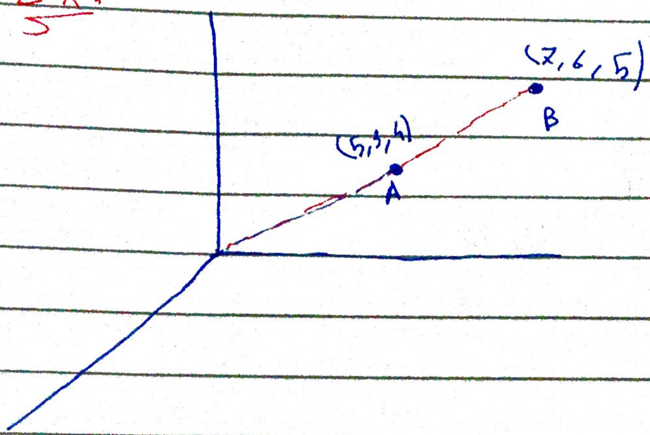
CH9 / Vector Diff Calculas

Def: position Vector from $\vec{A} = \langle a_1, a_2, a_3 \rangle$ to $B = (b_1, b_2, b_3)$

$$\overline{AB} = B - A = (\underbrace{B_1 - A_1}_{\hat{i}}, \underbrace{B_2 - A_2}_{\hat{j}}, \underbrace{B_3 - A_3}_{\hat{k}})$$

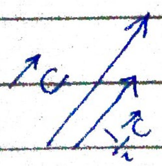
Ex:



$\overline{AB} = (5, 3, 4)$

Def: $\vec{U}$ & $\vec{V}$ are parallel if $\vec{U} = c\vec{V}$ for some scal
$c \in \mathbb{R}$

- if $c > 0$, u & V in the same Direction
- if $c < 0$, " " " " " opposit



$\boxed{1}$ addition vector

$\vec{U} = (2, 1, 0)$, $\vec{V} = (3, 4, 5)$, $\vec{W} = (2, 1, 3)$

$\vec{U} + \vec{V} + \vec{W} = (7, 6, 12)$

2 Dot product $(\bar{U} . \bar{V})$

Def: if $\vec{U} = (u_1, u_2, u_3)$ & $\vec{V} = (V_1, V_2, V_3)$ are Vector in $\mathbb{R}^3$ then

$$\bar{U} . \bar{V} = U_1 V_1 + U_2 V_2 + U_3 V_3$$

$$\bar{U} . \bar{U} = U_1^2 + U_2^2 + U_3^2$$

$$\left. \| U \| = \sqrt{\bar{U} . \bar{U}} \right\}$$ is the Norm of $\bar{U}$

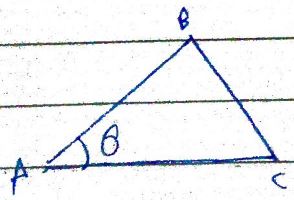
Propeties :

1) $\bar{U} . \bar{V} = \bar{V} . \bar{U}$
2) $| \bar{U} . \bar{V} | \leq \| \vec{U} \| \, \| \vec{V} \|$
3) $\vec{U} . (\vec{V} + \vec{W}) = \bar{U} . \bar{V} + \bar{U} . \bar{W}$
4) $\vec{u} . \vec{V} = \| u \| \, \| \vec{V} \| \cos \theta$

$\bar{U} \perp \bar{V}$ when $\bar{U} . \bar{V} = 0$

$\| \bar{u} \| = 0$ $\quad \bar{U} = 0$ Zero vector

Ex: Find the angle θ



$\overline{AB} = (-1, -4, -1) \Rightarrow \| \overline{AB} \| = \sqrt{18}$

$\overline{AC} = (5, -2, -1) \Rightarrow \| \overline{AC} \| = \sqrt{30}$

$$\overrightarrow{AB} . \overrightarrow{AC} = -5 + 8 + 1 = 4$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{30}\,\sqrt{18}} \right)$$

5 cross product

$\vec{U} = (u_1, u_2, u_3)$

$\bar{U} = (u_1, 0, 0) + (0, u_2, 0) + (0, 0, u_3)$

$= u_1 (1, 0, 0) + \bar{u}_2 (0, 1, 0) + u_3 (0, 0, 1)$

$= u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$

$\vec{U}$ is called a unit Vector if $\|\vec{u}\| = 1$

the cross product between $\vec{U}$ & $\vec{V}$ denoted by $\bar{U} \times \bar{V}$

$$U \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix}$$

properties:

1. $\bar{u} \times \bar{u} = 0$
2. $\bar{V} \times \bar{u} = -\bar{U} \times \bar{V}$
3. $U \times (\bar{V} + \bar{W}) = U \times V + \bar{V} \times \bar{W}$
4. $\|\bar{u} \times \bar{v}\| = \|u\| \, \|v\| \sin\theta$

