{jw})| = \frac{P_o}{d_o} = \frac{l_1 \cdot l_2}{l_3 \cdot l_4}$

→ $\measuredangle H(e^{jw}) = \frac{\angle P_o}{d_o} + w(\cancel{N-M}) + (\theta_1 + \theta_2) - (\theta_3 + \theta_4)$

example:

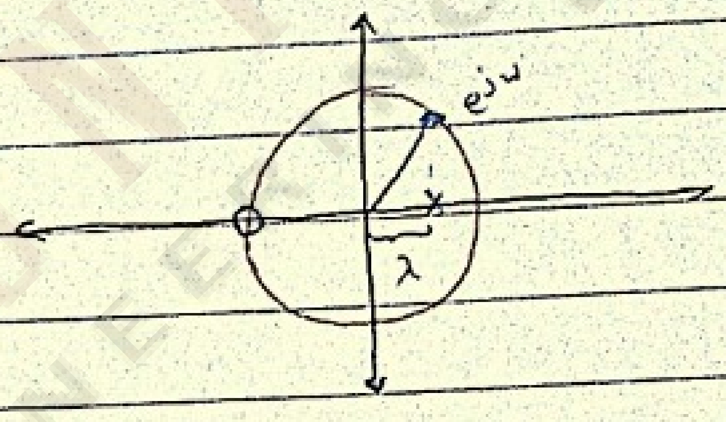
low pass filter:



find H(z).

solution:

$$H(z) = K \frac{\Pi (zeros)}{\Pi (poles)}$$ بدي اياه صفر تقريبي



→ at $w=0$ → الـ Magnitude عالي

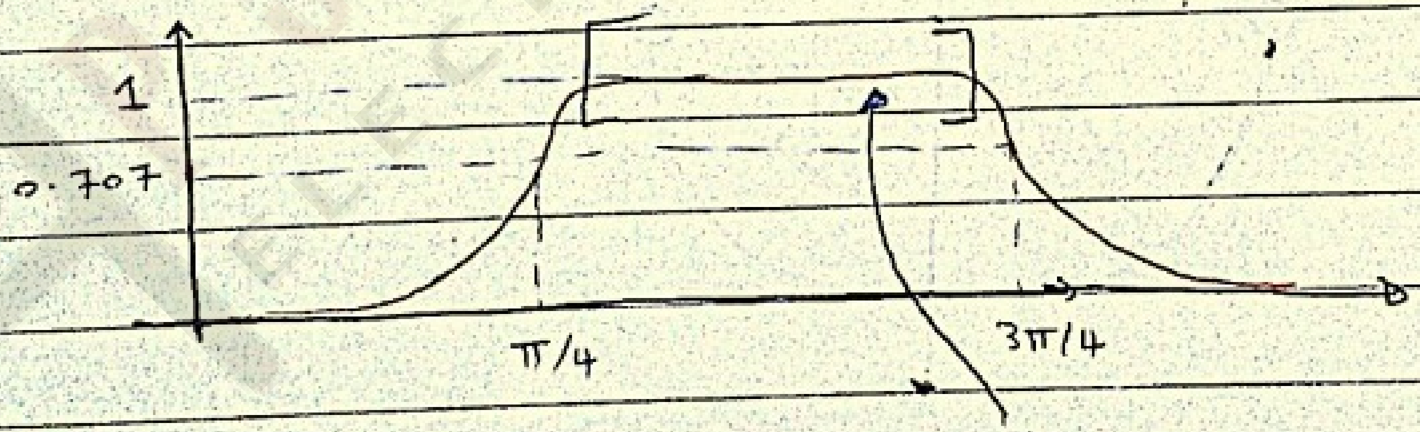
لأنه المقام صفر "pole"

→ at $w=\pi$ → الـ Magnitude صفر

لأنه البسط صفر "zero"