$\vec{U} \parallel \vec{V}$ when $\bar{u} \times \bar{V} = 0$

⇒ scaler function $f = f(p)$ where p is a point in space

Ex: euclidean distance is a scaler function

$$f(p) = f(x,y,z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Vector function $\bar{V} = \bar{V}(\bar{p}) = (V_1(p), V_2(p), V_3(p))$

→ Differantial Geometry

$$\bar{r}(t) = (x(t), y(t), z(t)) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

① ⇒ stright line

$$\bar{r}(t) = \bar{a} + \bar{b}t = (a_1 + b_1 t,\ a_2 + b_2 t,\ a_3 + b_3 t)$$

② ⇒ ellips circle

$$\bar{r}(t) = (x(t), y(t), 0) \leftarrow \text{in } xy \text{ plane}$$

$$= a\cos(t)\,\hat{i} + b\sin(t)\,\hat{j}$$

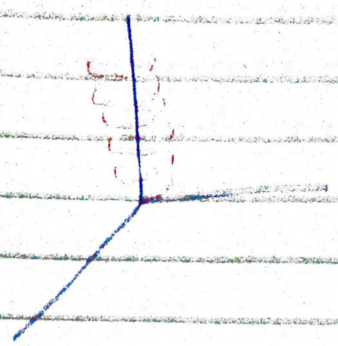
$$\|\bar{r}(t)\|^2 = x^2 + y^2$$

where $x = a\cos(t)\,\hat{i}$, $y = b\sin(t)\,\hat{j}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

④ Circular helix

$\vec{r}(t) = (a\cos t \,,\, b\sin t \,,\, ct)$



Def: if c is a curve given by $\vec{r}(t)$ then:

[1] tangent Vector of c is $r'(t)$

[2] unit tangent Vector of c is $\dfrac{r'(t)}{\|r'(t)\|}$

Ex: find UTV & TV for helix

$\vec{r}(t) = (a\cos(t) \,,\, b\sin(t) \,,\, ct)$

$r'(t) = (-a\sin(t) \,,\, b\cos(t) \,,\, c) \leftarrow$ T.V

$\|r'(t)\| = \sqrt{a^2+c^2}$

$\dfrac{\vec{r}'(t)}{\|r'(t)\|} = \dfrac{1}{\sqrt{a^2+c^2}}\left(-a\sin(t) \,,\, a\cos(t) \,,\, c\right) \leftarrow$ U.T.V

Partial derivative & chain rule

Theorem: let $w = f(x, y, z)$ be continuous & have cts first partial derivative in a (cts) domian xyz-space let $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ be functions that are cts & have first partial

$w = f(x(u,v), y(u,v), z(u,v))$ and the partial derivative are

$$\frac{dw}{du} = \frac{dw}{dx}\frac{dx}{dw} + \frac{dw}{dy}\frac{dy}{du} + \frac{dw}{dz}\frac{dz}{du}$$

Special case: partial derivative on a surface $z = g(x,y)$
let $w = f(x, y, z)$ and let $z = g(x, y)$ represent a surface S in a space then function becoms $w = f(x, y, g(x,y))$ and the partial derivative are

$$\frac{dw}{dx} = \frac{df}{dx} + \frac{df}{dz}\frac{dz}{dx}$$

$$\frac{dw}{dy} = \frac{df}{dy} + \frac{df}{dz}\frac{dz}{dy}$$

Ex: if $f = x^3 + y^3 + z^3$ & let $z = g = x^2 + y^2$
find $\frac{df}{dx}$

$$\frac{df}{dx} = \frac{df}{dx} + \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= 3x^2 + 3z^2 (2x)$$

$$= 3x^2 + 3(x^2 + y^2)^2 (2x)$$

→ Gradiant

DEF: let $f(x,y,z)$ scaler function defined and diff in a domain in 3-space the gradient of f is

$$\bar{\nabla} f = \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right)$$

$$\nabla = \frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{j}$$

Ex: find the gradient $f(x,y,z) = 4(x^2 + y^2) - z^2$

$$\bar{\nabla} f = (8x, 8y, -2z)$$

DEF: if $\bar{F} = \bar{\nabla} f$ then f is called a potential function for F

Ex: if $\nabla f(x,y,z) = 2x\tan^{-1}(y^3+3z^2)\hat{i} + \frac{3x^2 y^2}{1+(y^3+3z^2)^2}\hat{j} + \frac{6zx^2}{1+(y^3+3z^2)^2}\hat{k}$