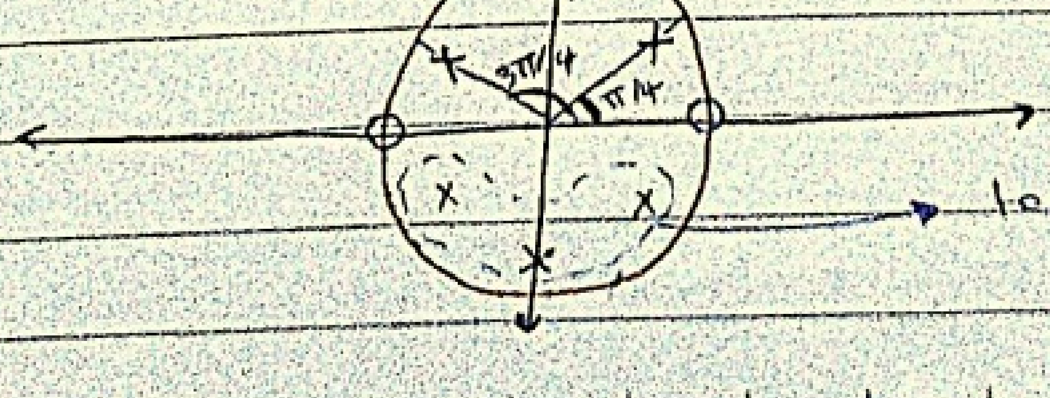
$$\text{so, } H(z) = K \frac{z+1}{z-\lambda}, \quad \underline{|H(e^{j0})| = 1}$$

→ now, for band-pass filter



مشان أضمن هاي المنطقة

sol:



→ to achieve real system

لكل pole الها conjugate

* Stability bases on transfer function :

→ For an LTI system, it's BIBO stable if

$$\rho = \sum |h[n]| < \infty$$

$$|H(z)| = |\sum h[n] z^{-n}| < \sum |h[n] z^{-n}|$$

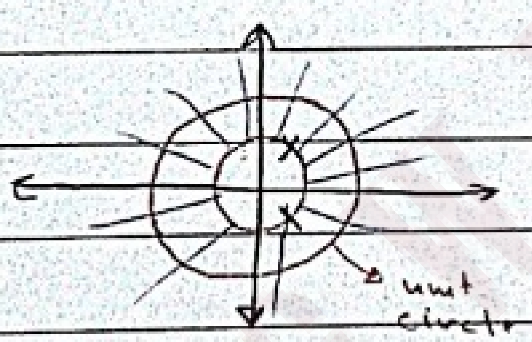
at the unit circle :

$$|H(z)|\Big|_{z=e^{j\omega}} = |H(e^{j\omega})| \leq |h[n] e^{-j\omega n}|$$

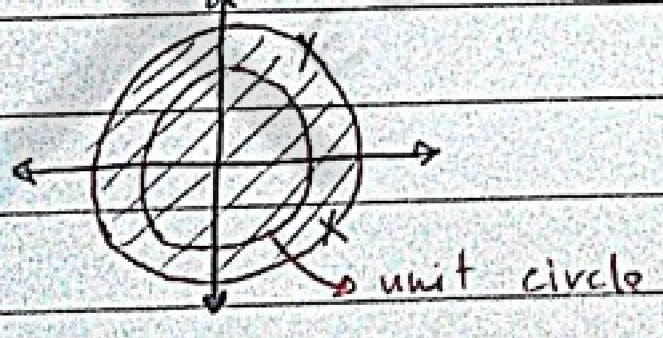
$$= |H(e^{j\omega})| \leq \sum |h[n]| < \infty$$

⇒ this implies that an LTI system is stable if the ROC includes the unit circle :

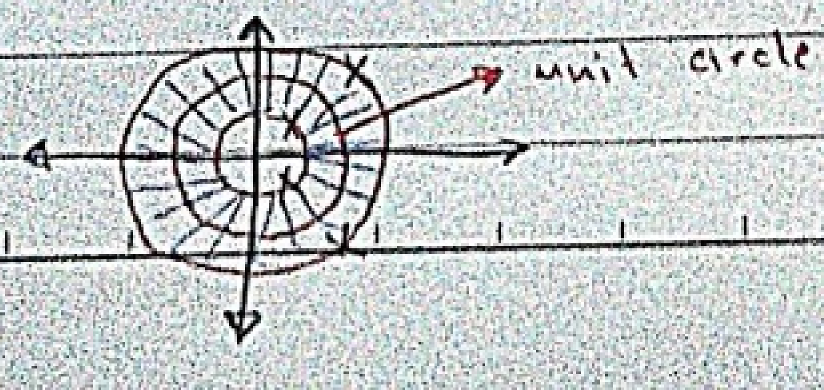
① if it's causal ⇒ all poles inside unit circle



② if it's not causal ⇒ all poles outside unit circle

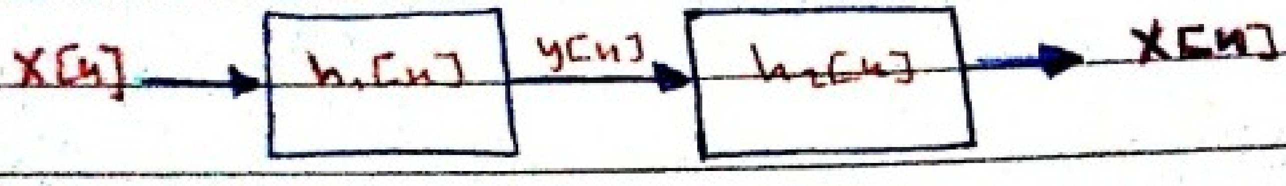


③ if it's double sided ⇒ poles should be inside & outside unit circle



lecture 19

* Inverse system :



$$y[n] = x[n] \circledast h_1[n]$$
$$x[n] = y[n] \circledast h_2[n]$$
$$= (x[n] \circledast h_1[n]) \circledast h_2[n]$$

$$X(z) = X(z)\ H_1(z)\ H_2(z)$$
$$\rightarrow H_1(z) \cdot H_2(z) = 1 \longrightarrow \boxed{H_2(z) = \frac{1}{H_1(z)}}$$

example

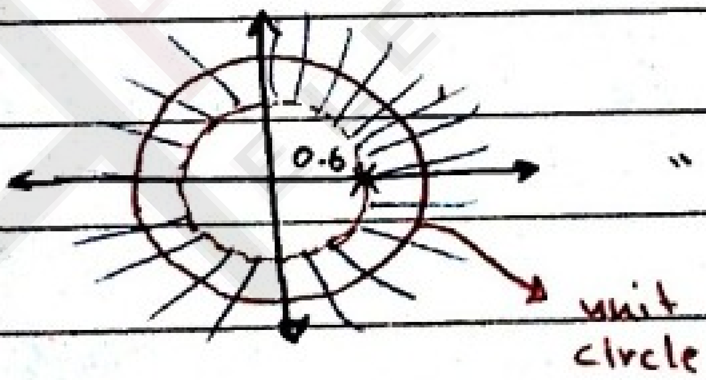
$$H(z) = \frac{(z+0.2)(z-0.6)}{(z-0.3)(z+0.5)} \quad , |z| > 0.5$$

a) find an inverse causal and stable system

b) find an anti-causal and stable inverse system

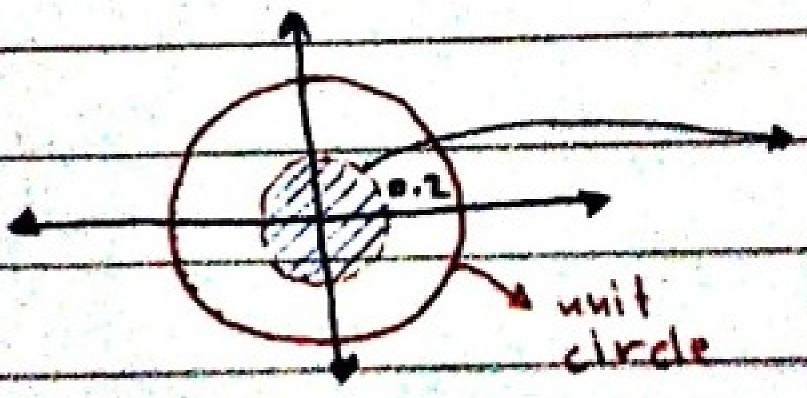
solution

a) $G(z) = \frac{1}{H(z)} = \frac{(z-0.3)(z+0.5)}{(z+0.2)(z-0.6)}$, Roc: $|z| > 0.6$



" causal & stable "

b) cant be defined !