We Know that

$$\bar{\nabla} f = \frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j} + \frac{df}{dz}\hat{k}$$

$$\frac{df}{dx} = 2x\tan^{-1}(y^3+3z^2)\hat{i} \quad --- \boxed{1}$$

$$\frac{df}{dy} = \frac{3x^2 y^2}{1+(y^3+3z^2)^2} \quad \boxed{2} \qquad \frac{df}{dz} = \frac{3x^2y^2}{1+(y^2+3z^2)^2} \quad --- \boxed{3}$$

from 1

$$f(x,y,z) = \int 2x\tan^{-1}(y^3+3z^2)\,dx = x^2\tan^{-1}(y^3+3z^2) + g(y,z) \quad \boxed{4}$$

$$\frac{df(x,y,z)}{dy} = \frac{x^2}{1+(y^3+3z^2)^2}\cdot 3y^2 = \frac{3y^2x^2}{1+(y^3+3z^2)^2} + \frac{dg}{dy}$$

$$\frac{dg}{dy} = 0 \quad \& \quad \frac{dg}{dz} = 0$$

So, $f(x,y,z) = x^2 + \tan^{-1}(y^3+3z^2) + c$

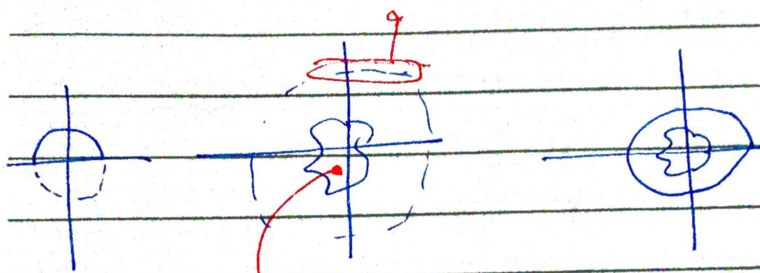
→ theorem :

$\bar{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$ is a gradient of a scaler function if

$$\frac{dP}{dy} = \frac{dQ}{dx} \quad \text{for all } x \,\&\, y$$

given that P & Q are both continiously differentiable on a simply connected open region ($\bar{F}$ is ~~const~~ conservative)

يعني ما الها حدود



not closed or open

open region
simply connected

closed region

$\bar{F}$ gradient for f

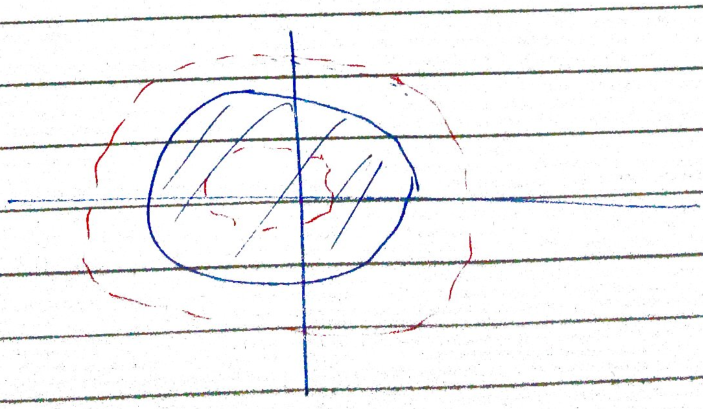
$\bar{F} = \bar{\nabla} f$ if

$$\frac{df}{dx} = P \qquad \frac{d^2 f}{dx\,dy} = \frac{dP}{dy}$$

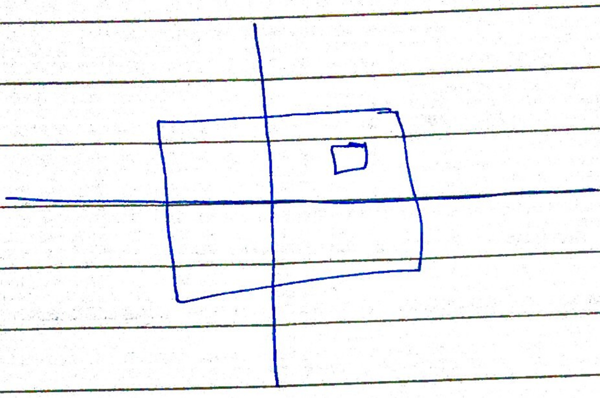
$$\frac{df}{dy} = Q \qquad \frac{d^2 f}{dx\,dy} = \frac{dQ}{dy}$$

حتى يكون simpl connected لازم لو أرسم curve مغلقة جواته يكون
ماخد كامل المنطقة

open region
not simply
connected



DEF: region Ω is said to be simply connected if for every closed curve c in Ω the inner region of c is containd in Ω



$\leftarrow$ not simply connected

Ex: the Vector function $\bar{F}(x,y) = 2x\sin y\, \hat{i} + x^2 \cos y\, \hat{j}$ is conservative or not

$$\frac{d(2x\sin y)}{dy} \stackrel{?}{=} \frac{d(x^2\cos(y))}{dx} \quad \checkmark$$

conservative

Ex: 1) $G(x,y) = xy\,\hat{i} + \frac{1}{2}(x+1)^2 y^2\,\hat{j}$ conservative or not

$$\frac{d(xy)}{dx} \stackrel{?}{=} \frac{d\left(\frac{1}{2}(x+1)^2 y^2\right)}{dy} \quad \times$$

Not conservative

2) $F(x,y) = \dfrac{y}{x^2+y^2}\,\hat{i} - \dfrac{x}{x^2+y^2}\,\hat{j}$

Not conservative