" dont include the unit circle to be stable "

example:

$h_1[n] = 1.9\,\delta[n] + 0.5\,(-0.2)^n\,u[n] - 0.6\,(0.7)^n\,u[n]$

→ find an inverse causal & stable system in time domain "$h_2[n]$".

solution

$$H_1(z) = 1.9 + \frac{0.5}{1+0.2z^{-1}} - \frac{0.6}{1-0.7z^{-1}}$$

$$H_1(z) = \frac{1.8 - 1.4z^{-1} - 0.266z^{-2}}{1 - 0.5z^{-1} - 0.14z^{-2}}$$

$$\rightarrow\; G(z) = H_2(z) = \frac{1}{H_1(z)} = \frac{1 - 0.5z^{-1} - 0.14z^{-2}}{1.8 - 1.4z^{-1} - 0.266z^{-2}}$$

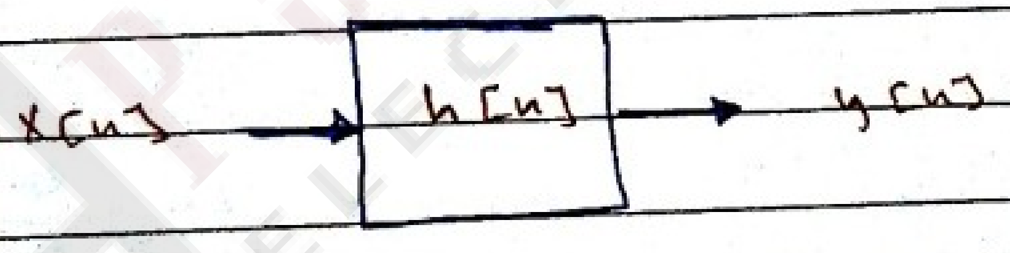
$$= \frac{(1-0.7z^{-1})(1+0.2z^{-1})}{1.8(1-0.9z^{-1})(1+0.1563z^{-1})}$$

$$G(z) = 0.5263 + \frac{0.149}{1-0.945z^{-1}} - \frac{0.1206}{1+0.1563z^{-1}},\quad |z| > 0.945$$

~~inverse z-transform~~

$g[n] = 0.5263\,\delta[n] + (0.945)^n(0.149)\,u[n] - (0.1563)^n(0.1206)\,u[n$

* example on Deconvolution:



if $y[n] = \{\underset{\uparrow}{6}, 10, 3, -2, 5, -6\}$

& $h[n] = \{2, 4, 1, 3\}$

what is $x[n]$?

cont. → next page

Solution

$y[n] = x[n] \quad h[n]$

Z trans

$Y(z) = X(z)\,H(z)$

So $X(z) = \frac{Y(z)}{H(z)} = \frac{6 + 10z^{-1} + 3z^{-2} + (-2)z^{-3} + 5z^{-4} + (-6)z^{-5}}{2 + 4z^{-1} + z^{-2} + (-3)z^{-3}}$

$X(z) = \frac{z^{-5}}{z^{-3}} \cdot \frac{6z^5 + 10z^4 + 3z^3 + (-2)z^2 + 5z + (-6)}{2z^3 + 4z^2 + z + (-3)}$

$X(z) = z^{-2}(3z^2 - z + 2)$

$X(z) = 3 - z^{-1} + 2z^{-2}$

inverse Z transform

$x[n] = 3\delta[n] - \delta[n-1] + 2\delta[n-2]$